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Jensen Δ^1_n Reals by Means of ZFC and Second-Order Peano Arithmetic

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Abstract: It was established by Jensen in 1970 that there is a generic extension $\mathbf{L}[a]$ of the constructible universe $\mathbf{L}$ by a non-constructible real $a \notin \mathbf{L}$, minimal over $\mathbf{L}$, such that a is Δ^1_3 in $\mathbf{L}[a]$. Our first main theorem generalizes Jensen's result by constructing, for each $n \geq 2$, a generic extension $\mathbf{L}[a]$ by a non-constructible real $a \notin \mathbf{L}$, still minimal over $\mathbf{L}$, such that a is Δ^1_{n+1} in $\mathbf{L}[a]$ but all Σ^1_n reals are constructible in $\mathbf{L}[a]$. Jensen's forcing construction has found a number of applications in modern set theory. A problem was recently discussed as to whether Jensen's construction can be reproduced entirely by means of second-order Peano arithmetic $\mathbf{PA}_2$, or, equivalently, $\mathbf{ZFC}^-$ (minus the power set axiom). The obstacle is that the proof of the key CCC property (whether by Jensen's original argument or the later proof using the diamond technique) essentially involves countable elementary submodels of $\mathbf{L}_{\omega_2}$, which is way beyond $\mathbf{ZFC}^-$. We demonstrate how to circumvent this difficulty by means of killing only definable antichains in the course of a Jensen-like transfinite construction of the forcing notion, and then use this modification to define a model with a minimal Δ^1_{n+1} real as required as a class-forcing extension of a model of $\mathbf{ZFC}^-$ plus $\mathbf{V} = \mathbf{L}$.

Keywords: forcing; projective well-orderings; projective classes; Peano arithmetic

MSC: 03E15; 03E35

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1. Introduction

This paper contains two main results. The following theorem is the first main result. Theorem 2 below is the second main result, along with the associated technique of consistency proofs by class forcing over the power-less set theory $\mathbf{ZFC}^-$, or, equivalently, the second-order Peano arithmetic $\mathbf{PA}_2$.

Studies of the relationship between Gödel's constructibility and the analytic definability of the reals (here: points of the Cantor discontinuum 2^ω or the Baire space ω^ω) began with a profound study by Shoenfield [1], in which it was established that all Σ^1_2 reals are constructible. With the development of the forcing method in the 1960s, various models of Zermelo–Fraenkel set theory $\mathbf{ZFC}$ were proposed, in which there exists a non-constructible real analytically definable above the Shoenfield level Σ^1_2; see a survey [2] by Mathias. Of those, the strongest result was obtained by Jensen [3], and it corresponds to the case $\mathbb{n} = 2$ of Theorem 1. A similar result for $\mathbb{n} = 2$, but in the absence of (ii) and (iii), was obtained by Jensen and Solovay [4] using a different technique. Further research in this direction included, in particular, studies of Solovay [5] on Δ^1_3 reals under large cardinal assumptions, Abraham [6] on definable reals coding minimal collapse functions $\omega \stackrel{\text{onto}}{\longrightarrow} \aleph_1^{\mathbf{L}}$, Harrington [7] on definable well-orderings of the reals in the absence of the continuum hypothesis, David [8] on non-constructible Δ^1_3 reals, Jensen and Johnsbraten [9] on Δ^1_3 reals, any pair of which entails a collapse function $\omega \stackrel{\text{onto}}{\longrightarrow} \aleph_1^{\mathbf{L}}$, and many more.

Theorem 1. *If $\mathbb{n} \geq 2$, then there exists a generic extension of the universe $\mathbf{L}$ in which it holds that:*

(i) *There is a nonconstructible* $\Delta^1_{\mathbb{n}+1}$ *real* $a \in 2^\omega$ *, such that:*
(ii) $\mathbf{V} = \mathbf{L}[a]$ *holds;*
(iii) a *is minimal over* $\mathbf{L}$*, in the sense that* $a \notin \mathbf{L}$ *but any real* $x \in \mathbf{L}[a]$ *either belongs to* $\mathbf{L}$ *or satisfies* $a \in \mathbf{L}[x]$ *;*
(iv) *But all* $\Sigma^1_{\mathbb{n}}$ *sets* $x \subseteq \omega$ *are constructible and* $\Sigma^1_{\mathbb{n}}$ *in* $\mathbf{L}$ *.*

We have recently succeeded in proving a weaker version of Theorem 1, again without (ii) and (iii), but for all $\mathbb{n} \geq 2$, in [10], using essentially the same technique of forcing by almost disjoined sets as in [4], but modified by the method of the definable generic construction of the notion of forcing. This led us to the problem of incorporating (ii) and (iii) into this general result, and Theorem 1 solves this problem. The first part of this paper (Sections 2–13) contains the full proof of Theorem 1, using a similar definable generic modification of the forcing notion originally developed in [3], so that the level of definability is determined by the value of the parameter $\mathbb{n}$ in Theorem 1. This innovation, on top of Jensen's forcing, is the sine qua non of our proof of Theorem 1. See Section 9 for a sketch of the construction.

The second main result of the paper concerns an important aspect of the result above in the context somewhat similar to the "reverse mathematics" approach. Indeed, Theorem 1 essentially asserts, for any given $\mathbb{n} \geq 2$, the consistency of the conjunction

$$\text{(i)} \wedge \text{(ii)} \wedge \text{(iii)} \wedge \text{(iv)} \qquad (*)$$

with the axioms of **ZFC**. We may note here that the conjunction $(*)$ can be adequately and rather straightforwardly represented by means of a suitable formula of the language $\mathscr{L}(\mathbf{PA}_2)$ of $\mathbf{PA}_2$, second-order Peano arithmetic.

We recall that, following [11–13], second-order Peano arithmetic $\mathbf{PA}_2$ is a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables: for natural numbers and for sets of them. We use j, k, m, n for variables over ω and x, y, z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets. The axioms are as follows in (1)–(5):

(1) *Peano's axioms* for numbers.
(2) *Induction* as one sentence: $\forall x\,\big(0 \in x \wedge \forall n\,(n \in x \implies n+1 \in x) \implies \forall n\,(n \in x)\big)$.
(3) *Extensionality* for sets of natural numbers.
(4) The *Comprehension* schema **CA**: $\exists x\,\forall k\,(k \in x \iff \Phi(k))$, for every formula Φ in which x does not occur, and, in Φ, we allow for parameter-free variables other than k.
(5) The schema $\mathbf{AC}_\omega$ of *Countable Choice*: $\forall k\,\exists x\,\Phi(k,x) \implies \exists x\,\forall k\,\Phi(k,(x)_k))$, for every formula Φ with parameters allowed, where $(x)_k = \{j : 2^k(2j+1) - 1 \in x\}$.

The theory $\mathbf{PA}_2$ is also known as A_2 (see, e.g., an early survey [11]), az Z_2 (in [14] or elsewhere). See also [15].

The analytical representation of Gödel's constructibility is well known since the 1950s; see, e.g., Addison [16], Apt and Marek [11], and Simpson's book [13]. This raises *the problem of the consistency* of (the analytical form of) $(*)$ under the assumption that only the consistency of $\mathbf{PA}_2$ as a premise is available, rather than the (much stronger) consistency of **ZFC**. This is why we consider and solve this problem in our paper.

The working technique of such a transformation of the consistency results related to **ZFC** to the basis of $\mathbf{PA}_2$ is also rather well known since some time ago. (See, e.g., Guzicki [17].) It makes use of $\mathbf{ZFC}^-$ as a proxy theory.

We recall that the power-less set theory $\mathbf{ZFC}^-$ is a subtheory of **ZFC** obtained so that:

(a) The power set axiom **PS** is excluded;
(b) The *well-orderability axiom* **WA**, which claims that every set can be well ordered, is substituted for the usual set-theoretic axiom of choice **AC** of **ZFC**;
(c) the separation schema is preserved, but the replacement schema (which is not sufficiently strong in the absence of **PS**) is substituted with the *collection* schema: $\forall X\,\exists Y\,\forall x \in X\,\big(\exists y\,\Phi(x,y) \implies \exists y \in Y\,\Phi(x,y)\big)$.

A comprehensive account of main features of $\mathbf{ZFC}^-$ is given in, e.g., [18–20].

Theories $\mathbf{PA}_2$ and $\mathbf{ZFC}^-$ are known to be equiconsistent (Kreisel [12], Apt and Marek [11]; see Section 18 for more details on this equiconsistency claim), so we can make use of $\mathbf{ZFC}^-$ as the background theory instead of $\mathbf{PA}_2$. If now we have established the consistency of a $\mathscr{L}(\mathbf{PA}_2)$-sentence S by means of a generic extension of $\mathbf{L}$, the constructible universe, via a forcing notion $P \in \mathbf{L}$, then we check if P can be defined in $\mathbf{ZFC}^-$ as a set or class in $\mathbf{L}$ and whether $\mathbf{ZFC}^-$ is strong enough to prove that P-generic extensions of $\mathbf{L}$ model S. And, if yes, then we have a proof of the consistency of S with $\mathbf{PA}_2$ on the basis of the consistency of $\mathbf{PA}_2$ alone.

Such a method (sketched, e.g., in [17]), however, does not seem to immediately work even for the result in [3] ($\mathbb{n} = 2$ of Theorem 1). Indeed, the construction of Jensen's forcing notion $\mathbb{P}$ (either using Jensen's [3] original method or via the diamond principle $\diamondsuit_{\omega_1}$ as in ([21] 28.A) in $\mathbf{L}$ does not directly work in $\mathbf{ZFC}^-$ because the proof of the key CCC property (the countable chain condition) and some other involved properties of $\mathbb{P}$, using either method, heavily depends on countable elementary submodels of $\mathbf{L}_{\omega_2}$, hence transitive models of $\mathbf{ZFC}^-$ itself, which is way beyond $\mathbf{ZFC}^-$. In the second part of this paper (Sections 14–18), we circumvent this difficulty by means of the method of killing only antichains that belong to a certain transitive model of the bounded separation axiom instead of the full separation as in $\mathbf{ZFC}^-$, in the course of a Jensen-like transfinite construction of the forcing notion. This innovation is not a trivial and easily seen modification, and we may observe that not all mathematically meaningful results about hereditarily countable sets, and countable ordinals in particular, can be rendered on the $\mathbf{ZFC}^-$ basis; see, e.g., [22]. The relevant changes are concentrated in Definition 10 and Condition 4^+ in Section 15.

Thereby, the following theorem is the second main result of this paper.

Theorem 2. *If* $\mathbb{n} \geq 2$, *then the conjunction* $(i) \wedge (ii) \wedge (iii) \wedge (iv)$ *of items of Theorem 1 is consistent with* $\mathbf{PA}_2$ *provided that* $\mathbf{PA}_2$ *itself is consistent.*

2. Preliminaries

Let $\omega^{<\omega}$ be the set of all strings (finite sequences) of natural numbers. Accordingly, $2^{<\omega} \subseteq \omega^{<\omega}$ is the set of all dyadic strings. If $t \in \omega^{<\omega}$ and $k < \omega$, then $t^\frown k$ is the extension of t by k as the rightmost term. If $s, t \in \omega^{<\omega}$, then $s \subseteq t$ means that t extends s, while $s \subset t$ means a proper extension of strings.

If $s \in \omega^{<\omega}$, then $\mathtt{lh}\, s$ is the length of s, and $\omega^n = \{s \in \omega^{<\omega} : \mathtt{lh}\, s = n\}$ (strings of length n), and, accordingly, $2^n = \omega^n \cap 2^{<\omega} = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$.

A set $T \subseteq \omega^{<\omega}$ is a *tree* iff, for any strings $s \subset t$ in $\omega^{<\omega}$, if $t \in T$ then $s \in T$. Thus, every non-empty tree $T \subseteq \omega^{<\omega}$ contains the empty string Λ. If $T \subseteq \omega^{<\omega}$ is a tree and $s \in T$, then put $T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$; this is a tree as well.

