



Article

A Model in Which Well-Orderings of the Reals Appear at a Given Projective Level

Vladimir Kanovei ^{*,†}  and Vassily Lyubetsky ^{*,†} 

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute),
127051 Moscow, Russia

* Correspondence: kanovei@iitp.ru (V.K.); lyubetsk@iitp.ru (V.L.)

† These authors contributed equally to this work.

Abstract: The problem of the existence of analytically definable well-orderings at a given level of the projective hierarchy is considered. This problem is important as a part of the general problem of the study of the projective hierarchy in the ongoing development of descriptive set theory. We make use of a finite support product of the Jensen-type forcing notions to define a model of set theory **ZFC** in which, for a given $n > 2$, there exists a good Δ_n^1 well-ordering of the reals but there are no such well-orderings in the class Δ_{n-1}^1 . Therefore the existence of a well-ordering of the reals at a certain level $n > 2$ of the projective hierarchy does not imply the existence of such a well-ordering at the previous level $n - 1$. This is a new result in such a generality (with $n > 2$ arbitrary), and it may lead to further progress in studies of the projective hierarchy.

Keywords: well-orderings; projective hierarchy; forcing; finite support product; generic models; definability

MSC: 03E35; 03E15



Citation: Kanovei, V.; Lyubetsky, V. A Model in Which Well-Orderings of the Reals Appear at a Given Projective Level. *Axioms* **2022**, *11*, 354. <https://doi.org/10.3390/axioms11080354>

Academic Editors: Andrei Alexandru, Andreja Tepavčević, Bogdan Aman and Gabriel Ciobanu

Received: 23 May 2022

Accepted: 18 July 2022

Published: 22 July 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The problem of the well-orderability of the continuum of real numbers \mathbb{R} has been known in set theory since the time of Cantor and Hilbert. Zermelo's axiom of choice **AC** directly postulates the existence of a well-ordering of \mathbb{R} (and of any other set of course), but this is far from an effective construction of a concrete, "nameable" well-ordering of \mathbb{R} . We refer to the famous "Sinq Lettres" [1] in matters of the discussion on these issues in early set theory.

Somewhat later, using the methods of the descriptive set theory that just emerged, it was established that no well-ordering \preceq of \mathbb{R} belong to the first-level projective classes Σ_1^1 , Π_1^1 —and then to Δ_1^1 since $x \preceq y$ iff $x = y$ or $y \not\preceq x$. This is an easy consequence of Luzin's theorem [2] that sets in $\Sigma_1^1 \cup \Pi_1^1$ are Lebesgue measurable; see, for example, Sierpinski [3]. (We use the modern notation Σ_n^1 , Π_n^1 , Δ_n^1 for projective classes and Σ_n^1 , Π_n^1 , Δ_n^1 for their effective subclasses, sometimes also called "lightface"; see, for example, monographs [4,5].)

The next key result was obtained by Gödel [6]: it is true that in the Gödel constructible universe \mathbf{L} , there exists a Δ_2^1 well-ordering $\leq_{\mathbf{L}}$ of the reals, or saying it differently, the existence of a Δ_2^1 well-ordering of the reals is a consequence of the axiom of constructibility $\mathbf{V} = \mathbf{L}$. It follows that the existence of a Δ_2^1 well-ordering of the reals is consistent with the axioms of the Zermelo–Fraenkel set theory **ZFC** (containing the axiom of choice **AC**) because the axiom of constructibility $\mathbf{V} = \mathbf{L}$ itself is consistent by [6].

Addison [7] singled out an important additional property of the Gödel well-ordering $\leq_{\mathbf{L}}$. Namely, let a Δ_n^1 -good well-ordering is any Δ_n^1 well-ordering \preceq such that for every binary Δ_n^1 relation $P(x, y)$ on the reals, the relations

$$Q(x, y) := \exists x' \preceq x P(x', y) \quad \text{and} \quad R(x, y) := \forall x' \preceq x P(x', y)$$

belong to Δ_n^1 as well, so that the class Δ_n^1 is closed under \preceq -bounded quantification (see Moschovakis [5]). In these terms, the Gödel–Addison result says that \leq_L is a Δ_2^1 -good well-ordering of the reals in L , and hence the existence of such a well-ordering follows from $V = L$ and is consistent with ZFC. The property of Δ_n^1 -goodness of \leq_L is behind many key results on projective sets in Gödel’s universe L , see Section 5A in [5].

In the opposite direction, it was established in the early years of modern set theory (see, for example, Levy [8] and Solovay [9]) that the statement of the non-existence of a well-ordering of the reals of any projective class is consistent as well.

Recent studies on projective well-orderings explore various topics concentrated around the general problem formulated by Moschovakis [5] (Introduction) as follows:

[T]he central problem of descriptive set theory and definability theory in general [is] to find and study the characteristic properties of definable objects.

For instance, it is established in [10] that the bounded proper forcing axiom BPFA combined with $\omega_1 = \omega_1^L$ implies the existence of a Δ_3^1 well-ordering of the reals. Studies in [11–13] presented different constructions of countable support-iterated generic models which, first, admit controlled cardinal characteristics of the continuum, and second, admit a Δ_3^1 well-ordering of the reals. A model of ZFC in which the nonstationary ideal on ω_1 is \aleph_1 -saturated and whose reals admit a Δ_4^1 well-ordering, is defined in [14] under a large-cardinal hypothesis. A finite support product of clones of Jensen’s minimal singleton forcing [15] is used in [16] to define a model in which any non-empty analytically definable set of reals contains an analytically definable real (the full basis theorem), but there is no analytically definable well-ordering of the reals of any class Δ_n^1 .

However one of principal questions related to projective well-orderings remained unsolved by those studies. We let $\mathbf{WO}(\Delta_n^1)$, respectively, $\mathbf{WO}(\Delta_{n-1}^1)$ be the following statement:

there is a well-ordering of the reals which, as a set of pairs, belongs to, respectively, $\Delta_n^1, \Delta_{n-1}^1$,

for the sake of brevity. As the strict inclusions

$$\Delta_{n-1}^1 \subsetneq \Delta_n^1 \quad \text{and} \quad \Delta_{n-1}^1 \subsetneq \Delta_n^1 \tag{1}$$

hold for all $n \geq 2$, we have accordingly

$$\mathbf{WO}(\Delta_{n-1}^1) \implies \mathbf{WO}(\Delta_n^1) \quad \text{and} \quad \mathbf{WO}(\Delta_{n-1}^1) \implies \mathbf{WO}(\Delta_n^1), \tag{2}$$

and the ensuing principal problem is as follows.

Problem 1. *Are implications (2) irreversible in ZFC, similar to inclusions (1)?*

In other words, for a given $n \geq 3$, are there models of ZFC in which $\mathbf{WO}(\Delta_n^1)$ holds but $\mathbf{WO}(\Delta_{n-1}^1)$ fails, as well as those in which $\mathbf{WO}(\Delta_n^1)$ holds but $\mathbf{WO}(\Delta_{n-1}^1)$ fails?

This problem is a version of a well-known problem posed by S. D. Friedman, one of the leading experts in set theory, in [17] (Problem 11 on page 209) and [18] (Problem 9 in Section 9). Friedman’s problem asks for a model for $\mathbf{WO}(\Delta_n^1)$ plus the Lebesgue measurability and the Baire property of all Σ_{n-1}^1 sets of reals, which is somewhat stronger than the related requirements of the failure of $\mathbf{WO}(\Delta_{n-1}^1)$ and $\mathbf{WO}(\Delta_{n-1}^1)$ in Problem 1.

The following theorem (our main result here) contributes to the studies of these problems. It gives a partial positive solution of the “lightface” part of Problem 1 that uniformly works for all values of the index. No such result has ever been obtained before.

Theorem 1. *Let $n \geq 3$. There exists a generic extension of L , in which it is true that*

- (i) *there is a Δ_n^1 -good well-ordering of the reals, of length ω_1 ;*
- (ii) *there are no Δ_{n-1}^1 -good well-orderings of the reals.*

2. Outline of the Proof

Given $n \geq 3$, our plan is to make use of a generic extension of \mathbf{L} defined in [19] in order to get a model where the separation principle fails for both classes Σ_n^1 and Π_n^1 . This extension utilizes a sequence of forcing notions $\mathbb{P}(\xi)$, $\xi < \omega_1$, defined in \mathbf{L} so that the finite-support product $\mathbb{P} = \prod_{\xi} \mathbb{P}(\xi)$ satisfies CCC and adjoins a sequence of generic reals $x_{\xi} \in 2^{\omega}$, satisfying the following crucial definability property: the binary relation “ $x \in 2^{\omega}$ is a real \mathbb{P}_{ξ} -generic over \mathbf{L} ” (with arguments x, ξ) is Π_{n-1}^1 in $\mathbf{L}[G] = \mathbf{L}[\langle x_{\xi} \rangle_{\xi < \omega_1}]$. This will suffice to define a well-ordering satisfying Theorem 1(i).