Let $\mathbf{PT}$ be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus, a non-empty tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ iff it has no endpoints and no isolated branches. In this case, there is a largest string $s \in T$ such that $T = T \restriction_s$; it is denoted by $s = \mathtt{root}(T)$ (the *root* of a perfect tree T). If $s = \mathtt{root}(T)$, then s is a *branching node* of T; that is, $s^\frown 1 \in T$ and $s^\frown 0 \in T$.

Each perfect tree $T \in \mathbf{PT}$ defines a perfect set $[T] = \{a \in 2^\omega : \forall n\, (a \restriction n \in T)\} \subseteq 2^\omega$ of all paths through T; then, accordingly, $T = \mathbf{tree}\,([T])$, where

$$\mathbf{tree}\,(X) = \{a \restriction n : a \in X \wedge n \in \omega\} \subseteq 2^{<\omega} \quad \text{for any set} \quad X \subseteq 2^\omega.$$

If $S \subseteq T$ are trees in $\mathbf{PT}$ and there is a finite set $A \subseteq T$ such that $S = \bigcup_{s \in A} T \restriction_s$, then we say that S is *clopen* in T; then, $[S]$ is a relatively clopen subset of $[T]$. Trees clopen in $2^{<\omega}$ itself will be called simply *clopen*; thus, clopen trees are those of the form $S = \bigcup_{s \in A} [s]$, where $A \subseteq 2^{<\omega}$ is a finite set and $[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$ for each $s \in 2^{<\omega}$.

A set $A \subseteq \mathbf{PT}$ is a *true antichain* iff $[T] \cap [S] = \varnothing$ (or, equivalently, $S \cap T$ is finite) for all $S \neq T$ in A. If $X \subseteq \mathbf{PT}$, then a set $D \subseteq X$ is:

– *Dense in* X, iff, for every tree $T \in X$, there is a subtree $S \in D$, $S \subseteq T$;

– *Open dense in* X, iff it is dense in X and $T \in D$ holds whenever $T \in X$, $S \in D$, and $T \subseteq S$;
– *Pre-dense in* X, iff the set $D^+ = \{T \in X : \exists S \in D\,(T \subseteq S)\}$ is dense in X.

As usual, if $T \in \mathbf{PT}$, $D \subseteq \mathbf{PT}$, and there is a finite set $A \subseteq D$ such that $T \subseteq \bigcup A$ (or, equivalently, $[T] \subseteq \bigcup_{S \in A}[S]$) then we write $T \subseteq^{\mathtt{fin}} \bigcup D$, and if, in addition, A is a true antichain, then we write $T \subseteq^{\mathtt{fd}} \bigcup D$.

Thus, perfect sets in the Cantor space $\mathbb{C} = 2^\omega$ are straightforwardly coded by perfect trees in $\mathbf{PT}$. It takes more effort to introduce a reasonable coding system for continuous functions $F : 2^\omega \to \omega^\omega$. Let $\mathbf{FPT}$ (functional perfect trees) be the set of all sets $c \subseteq 2^{<\omega} \times \omega^{<\omega}$ such that

(a) If $\langle s, u\rangle \in c$, then $\mathtt{lh}\, s = \mathtt{lh}\, u$;
(b) c is a tree; that is, if $\langle s, u\rangle \in c$ and $n < \mathtt{lh}\, s = \mathtt{lh}\, u$, then $\langle s \restriction n, u \restriction n\rangle \in c$;
(c) $\mathtt{dom}\, c = 2^{<\omega}$; that is, $\forall\, s \in 2^{<\omega}\, \exists\, u \in \omega^{<\omega}\, (\langle s, u\rangle \in c)$;
(d) c has no endpoints; that is, if $\langle s, u\rangle \in c$ and $\ell \in \{0, 1\}$, then there is $k < \omega$ such that $\langle s^\frown \ell, u^\frown k\rangle \in c$;
(e) For every m, there exists $k \geq m$ such that if $s \in 2^k$, then there is a string $u_s \in \omega^m$ satisfying $\forall\, u \in \omega^{<\omega}\, (\langle s, u\rangle \in c \Longrightarrow u_s \subseteq u)$.

If $F : 2^\omega \to \omega^\omega$ is continuous, then the set $c = \mathbf{cod}(F)$, where

$$\mathbf{cod}(F) = \{\langle a \restriction n, F(a) \restriction n\rangle : a \in 2^\omega \wedge n < \omega\} \quad \text{for any map} \quad F : 2^\omega \to \omega^\omega,$$

belongs to $\mathbf{FPT}$ (condition (e) represents the uniform continuity of F defined on a compact space), and $\mathbb{f}_{\mathbf{cod}(F)} = F$, where

$$\mathbb{f}_c = \{\langle a, b\rangle \in 2^\omega \times \omega^\omega : \forall\, m\, \langle a \restriction m, b \restriction m\rangle \in c)\}, \quad \text{for every} \quad c \in \mathbf{FPT}.$$

(a function coded by c). Conversely, if $c \in \mathbf{FPT}$, then $\mathbf{cod}(\mathbb{f}_c) = c$.

Lemma 1 (well known)**.** *If $T \in \mathbf{PT}$ and $c \in \mathbf{FPT}$, then either there is a string $s \in T$ such that the restriction $\mathbb{f}_c \restriction [T \restriction_s]$ is a constant, or there is a subtree $S \in \mathbf{PT}$, $S \subseteq T$, such that the restriction $\mathbb{f}_c \restriction [S]$ is an injection.*

3. Splitting Systems of Trees

If $T \in \mathbf{PT}$ and $i = 0, 1$, then let $T[\to i] = T \restriction_{r^\frown i}$, where $r = \mathtt{root}(T)$; obviously, $T[\to i]$ are trees in $\mathbf{PT}$ as well. Define $T[\to s]$ for $s \in 2^{<\omega}$ by induction on $\mathtt{lh}\, s$ so that $T[\to \Lambda] = T$ and $T[\to s^\frown i] = T[\to s][\to i]$.

A *splitting system* is any indexed set $\langle T_s\rangle_{s \in 2^{<\omega}}$ of trees $T_s \in \mathbf{PT}$ satisfying

(A) If $s \in 2^{<\omega}$ and $i = 0, 1$, then $T_{s^\frown i} \subseteq T_s[\to i]$.

It easily follows from (A) that

(B) $s \subseteq s' \Longrightarrow T_{s'} \subseteq T_s$; and
(C) If $n < \omega$ and strings $s \neq t$ belong to 2^n, then $[T_s] \cap [T_t] = \varnothing$.

Lemma 2 (routine)**.** *If $\langle T_s\rangle_{s \in 2^{<\omega}}$ is a splitting system, then $T = \bigcap_n \bigcup_{s \in 2^n} T_s$ is a perfect subtree of T_Λ, and $[T] = \bigcap_n \bigcup_{s \in 2^n} [T_s]$. In addition, $[T \restriction_s] = [T] \cap [T_s]$ for all s.*

We proceed to several slightly more complicated applications.

Lemma 3. *If $\{T^n : n < \omega\} \subseteq \mathbf{PT}$, then there exists a sequence of trees $S^n \in \mathbf{PT}$ such that $S^n \subseteq T^n$ for all n and $[S^k] \cap [S^n] = \varnothing$ whenever $k \neq n$.*

Proof. If $T, T' \in \mathbf{PT}$, then there are perfect trees $S \subseteq T$ and $S' \subseteq T'$ such that $[S] \cap [S'] = \varnothing$. This allows us to easily define a system $\langle T_s(k)\rangle_{s \in 2^{<\omega}, k < \omega}$ of trees $T_s(k) \in \mathbf{PT}$ such that

(1) If $k < \omega$, then $\langle T_s(k)\rangle_{s \in 2^{<\omega}}$ is a splitting system consisting of subtrees of T^k;

(2) If $k < \ell < n < \omega$ and $s, t \in 2^n$, then $[T_s(k)] \cap [T_t(\ell)] = \varnothing$.

(The inductive construction is arranged so that, at each step n, we define all trees $T_s(k)$ with $k < n$ and $s \in 2^n$ and also all trees $T_s(n)$ with $\mathtt{lh}\, s \leq n$.) Now, we simply put $S^k = \bigcap_n \bigcup_{s \in 2^n} T_s(k)$ for all k. □

Lemma 4. *If $\{T^n : n < \omega\} \subseteq \mathbf{PT}$ and $F : 2^\omega \to \omega^\omega$ is continuous, then there exist perfect trees $S^n \subseteq T^n$ such that either $F(a) \notin \bigcup_n [S^n]$ for all $a \in [S^0]$ or $F(a) = a$ for all $a \in [S^0]$.*

Proof. Suppose that $F(a_0) \neq a_0$ for some $a_0 \in [T^0]$. By continuity of F, there are a clopen subtree $S \subseteq T^0$ and a clopen neighborhood A of $F(a_0)$ such that $F([S]) \subseteq A$ and $[S] \cap A = \varnothing$. Hence, $F(a) \notin [S]$ for all $a \in [S]$. The compact set $X = F([S])$ is either countable or has a perfect subset. If X is countable, then let $S^0 = S$ and, for every, $n \geq 1$ let $S^n \subseteq T^n$ be an arbitrary perfect tree such that $[S^n] \subseteq [T^n] \smallsetminus X$.

Assume that there is a perfect tree T such that $[T] \subseteq X$. By Lemma 3, there are trees $U^n \in \mathbf{PT}$ such that $U^0 \subseteq T$, $U^{n+1} \subseteq T^{n+1}$, and $[U^k] \cap [U^n] = \varnothing$ whenever $k \neq n$. Choose $S^0 \in \mathbf{PT}$ such that $[S^0] \subseteq [S] \cap F^{-1}([U^0])$ and let $S^{n+1} = U^{n+1}$. □

4. Jensen's Construction: Overview

Beginning the proof of case $\mathbb{n} = 2$ of Theorem 1, we list essential properties of Jensen's forcing $\mathbb{P} \in \mathbf{L}$:

(1) $\mathbb{P}$ consists of perfect trees $T \subseteq 2^{<\omega}$ (a subset of the Sacks forcing);

(2) $\mathbb{P}$ forces that there is a unique $\mathbb{P}$-generic real;

(3) "being a $\mathbb{P}$-generic real" is a Π^1_2 property;

(4) $\mathbb{P}$ forces that the generic real is (nonconstructible and) minimal.

Thus, $\mathbb{P}$ forces a nonconstructible Π^1_2 real singleton $\{a\}$ over $\mathbf{L}$, whose only element is, therefore, a Δ^1_3 real in $\mathbf{L}[a]$.

Jensen [3] defined a forcing $\mathbb{P}$ in $\mathbf{L}$ in the form $\mathbb{P} = \bigcup_{\alpha < \omega_1} \mathbb{P}_\alpha$, where each $\mathbb{P}_\alpha$ is a countable collection of perfect trees $T \subseteq 2^{<\omega}$. The construction of the ω_1-sequence of sets $\mathbb{P}_\alpha$ is arranged so that each $\mathbb{P}_\alpha$ is generic, in a certain sense, over the least transitive model of a suitable fragment of $\mathbf{ZFC}$, containing the subsequence $\langle \mathbb{P}_\gamma \rangle_{\gamma < \alpha}$. A striking corollary of such a genericity is that $\mathbb{P}$ forces that there is only one $\mathbb{P}$-generic real. Another corollary consists in the fact that, for a real $x \in 2^\omega$, being $\mathbb{P}$-generic is equivalent to $x \in \bigcap_{\alpha < \omega_1} \bigcup_{T \in \mathbb{P}_\alpha} [T]$. The construction can be managed so that the whole sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1}$ is Δ^1_2, or, more exactly, Δ^{HC}_1 in $\mathbf{L}$. (We recall that HC = all *hereditarily countable* sets. A set x is hereditarily countable iff its transitive closure is at most countable.) Altogether, it follows that if $a \in 2^\omega$ is a $\mathbb{P}$-generic real, then $\{a\} \in \Pi^1_2$ in $\mathbf{L}[a]$; that is, $a \in \Delta^1_3$ in $\mathbf{L}[a]$, which is obviously the lowest possible level for a nonconstructible real. The minimality of $\mathbb{P}$-generic reals follows from another property of $\mathbb{P}$: given a tree $S \in \mathbb{P}$ and a continuous $F : 2^\omega \to \omega^\omega$, there is a tree $T \in \mathbb{P}$, $T \subseteq S$ (a stronger condition) such that $F \restriction [T]$ is either a bijection or a constant.

Now, we consider this construction in detail.

5. Jensen's Sequences

In this section, *we argue in* $\mathbf{L}$.

See Section 1 regarding matters of the power-less set theory $\mathbf{ZFC}^-$. Let $\mathbf{ZFC}^-_{\mathscr{P}(\omega)}$ be the theory: $\mathbf{ZFC}^-$ + "the set $\mathscr{P}(\omega)$ exists" (then ω_1 exists as well) + "$\mathbf{V} = \mathbf{L}$". Note that $\mathbf{L}_{2^\omega}$ (all sets constructible up to 2^ω) is a natural model of $\mathbf{ZFC}^-_{\mathscr{P}(\omega)}$.

Definition 1 (in $\mathbf{L}$). *Suppose that $\alpha < \omega_1$ and $\langle X_\beta \rangle_{\beta < \alpha}$ is a sequence of hereditarily countable sets. We let $\mathbf{M}(\langle X_\beta \rangle_{\beta < \alpha})$ be the least CTM $M \models \mathbf{ZFC}^-_{\mathscr{P}(\omega)}$, necessarily of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\langle X_\beta \rangle_{\beta < \alpha}$ and such that $\alpha < \omega_1^M$ strictly and still all sets X_β, $\beta < \alpha$, are, at most, countable in M.*

Definition 2 (in $\mathbf{L}$)**.** *If $\alpha < \omega_1$, then let $\langle T_\alpha, c_\alpha, \nu_\alpha \rangle$ be the αth element of the set $\mathbf{PT} \times \mathbf{FPT} \times \omega_1$ in the sense of the Gödel canonical well-ordering of $\mathbf{L}$.*

Thus, for any $T \in \mathbf{PT}$ and $c \in \mathbf{FPT}$, there exist uncountably many indices $\alpha < \omega_1$ such that $T = T_\alpha$ and $c = c_\alpha$.

For any ordinal $\lambda \le \omega_1$, we let $\mathbf{J}_\lambda$ (Jensen's sequences of length λ) be the set of all sequences $\langle P_\alpha \rangle_{\alpha<\lambda}$, of countable sets $P_\alpha \subseteq \mathbf{PT}$, satisfying the following conditions 1°–6°.

1°. P_0 consists of all clopen trees $\varnothing \ne S \subseteq 2^{<\omega}$, including the full tree $2^{<\omega}$ itself.

2°. If $\alpha < \lambda$, $T \in P_\alpha$, and $S \subseteq T$ is a perfect tree clopen in T, then $S \in P_\alpha$.

3°. If $\alpha < \lambda$ and $S \in P_{<\alpha} = \bigcup_{\beta<\alpha} P_\beta$, then there is a tree $T \in P_\alpha$, $T \subseteq S$.

4°. If $\alpha < \lambda$, $T \in P_\alpha$, $D \in \mathbf{M}(\langle P_\beta \rangle_{\beta<\alpha})$, $D \subseteq P_{<\alpha}$ is open dense in $P_{<\alpha}$, then $T \subseteq^{\mathtt{fd}} \bigcup D$.

5°. If $\alpha < \omega_1$, $c = c_\alpha$, and $S = T_\alpha \in P_{<\alpha}$, then there is $T \in P_\alpha$ such that $T \subseteq S$ and:

either we have $\mathbb{f}_c(a) \notin \bigcup_{T' \in P_\alpha} [T']$ for all $a \in [T]$,

or we have $\mathbb{f}_c(a) = a$ for all $a \in [T]$.

6°. If $\alpha < \omega_1$, $c = c_\alpha$, and $S = T_\alpha \in P_{<\alpha}$, then there exists $T \in P_\alpha$ such that $T \subseteq S$ and the restricted function $\mathbb{f}_c \restriction [T]$ is either a bijection or a constant.

Let $\mathbf{J}_{<\lambda} = \bigcup_{\alpha<\omega_1} \mathbf{J}_\alpha$. (Jensen's sequences of any countable length).

Lemma 5 (in $\mathbf{L}$)**.** *Suppose that $\beta < \lambda \le \omega_1$ and $\langle P_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}_\lambda$. Then, P_β is pre-dense in the set $P_{<\lambda} = \bigcup_{\alpha<\lambda} P_\alpha$.*

Proof. First, P_β is dense in $P_{<\beta+1}$ by 3°. Now, by induction on λ, suppose that P_β is pre-dense in $P_{<\lambda}$. To check that P_β remains pre-dense in $P_{<\lambda+1} = P_{<\lambda} \cup P_\lambda$, consider any tree $T \in P_\lambda$. By definition, $P_\beta \in \mathbf{M}(\langle P_\beta \rangle_{\beta<\lambda})$, and hence we have $T \subseteq^{\mathtt{fd}} \bigcup P_\beta$ by 4°. (Note that the set $P_\beta^+ = \{S \in P_{<\lambda} : \exists S' \in P_\beta\, (S \subseteq S')\}$ belongs to $\mathbf{M}(\langle P_\beta \rangle_{\beta<\lambda})$ and is open dense.) It follows that there exist a tree $S \in P_\beta$ and a string $s \in T$ such that $T \restriction_s \subseteq S$. Finally, $T' = T \restriction_s \in P_\lambda$ by 2°, so T is compatible with $S \in P_\beta$, as required. □

Lemma 6 (in $\mathbf{L}$)**.** *Assume that $\langle P_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$. Then, the forcing $P = \bigcup_{\alpha<\omega_1} P_\alpha \in \mathbf{L}$ satisfies CCC in $\mathbf{L}$. Therefore the cardinals are preserved in P-generic extensions of $\mathbf{L}$.*

We recall that CCC, or the countable chain condition, is the claim that every antichain in a given partially ordered set is at most countable.

Proof. Arguing in $\mathbf{L}$, suppose that $A \subseteq P$ is a maximal P-antichain, that is, a pre-dense set, and, if $S \ne S'$ belongs to A, then there is no tree $T \in P$, $T \subseteq S \cap S'$. Consider a countable elementary submodel $\mathscr{M} \subseteq \mathbf{L}_{\omega_2}$ containing A. Let $\varphi : \mathscr{M} \xrightarrow{\text{onto}} \mathbf{L}_\gamma$ be the Mostowski collapse; $\gamma < \omega_1$. Let $\alpha = \varphi(\omega_1)$. Thus, $\mathscr{M} \models \mathbf{ZFC}^-_{\mathscr{P}(\omega)}$ and $\alpha = \omega_1^{\mathscr{M}}$. The set $A' = \varphi(A)$ satisfies $A' = A \cap P_{<\alpha}$ and is pre-dense in $P_{<\alpha} = \bigcup_{\beta<\alpha} P_\beta$. It remains to prove that $A = A'$.

Suppose toward the contrary that $T \in A \smallsetminus A' = A \smallsetminus P_{<\alpha}$. Then, T is compatible with some $T' \in P_\alpha$ by Lemma 5; that is, there is a tree $T'' \in P$, $T'' \subseteq T' \cup T$.

On the other hand, $\alpha = \omega_1^{\mathbf{L}_\gamma}$; hence, we have $\mathscr{M} \subseteq \mathbf{M}(\langle P_\beta \rangle_{\beta<\alpha})$ and $A' \in \mathbf{M}(\langle P_\beta \rangle_{\beta<\alpha})$. It easily follows from 4° that $T' \subseteq^{\mathtt{fd}} A'$. Then, $T'' \subseteq^{\mathtt{fd}} A'$ as well, and hence there exist $s \in T''$ and $S \in A'$ such that the tree $U = T'' \restriction_s$ satisfies $U \subseteq T'' \cap S$; therefore, $U \subseteq T \cap S$. However, $U \in P$ by 2°, and $S \in A'$ but $T \in A \smallsetminus A'$, contrary to A being a P-antichain. □

The following rather obvious lemma demonstrates that the top level of a Jensen sequence of successor length can be freely enlarged by adding smaller trees, with only care of the property 2°.

Lemma 7 (in $\mathbf{L}$)**.** *Suppose that $\lambda = \xi + 1 < \omega_1$ and $\langle P_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}_\lambda$, so that P_ξ is the last term in this sequence. Let $S \subseteq T$ be trees in $\mathbf{PT}$ and $T \in P_\xi$. Let P'_ξ consist of all trees in P_ξ and all trees $S' \in \mathbf{PT}$, $S' \subseteq S$, clopen in S. Then, the sequence $\langle P_\alpha \rangle_{\alpha<\xi}{}^\frown P'_\xi$ belongs to $\mathbf{J}_\lambda$, too.*

6. Extension of Jensen's Sequences

Now, we prove a theorem that shows that Jensen's sequences of any countable length are extendable to longer sequences in $\mathbf{L}$.

Theorem 3 (in $\mathbf{L}$)**.** *Suppose that $\lambda < \omega_1$. Then, any sequence $\langle P_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}_\lambda$ has an extension $\langle P_\alpha \rangle_{\alpha\le\lambda} \in \mathbf{J}_{\lambda+1}$.*

Proof. We argue in $\mathbf{L}$. Basically, we have to appropriately define the *top level* P_λ $(\lambda > 0)$ of the extended sequence. The definition goes on in four steps.

Step 1: we define a provisional set P_λ satisfying only requirements 3°, 4°. Put $\boldsymbol{M}_\lambda = \mathbf{M}(\langle P_\beta \rangle_{\beta<\lambda})$. Fix an arbitrary enumeration $\{D_n : n < \omega\}$ of all sets $D \in \boldsymbol{M}_\lambda$, $D \subseteq P_{<\lambda}$, open dense in $P_{<\lambda}$, and any enumeration $P_{<\lambda} = \{S^k : k < \omega\}$. For any k, there is a system $\langle T_s(k) \rangle_{s\in 2^{<\omega}}$ of trees $T_s(k) \in P_{<\lambda}$ satisfying the following conditions (i)–(iii):

(i) If $S = S^k \in P_{<\lambda}$, then $T_\Lambda(k) \subseteq S$;

(ii) For each k, $\langle T_s(k) \rangle_{s\in 2^{<\omega}}$ is a splitting system in the sense of Section 3;

(iii) If $n \ge 1$ and $s \in 2^n$, then $T_s(k) \in D_n$.

Indeed, if some $T_s(k) \in P_{<\lambda}$ is already defined and $n = \mathtt{lh}\, s$, then the trees $U_0 = T_s(k){\restriction}_0$ and $U_1 = T_s(k){\restriction}_1$ belong to $P_{<\lambda}$ as well, and hence there are trees $T_{s^\frown 0} \subseteq U_0$ and $T_{s^\frown 1} \subseteq U_1$ in $P_{<\lambda}$, which belong to D_{n+1}.

It remains to define $P_\lambda = \{T^k : k < \omega\}$, where $T^k = \bigcap_n \bigcup_{s\in 2^n} T_s(k)$.

Step 2. We are going to shrink the trees T^k obtained at Step 1 in order to satisfy requirement 5°. Suppose that $c = c_\lambda$ and $S = T_\lambda \in P_{<\lambda}$, as in 5°. (If $S \notin P_{<\alpha}$, then we skip this step.) We may assume that the enumeration $\langle T^k \rangle_{k<\omega}$ is chosen so that $T^0 \subseteq S$. Let $G = \mathbb{f}_c$ (a continuous map $2^\omega \to \omega^\omega$). By Lemma 4, there exist perfect trees $U^n \subseteq T^n$ such that either $G(a) \notin \bigcup_n [U^n]$ for all $a \in [U^0]$ or $G(a) = a$ for all $a \in [U^0]$. The new set $P_\lambda = \{U^k : k < \omega\}$ still satisfies 3° and 4°, of course.