On the other hand, Claim (ii) of Theorem 1 involves another crucial property: the \mathbb{P} -forcing relation of Σ_n^1 formulas is equivalent to an auxiliary forcing relation *forc* invariant w.r.t. permutations of indices $\xi < \omega_1$.

Each factor forcing $\mathbb{P}(\xi)$ consists of *perfect trees* in $2^{<\omega}$ and is a clone of Jensen’s minimal forcing defined in [15]; see also [20] (28A) on this forcing. The technique of finite-support products of Jensen’s forcing, which we owe to Enayat [21], was exploited recently to obtain generic models with counterexamples to the separation theorem for both Σ_3^1 and Π_3^1 [22], and some counterexamples to the axiom of choice [23], to name a few applications.

Section 3 introduces perfect trees in $2^{<\omega}$, arboreal forcing notions, multitrees (finite products of trees), multiforcings (countable products of arboreal forcing notions).

Section 4 defines the refinement relation and presents the principal properties of refinements. We define the set $\overrightarrow{\mathbf{MF}}$ of all countable sequences $\vec{\pi}$ of small multiforcings, increasing in the sense of the refinement relation.

Then, following our earlier paper [19], we introduce the key forcing notion $\mathbb{P} = \mathbb{P}_n$ for Theorem 1 with a fixed $n \geq 3$, and study the main properties of \mathbb{P} -generic models in Section 5. Theorem 2 in Section 6 shows that condition (i) of Theorem 1 holds in \mathbb{P} -generic extensions of \mathbf{L} .

Sections 7 and 8 introduce an auxiliary forcing relation *forc*, which approximates the truth in \mathbb{P} -generic extensions for Σ_{n-1}^1 -formulas and below, so that the relation *forc* restricted to any class Σ_m^1 or Π_m^1 , $m \geq 2$, is Σ_m^1 , respectively, Π_m^1 itself. The tail invariance and permutation invariance of the relation *forc* is established in Section 9. (We may note in brackets that the product forcing notion \mathbb{P} itself is not permutation invariant.)

Using these results, we finally prove that condition (ii) of Theorem 1 holds in \mathbb{P} -generic extensions of \mathbf{L} in Section 10. This completes the proof of Theorem 1.

This paper is a sequel of [19] in many technical details, and hence some intermediate results involved in the proof of Theorem 1 are taken from [19] without proof.

3. Arboreal Forcing Notations and Multiforcings

Let $2^{<\omega}$ be the set of all tuples (finite sequences) of numbers 0, 1. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that t extends s , while $s \subset t$ means proper extension. If $t \in 2^{<\omega}$ then $\text{lh}(t)$ is the length of t , and $2^n = \{t \in 2^{<\omega} : \text{lh}(t) = n\}$ (tuples of length n).

PT is the set of all *perfect trees* $\emptyset \neq T \subseteq 2^{<\omega}$. Thus a tree $\emptyset \neq T \subseteq 2^{<\omega}$ belongs to **PT** if it has no endpoints and no isolated branches. In this case,

$$[T] = \{a \in 2^{\omega} : \forall n (a \upharpoonright n \in T)\} \subseteq 2^{\omega}.$$

is a perfect set. If $s \in T \in \mathbf{PT}$ then put $T \upharpoonright_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$; then $T \upharpoonright_s \in \mathbf{PT}$.

Let an *arboreal forcing* be any set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $u \in T \in \mathbb{P}$ then $T \upharpoonright_u \in \mathbb{P}$. Let **AF** be the set of all arboreal forcings \mathbb{P} .

A forcing $\mathbb{P} \in \mathbf{AF}$ is *special*, if there is a finite or countable antichain $\mathbb{A} \subseteq \mathbb{P}$ such that $\mathbb{P} = \{T \upharpoonright_s : s \in T \in \mathbb{A}\}$ —the antichain \mathbb{A} is unique and \mathbb{P} is countable in this case.

Let a **multiforcing** be any map $\pi : |\pi| \rightarrow \mathbf{AF}$, where $|\pi| = \text{dom } \pi \subseteq \omega_1$. Let **MF** be the collection of all multiforcings. Every $\pi \in \mathbf{MF}$ can be presented as an indexed set $\pi = \langle \mathbb{P}_{\xi} \rangle_{\xi \in |\pi|}$, where $\mathbb{P}_{\xi} \in \mathbf{AF}$ for all $\xi \in |\pi|$, so that each component $\mathbb{P}_{\xi} = \mathbb{P}_{\xi}^{\pi} = \pi(\xi)$, $\xi \in |\pi|$, is an arboreal forcing.

Accordingly, let a **multitree** be any function $p : |p| \rightarrow \mathbf{PT}$, with a finite support $|p| = \text{dom } p$; \mathbf{MT} will be the collection of all multitrees. Every $p \in \mathbf{MT}$ can be seen as an indexed set $p = \langle T_\xi^p \rangle_{\xi \in |p|}$, where $T_\xi^p = p(\xi) \in \mathbf{PT}$ for all $\xi \in |p|$. We order \mathbf{MT} componentwise: $q \leq p$ (q is stronger than p) if $|p| \subseteq |q|$ and $T_\xi^q \subseteq T_\xi^p$ for all $\xi \in |p|$.

Assume that $\pi = \langle \mathbb{P}_\xi \rangle_{\xi \in |\pi|}$ is a multiforcing. Let a π -multitree be any multitree $p \in \mathbf{MT}$ such that $|p| \subseteq |\pi|$, and if $\xi \in |p|$, then the tree $p(\xi) = T_\xi^p$ belongs to \mathbb{P}_ξ . The set $\mathbf{MT}(\pi)$ of all π -multitrees can be identified with the finite support product $\prod_{\xi \in |\pi|} \mathbb{P}_\xi$ of the arboreal forcings \mathbb{P}_ξ involved.

Any arboreal forcing $\mathbb{P} \in \mathbf{AF}$ is considered a forcing notion (if $T \subseteq T'$, then T is a stronger condition); such a forcing \mathbb{P} adjoins a real in 2^ω .

Accordingly, any forcing notion of the form $\mathbf{MT}(\pi)$, where $\pi = \langle \mathbb{P}_\xi \rangle_{\xi \in |\pi|} \in \mathbf{MF}$, adds a generic sequence $\langle x_\xi \rangle_{\xi \in |\pi|}$, where each $x_\xi = x_\xi[G] \in 2^\omega$ is a \mathbb{P}_ξ -generic real. Reals of the form $x_\xi[G]$ are called *principal generic reals* in $\mathbf{V}[G]$.

4. Refinements and Increasing Sequences of Multiforcings

Here we present an important notion of refinement \sqsubset and a construction of \sqsubset -increasing sequences of multiforcings.

Recall that if $\mathbb{P} \subseteq \mathbb{R} \subseteq \mathbf{PT}$ then the set \mathbb{P} is *dense* in \mathbb{R} if $\forall T \in \mathbb{R} \exists S \in \mathbb{P} (S \subseteq T)$.

The following definition introduces a relation of refinement between arboreal forcings. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{AF}$ be arboreal forcings. Say that \mathbb{Q} is a *refinement* of \mathbb{P} (symbolically $\mathbb{P} \sqsubset \mathbb{Q}$) if

- (1) the set \mathbb{Q} is dense in $\mathbb{P} \cup \mathbb{Q}$, so that if $T \in \mathbb{P}$ then $\exists Q \in \mathbb{Q} (Q \subseteq T)$;
- (2) if $Q \in \mathbb{Q}$ then there is a finite set $D \subseteq \mathbb{P}$ such that $T \subseteq \bigcup D$, or equivalently $[T] \subseteq \bigcup_{S \in D} [S]$;
- (3) if $Q \in \mathbb{Q}$ and $T \in \mathbb{P}$ then $[Q] \cap [T]$ is clopen in $[Q]$ and $T \not\subseteq Q$.

Let π, φ be multiforcings. Say that φ is a *refinement* of π , symbolically $\pi \sqsubset \varphi$, if $|\pi| \subseteq |\varphi|$ and $\pi(\xi) \sqsubset \varphi(\xi)$ in \mathbf{AF} for all $\xi \in |\pi|$.

Remark 1. The relations \sqsubset and \sqsubset are strict partial orders on sets, respectively, \mathbf{AF} , \mathbf{MF} ; see Lemma 5.2 and Corollary 6.1 in [19]. We can also note that if π, φ are multiforcings and $|\pi| \subseteq |\varphi|$, then $\pi \sqsubset \varphi$ is equivalent to $\pi \sqsubset (\varphi \upharpoonright |\pi|)$.

Recall that \mathbf{MF} is the collection of all multiforcings. By [19], a multiforcing π is *small*, if both $|\pi|$ and each component forcing $\mathbb{P}_\xi^\pi = \pi(\xi)$, $\xi \in |\pi|$, are countable. A multiforcing π is *special* if each component $\pi(\xi)$ is special in the sense defined in Section 3. Let

$$\mathbf{spMF} = \{ \pi \in \mathbf{MF} : \pi \text{ is a special and small multiforcing} \}.$$

Thus a multiforcing $\pi \in \mathbf{MF}$ belongs to \mathbf{spMF} if $|\pi| \subseteq \omega_1$ is (at most) countable and if $\xi \in |\pi|$ then $\pi(\xi)$ is an special, hence countable forcing in \mathbf{AF} .

If $\kappa \leq \omega_1$ then let $\overrightarrow{\mathbf{MF}}_\kappa$ be the set of all \sqsubset -increasing sequences $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \kappa}$ of multiforcings $\pi_\alpha \in \mathbf{spMF}$, of length $\text{dom}(\vec{\pi}) = \kappa$, *domain-continuous* in the sense that if $\lambda < \kappa$ is a limit ordinal then $|\pi_\lambda| = \bigcup_{\alpha < \lambda} |\pi_\alpha|$.

If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \lambda} \in \overrightarrow{\mathbf{MF}}_\kappa$ then define the *component-wise union* $\pi = \bigcup^{\text{cw}} \vec{\pi} = \bigcup_{\alpha < \lambda} \pi_\alpha \in \mathbf{MF}$ so that $|\pi| = \bigcup_{\alpha < \lambda} |\pi_\alpha|$ and $\pi(\xi) = \bigcup_{\alpha < \lambda, \xi \in |\pi_\alpha|} \pi_\alpha(\xi)$ for all indices $\xi \in |\pi|$, and define $\mathbf{MT}(\vec{\pi}) = \mathbf{MT}(\pi)$ (the set of all π -multitrees).

We put $\overrightarrow{\mathbf{MF}} = \bigcup_{\kappa < \omega_1} \overrightarrow{\mathbf{MF}}_\kappa$ (\sqsubset -increasing sequences of countable length).

The set $\overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$ is ordered by the relations \subseteq, \subset of extension of sequences.

Lemma 1 (Lemma 14.4(ii) in [19]). *If $\kappa < \lambda \leq \omega_1$ and $\vec{\pi} \in \overrightarrow{\mathbf{MF}}_\kappa$ then there exists a sequence $\vec{\varphi} \in \overrightarrow{\mathbf{MF}}_\lambda$ satisfying $\vec{\pi} \subset \vec{\varphi}$.*

5. The Key Sequence, Key Forcing Notation, and Key Model

In this section, we introduce the forcing notion to prove Theorem 1, defined in our earlier paper [19]. It has the form $\mathbf{MT}(\mathbb{P})$, for a certain multiforcing \mathbb{P} with $|\mathbb{P}| = \omega_1$. The multiforcing \mathbb{P} is equal to the componentwise union of terms of a certain sequence $\vec{\mathbb{P}} \in \overline{\mathbf{MF}}_{\omega_1}$ which we present in Definition 1. Yet we need to recall one more concept.

Let HC be the set of all *hereditarily countable* sets. Thus $X \in \text{HC}$ if the *transitive closure* $\text{TC}(X)$ is at most countable.

We use standard notation $\Sigma_n^{\text{HC}}, \Pi_n^{\text{HC}}, \Delta_n^{\text{HC}}$ (slanted Σ, Π, Δ) for classes of *lightface* definability over HC (no parameters allowed), and $\Sigma_n(\text{HC}), \Pi_n(\text{HC}), \Delta_n(\text{HC})$ for *boldface* definability over HC (parameters in HC allowed). The following useful result connects projective hierarchy with the definability classes over HC.

Lemma 2 (Lemma 25.25 in [20]). *If $n \geq 1$ and $X \subseteq 2^\omega$ then*

$$X \in \Sigma_n^{\text{HC}} \iff X \in \Sigma_{n+1}^1, \quad \text{and} \quad X \in \Sigma_n(\text{HC}) \iff X \in \Sigma_{n+1}^1,$$

and the same for Π, Π, Δ, Δ .

Definition 1 (in \mathbf{L}). From now on, we fix a number $n \geq 3$ as in Theorem 1. We also fix a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1} \in \overline{\mathbf{MF}}_{\omega_1}$ satisfying Theorem 15.3 in [19] for this n . This includes the equality $\bigcup_\alpha |\mathbb{P}_\alpha| = \omega_1$ and the following conditions (in \mathbf{L}):

- (A) the sequence $\vec{\mathbb{P}}$ belongs to the definability class Δ_{n-2}^{HC} ;
- (B) if $n \geq 4$ and $W \subseteq \overline{\mathbf{MF}}$ is a boldface $\Sigma_{n-3}(\text{HC})$ set, then there is an ordinal $\gamma < \omega_1$ such that the sequence $\vec{\mathbb{P}} \upharpoonright \gamma$ **blocks** W , in the sense that *either* $\vec{\mathbb{P}} \upharpoonright \gamma \in W$, *or* there is no sequence $\mathfrak{P} \in W$ extending $\vec{\mathbb{P}} \upharpoonright \gamma$.

We call this fixed $\vec{\mathbb{P}} \in \mathbf{L}$ **the key sequence**. The construction of $\vec{\mathbb{P}}$ in [19] is rather long and too technical, so we do not reproduce it here. It employs some ideas related to diamond-style constructions, as well as to some sort of *definable generic* inductive constructions. This method is realized by a special transfinite construction of the sequence $\vec{\mathbb{P}}$ in \mathbf{L} from countable subsequences. The construction can be viewed as a maximal branch in a certain mega-tree, say \mathcal{P} , whose nodes are such countable subsequences. A suitable character of extension in the mega-tree allows to define a maximal branch in \mathcal{P} that blocks all sets in \mathcal{P} as in (B) of Definition 1, and still satisfies (A).

The following definition introduces some derived notions.

Definition 2. *Using the key sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1}$ as in Definition 1, we define the multiforcing $\mathbb{P} = \bigcup_{\alpha < \omega_1} \mathbb{P}_\alpha \in \mathbf{MF}$, and the forcing notion $\mathbf{P} = \mathbf{MT}(\mathbb{P}) = \mathbf{MT}(\vec{\mathbb{P}})$.*

If $\xi < \omega_1$ then let $\alpha(\xi) < \omega_1$ be the least ordinal α satisfying $\xi \in |\mathbb{P}_\alpha|$.

If $\alpha(\xi) \leq \alpha < \omega_1$ then a special forcing notion $\mathbb{P}_\alpha(\xi) \in \mathbf{AF}$ is defined by construction and $\langle \mathbb{P}_\alpha(\xi) \rangle_{\alpha(\xi) \leq \alpha < \omega_1}$ is a \square -increasing sequence; hence $\mathbb{P}(\xi) = \bigcup_{\alpha(\xi) \leq \alpha < \omega_1} \mathbb{P}_\alpha(\xi) \in \mathbf{AF}$.

In the remainder, \mathbb{P} is referred to as **the key multiforcing**, whereas the set $\mathbf{P} = \mathbf{MT}(\mathbb{P})$ is our **key forcing notion**. As established by 16.2 in [19], \mathbb{P} is a regular multiforcing and $|\mathbb{P}| = \omega_1$, thus $\mathbf{P} = \prod_{\xi < \omega_1} \mathbb{P}(\xi)$ (with finite support).

Lemma 3 ([19], 16.7). *The forcing notion \mathbf{P} satisfies the countable chain condition, CCC, in \mathbf{L} . Therefore, \mathbf{P} -generic extensions of \mathbf{L} preserve cardinals.*

Our final goal is to prove Theorem 1 by means of \mathbf{P} -generic extensions of \mathbf{L} . We call these extensions **key models**.

From now on, we will typically argue in \mathbf{L} and in $\omega_1^{\mathbf{L}}$, preserving generic extensions of \mathbf{L} , in particular, in \mathbf{P} -generic extensions (see above). Thus it will always be the case that $\omega_1^{\mathbf{L}} = \omega_1$. This allows us to think that $|\mathbb{P}| = \omega_1$ (rather than $\omega_1^{\mathbf{L}}$).