Step 3. We shrink the trees $U^k \in \mathbf{PT}$ obtained at Step 2 in order to satisfy 6°. This is similar to Step 2, with the only difference being that we apply Lemma 1 instead of Lemma 4.

Step 4. If $V^k \in \mathbf{PT}$ is one of the trees in P_λ obtained at Step 3, then we adjoin all trees $\varnothing \ne S \subseteq V^k$ clopen in V^k to P_λ in order to satisfy 2°. □

7. Definable Jensen's Sequence

Each of the conditions 4°, 5°, 6° (Section 5) will have its own role. Namely, 4° implies CCC and continuous reading of names (Lemma 10) and 5° is responsible for the generic uniqueness of a_G as in Lemma 11, while 6° yields the minimality of a_G. However, to obtain the required type of definability of $\mathbb{J}$-generic reals in the extensions, we need to take care of the appropriate definability of a Jensen's sequence in $\mathbf{L}$.

Definition 3. *Recall that* HC *is the collection of all hereditarily countable sets.*

Σ_n^{HC} = *all sets $X \subseteq$ HC, definable in HC by a parameter-free Σ_n formula.*

$\mathbf{\Sigma}_n(\mathrm{HC})$ = *all $X \subseteq$ HC definable in HC by a Σ_n formula with sets in HC as parameters.*

Collections Π_n^{HC}, $\mathbf{\Delta}_n(\mathrm{HC})$, etc. are defined similarly. Something like $\Sigma_n^{\mathrm{HC}}(x)$, $x \in$ HC means that only x is admitted as a parameter. It is known that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ under $\mathbf{V} = \mathbf{L}$, and that Σ_n^{HC}, Π_n^{HC}, Δ_n^{HC} is the same as Σ_{n+1}^1, Π_{n+1}^1, Δ_{n+1}^1 for reals and sets of reals, modulo any appropriate coding, and the same with parameters.

Lemma 8 (in $\mathbf{L}$)**.** *The set $\{\langle \alpha, \boldsymbol{p} \rangle : \alpha < \omega_1 \wedge \boldsymbol{p} \in \mathbf{J}_\alpha\}$ is Δ_1^{HC}.*

Proof. Suppose that $J = \langle P_\beta \rangle_{\beta<\alpha}$ is a sequence (of any kind) of length $\lambda < \omega_1$, $\vartheta < \omega_1$, the set $\mathbf{L}_\vartheta$ contains J and is a model of $\mathbf{ZFC}^-$, and, for every $\alpha < \lambda$, the model $\boldsymbol{M}_\alpha = \mathbf{M}(\langle P_\beta \rangle_{\beta<\alpha})$ (defined in Definition 1) also belongs to $\mathbf{L}_\vartheta$. Then, the property of J being a Jensen sequence is absolute for $\mathbf{L}_\vartheta$. This yields a Σ_1 definition for the statement "J is a Jensen sequence" in the form: there is such-and-such ordinal ϑ such that $J \in \mathbf{L}_\vartheta$ and "J is a Jensen sequence" holds in $\mathbf{L}_\vartheta$. □

Corollary 1 (in $\mathbf{L}$). *There exists a Δ^{HC}_1 sequence $\boldsymbol{p} = \langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$.*

Proof. For every α, we define, by transfinite induction, $\mathbb{J}_\alpha$ to be the least set, in the sense of the Gödel Δ_1 well-ordering of $\mathbf{L}$, such that $\langle \mathbb{P}_\beta \rangle_{\beta\le\alpha} \in \mathbf{J}_{\alpha+1}$. To establish the definability type Δ^{HC}_1 of the sequence obtained, use Lemma 8. □

8. Adding One Jensen Real: Theorem 1, Case $\mathtt{n} = 2$

Here, we prove the case $\mathtt{n} = 2$ of Theorem 1.

Definition 4. *By Corollary 1, fix a sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1^{\mathbf{L}}} \in \mathbf{L}$ such that it is true in $\mathbf{L}$ that*

(1) $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$—will be used in Lemmas 9, 10, 11 and Corollary 2; and

(2) $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1}$ is a Δ^{HC}_1 sequence—will be used only in Corollary 2.

Put $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$.

Consider such a set $\mathbb{P} \subseteq \mathbf{PT}$ as a forcing notion over $\mathbf{L}$, the ground universe. It is ordered so that $S \subseteq T$ means that S is stronger as a forcing condition. Thus, $\mathbb{P}$, *Jensen's forcing* of [3] (see also [21], 28.A), consists of (some, not all) perfect trees by construction.

Lemma 9. *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathbf{L}$, then the intersection $\bigcap_{T\in G}[T]$ is a singleton $\{a_G\}$, $a_G \in 2^\omega$, and $G = \{T \in \mathbb{P} : a_G \in T\}$; hence, $\mathbf{L}[a_G] = \mathbf{L}[G]$.*

Proof. Make use of 2° of Section 5. □

Reals a_G, $G \subseteq \mathbb{P}$ being a $\mathbb{P}$-generic set over $\mathbf{L}$, are called *$\mathbb{P}$-generic over* $\mathbf{L}$. The next lemma provides a useful tool of representation for reals in $\mathbb{P}$-generic extensions.

Lemma 10 (continuous reading of names). *Suppose that $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Let $x \in \mathbf{L}[G] \cap \omega^\omega$. There exists $c \in \mathbf{L} \cap \mathbf{FPT}$ such that $x = \mathbb{f}_c(a_G)$.*

Proof. Let $T_0 \in \mathbb{P}$. Let $\dot{x}$ be a name for x in the forcing language; then, every $T \in \mathbb{P}$ forces

$$\dot{x} \in \omega^\omega, \quad \text{and} \quad x(k) = l \iff \exists\, T \in G\; (T \Vdash_{\mathbb{P}} \dot{x}(k) = l).$$

We argue in $\mathbf{L}$. Let $D_{kl} = \{T \in \mathbb{P} : T \Vdash_{\mathbb{P}} \dot{x}(k) = l\}$. Each set $D_k = \bigcup_{l\in\omega} D_{kl}$ is dense in $\mathbb{P}$. Let $A_k \subseteq D_k$ be a maximal $\mathbb{P}$-antichain. Then, every $A_k = \{T^m_k : m < \omega\}$ is countable by Lemma 6; hence, there is an ordinal $\alpha < \omega_1$ such that $T_0 \in \mathbb{P}_{<\alpha}$, and, for each k, $A_k \subseteq \mathbb{P}_{<\alpha}$, and the set $D_k(\alpha) = D_k \cap \mathbb{P}_{<\alpha}$ belongs to $\mathbf{M}(\langle \mathbb{P}_\beta \rangle_{\beta<\alpha})$. Note that $D_k(\alpha)$ is dense in $\mathbb{P}_{<\alpha}$ by the maximality of A_k.

By 3° of Section 5, there exists $T \in \mathbb{P}_\alpha$, $T \subseteq T_0$. By 4°, we have $T \subseteq^{\mathtt{fd}} \bigcup D_k(\alpha)$ for every k, so that there are finite sets $D'_k \subseteq D_k(\alpha)$ such that $T \subseteq \bigcup D'_k$ and, if $S \neq S'$ belongs to the same set D'_k, then $[S] \cap [S'] = \varnothing$.

Put $D'_{kl} = D'_k \cap D_{kl}$. For any k, there is a finite set of values l such that $D'_{kl} \neq \varnothing$. Thus, a continuous function $F' : [T] \to \omega^\omega$ can be defined in $\mathbf{L}$ as follows: $F'(x)(k) = l$ iff $x \in [T]$ for some $T \in D'_{kl}$. Let $F : 2^\omega \to \omega^\omega$ be a continuous extension of F'; $F = \mathbb{f}_c$ for some $c \in \mathbf{FPT} \cap \mathbf{L}$. Then, T forces $\dot{x} = \mathbb{f}_c(\dot{a})$, where $\dot{a}$ is the canonical name for a_G. □

Lemma 11. *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathbf{L}$, then $a = a_G$ is the only element of the set $\bigcap_{\alpha<\omega_1} \bigcup_{T\in\mathbb{P}_\alpha}[T]$ in $\mathbf{L}[G]$. Moreover, a_G is minimal over $\mathbf{L}$.*

Proof. If $\alpha < \omega_1$, then the real $a = a_G$ belongs to $\bigcup_{T \in \mathbb{P}_\alpha} [T]$ since all sets $\mathbb{P}_\alpha$ are pre-dense by Lemma 5. To prove the opposite direction, consider any $S \in \mathbb{P}$ and $b \in 2^\omega \cap \mathbf{L}[G]$. By Lemma 10, there exists $c \in \mathbf{L} \cap \mathbf{FPT}$ such that $b = \mathbb{f}_c(a_G)$. There is an ordinal $\alpha < \omega_1$ in $\mathbf{L}$ such that $T = T_\alpha$ and $c = c_\alpha$. Let $T \in \mathbb{P}_\alpha$ witness 5°. In the "either" case of 5°, T obviously forces that $\mathbb{f}_c(a_G) \notin \bigcup_{T' \in \mathbb{P}_\alpha} [T']$, while, in the "or" case, T forces $\mathbb{f}_c(a_G) = a_G$.

To prove the minimality, consider any real $b \in 2^\omega \cap \mathbf{L}[a_G]$. By Lemma 10, we have $b = \mathbb{f}_c(a_G)$, where $c \in \mathbf{FPT} \cap \mathbf{L}$. It follows from 6° that there exists $T \in G$ such that $\mathbb{f}_c \restriction [T]$ is either a bijection or a constant. If $\mathbb{f}_c \restriction [T]$ is a bijection, then $a_G \in \mathbf{L}[x]$ by means of the inverse map. If $\mathbb{f}_c \restriction [T]$ is a constant z, say $\mathbb{f}_c(x) = z$ for all $x \in [T]$ in $\mathbf{L}$, then obviously $b = \mathbb{f}_c(a_G) = z \in \mathbf{L}$. □

Corollary 2 (= Theorem 1, case $\mathbb{n} = 2$)**.** *Assume that $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then, $\mathbf{L}[G]$ satisfies Theorem 1 for $\mathbb{n} = 2$.*

Proof. Lemma 11 implies that $\{a_G\} \in \Pi^{\mathrm{HC}}_1$; hence, $\in \Pi^1_2$, in $\mathbf{L}[G] = \mathbf{L}[a_G]$. Thus $a_G \in \Delta^1_3$ in $\mathbf{L}[a_G]$, as required by (i) of Theorem 1. The minimality claim (iii) follows from Lemma 11, whereas the equality $\mathbf{V} = \mathbf{L}[a_G]$ of (ii) of Theorem 1 in $\mathbf{L}[G]$ is implied by Lemma 9. Finally, (iv) holds since all Σ^1_2 sets $x \subseteq \omega$ are constructible by Shoenfield's absoluteness. □

9. Warmup: Definable Generic Forcing Construction

To solve the general case of Theorem 1, we employ one more idea. Jensen's ω_1-sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1}$ as in 4 can be seen as an ω_1-branch of type Δ^{HC}_1 through the set $\mathbf{J}_{<\omega_1}$ of all countable (transfinite) sequences satisfying conditions 1°–6° above.

The idea behind the general case $\mathbb{n} \geq 3$ is to maintain the choice of $\mathbb{P}_\alpha$ in such a way that the final ω_1-long sequence of (countable sets of trees) $\mathbb{P}_\alpha$ intersects all suitably definable (depends on $\mathbb{n}$!) "dense" sets. In this way, we will obtain a version of Jensen's forcing that allows us to prove Theorem 1. The main cog in this construction is that, because of the "definable genericity", the resulting set $\mathbb{P} = \bigcup_{\alpha < \omega} \mathbb{P}_\alpha$ resolves every boldface $\mathbf{\Sigma}^1_{\mathbb{n}-1}$ set D of perfect trees, in the sense that either it contains a tree in D or it contains a tree non-extendable to a tree in D. This makes $\mathbb{P}$ similar to the Sacks forcing up to level $\mathbb{n}$, leading to claim (iv) of Theorem 1 because of the homogeneity of the Sacks forcing.