Definition 3. Let a set $G \subseteq \mathbb{P}$ be generic over the constructable universe \mathbf{L} . If $\xi < \omega_1$ then following the remark in the end of Section 3, we define

$$G(\xi) = \{T_\xi^p : p \in G \wedge \xi \in |p|\} \subseteq \mathbb{P}(\xi),$$

and let $x_\xi[G] \in 2^\omega$ be the only real in $\bigcap_{T \in G(\xi)} [T]$. Then we put

$$\mathbf{X}[G] = \langle x_\xi[G] \rangle_{\xi < \omega_1} = \{ \langle \xi, x_\xi[G] \rangle : \xi < \omega_1 \}.$$

Thus the forcing notion \mathbb{P} adjoins an array $\mathbf{X}[G]$ of reals $x_\xi[G]$ to \mathbf{L} , where each $x_\xi[G] \in 2^\omega \cap \mathbf{L}[G]$ is a $\mathbb{P}(\xi)$ -generic real over \mathbf{L} , and $\mathbf{L}[G] = \mathbf{L}[\mathbf{X}[G]]$. The following important claim is essentially a corollary of condition (A) of Definition 1.

Lemma 4 (Corollary 18.2 in [19]). Assume that $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} . Then it is true in $\mathbf{L}[G]$ that $\mathbf{X}[G]$ is a set of definability class Π_{n-2}^{HC} , hence, of class Π_{n-1}^1 by Lemma 2 above.

6. Δ_n^1 -Good Well-Ordering in the Key Model

The next theorem proves that the key model $\mathbf{L}[G]$ satisfies condition (i) of Theorem 1. The reals are treated here as points of the Cantor space 2^ω .

Theorem 2. If $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} then it holds in $\mathbf{L}[G]$ that there is a Δ_n^1 -good well-ordering of 2^ω of length ω_1 , hence (i) of Theorem 1 holds.

Proof. We argue in $\mathbf{L}[G]$. Lemma 4 will be the principal ingredient of the proof.

Let $\mathbf{X} = \mathbf{X}[G]$. If $\gamma < \omega_1$ then let $\mathbf{X} \upharpoonright \gamma = \langle x_\xi[G] \rangle_{\xi < \gamma}$. The map $\gamma \mapsto \mathbf{X} \upharpoonright \gamma$ is Π_{n-2}^{HC} (in $\mathbf{L}[G]$) by Lemma 4, because

$$Y = \mathbf{X} \upharpoonright \gamma \iff Y \text{ is a function on } \gamma \wedge \forall \xi < \gamma (\langle \xi, Y(\xi) \rangle \in \mathbf{X}).$$

Now if $x \in 2^\omega$ (in $\mathbf{L}[G]$) then $x \in \mathbf{L}[\mathbf{X} \upharpoonright \gamma]$ for some $\gamma < \omega_1$ by Lemma 4, hence we let $\gamma(x)$ be the least $\gamma < \omega_1$ such that $x \in \mathbf{L}[\mathbf{X} \upharpoonright \gamma]$, and $\nu(x) < \omega_1$ be the index of x in the canonical $\Delta_1^{\text{HC}}(\{\mathbf{X} \upharpoonright \gamma\})$ well-ordering $\leq_{\mathbf{X} \upharpoonright \gamma}$ of 2^ω in $\mathbf{L}[\mathbf{X} \upharpoonright \gamma]$ by Gödel. We claim that the maps $x \mapsto \gamma(x)$ and $x \mapsto \nu(x)$ are Δ_{n-1}^{HC} . Indeed,

$$\begin{aligned} \gamma = \gamma(x) &\iff \exists Y (Y = \mathbf{X} \upharpoonright \gamma \wedge x \in \mathbf{L}[Y] \wedge \forall \gamma' < \gamma (x \notin \mathbf{L}[Y \upharpoonright \gamma'])) \\ &\iff \forall Y (Y = \mathbf{X} \upharpoonright \gamma \implies x \in \mathbf{L}[Y] \wedge \forall \gamma' < \gamma (x \notin \mathbf{L}[Y \upharpoonright \gamma'])). \end{aligned}$$

This easily yields the result for the map $x \mapsto \gamma(x)$. The result for the other map follows by a similar rather routine estimation.

Now let \preceq be the well-ordering of the set $2^\omega \cap \mathbf{L}[G]$, according to the lexicographical well-ordering of the triples $\langle \max\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x) \rangle$. It easily follows from the results for maps $x \mapsto \gamma(x)$ and $x \mapsto \nu(x)$ that \preceq is Δ_{n-1}^{HC} , and hence Δ_n^1 by Lemma 2 of Section 5.

Finally to check the Δ_n^1 -goodness, by definition it remains to prove that, given a Δ_n^1 set $P \subseteq 2^\omega \times 2^\omega$, the set $Q = \{ \langle z, x \rangle : \forall y \preceq x \neg P(z, y) \}$ is Δ_n^1 as well. The class Π_n^1 is obvious, as \preceq is already shown to be Δ_n^1 . Thus we have to verify the definability class Σ_n^1 , or equivalently, class Σ_{n-1}^{HC} , for Q . However, this is true, as $Q(z, x)$ is equivalent to the following:

for all $\gamma', \nu' \leq \max\{\gamma(x), \nu(x)\}$, if the triple $\langle \max\{\gamma', \nu'\}, \gamma', \nu' \rangle$ non-strictly precedes $\langle \max\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x) \rangle$ lexicographically, then there is a real $y \in 2^\omega$ such that $\gamma' = \gamma(y)$, $\nu' = \nu(y)$, and $\neg P(z, y)$.

It remains to note that the quoted formula is essentially Σ_{n-1}^{HC} since the bounded quantifiers $\forall \gamma', \nu' \leq \max\{\gamma(x), \nu(x)\}$ do not destroy Σ -classes over HC. \square

Our final step is to prove that the key model also satisfies condition (ii) of Theorem 1. However, this will involve much more work and will be carried out under the following assumption.

Assumption 1. We shall assume that $\aleph \geq 4$ henceforth.

This leaves aside the case $\aleph = 3$ in (ii) of Theorem 1 which thus needs a separate consideration to justify the assumption. Thus suppose for a moment that $\aleph = 3$. We claim that (ii) of Theorem 1 holds in the key model $\mathbf{L}[G]$, where G is \mathbb{P} -generic over \mathbf{L} . Suppose to the contrary that (ii) of Theorem 1 fails, so that there is a Δ^1_2 well-ordering of the reals (even not necessarily good). Then by Theorem 25.39 in [20], we have $2^\omega \subseteq \mathbf{L}[x]$ in $\mathbf{L}[G]$ for some $x \in 2^\omega$ in $\mathbf{L}[G]$. However, this is definitely not the case for the key model $\mathbf{L}[G]$ we consider.

Indeed, arguing in $\mathbf{L}[G]$, suppose to the contrary that $x \in 2^\omega \cap \mathbf{L}[G] = \mathbf{L}[\langle x_\zeta[G] \rangle_{\zeta < \omega_1}]$ satisfies $2^\omega \cap \mathbf{L}[Gsq\mathbf{L}[x]]$. It follows by Lemma 3 that there is an ordinal $\lambda < \omega_1 = \omega^{\mathbf{L}}_1$ such that $x \in \mathbf{L}[\langle x_\zeta[G] \rangle_{\zeta < \lambda}]$. However the real $y = x_\lambda[G]$ does not belong to $\mathbf{L}[\langle x_\zeta[G] \rangle_{\zeta < \lambda}]$ by the product forcing theory. Therefore $y \notin \mathbf{L}[x]$, contrary to the choice of x .

7. Real Names

We begin with a technical concept. The goal of the following definitions is to give a suitable notation for names of reals in 2^ω in the context of forcings of the form $\mathbf{MT}(\pi)$.

Let a *real name* be any set $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times 2)$ such that the sets

$$K_{ni}^{\mathbf{c}} = \{p \in \mathbf{MT} : \langle p, n, i \rangle \in \mathbf{c}\}$$

satisfy the following: if $n < \omega$ and $p \in K_{n0}^{\mathbf{c}}, q \in K_{n1}^{\mathbf{c}}$, then p, q are *somewhere almost disjoint*, in the sense that there is an index $\zeta \in |p| \cap |q|$ such that $T_\zeta^p \cap T_\zeta^q$ is finite (or equivalently, $[T_\zeta^p] \cap [T_\zeta^q] = \emptyset$) — and then p, q are obviously incompatible in \mathbf{MT} .