Such a *definably generic* forcing construction was applied to great effect by Harrington [23] with the almost disjoint forcing. We will overview some new results in this direction in the concluding section.

Now, let us present the definably generic forcing construction in detail.

10. Complete Sequences and Forcing Notions

Approaching the general case of Theorem 1, we begin with a few definitions.

Definition 5. *Let $P = \langle P\,; \preccurlyeq \rangle$ be a partially ordered set. For any $D \subseteq P$, let $D^{\mathtt{solv}} = D^{\mathtt{solv}}_P$ be the set of all $p \in P$ that solve D in the sense that either $p \in D$ or there are no elements $q \in D$, $q \preccurlyeq p$.*

Recall Definition 3 on the definability types like $\mathbf{\Sigma}_n(\mathrm{HC})$ and Σ^{HC}_n.

Definition 6 (in $\mathbf{L}$)**.** *Suppose that $n \geq 3$. A sequence $\langle P_\alpha \rangle_{\alpha < \omega_1} \in \mathbf{J}_{\omega_1}$ is n-complete if, for any $\mathbf{\Sigma}_{n-2}(\mathrm{HC})$ set $D \subseteq \mathbf{J}_{<\omega_1}$, there is $\gamma < \omega_1$ such that $\langle P_\alpha \rangle_{\alpha < \gamma} \in D^{\mathtt{solv}}$—meaning that either $\langle P_\alpha \rangle_{\alpha < \gamma} \in D$ or there is no sequence in D extending $\langle P_\alpha \rangle_{\alpha < \gamma}$.*

A set $P \subseteq \mathbf{PT}$ of perfect trees is n-complete if, for any $\mathbf{\Sigma}_{n-2}(\mathrm{HC})$ set $W \subseteq \mathbf{PT}$, the set $W^{\mathtt{solv}} \cap P = \{S \in P : S \in W \vee \neg \exists T \in W\, (T \subseteq S)\}$ is dense in P.

Thus, n-completeness is a property of "generic" nature, where genericity is related to a family of sets distinguished by a definability property.

Lemma 12 (in $\mathbf{L}$)**.** *If a sequence $\langle P_\alpha \rangle_{\alpha < \omega_1} \in \mathbf{J}_{\omega_1}$ is n-complete, then $\bigcup_{\alpha < \omega_1} P_\alpha$ is n-complete.*

Proof. Suppose that $W \subseteq \mathbf{PT}$ is a $\mathbf{\Sigma}_{n-2}(\mathrm{HC})$ set, and $S \in P = \bigcup_{\alpha<\omega_1} P_\alpha$; that is, $S \in P_\vartheta$ for some $\vartheta < \omega_1$. We prove that there is $T \in W^{\mathtt{solv}} \cap P$ such that $T \subseteq S$. The set D of all sequences $\langle P'_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}_{<\omega_1}$, $\lambda < \omega_1$, such that there exists $T \in \bigcup_{\alpha<\lambda} P'_\alpha \cap W$, $T \subseteq S$, is $\mathbf{\Sigma}_{n-2}(\mathrm{HC})$. It follows that $\langle P_\alpha \rangle_{\alpha<\lambda} \in D^{\mathtt{solv}}$ for some $\lambda < \omega_1$, *i.e.*, either $\langle P_\alpha \rangle_{\alpha<\lambda} \in D$, or there is no sequence in D that extends $\langle P_\alpha \rangle_{\alpha<\lambda}$.

If $\langle P_\alpha \rangle_{\alpha<\lambda} \in D$, then, by definition, there is a tree $T \in P \cap W$ with $T \subseteq S$, as required.

Suppose that $\langle P_\alpha \rangle_{\alpha<\lambda}$ is not extendable to a sequence in D, and denote $\xi = \max\{\lambda, \vartheta + 1\}$. Then, the extended sequence $\boldsymbol{p} = \langle P_\alpha \rangle_{\alpha\le\xi}$ is not extendable to a sequence in D because $\langle P_\alpha \rangle_{\alpha<\lambda}$ is not extendable. By 3°, there is a tree $T \in P_\xi$, $T \subseteq S$. We claim that $T \in W^{\mathtt{solv}}$.

Suppose, to the contrary, that $T \notin W$ and there is $T' \in W$ such that $T' \subseteq T$. Then, by Lemma 7, there is a set $P'_\xi \subseteq \mathbf{PT}$ containing T' and such that $\boldsymbol{p}' = \boldsymbol{p}^\frown P'_\xi$ is still a sequence in $\mathbf{J}_{\xi+1}$ extending $\boldsymbol{p}$, and $\boldsymbol{p}' \in D$ by the choice of T'. But, this contradicts the non-extendability of $\boldsymbol{p}$, and therefore $T \in W^{\mathtt{solv}}$. □

Lemma 13 (in $\mathbf{L}$). *If $n \ge 3$, then there exists an n-complete $\Delta^{\mathrm{HC}}_{n-1}$ sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$.*

Proof. Let $U \subseteq \mathrm{HC} \times \mathrm{HC}$ be a universal $\Sigma^{\mathrm{HC}}_{n-2}$ set. That is, U itself is $\Sigma^{\mathrm{HC}}_{n-2}$, and if $X \subseteq \mathrm{HC}$ is a (boldface) $\mathbf{\Sigma}_{n-2}(\mathrm{HC})$ set, then there is a parameter $p \in \mathrm{HC}$ such that $X = U_p := \{x \in \mathrm{HC} : \langle p, x \rangle \in U\}$. As we argue in $\mathbf{L}$, for any $\alpha < \omega_1$, let p_α be the αth element of $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in the sense of Gödel's Δ^{HC}_1 well-ordering of $\mathrm{HC} = \mathbf{L}_{\omega_1}$. Then, $\mathrm{HC} = \{p_\alpha : \alpha < \omega_1\}$ and the sequence $\langle p_\alpha \rangle_{\alpha<\omega_1}$ is Σ^{HC}_1.

To prove the lemma, we define a strictly $\subset$-increasing sequence $\langle \boldsymbol{j}[\alpha] \rangle_{\alpha<\omega_1}$ of sequences $\boldsymbol{j}[\alpha] \in \mathbf{J}_{<\omega_1}$ as follows. Let $\boldsymbol{j}[0]$ be the empty sequence.

Let $\boldsymbol{j}[\lambda] = \bigcup_{\alpha<\lambda} \boldsymbol{j}[\alpha]$ whenever $\lambda < \omega_1$ is a limit.

For every α, if $\boldsymbol{j}[\alpha] \in \mathbf{J}_{<\omega_1}$ is defined, then let $\boldsymbol{j}[\alpha + 1]$ be the Gödel-least sequence $\boldsymbol{j} \in \mathbf{J}_{<\omega_1}$ such that $\boldsymbol{j}[\alpha] \subseteq \boldsymbol{j}$ and $\boldsymbol{j} \in {U_{p_\alpha}}^{\mathtt{solv}}$.

The limit sequence $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} = \bigcup_{\alpha<\omega_1} \boldsymbol{j}[\alpha] \in \mathbf{J}_{\omega_1}$ is n-complete by construction, and, by an easy estimation, based on the assumption that U is $\Sigma^{\mathrm{HC}}_{n-2}$, it belongs to $\Delta^{\mathrm{HC}}_{n-1}$. □

The next theorem is the conclusive step in the proof of Theorem 1.

Theorem 4 (in $\mathbf{L}$). *Assume that $\mathfrak{n} \ge 3$, $\langle \mathbb{P}_\alpha(\mathfrak{n}) \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$ is an $\mathfrak{n}$-complete $\Delta^{\mathrm{HC}}_{\mathfrak{n}-1}$ sequence* (Lemma 13), *and $\mathbb{P}(\mathfrak{n}) = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha(\mathfrak{n})$. Then, $\mathbb{P}(\mathfrak{n})$-generic extensions of $\mathbf{L}$ prove Theorem 1.*

Its proof will be accomplished in Section 13. A few remarks follow before the proof starts.

Lemma 11 implies that if $G \subseteq \mathbb{P}(\mathfrak{n})$ is $\mathbb{P}(\mathfrak{n})$-generic over $\mathbf{L}$, then the corresponding real a_G is minimal. It also follows from the same lemma and the fact that the sequence $\langle \mathbb{P}_\alpha(\mathfrak{n}) \rangle_{\alpha<\omega_1} \in \mathbf{J}_{\omega_1}$ is $\Delta^{\mathrm{HC}}_{\mathfrak{n}-1}$ in $\mathbf{L}$ that the singleton $\{a_G\}$ is $\Pi^1_{\mathfrak{n}}$ and hence a_G is $\Delta^1_{\mathfrak{n}+1}$ in $\mathbf{L}[G]$. It is a more difficult problem to prove the remaining claim of Theorem 1, that is, that any Σ^1_n set $x \subseteq \omega$ in $\mathbf{L}[G]$ is constructible. We will establish this fact in the remainder; the result will be based on the n-completeness property and on some intermediate claims.

11. Digression: Definability of the Sacks Forcing

Our next goal is to estimate the definability of the Sacks forcing relation, restricted to formulas of a certain ramified version of the second-order Peano language.

Definition 7. *Let $\mathcal{L}$ be the ordinary language of the second-order Peano arithmetic, with variables of type 1 for functions in ω^ω. Extend this language so that some type 1 variables can be substituted by symbols of the form $\widehat{c}$, $c \in \mathbf{FPT}$, and each $\widehat{c}$ is viewed as a name for $\mathbb{f}_c(a)$, where a means a generic real of any kind. (Recall that $\mathbb{f}_c : 2^\omega \to \omega^\omega$ is a continuous map coded by $c \in \mathbf{FPT}$.) Let ${}^{\mathrm{s}}\mathcal{L}$ be the extended language; the index s is from Sacks. Accordingly, ${}^{\mathrm{s}}\Sigma^1_n$ and ${}^{\mathrm{s}}\Pi^1_n$ will denote the standard types of formulas of ${}^{\mathrm{s}}\mathcal{L}$.*

If $a \in 2^\omega$ and φ is a formula of ${}^{\mathrm{s}}\mathcal{L}$, then $\varphi[a]$ is the result of the substitution of $\mathbb{f}_c(a)$ for any name $\widehat{c}$ in φ; $\varphi[a]$ is a formula of $\mathcal{L}$ with real parameters.

Definition 8. *Let $\Vdash$ be the Sacks forcing relation (that is,* **PT** *is the forcing notion). Define an auxiliary relation of "strong" forcing* `forc`*, restricted to* ${}^{\mathsf{s}}\Sigma^1_k$ *formulas,* $k \geq 1$*, generally, to all existential formulas of* ${}^{\mathsf{s}}\mathcal{L}$*, as follows:*

($*$) *If* $\varphi(x)$ *is a formula of* ${}^{\mathsf{s}}\mathcal{L}$ *with the only free variable* x *(over* ω^ω*), and* $T \in \mathbf{PT}$*, then* T `forc` $\exists\, x\, \varphi(x)$ *if there exists* $c \in \mathbf{FPT}$ *such that* $T \Vdash \varphi(\widehat{c})$*.*

But, if φ *is a* ${}^{\mathsf{s}}\Pi^1_k$ *formula, then we define:* T `forc` φ *iff* $T \Vdash \varphi$*.*

It is a known property of the Sacks forcing that any real x in the **PT**-generic extension $\mathbf{V}[G]$ of the universe $\mathbf{V}$ has the form $x = \mathbb{f}_c(a_G)$, where $c \in \mathbf{FPT} \cap \mathbf{V}$; see, e.g., [24]. Therefore, the forcing relation `forc` as in Definition 8 is still adequate. In particular, the following lemma holds:

Lemma 14. *Suppose that* φ *is a closed formula in* ${}^{\mathsf{s}}\Pi^1_k$*,* $k \geq 1$*, and* $T \in \mathbf{PT}$*. Then,* T `forc` φ *iff there is no* $S \in \mathbf{PT}$*,* $S \subseteq T$*, such that* S `forc` $\varphi^\neg$*.*

Here, $\varphi^\neg$ is the result of the canonical transformation of $\neg\,\varphi$ to a ${}^{\mathsf{s}}\Sigma^1_k$ form.

Now, let us address the descriptive complexity of `forc`.

Lemma 15. *The relation* `forc` *restricted to* ${}^{\mathsf{s}}\Pi^1_1$ *formulas is* Π^{HC}_1*. If* $k \geq 2$*, then the relation* `forc` *restricted to* ${}^{\mathsf{s}}\Sigma^1_k$ *formulas is* $\Sigma^{\mathrm{HC}}_{k-1}$ *while* `forc` *restricted to* ${}^{\mathsf{s}}\Pi^1_k$ *formulas is* Π^{HC}_{k-1}*.*

Proof. We argue by induction. Suppose that $\varphi = \varphi(\widehat{c}_1, \ldots, \widehat{c}_m)$ is a closed formula in ${}^{\mathsf{s}}\Pi^1_1$. It follows from the Shoenfield absoluteness and the perfect set theorem for $\boldsymbol{\Sigma}^1_1$ sets that, for any $T \in \mathbf{PT}$, T `forc` φ is equivalent to the set $T_\varphi = \{a \in [T] : \neg\,\varphi[a]\}$ being countable, and then to

$$\forall\, a \in [T]\; (\varphi[a] \vee a \in \Delta^1_1(c_1, \ldots, c_m))$$

as any countable $\Sigma^1_1(c)$ set $X \subseteq \omega^\omega$ consists of elements of type $\Delta^1_1(c)$. Yet, the displayed formula is Π^1_1, hence Δ^{HC}_1, as $x \in \Delta^1_1(c)$ is a Π^1_1 relation.

The step $\Pi^1_k \to \Sigma^1_{k+1}$: make use of Definition 8($*$).

Now, the step $\Sigma^1_k \to \Pi^1_k$. Suppose that $k \geq 2$, φ is a closed formula in ${}^{\mathsf{s}}\Sigma^1_k$, and $T \in \mathbf{PT}$. Then, by Lemma 14, T `forc` φ is equivalent to

$$\forall\, S \in \mathbf{PT}\; (S \subseteq T \implies \neg\, S \;\texttt{forc}\; \varphi^\neg)\,,$$

and hence we obtain Π^1_k using the inductive hypothesis for $\varphi^\neg$. □

12. Back to the $\mathfrak{n}$-Complete Jensen's Forcing

Let $\mathfrak{n}$ and $\mathbb{P}(\mathfrak{n})$ be the same as in Theorem 4. We begin with the following.

Lemma 16 (in **L**)**.** *For any closed formula* φ *in* ${}^{\mathsf{s}}\Sigma^1_k$*,* $1 \leq k \leq n-1$*, the set of all* $T \in \mathbb{P}(\mathfrak{n})$ *such that* T `forc` φ *or* T `forc` $\varphi^\neg$ *is dense in* $\mathbb{P}(\mathfrak{n})$*.*

Proof. The set $W = \{T \in \mathbf{PT} : T \;\texttt{forc}\; \varphi\}$ is $\Sigma^{\mathrm{HC}}_{n-2}$ by Lemma 15. Therefore, the set $\mathbb{P}(\mathfrak{n}) \cap W^{\texttt{solv}}$ is dense in $\mathbb{P}(\mathfrak{n})$ by Lemma 12. However, it follows from Lemma 14 that $W^{\texttt{solv}}$ is equal to the set of all $T \in \mathbf{PT}$ such that T `forc` φ or T `forc` $\varphi^\neg$. □

It is a basic fact of forcing theory that the truth in generic extensions is, in a certain way, connected with the forcing relation. Thus, the truth in $\mathbb{P}(\mathfrak{n})$-generic extensions $\mathbf{L}[G]$ of $\mathbf{L}$ corresponds to the $\mathbb{P}(\mathfrak{n})$-forcing relation. However—and this is the key moment—the following theorem shows that the truth in $\mathbb{P}(\mathfrak{n})$-generic extensions is also in tight connection with **PT**, the Sacks forcing notion, up to the level $\Sigma^1_{\mathfrak{n}}$. This is a consequence of $\mathfrak{n}$-completeness, of course: in some sense, the $\mathfrak{n}$-completeness means that $\mathbb{P}(\mathfrak{n})$ is an *elementary submodel* of **PT** with respect to formulas of a certain level of complexity.

Theorem 5. *Let* $\mathbb{n}$ *and* $\mathbb{P}(\mathbb{n})$ *be the same as in Theorem 4. Suppose that* Φ *is a closed formula in* ${}^{\mathsf{s}}\Sigma^1_k$, $1 \le k \leqslant \mathbb{n}$, *or* ${}^{\mathsf{s}}\Pi^1_k$, $1 \le k < \mathbb{n}$, *and a set* $G \subseteq \mathbb{P}(\mathbb{n})$ *is* $\mathbb{P}(\mathbb{n})$*-generic over* $\mathbf{L}$. *Then,* $\Phi[a_G]$ *holds in* $\mathbf{L}[G]$ *if there is* $T \in G$ *such that* T `forc` Φ.

Proof. We argue by induction on k. Let Φ be a closed ${}^{\mathsf{s}}\Pi^1_1$ formula. If $T \in G$ and T `forc` Φ, then, in $\mathbf{L}$, $\Phi[a]$ is true for all $a \in [T]$ with, at most, a countable set of exceptions; see the proof of Lemma 15. And, all exceptions are Δ^1_1, hence absolutely defined and belong to $\mathbf{L}$. Therefore, the generic real $a_G \in [T]$ cannot be an exception, thus $\Phi[a_G]$ holds in $\mathbf{L}[G]$. If Φ is ${}^{\mathsf{s}}\Sigma^1_1$, then Φ is $\exists\, x\, \varphi(x)$, φ being ${}^{\mathsf{s}}\Pi^1_0$, and if T `forc` Φ, then, by definition, T `forc` $\varphi(\widehat{c})$ for some $c \in \mathbf{FPT}$, and so on. On the other hand, it follows from Lemma 16 that there is $T \in G$ such that T `forc` Φ or T `forc` $\Phi^{\neg}$. This easily implies the result for ${}^{\mathsf{s}}\Sigma^1_1 \cup {}^{\mathsf{s}}\Pi^1_1$.

Step ${}^{\mathsf{s}}\Sigma^1_k \to {}^{\mathsf{s}}\Pi^1_k$, $2 \le k < \mathbb{n}$. Let Φ be a ${}^{\mathsf{s}}\Pi^1_k$ formula. Suppose that $\Phi[a_G]$ fails in $\mathbf{L}[a_G]$. Then, $\Phi^{\neg}[a_G]$ holds in $\mathbf{L}[a_G]$, and hence, by the inductive hypothesis, there is a condition $S \in G$ satisfying S `forc` $\Phi^{\neg}$. Then, by Lemma 14, there is no $T \in G$ with T `forc` Φ. Conversely, suppose that there is no $T \in G$ with T `forc` Φ. Then, by Lemma 16, there is a condition $S \in G$ satisfying S `forc` $\Phi^{\neg}$. It follows that $\Phi^{\neg}[a_G]$ holds in $\mathbf{L}[a_G]$, and, subsequently, $\Phi[a_G]$ fails, as required.

Step ${}^{\mathsf{s}}\Pi^1_k \to {}^{\mathsf{s}}\Sigma^1_{k+1}$, $1 \le k < \mathbb{n}$. Thus, let Φ be a formula $\exists\, x\, \varphi(x)$, where φ is Π^1_k. Assume that $T \in G$ satisfies T `forc` Φ. This entails, by $(*)$ of Definition 8, that T `forc` $\varphi(\widehat{c})$ for some $c \in \mathbf{FPT} \cap \mathbf{L}$, a code of the continuous map $\mathbb{f}_c : 2^\omega \to \omega^\omega$. Apply the induction hypothesis to the formula $\varphi(\widehat{c})$: it says that $\varphi(\widehat{c})[a_G]$ holds in $\mathbf{L}[G]$. But, $\varphi(\widehat{c})[a_G]$ is $\varphi[a_G](x)$, where $x = \mathbb{f}_c(a_G) \in \omega^\omega \cap \mathbf{L}[G]$. Therefore, $\Phi[a_G]$ holds in $\mathbf{L}[G]$, as required.

In the opposite direction, let $\Phi[a_G]$ be true in $\mathbf{L}[G]$; that is, $\varphi[a_G](x)$ holds for some $x \in \omega^\omega \cap \mathbf{L}[G]$. By Lemma 10, there is $c \in \mathbf{FPT} \cap \mathbf{L}$ such that $x = \mathbb{f}_c(a_G)$. The formula $\varphi(\widehat{c})[a_G]$ coincides with $\varphi[a_G](x)$ and hence holds in $\mathbf{L}[G]$. Therefore, by the induction hypothesis, there is $T \in G$ such that T `forc` $\varphi(\widehat{c})$. But, then, T `forc` Φ by $(*)$ of Definition 8, as required. □

13. Proof of Theorem 1: General Case

Here, we accomplish the proof of Theorems 4 and 1. We fix $\mathbb{n} \ge 3$.

Let $\mathbb{P}(\mathbb{n}) \in \mathbf{L}$ be the same as in Theorem 4. If a set $G \subseteq \mathbb{P}(\mathbb{n})$ is $\mathbb{P}(\mathbb{n})$-generic over $\mathbf{L}$, then all Σ^1_n sets $x \subseteq \omega$ in $\mathbf{L}[G]$ are constructible by Theorem 5 because, by the homogeneity of the Sacks forcing, for any parameter-free formula Φ and any trees $T, T' \in \mathbf{PT}$, we have

$$T \texttt{ forc } \Phi \iff S \texttt{ forc } \Phi.$$

Let us present this final argument in more detail.

If $S, T \in \mathbf{PT}$, then let $\textsc{Hom}_{ST}$ be the set of all homeomorphisms $h : [S] \xrightarrow{\text{onto}} [T]$; clearly, $\textsc{Hom}_{ST}$ is non-empty. Suppose that $h \in \textsc{Hom}_{ST}$. Recall that continuous functions $F : 2^\omega \to \omega^\omega$ are coded so that $\mathbb{f}_c$ is the function coded by $c \in \mathbf{FPT}$. If $c, d \in \mathbf{FPT}$, then write $c \longrightarrow hd$ iff $\mathbb{f}_d(h(a)) = \mathbb{f}_c(a)$ for all $a \in [S] = \mathtt{dom}\, h$. If $\varphi = \Phi(\widehat{c}_1, \dots, \widehat{c}_m)$ and $\psi = \Phi(\widehat{d}_1, \dots, \widehat{d}_m)$ are formulas of ${}^{\mathsf{s}}\mathcal{L}$ (see Section 11), and $c_i \longrightarrow hd_i$ for all i, then write $\varphi \longrightarrow h\psi$. In this case, the formulas $\varphi[a]$ and $\psi[h(a)]$ coincide for any $a \in [S]$.

Lemma 17. *Suppose that* $S, T \in \mathbf{PT}$, $h \in \textsc{Hom}_{ST}$, Φ *and* Ψ *are closed formulas in one and the same type,* ${}^{\mathsf{s}}\Sigma^1_k$ *or* ${}^{\mathsf{s}}\Pi^1_k$, *and* $\Phi \longrightarrow h\Psi$. *Then,* S `forc` Φ *if and only if* T `forc` Ψ.

Proof. Routinely argue by induction on the complexity of the formulas. □

Corollary 3. *If* $S, T \in \mathbf{PT}$ *and* Φ *is a formula in* Σ^1_k *or* Π^1_k, *then* S `forc` Φ *iff* T `forc` Φ.