Let $K_n^{\mathbf{c}} = K_{n0}^{\mathbf{c}} \cup K_{n1}^{\mathbf{c}}$; then $K_n^{\mathbf{c}} \subseteq \mathbf{MT}$.

A real name \mathbf{c} is *small* if each $K_n^{\mathbf{c}}$ is at most countable—then the set $|\mathbf{c}| = \bigcup_n \bigcup_{p \in K_n^{\mathbf{c}}} |p|$, and \mathbf{c} itself as a set are countable, too.

Now let π be a multiforcing. A real name \mathbf{c} is π -complete if the set

$$K_n^{\mathbf{c}} \uparrow \pi = \{p \in \mathbf{MT}(\pi) : \exists q \in K_n^{\mathbf{c}} (p \leq q)\} \tag{3}$$

is dense in $\mathbf{MT}(\pi)$ for each n . In this case, if a set (a filter) $G \subseteq \mathbf{MT}(\pi)$ is $\mathbf{MT}(\pi)$ -generic over the family of all sets $K_n^{\mathbf{c}} \uparrow \pi, n < \omega$, then we define a real $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ if $G \cap (K_{ni}^{\mathbf{c}} \uparrow \pi) \neq \emptyset$, where $K_{ni}^{\mathbf{c}} \uparrow \pi$ is defined from $K_{ni}^{\mathbf{c}}$ similarly to (3).

We do not require here that $\mathbf{c} \subseteq \mathbb{P} \times (\omega \times 2)$, or equivalently, $K_n^{\mathbf{c}} \subseteq \mathbb{P}$ for all n .

Finally, if $\vec{\pi}$ is a sequence in $\overline{\mathbf{MF}} \cup \overline{\mathbf{MF}}_{\omega_1}$, then a $\vec{\pi}$ -complete real name will mean a π -complete real name, where $\pi = \bigcup^{\text{cw}} \vec{\pi}$ (the componentwise union).

As an elementary example, we let $\zeta < \omega_1$ and define a real name \dot{x}_ζ such that each set $K_{ni}^{\dot{x}_\zeta}$ consists of a single multitree p_{ni}^ζ , defined as follows. We let $|p_{ni}^\zeta| = \{\zeta\}$ (the domain), and let $p_{ni}^\zeta(\zeta) = T_{ni}$, where $T_{ni} = \{s \in 2^{<\omega} : \text{lh}(s) \leq n \vee s(n) = i\}$.

We leave it as a routine exercise to prove that \dot{x}_ζ is a small real name, π -complete for any multiforcing π , and if a set $G \subseteq \mathbf{MT}(\pi)$ is $\mathbf{MT}(\pi)$ -generic over \mathbf{L} , then the real $\dot{x}_\zeta[G]$ is identical to $x_\zeta[G]$ defined by Definition 3. Thus, \dot{x}_ζ is a *canonical name* for the real $x_\zeta[G]$.

8. An Auxiliary Forcing Relation

We begin a lengthy proof of the non-existence of $\Delta^1_{\aleph-1}$ -good well-orderings of the reals in the generic models considered. The proof involves an auxiliary forcing relation, not explicitly connected with any particular forcing notion, in particular, not explicitly connected with the key forcing \mathbb{P} .

We argue in L. Consider the 2nd order arithmetic language, with variables k, l, m, n, \dots of type 0 over ω and variables a, b, x, y, \dots of type 1 over 2^ω , whose atomic formulas are those of the form $x(k) = n$. Let \mathcal{L} be the extension of this language, which allows to substitute variables of type 0 with natural numbers and variables of type 1 with small real names (see Section 7) $\mathbf{c} \in \mathbf{L}$.

We consider natural classes $\mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ ($n \geq 1$) of \mathcal{L} -formulas. Let $\mathcal{L}(\Sigma\Pi)_1^1$ be the closure of $\mathcal{L}\Sigma_1^1 \cup \mathcal{L}\Pi_1^1$ under \neg, \wedge, \vee and quantifiers over ω .

A relation $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$ between multitrees \mathbf{p} , sequences $\vec{\pi} \in \overline{\mathbf{MF}}$, and closed \mathcal{L} -formulas φ in $\mathcal{L}(\Sigma\Pi)_1^1$ or $\mathcal{L}\Sigma_n^1 \cup \mathcal{L}\Pi_n^1, n \geq 2$, was defined in [19] (§22) by induction on the complexity of φ . We skip here the initial step of the definition (the case of $\mathcal{L}(\Sigma\Pi)_1^1$ formulas, 1° in [19] (§22)), as it involves technical issues not considered in this paper. The following inductive steps 2° and 3° in [19] (§22) demonstrate obvious similarities with various conventional forcing notions.

- 2°. If $\varphi(x)$ is a $\mathcal{L}\Pi_n^1$ formula, $n \geq 1$, then $\mathbf{p} \text{ forc}_{\vec{\pi}} \exists x \varphi(x)$ if there is a small real name \mathbf{c} such that $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi(\mathbf{c})$.
- 3°. If φ is a closed $\mathcal{L}\Pi_n^1$ formula, $n \geq 2$, then $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$ if there is no sequence $\vec{\tau} \in \overline{\mathbf{MF}}$ and multitree $\mathbf{p}' \in \mathbf{MT}(\vec{\tau})$ such that $\vec{\pi} \subseteq \vec{\tau}, \mathbf{p}' \leq \mathbf{p}$, and $\mathbf{p}' \text{ forc}_{\vec{\tau}} \varphi^-$, where φ^- is the result of the canonical transformation of $\neg \varphi$ to $\mathcal{L}\Sigma_n^1$ form.

The principal properties of the relation forc are presented in Propositions 1–5 below, with references to according claims in [19].

Proposition 1 (Lemma 22.3 in [19]). *Assume that sequences $\vec{\pi} \subseteq \vec{\varphi}$ belong to $\overline{\mathbf{MF}}$, $\mathbf{q}, \mathbf{p} \in \mathbf{MT}$, $\mathbf{q} \leq \mathbf{p}$, φ is a formula in one of the classes $\mathcal{L}(\Sigma\Pi)_1^1$ or $\mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ ($n \geq 2$), and $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$. Then $\mathbf{q} \text{ forc}_{\vec{\varphi}} \varphi$.*

If K is one of the classes $\mathcal{L}(\Sigma\Pi)_1^1, \mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ ($n \geq 2$), then let $\mathbf{FORC}[K]$ consist of all triples $\langle \vec{\pi}, \mathbf{p}, \varphi \rangle$ such that $\vec{\pi} \in \overline{\mathbf{MF}}, \mathbf{p} \in \mathbf{MT}$, φ is a formula in K , and $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$. Note that $\mathbf{FORC}[K]$ is a subset of \mathbf{HC} , the set of all hereditarily countable sets.

Proposition 2 (Lemma 22.5 in [19]). *It is true in L that $\mathbf{FORC}[\mathcal{L}(\Sigma\Pi)_1^1] \in \Delta_1^{\mathbf{HC}}$, whereas if $n \geq 2$ then $\mathbf{FORC}[\mathcal{L}\Sigma_n^1]$ belongs to $\Sigma_{n-1}^{\mathbf{HC}}$ and $\mathbf{FORC}[\mathcal{L}\Pi_n^1]$ belongs to $\Pi_{n-1}^{\mathbf{HC}}$.*

Proposition 3 just below demonstrates that the forcing relation $\text{forc}_{\vec{\pi}}$, considered with countable initial segments $\vec{\pi} = \vec{\pi} \upharpoonright \alpha$ of the key sequence $\vec{\pi}$ (introduced by Definition 1), coincides with the true \mathbb{P} -forcing relation (see Definition 2) up to the level $n - 1$.

Recall that $n \geq 4$ by Assumption 1.

We write $\mathbf{p} \text{ forc}_\alpha \varphi$ instead of $\mathbf{p} \text{ forc}_{\vec{\pi} \upharpoonright \alpha} \varphi$, for the sake of brevity. Let $\mathbf{p} \text{ forc} \varphi$ mean: $\mathbf{p} \text{ forc}_\alpha \varphi$ for some $\alpha < \omega_1$. The next result makes use of (B) of Definition 1.

Proposition 3 ([19], 25.3). *If φ is a closed \mathcal{L} -formula in $\mathcal{L}(\Sigma\Pi)_1^1$ or $\mathcal{L}\Pi_k^1 \cup \mathcal{L}\Sigma_{k+1}^1, 1 \leq k \leq n - 2$, and $\mathbf{p} \in \mathbb{P}$, then $\mathbf{p} \mathbb{P}$ -forces $\varphi[\underline{G}]$ over L in the usual sense, if and only if $\mathbf{p} \text{ forc} \varphi$.*

9. Invariance

Invariance theorems are very typical for all kinds of forcing. We present here two major invariance theorems on the auxiliary forcing forc , established in [19]. The first one shows tail invariance, while the other one explores the permutational invariance.

If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \lambda} \in \overline{\mathbf{MF}}$ and $\gamma < \lambda = \text{dom } \vec{\pi}$, then let the γ -tail $\vec{\pi} \upharpoonright_{\geq \gamma}$ be the restriction $\vec{\pi} \upharpoonright [\gamma, \lambda)$ to the ordinal semiinterval $[\gamma, \lambda) = \{\alpha : \gamma \leq \alpha < \lambda\}$. Then the set $\mathbf{MT}(\vec{\pi} \upharpoonright_{\geq \gamma}) = \bigcup_{\gamma \leq \alpha < \lambda} \mathbf{MT}(\vec{\pi} \upharpoonright \alpha)$ is dense in $\mathbf{MT}(\vec{\pi})$. Therefore it can be expected that if $\vec{\varphi}$ is another sequence of the same length $\lambda = \text{dom } \vec{\varphi}$, and $\vec{\varphi} \upharpoonright_{\geq \gamma} = \vec{\pi} \upharpoonright_{\geq \gamma}$, then the relation $\text{forc}_{\vec{\pi}}$ coincides with $\text{forc}_{\vec{\varphi}}$. Indeed this turns out to be the case.

Proposition 4 (Theorem 23.1 in [19]). Assume that $\vec{\pi}, \vec{\varphi}$ are sequences in $\overline{\mathbf{MF}}$, $\gamma < \lambda = \text{dom } \vec{\pi} = \text{dom } \vec{\varphi}$, $\vec{\varphi} \upharpoonright_{\geq \gamma} = \vec{\pi} \upharpoonright_{\geq \gamma}$, $\mathbf{p} \in \mathbf{MT}$, $n \geq 2$, and φ is a formula in $\mathcal{L}\Pi_n^1 \cup \mathcal{L}\Sigma_{n+1}^1$. Then $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$ iff $\mathbf{p} \text{ forc}_{\vec{\varphi}} \varphi$.

The other invariance result treats permutations of indices. Arguing in \mathbf{L} , let PERM be the set of all bijections $h : \omega_1 \xrightarrow{\text{onto}} \omega_1$ such that $h = h^{-1}$ and the non-identity domain $\text{NID}(h) = \{\xi : h(\xi) \neq \xi\}$ is at most countable. Elements of PERM are called *permutations*. Let $h \in \text{PERM}$. The action of h is extended as follows. (See [19], Section 24.)

1. If \mathbf{p} is a multitree then $h\mathbf{p}$ is a multitree defined so that $|h\mathbf{p}| = h''|\mathbf{p}| = \{h(\xi) : \xi \in |\mathbf{p}|\}$, and $(h\mathbf{p})(h(\xi)) = \mathbf{p}(\xi)$ whenever $\xi \in |\mathbf{p}|$.
2. If $\pi \in \mathbf{MT}$ is a multiforcing then $h \cdot \pi = \pi \circ (h^{-1})$ is a multiforcing defined so that $|h \cdot \pi| = h''|\pi|$ and $(h \cdot \pi)(h(\xi)) = \pi(\xi)$ whenever $\xi \in |\pi|$.
3. If $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then put $h\mathbf{c} = \{\langle h\mathbf{p}, n, i \rangle : \langle \mathbf{p}, n, i \rangle \in \mathbf{c}\}$, thus easily $h\mathbf{c}$ is a real name as well.
4. If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \kappa} \in \overline{\mathbf{MF}}$, then put $h\vec{\pi} = \langle h \cdot \pi_\alpha \rangle_{\alpha < \kappa}$, this is still a sequence in $\overline{\mathbf{MF}}$.
5. If $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is a \mathcal{L} -formula (with all names explicitly indicated), then let $h\varphi$ be accordingly the formula $\varphi(h\mathbf{c}_1, \dots, h\mathbf{c}_n)$.

Many notions and relations defined above are clearly PERM -invariant, e.g., $\mathbf{p} \in \mathbf{MT}(\pi)$ if $h\mathbf{p} \in \mathbf{MT}(h \cdot \pi)$, $\pi \sqsubset \varphi$ if $h \cdot \pi \sqsubset h \cdot \varphi$, et cetera. The invariance also takes place with respect to the relation *forc*.

Proposition 5 (Theorem 24.1 in [19]). Assume that $\vec{\pi} \in \overline{\mathbf{MF}}$, $\mathbf{p} \in \mathbf{MT}(\vec{\pi})$, $h \in \text{PERM}$, $n \geq 2$, and φ belongs to $\mathcal{L}\Pi_n^1 \cup \mathcal{L}\Sigma_{n+1}^1$. Then $\mathbf{p} \text{ forc}_{\vec{\pi}} \varphi$ iff $(h\mathbf{p}) \text{ forc}_{h\vec{\pi}} (h\varphi)$.

10. No Δ_{n-1}^1 -Good Well-Orderings in the Key Model

In this section, we accomplish the proof of Theorem 1 by verifying that the key model $\mathbf{L}[G]$ of Section 5 satisfies (ii) of Theorem 1. That the key model satisfies (i) of Theorem 1 was already established by Theorem 2. The following lemma is the principal step.

Lemma 5. If $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} then it holds in $\mathbf{L}[G]$ that every Σ_{n-1}^1 set $S \subseteq \omega$ is constructable.

Proof. There is a parameter-free Σ_{n-1}^1 formula $\varphi(j)$, such that $S = \{j < \omega : \varphi(j)\}$ in $\mathbf{L}[G]$. We claim that

- (A) if $j < \omega$ then $j \in X$ if and only if it is true in \mathbf{L} that there exists a sequence $\vec{\sigma} \in \overline{\mathbf{MF}}$ and a multitree $\mathbf{s} \in \mathbf{MT}(\vec{\sigma})$ such that $\mathbf{s} \text{ forc}_{\vec{\sigma}} \varphi(j)$.

In the easy direction, assume that $j \in X$. There is a condition $s \in \mathbb{P}$ which \mathbb{P} -forces $\varphi(j)$ over \mathbf{L} . Then $s \text{ forc } \varphi(j)$ by Proposition 3, that is, $s \text{ forc}_{\vec{\pi} \upharpoonright_\alpha} \varphi$ for some $\alpha < \omega_1$. We can increase α if necessary to guarantee that $\mathbf{s} \in \mathbf{MT}(\vec{\pi} \upharpoonright_\alpha)$. It remains to take $\vec{\sigma} = \vec{\pi} \upharpoonright_\alpha$.

In the difficult direction, suppose that $\mathbf{s} \text{ forc}_{\vec{\sigma}} \varphi(j)$, where $\vec{\sigma} \in \overline{\mathbf{MF}}$ and $\mathbf{s} \in \mathbf{MT}(\vec{\sigma})$; we have to prove that $j \in X$. Suppose toward the contrary that $j \notin X$. Then there is a multitree $\mathbf{p}_0 \in G$ such that

- (B) \mathbf{p}_0 \mathbb{P} -forces $\neg \varphi(j)$ over \mathbf{L} .

We argue in \mathbf{L} . Let U be the set of all sequences $\vec{\pi} \in \overline{\mathbf{MF}}$, such that

- (C) there exist (1) a sequence $\vec{\varphi} \in \overline{\mathbf{MF}}$ with $\vec{\sigma} \subset \vec{\varphi}$ and with double-successor length $\text{dom}(\vec{\varphi}) = \nu + 2 < \lambda = \text{dom}(\vec{\pi})$, and (2) a permutation $h \in \text{PERM}$ such that $\varphi' \subseteq \vec{\pi}(\nu + 1)$ and $|\varphi'| \cap |\varphi| = \emptyset$, where $\varphi' = h \cdot \varphi$ and $\varphi = \vec{\varphi}(\nu + 1)$, the last term.

Note that the inclusion $\varphi' \subseteq \pi$ between multiforcings $\varphi' = h \cdot \varphi$ and $\pi = \vec{\pi}(\nu + 1)$ as in (C)(2) means simply that $|\varphi'| \subseteq |\pi|$ and $\varphi' = \pi \upharpoonright |\varphi'|$, that is, $\varphi'(\xi) = \pi(\xi)$ for all $\xi \in |\varphi'|$.