Proof. Pick $h \in \textsc{Hom}_{ST}$, note that $\Phi \longrightarrow h\Phi$ (as Φ contains no symbols of the form $\widehat{c}$), and apply Lemma 17. □

Lemma 18. *If* $G \subseteq \mathbb{P}(\mathfrak{n})$ *is* $\mathbb{P}(\mathfrak{n})$*-generic over* $\mathbf{L}$*, and* $x \subseteq \omega$*,* $x \in \mathbf{L}[G]$ *is* $\Sigma^1_{\mathfrak{n}}$ *in* $\mathbf{L}[G]$*, then* $x \in \mathbf{L}$ *and* x *is* $\Sigma^1_{\mathfrak{n}}$ *in* $\mathbf{L}$*.*

Proof. Let $\varphi(m)$ be a parameter-free $\Sigma^1_{\mathfrak{n}}$ formula such that $x = \{m : \varphi(m)\}$ in $\mathbf{L}[G]$. Consider the tree $S = 2^{<\omega} \in \mathbf{PT}$. Then,

$$m \in x \iff \mathbf{L}[G] \models \varphi(m) \iff \exists T \in G\,(T \texttt{ forc } \varphi(m)) \iff S \texttt{ forc } \varphi(m),$$

by Theorem 5 and Corollary 3. It remains to refer to Lemma 15. □

This ends the proof of Theorems 4 and 1.

14. Theorem 2: Outline

As the proof of Theorem 1, given above, contains a heavy dose of the forcing technique, first of all we have to adequately replace $\mathbf{PA}_2$ with a more **ZFC**-like, forcing-friendly set theory, dealing with Theorem 2. We will make use of the theory

$$\mathbf{ZFC}^-_{\mathbf{lc}} \;:=\; \mathbf{ZFC}^- \text{ plus } \text{“}\mathbf{V} = \mathbf{L}\text{” plus “all sets are countable”}, \qquad (\dagger)$$

as such a proxy theory. (The upper minus stands for the absence of the power sets axiom, whereas l and c in the lower index stand for the constructibility ($\mathbf{L}$) and countability.) The following is the according proxy theorem (compared to Theorem 1).

Theorem 6. *If* $\mathfrak{n} \geq 2$*, then there exists a generic extension of the universe of* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, in which all axioms of* $\mathbf{ZFC}^-$ *hold, along with the following:*

(i) *There is a nonconstructible* $\Delta^1_{\mathfrak{n}+1}$ *real* $a \in 2^\omega$ *such that:*
(ii) $\mathbf{V} = \mathbf{L}[a]$ *holds;*
(iii) a *is minimal over the ground universe of* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, in the sense similar to (iii) of Theorem 1;*
(iv) *But, all* $\Sigma^1_{\mathfrak{n}}$ *sets* $x \subseteq \omega$ *are constructible and* $\Sigma^1_{\mathfrak{n}}$ *in the ground universe of* $\mathbf{ZFC}^-_{\mathbf{lc}}$*.*

The universe of $\mathbf{ZFC}^-_{\mathbf{lc}}$ is naturally identified with $\mathbf{L}_{\omega_1^{\mathbf{L}}}$. It will take some effort to obtain the proof of Theorem 1 relativized to $\mathbf{L}_{\omega_1^{\mathbf{L}}}$ so that it can be executed in the universe of $\mathbf{ZFC}^-_{\mathbf{lc}}$, denoted by $\mathbf{L}^{\dagger}_{\omega_1}$ below for the sake of convenience.

To establish Theorem 6, we will make use of a suitable version of the forcing notion $\mathbb{P}(\mathfrak{n})$ as a definable class in $\mathbf{L}^{\dagger}_{\omega_1}$, and a class-forcing notion, CCC, with regard to all definable class-antichains, and then we will show that $\mathbb{P}(\mathfrak{n})$-generic extensions of $\mathbf{L}^{\dagger}_{\omega_1}$ prove Theorem 2.

Yet, there is a serious obstacle: the treatment of $\mathbb{P}(\mathfrak{n})$ involves ordinals and some other objects in $\mathbf{L}_{\omega_2}$ (rather than $\mathbf{L}_{\omega_1}$) in the proof of the key CCC result by Lemma 6, and this is not admissible in $\mathbf{ZFC}^-$. We overcome this difficulty, following the idea of a recent construction of definable-$\diamond$ sequences by Enayat and Hamkins [25].

Definition 9. The ground set universe of $\mathbf{ZFC}^-_{\mathbf{lc}}$ is denoted by $\mathbf{L}^{\dagger}_{\omega_1}$. We use ω_1 to denote the collection (a proper class) of all ordinals in $\mathbf{L}^{\dagger}_{\omega_1}$; all of them are countable.

Remark 1. *Arguing in* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, we will often consider (definable) proper classes as they will play a more essential role than is common in* **ZFC***. We will also consider such things as* class-size collections of proper classes*, e.g., class-long sequences* $\langle X_\alpha \rangle_{\alpha<\omega_1}$ *of proper classes* X_α*, with the understanding that the real thing considered in this case is some (definable) class* $Y \subseteq \omega_1 \times \mathbf{L}^{\dagger}_{\omega_1}$ *whose slices* $Y_\alpha = \{x : \langle \alpha, x \rangle \in Y\}$ *are equal to the given classes* X_α*.*

15. Jensen's Sequences, ZFC$^-$ Version

Adapting the proof of Theorem 1 above for the proof of Theorem 6, we are going to introduce $\mathbb{P}$ as a definable class forcing under $\mathbf{ZFC}^-_{\mathbf{lc}}$. In this section, we argue in $\mathbf{ZFC}^-_{\mathbf{lc}}$.

Definition 10 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *If $\alpha < \omega_1$ and $\langle X_\beta \rangle_{\beta<\alpha}$ is a sequence of any sets, then let $\mathbf{M}^\dagger(\langle X_\beta \rangle_{\beta<\alpha})$ be the least CTM M, necessarily of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, which*

(1) *Models $\mathbf{ZFC}^-_{\mathbf{lc}}(\text{bound})$, i.e., $\mathbf{ZFC}^-_{\mathbf{lc}}$ with the collection and separation schemata (see Section 1) restricted to bounded $\in$-formulas;*
(2) *Contains $\langle X_\beta \rangle_{\beta<\alpha}$; and*
(3) *Contains the set $\mathbf{Tru}(\mathbf{L}_\alpha) =$ all $\in$-formulas, with parameters in $\mathbf{L}_\alpha$, true in $\mathbf{L}_\alpha$.*

Compared to Definition 1, we may note that, arguing in $\mathbf{ZFC}^-_{\mathbf{lc}}$, it is not suitable to refer to models of $\mathbf{ZFC}^-$. This is the reason for passing to $\mathbf{ZFC}^-_{\mathbf{lc}}(\text{bound})$ here.

Definition 11 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *If $\alpha < \omega_1$, then let $\langle T_\alpha, c_\alpha, \nu_\alpha \rangle$ be the αth element of the set $\mathbf{PT} \times \mathbf{FPT} \times \omega_1$ in the sense of the Gödel canonical well-ordering of $\mathbf{L}$.*

For any ordinal $\lambda \le \omega_1$, we let $\mathbf{J}^\dagger_\lambda$ (Jensen's sequences of length λ, $\mathbf{ZFC}^-_{\mathbf{lc}}$ version) be the set of all sequences $\langle P_\alpha \rangle_{\alpha<\lambda}$ of length λ, of **countable** sets $P_\alpha \subseteq \mathbf{PT}$, satisfying conditions 1°, 2°, 3°, 5°, 6° of Definition 2, and the following condition instead of 4°.

$4^\dagger$. If $\alpha < \lambda$, $T \in P_\alpha$, $D \in \mathbf{M}^\dagger(\langle P_\beta \rangle_{\beta<\alpha})$, $D \subseteq P_{<\alpha}$ is open dense in $P_{<\alpha}$, then $T \subseteq^{\mathtt{fd}} \bigcup D$.

Let $\mathbf{J}^\dagger_{<\lambda} = \bigcup_{\alpha<\lambda} \mathbf{J}^\dagger_\alpha$.

Lemma 19 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *Suppose that $\beta < \lambda \le \omega_1$ and $\langle P_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}^\dagger_\lambda$. Then, P_β is pre-dense in the set $P_{<\lambda} = \bigcup_{\alpha<\lambda} P_\alpha$*—the proof is similar to Lemma 5.

Lemma 20 (in $\mathbf{L}$)**.** *Assume that $\langle P_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}^\dagger_{\omega_1}$. Then, the forcing notion $P = \bigcup_{\alpha<\omega_1} P_\alpha \in \mathbf{L}$ satisfies CCC in $\mathbf{L}$ with regard to all antichains $A \subseteq P$ definable in $\mathbf{L}^\dagger_{\omega_1}$ with parameters.*

In this lemma, CCC is naturally understood in the class form: *every class-size definable antichain is a countable set.*

Proof. Suppose that $A \subseteq P$ is a maximal P-antichain. As A is definable, assume that $A = \{T \in \mathbf{L}_{\omega_1} : \mathbf{L}_{\omega_1} \models \varphi(p, T)\}$, where $p \in \mathbf{L}^\dagger_{\omega_1}$ is a parameter and φ any $\in$- formula.

There exists a limit ordinal α such that $p \in \mathbf{L}_\alpha$, the set $P_{<\alpha} = \bigcap_{\gamma<\alpha} P_\gamma$ satisfies $P_{<\alpha} = P \cap \mathbf{L}_\alpha$, the set $A_{<\alpha} = A \cap P_{<\alpha}$ is a maximal $P_{<\alpha}$-antichain, and therefore pre-dense in $P_{<\alpha}$, and, finally, $\mathbf{L}_\alpha$ is elementarily equivalent to $\mathbf{L}^\dagger_{\omega_1}$ with regard to φ, so that, overall, we have: $A_{<\alpha} = \{T \in \mathbf{L}_\alpha : \mathbf{L}_\alpha \models \varphi(p, T)\}$.

Let $M = \mathbf{M}^\dagger(\langle P_\gamma \rangle_{\gamma<\alpha})$. We assert that $A_{<\alpha} \in M$. Indeed, by definition, the truth set $\boldsymbol{\mathcal{T}} = \mathbf{Tru}(\mathbf{L}_\alpha)$ belongs to $\mathbf{L}_\mu$. On the other hand, $A_{<\alpha} = \{T : \varphi(p, T) \in \boldsymbol{\mathcal{T}}\}$ by the above. It follows that $A_{<\alpha} \in M$ since M models $\mathbf{ZFC}^-_{\mathbf{lc}}(\text{bound})$.

Now, it suffices to prove that $A = A_{<\alpha}$. Suppose, to the contrary, that $T \in A \smallsetminus A_{<\alpha} = A \smallsetminus P_{<\alpha}$. Then, T is compatible with some $T' \in P_\alpha$ by Lemma 19; that is, there is a tree $T'' \in J$, $T'' \subseteq T' \cap T$. On the other hand, it follows from $4^\dagger$ that $T' \subseteq^{\mathtt{fd}} A_{<\alpha}$. Then, $T'' \subseteq^{\mathtt{fd}} A_{<\alpha}$ as well, and hence there exist $s \in T''$ and $S \in A_{<\alpha}$ such that the tree $U = T'' \restriction_s$ satisfies $U \subseteq T'' \cap S$; therefore, $U \subseteq T \cap S$. However, $U \in P$ by 2°, and $S \in A_{<\alpha}$ but $T \in A \smallsetminus A_{<\alpha}$, contrary to the assumption that A is a P-antichain. □

The following extendability theorem is proved in a similar way to Theorem 3, so we skip the proof.

Theorem 7 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *Suppose that $\lambda < \omega_1$. Then, any sequence $\langle P_\alpha \rangle_{\alpha<\lambda} \in \mathbf{J}^\dagger_\lambda$ has an extension $\langle P_\alpha \rangle_{\alpha\le\lambda} \in \mathbf{J}^\dagger_{\lambda+1}$.*

16. Definable Jensen's Sequence and the Forcing Engine, $\mathbf{ZFC}^-$ Version

We deal with the issue of the definability of Jensen's sequences in $\mathbf{ZFC}^-_{\mathbf{lc}}$.