By routine estimation, U is a Σ_1 (HC) set (with δ, σ as the only parameters of a Σ_1 definition in HC), hence a $\Sigma_{\aleph-3}$ (HC) set as $\aleph \geq 4$ by Assumption 1. Therefore by Definition 1(B) there is an ordinal $\lambda < \omega_1$ such that $\vec{\pi} \upharpoonright \lambda$ blocks U .

Case 1: $\vec{\pi} = \vec{\pi} \upharpoonright \lambda \in U$. Let this be witnessed by $\vec{\varphi}, \nu, \lambda, \varphi, h, \varphi'$ as in (C)(1,2). In addition, fix a multitree $q \in \mathbf{MT}(\varphi)$, $q \leq s$, put $\vec{\sigma}' = h\vec{\sigma}$, $s' = hs$, $q' = hq$, $\vec{\varphi}' = h\vec{\varphi}$. Then clearly

$$s' \in \mathbf{MT}(\vec{\sigma}'), \quad \varphi' = \vec{\varphi}'(\nu + 1), \quad q' \in \mathbf{MT}(\varphi') \subseteq \mathbf{MT}(\pi), \quad q' \leq s'.$$

Our next goal will be to prove that q' forc $_{\vec{\pi}}$ $\varphi(j)$.

First of all, we have s' forc $_{\vec{\sigma}'}$ $\varphi(j)$ by Proposition 5 since $h\varphi(j)$ coincides with $\varphi(j)$ for any parameter-free φ .

Now consider a sequence $\vec{\varphi}''$ with $\text{dom}(\vec{\varphi}'') = \text{dom}(\vec{\varphi}') = \nu + 2$, defined so that $\vec{\varphi}'' \upharpoonright \nu + 1 = \vec{\varphi}' \upharpoonright \nu + 1$, in particular, still $\vec{\sigma}' \subset \vec{\varphi}''$, but $\vec{\varphi}''(\nu + 1) = \pi$ (instead of the value $\varphi' = \vec{\varphi}'(\nu + 1)$). To see that $\vec{\varphi}''$ is still \sqsubset -increasing, recall that $\varphi' \subseteq \pi$ and apply Remark 1. As s' forc $_{\vec{\sigma}'}$ $\varphi(j)$ (see above), we have q' forc $_{\vec{\varphi}''}$ $\varphi(j)$ by Proposition 1.

Consider a sequence $\vec{\pi}''$ with $\text{dom}(\vec{\pi}'') = \text{dom}(\vec{\pi}) = \lambda > \nu + 2$, defined so that $\vec{\varphi}'' \subset \vec{\pi}''$ and $\vec{\pi}''(\alpha) = \vec{\pi}(\alpha)$ whenever $\nu + 2 \leq \alpha < \lambda$. Then we have q' forc $_{\vec{\pi}''}$ $\varphi(j)$ still by Proposition 1.

Note that $\vec{\pi}'' \upharpoonright_{\geq \nu+1} = \vec{\pi} \upharpoonright_{\geq \nu+1}$ by construction. In particular, $\vec{\pi}''(\nu + 1) = \vec{\varphi}''(\nu + 1) = \pi = \vec{\pi}(\nu + 1)$. We conclude by Proposition 4 that q' forc $_{\vec{\pi}}$ $\varphi(j)$ as well. Then q' \mathbb{P} -forces $\varphi(j)$ over \mathbf{L} by Proposition 3. Now to get a contradiction with (B), it suffices to check that p_0 and q' are compatible in \mathbb{P} . However, this is easy: $|p_0| \subseteq |\varphi|$ and $|q'| \subseteq |\varphi'|$ by construction, whereas $|\varphi'| \cap |\varphi| = \emptyset$ by (C), and hence the ordinary union $r = p_0 \cup q'$ witnesses the compatibility.

The contradiction obtained closes Case 1.

Case 2: no sequence in U extends $\vec{\pi} \upharpoonright \lambda$. Let $\nu = \max\{\text{dom}(\vec{\sigma}), \lambda\}$. By Lemma 1, there is a sequence $\vec{\varphi} \in \overline{\mathbf{MF}}$ satisfying $\text{dom}(\vec{\varphi}) = \nu + 2$ and $\vec{\sigma} \subset \vec{\varphi}$. Let $\varphi = \vec{\varphi}(\nu + 1)$ (the last term). Let $\vec{\tau} \in \overline{\mathbf{MF}}$ be any extension of $\vec{\pi} \upharpoonright \lambda$ of length $\text{dom}(\vec{\tau}) = \nu + 2 = \text{dom}(\vec{\varphi})$.

There is a permutation $h \in \text{PERM}$ such that the derived multiforcing $\varphi' = h \cdot \varphi$ satisfies $|\varphi'| \cap (|\varphi| \cup |\vec{\tau}|) = \emptyset$.

Consider a sequence $\vec{\tau}_1$ still with $\text{dom}(\vec{\tau}_1) = \text{dom}(\vec{\tau}) = \nu + 2$, defined so that $\vec{\tau}_1 \upharpoonright (\nu + 1) = \vec{\tau} \upharpoonright (\nu + 1)$, in particular, still $\vec{\pi} \upharpoonright \lambda \subset \vec{\tau}_1$, but $\vec{\tau}_1(\nu + 1) = \vec{\tau}(\nu + 1) \cup \varphi'$. (Note that the union $\vec{\tau}(\nu + 1) \cup \varphi'$ of multiforcings $\vec{\tau}(\nu + 1)$ and φ' with disjoint domains is a multiforcing as well.) To see that $\vec{\tau}_1$ is \sqsubset -increasing, we note that $\vec{\tau}(\nu + 1) \subseteq \vec{\tau}_1(\nu + 1)$ by construction, and refer to Remark 1.

Finally, let $\vec{\pi}_1 \in \overline{\mathbf{MF}}$ be any extension of $\vec{\tau}_1$ of length $\lambda' = \text{dom}(\vec{\pi}_1) = \nu + 3 > \text{dom}(\vec{\varphi})$. We assert that $\vec{\pi}_1 \in U$, and this is witnessed by $\vec{\varphi}$ and h .

Indeed we have $\vec{\sigma} \subset \vec{\varphi}$, $\text{dom}(\vec{\varphi}) = \nu + 2 < \lambda' = \text{dom}(\vec{\pi}_1)$, and $\varphi' \subseteq \vec{\pi}_1(\nu + 1) = \vec{\tau}_1(\nu + 1) = \vec{\tau}(\nu + 1) \cup \varphi'$ by construction. Thus $\vec{\pi}_1 \in U$.

On the other hand, $\vec{\pi} \upharpoonright \lambda \subset \vec{\pi}_1$. However, this contradicts the Case 2 assumption.

To conclude, either case leads to a contradiction. This ends the proof of (A).

To accomplish the proof of Lemma 5, it remains to make use of (A) in view of the fact that the relation forc is defined inside \mathbf{L} . \square

Theorem 3. *If a set $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} then it is true in $\mathbf{L}[G]$ that there is no $\Delta^1_{\aleph-1}$ -good well-ordering of the reals, so that (ii) of Theorem 1 holds.*

Proof. We argue in $\mathbf{L}[G]$. Suppose to the contrary that there exists a $\Delta^1_{\aleph-1}$ -good well-ordering of 2^ω . It follows that any non-empty $\Sigma^1_{\aleph-1}$ set $X \subseteq 2^\omega$ contains a $\Delta^1_{\aleph-1}$ element. (The basis theorem, see Section 5A in [5].) Recall that $\aleph - 1 > 2$ by Assumption 1, and hence $\Pi^1_2 \subseteq \Sigma^1_{\aleph-1}$. It follows that the Π^1_2 set $X = 2^\omega \setminus \mathbf{L}$ of all nonconstructable reals contains a (nonconstructable) $\Delta^1_{\aleph-1}$ real $x \in 2^\omega$ in $\mathbf{L}[G]$. We conclude that $S = \{j : x(j) = 0\} \subseteq \omega$ is a nonconstructable $\Sigma^1_{\aleph-1}$ set in $\mathbf{L}[G]$, which contradicts Lemma 5. \square

Combining Theorem 3 with the result of Theorem 2, we conclude that $L[G]$ is a model for Theorem 1.

11. Conclusions and Problems

In this study, the method of finite-support products of Jensen's forcing was employed to the problem of obtaining a model of ZFC in which, for a given $n \geq 3$, good well-orderings of the reals exist in the class Δ_n^1 but do not exist in Δ_{n-1}^1 . This result (Theorem 1 of this paper) continues our series of recent research, such as a model defined in [24] for a given n , in which there is a Π_n^1 Vitali equivalence class containing no ordinal-definable elements, whereas every countable Σ_n^1 -set of reals contains only ordinal-definable reals, or a model defined in [25] in which there is a Π_n^1 real singleton $\{a\}$ such that a codes a cofinal map $f: \omega \rightarrow \omega_1^L$, whereas every Σ_n^1 set $X \subseteq \omega$ is constructible and hence cannot code a cofinal map $\omega \rightarrow \omega_1^L$, or a very recent model defined in [26] in which the separation principle holds for a given class Σ_n^1 for sets of integers. Theorem 1 may also be a step towards the solution of the all-important problem by S. D. Friedman mentioned in the introduction above (Section 1).

From our study, it is concluded that the technique of *definable generic* inductive constructions of forcing notions in L , developed for Jensen-type product forcing in our earlier papers [16,19], leads to a new result (Theorem 1), which is a significant advance toward solving an important set theoretic problem formulated in the introduction as Problem 1.

From the result of Theorem 1, we immediately come to the following problems.

Problem 2. *Prove that it is true in the key model $L[G]$ of Section 5 that there is no “boldface” Δ_{n-1}^1 well-ordering of the reals of any kind (that is, not necessarily Δ_{n-1}^1 -good).*

Such a strengthening of Theorem 1 would solve Problem 1 of Section 1 completely.

Problem 3. *Prove a version of Theorem 1 with the additional requirement that the negation $2^{\aleph_0} > \aleph_1$ of the continuum hypothesis holds in the generic extension considered.*

To comment upon Problem 3, note that the model for Theorem 1 introduced in Section 5 (the key model) definitely satisfies the continuum hypothesis $2^{\aleph_0} = \aleph_1$. The problem of obtaining models of ZFC in which $2^{\aleph_0} > \aleph_1$ and there is a projective well-ordering of the continuum, has been known since the early years of modern set theory. See, for example, problem 3214 in an early survey [27] by Mathias. Harrington [28] solved this problem by a generic model in which $2^{\aleph_0} > \aleph_1$ and there is a Δ_3^1 well-ordering of the continuum, by a combination of methods based on different forcing notions, such as the almost-disjoint forcing [29] and the forcing notion by Jensen and Johnsbråten [30]. See [13] for further remarkable progress in forcing constructions of models with long projective well-orderings of low projective classes. Solving Problem 3 would be a further significant step in this direction.

Author Contributions: Conceptualization, V.K. and V.L.; methodology, V.K. and V.L.; validation, V.K.; formal analysis, V.K. and V.L.; investigation, V.K. and V.L.; writing—original draft preparation, V.K.; writing—review and editing, V.K. and V.L.; project administration, V.L.; funding acquisition, V.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by Russian Foundation for Basic Research RFBR grant number 20-01-00670.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable. The study does not report any data.

Acknowledgments: We thank the anonymous reviewers for their thorough reviews and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the publication.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Hadamard, J.; Baire, R.; Lebesgue, H.; Borel, E. Cinq lettres sur la théorie des ensembles. *Bull. Soc. Math. Fr.* **1905**, *33*, 261–273. [[CrossRef](#)]
2. Lusin, N. Sur la classification de M. Baire. *C. R. Acad. Sci. Paris* **1917**, *164*, 91–94.
3. Sierpiński, W. L'axiome de M. Zermelo et son rôle dans la théorie des ensembles et l'analyse. *Bull. Acad. Sci. Cracovie* **1918**, 97–152.
4. Kechris, A.S. *Classical Descriptive Set Theory*; Springer: New York, NY, USA, 1995; pp. xviii+402.
5. Moschovakis, Y.N. Descriptive set theory. In *Studies in Logic and the Foundations of Mathematics*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1980; Volume 100, pp. XII+637.
6. Gödel, K. *The Consistency of the Continuum Hypothesis*; Annals of Mathematics Studies, no. 3; Princeton University Press: Princeton, NJ, USA, 1940; p. 66.
7. Addison, J.W. Some consequences of the axiom of constructibility. *Fundam. Math.* **1959**, *46*, 337–357. [[CrossRef](#)]
8. Levy, A. Definability in Axiomatic Set Theory II. In *Studies in Logic and the Foundations of Mathematics*; Bar-Hillel, Y., Ed.; North-Holland: Amsterdam, The Netherlands; London, UK, 1970; pp. 129–145.
9. Solovay, R.M. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math.* **1970**, *92*, 1–56. [[CrossRef](#)]
10. Caicedo, A.E.; Friedman, S.D. BPFA and projective well-orderings of the reals. *J. Symb. Log.* **2011**, *76*, 1126–1136. [[CrossRef](#)]
11. Fischer, V.; Friedman, S.D. Cardinal characteristics and projective wellorders. *Ann. Pure Appl. Logic* **2010**, *161*, 916–922. [[CrossRef](#)]
12. Fischer, V.; Friedman, S.D.; Zdomskyy, L. Cardinal characteristics, projective wellorders and large continuum. *Ann. Pure Appl. Log.* **2013**, *164*, 763–770. [[CrossRef](#)]
13. Fischer, V.; Friedman, S.D.; Khomskii, Y. Measure, category and projective wellorders. *J. Log. Anal.* **2014**, *6*, 1–25.
14. Friedman, S.; Hoffelner, S. A Σ_4^1 wellorder of the reals with NS_{ω_1} saturated. *J. Symb. Log.* **2019**, *84*, 1466–1483. [[CrossRef](#)]
15. Jensen, R. Definable sets of minimal degree. In *Studies in Logic and the Foundations of Mathematics*; Bar-Hillel, Y., Ed.; North-Holland: Amsterdam, The Netherlands; London, UK, 1970; Volume 59, pp. 122–128.
16. Kanovei, V.; Lyubetsky, V. The full basis theorem does not imply analytic wellordering. *Ann. Pure Appl. Log.* **2021**, *172*, 46. [[CrossRef](#)]
17. Friedman, S.D. *Fine Structure and Class Forcing*. In *De Gruyter Series in Logic and Its Applications*; de Gruyter: Berlin, Germany, 2000; Volume 3, pp. x+221.
18. Friedman, S.D. Constructibility and class forcing. In *Handbook of Set Theory*; Springer: Dordrecht, The Netherlands, 2010; Volume 3, pp. 557–604.
19. Kanovei, V.; Lyubetsky, V. Models of set theory in which separation theorem fails. *Izv. Math.* **2021**, *85*, 1181–1219. [[CrossRef](#)]
20. Jech, T. *Set Theory*; The Third Millennium Revised and Expanded ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2003; pp. xiii+769. [[CrossRef](#)]
21. Enayat, A. On the Leibniz – Mycielski axiom in set theory. *Fundam. Math.* **2004**, *181*, 215–231. [[CrossRef](#)]
22. Kanovei, V.; Lyubetsky, V. Counterexamples to countable-section Π_2^1 uniformization and Π_3^1 separation. *Ann. Pure Appl. Logic* **2016**, *167*, 262–283. [[CrossRef](#)]
23. Friedman, S.D.; Gitman, V.; Kanovei, V. A model of second-order arithmetic satisfying AC but not DC. *J. Math. Log.* **2019**, *19*, 850013. [[CrossRef](#)]
24. Kanovei, V.; Lyubetsky, V. Definable E_0 classes at arbitrary projective levels. *Ann. Pure Appl. Log.* **2018**, *169*, 851–871. [[CrossRef](#)]
25. Kanovei, V.; Lyubetsky, V. Definable minimal collapse functions at arbitrary projective levels. *J. Symb. Log.* **2019**, *84*, 266–289. [[CrossRef](#)]
26. Kanovei, V.; Lyubetsky, V. A model in which the separation principle holds for a given effective projective sigma-class. *Axioms* **2022**, *11*, 122. [[CrossRef](#)]
27. Mathias, A.R.D. Surrealist landscape with figures (a survey of recent results in set theory). *Period. Math. Hung.* **1979**, *10*, 109–175. (The Original Preprint of This Paper Is Known in Typescript Since 1968 under the Title “A Survey of Recent Results in Set Theory”). [[CrossRef](#)]
28. Harrington, L. Long projective wellorderings. *Ann. Math. Log.* **1977**, *12*, 1–24. [[CrossRef](#)]
29. Jensen, R.B.; Solovay, R.M. Some applications of almost disjoint sets. In *Studies in Logic and the Foundations of Mathematics*; Bar-Hillel, Y., Ed.; North-Holland: Amsterdam, The Netherlands; London, UK, 1970; Volume 59, pp. 84–104. [[CrossRef](#)]
30. Jensen, R.B.; Johnsbraten, H. A new construction of a non-constructible Δ_3^1 subset of ω . *Fundam. Math.* **1974**, *81*, 279–290. [[CrossRef](#)]