Remark 2. *Note that* $\mathrm{HC} = \mathbf{L}^{\dagger}_{\omega_1}$ *= all sets, in* $\mathbf{ZFC}^{-}_{\mathbf{lc}}$*. The definability types* Σ^{HC}_n*,* Π^{HC}_n*,* Δ^{HC}_n *consist of definable classes* $X \subseteq \mathbf{L}^{\dagger}_{\omega_1}$ *in* $\mathbf{ZFC}^{-}_{\mathbf{lc}}$*, of course.*

Lemma 21 (in $\mathbf{ZFC}^{-}_{\mathbf{lc}}$, note similarities to Corollary 1)**.** *There exists a* $(\Delta_1^{\mathbf{L}^{\dagger}_{\omega_1}} = \Delta^{\mathrm{HC}}_1)$ *sequence* $\boldsymbol{p} = \langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}^{\dagger}_{\omega_1}$.

Definition 12 (in $\mathbf{ZFC}^{-}_{\mathbf{lc}}$)**.** *By Lemma 21, fix a sequence* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1}$ *of sets* $\mathbb{P}_\alpha \in \mathbf{L}^{\dagger}_{\omega_1}$*, such that it holds in* $\mathbf{L}^{\dagger}_{\omega_1}$ *that 1)* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}^{\dagger}_{\omega_1}$*, and 2)* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1}$ *is a* $\Delta_1^{\mathbf{L}^{\dagger}_{\omega_1}}$ *sequence.*

Put $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$.

Consider such a set $\mathbb{P} \subseteq \mathbf{PT}$ as a forcing notion (here, a proper class) over $\mathbf{L}^{\dagger}_{\omega_1}$.

The forcing engine does not necessarily work in $\mathbf{ZFC}^{-}_{\mathbf{lc}}$ for an arbitrary class-size forcing notion. But, there is a type of forcing notions that admits adequate treatment of forcing similar to the standard **ZFC** case. This is the class forcing theory of S. D. Friedman [26,27], further developed by Antos and Gitman [19] to be applicable over $\mathbf{ZFC}^{-}$.

Definition 13 (S. D. Friedman, see [19,26])**.** A forcing notion (a partially ordered definable class) $P = \langle P\,;\leq \rangle$ is *pre-tame* if, for every class sequence $\langle D_x \rangle_{x\in a}$ of dense classes $D_x \subseteq P$, parametrized by elements of a set a (so that $D = \{\langle x,z\rangle : x \in a \wedge z \in D_x\}$ is a definable class), and every condition $p \in P$, there is a condition $q \leq p$ and a sequence $\langle d_x \rangle_{x\in a}$ of sets $d_x \subseteq P$ such that each $d_x \subseteq D_x$ is pre-dense below q in P.

Theorem 8 (S. D. Friedman, see [19,26])**.** *In* $\mathbf{ZFC}^{-}$*, let* P *be a pre-tame class-forcing notion. Then,* P *preserves* $\mathbf{ZFC}^{-}$ *and satisfies the main forcing principles, including the truth forcing and forcing definability theorems.*

Remark 3. *The forcing notion* $\mathbb{P}$*, introduced by Definition 12, is a class forcing satisfying CCC by Theorem 20. Therefore,* $\mathbb{P}$ *is pre-tame under* $\mathbf{ZFC}^{-}_{\mathbf{lc}}$*, as, obviously, is any CCC forcing. We conclude that Theorem 8 is applicable, and hence usual forcing theorems are valid for* $\mathbf{P}$*-generic extensions of* $\mathbf{L}^{\dagger}_{\omega_1}$*, the* $\mathbf{ZFC}^{-}_{\mathbf{lc}}$ *set universe.*

This justifies all forcing results in Sections 7 and 8 above, on the basis of $\mathbf{ZFC}^{-}_{\mathbf{lc}}$.

In particular, we have:

Corollary 4 (in $\mathbf{ZFC}^{-}_{\mathbf{lc}}$, = Theorem 6, case $\mathtt{n} = 2$)**.** *Assume that* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}^{\dagger}_{\omega_1}$*. Then,* $\mathbf{L}^{\dagger}_{\omega_1}[G]$ *satisfies Theorem 6 for* $\mathtt{n} = 2$.

This completes the proof of Theorem 6, case $\mathtt{n} = 2$.

17. Theorem 6: General Case

The proof of the general case of Theorem 6 follows the arguments in Sections 10–13 mutatis mutandis. We sketch it here without going into details.

Recall Definition 2 on the definability types like $\Sigma^{\mathrm{HC}}_n = \Sigma_n^{\mathbf{L}^{\dagger}_{\omega_1}}$.

Definition 14 (in $\mathbf{ZFC}^{-}_{\mathbf{lc}}$)**.** *Suppose that* $n \geq 3$*. Similarly to Definition 6, a sequence* $\langle P_\alpha \rangle_{\alpha<\omega_1}$ *in* $\mathbf{J}^{\dagger}_{\omega_1}$ *is* n-complete *if, for any* $\boldsymbol{\Sigma}_{n-2}(\mathbf{L}^{\dagger}_{\omega_1})$ *set* $D \subseteq \mathbf{J}^{\dagger}_{<\omega_1}$*, there is* $\gamma < \omega_1$ *such that* $\langle P_\alpha \rangle_{\alpha<\gamma} \in D^{\mathtt{solv}}$*, i.e., either* $\langle P_\alpha \rangle_{\alpha<\gamma} \in D$ *or no sequence in* D *extends* $\langle P_\alpha \rangle_{\alpha<\gamma}$*.*

A set $P \subseteq \mathbf{PT}$ *of perfect trees is* n*-complete if, for any* $\boldsymbol{\Sigma}_{n-2}(\mathbf{L}^{\dagger}_{\omega_1})$ *set* $W \subseteq \mathbf{PT}$*, the set* $W^{\mathtt{solv}} \cap P = \{S \in P : S \in W \vee \neg\, \exists\, T \in W\, (T \subseteq S)\}$ *is dense in* P*.*

The two following results are the conclusive steps in the proof of Theorem 6.

Lemma 22 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$, similar to Lemma 13)**.** *If* $n \geq 3$*, then there exists an* n*-complete* $\Delta^{\mathbf{L}^\dagger_{\omega_1}}_{n-1}$ *sequence* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{J}^\dagger_{\omega_1}$*.*

Theorem 9 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$, similar to Theorem 4)**.** *Assume that* $\mathfrak{n} \geq 3$*,* $\langle \mathbb{P}_\alpha(\mathfrak{n}) \rangle_{\alpha<\omega_1} \in \mathbf{J}^\dagger_{\omega_1}$ *is an* $\mathfrak{n}$*-complete* $\Delta^{\mathbf{L}^\dagger_{\omega_1}}_{n-1}$ *sequence* (Lemma 22)*, and* $\mathbb{P}(\mathfrak{n}) = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha(\mathfrak{n})$*. Then,* $\mathbb{P}(\mathfrak{n})$*-generic extensions of* $\mathbf{L}$ *prove Theorem 6.*

This completes the proof of Theorem 6 (general case).

Corollary 5. *If* $\mathfrak{n} \geq 2$*, then the conjunction* $(i) \wedge (ii) \wedge (iii) \wedge (iv)$ *of items of Theorem 6 is consistent with* $\mathbf{ZFC}^-$ *provided that* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *is consistent.*

Proof. This is a usual metamathematical corollary of Theorems 9 and 8 and Remark 3. □

18. Reduction to Second-Order Peano Arithmetic

Corollary 5 reduces Theorem 2 to the statement

$$\text{Consis}(\mathbf{PA}_2) \implies \text{Consis}(\mathbf{ZFC}^-_{\mathbf{lc}}).$$

We recall that the consistency of $\mathbf{PA}_2$ is the blanket assumption in Theorem 2. Yet, we can use the following equiconsistency result:

Theorem 10. *Theories* $\mathbf{PA}_2$ *and* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *are equiconsistent.*

Proof. The theorem has been a well-known fact since some while ago; see, e.g., Theorem 5.25 in [11]. A rather natural way of proof is as follows.

Step 1. Theory $\mathbf{ZFC}^-$ + "all sets are countable" is interpreted in $\mathbf{PA}_2$ by the tree interpretation described in [11], § 5, especially Theorem 5.11, or in [13], Definition VII.3.10 ff. Kreisel [12], VI(a)(ii), attributed this interpretation to the type of "crude" results.

Step 2. Arguing in $\mathbf{ZFC}^-$ + "all sets are countable", we define the transitive class $\mathbf{L}$ of all constructible sets, which models $\mathbf{ZFC}^-$ + "all sets are constructible".

Step 3. We argue in $\mathbf{ZFC}^-$ + "all sets are constructible". If every ordinal is countable, then immediately all sets are countable; that is, we have $\mathbf{ZFC}^-_{\mathbf{lc}}$. If there exist uncountable ordinals, then let ω_1 be the least of them. Then, $\mathbf{L}_{\omega_1}$ is a transitive set that models $\mathbf{ZFC}^-_{\mathbf{lc}}$.

We conclude from Steps 1,2,3 that $\mathbf{PA}_2$ and $\mathbf{ZFC}^-_{\mathbf{LC}}$ are equiconsistent. □

Combining Theorem 10 and Corollary 5, we finalize the proof of Theorem 2.

19. Conclusions and Problems

In this study, the method of definable generic forcing notions was employed to the construction of a model in which, for a given $\mathfrak{n} \geq 2$, there is a nonconstructible $\Delta^1_{\mathfrak{n}+1}$ real a, minimal over $\mathbf{L}$ and satisfying $\mathbf{V} = \mathbf{L}[a]$, but all $\Sigma^1_{\mathfrak{n}}$ reals are constructible (Theorem 1). This essentially strengthens and extends our earlier results in [10] by $\mathbf{V} = \mathbf{L}[a]$ and the minimality claim. In addition, we established (Theorem 2) the ensuing consistency result on the basis of second-order Peano arithmetic $\mathbf{PA}_2$, instead of the much stronger theory $\mathbf{ZFC}$ typically assumed as a premise in independence results obtained by the forcing method. This is a new result and a valuable improvement upon much of known independence results in modern set theory.

The technique developed in this paper may lead to further progress in studies of different aspects of the projective hierarchy. We hope that this study will contribute to the following crucial problem by S. D. Friedman; see [26] (P. 209) and [27] (P. 602): find a model of $\mathbf{ZFC}$, for a given $\mathfrak{n}$, in which all $\mathbf{\Sigma}^1_{\mathfrak{n}}$ sets of reals are Lebesgue measurable and have the Baire and perfect set properties, and, at the same time, there exists a $\mathbf{\Delta}^1_{\mathfrak{n}+1}$ well-ordering of the reals.

From our study, it is concluded that the technique of transitive models of bounded Separation in $\mathbf{ZFC}^-$, as in Section 15, will lead to similar consistency and independence results, related to second-order Peano arithmetic $\mathbf{PA}_2$ and similar to our Theorem 2, on the basis of the consistency of $\mathbf{PA}_2$ itself.

The following problems arise from our study.

Problem 1. Iterations of Jensen's forcing were developed by Abraham [28]. Combining this technique with the finite-support Jensen products technique and some earlier forcing constructions used in the theory of generic choiceless models, a model of **ZF** is presented in [29] in which the countable **AC** holds but the dependent choices scheme **DC** fails for some $\mathit{\Pi}^1_2$ relation (which is the best possible). This leads to two different problems:

(I) Reprove the consistency results in [29] on the basis of the consistency of theory $\mathbf{PA}_2$, similar to Theorem 2.

(II) Generalize the mentioned consistency result of [29] to higher projective levels by means of a suitable definable generic forcing notion. That is, given $\mathtt{n} \geq 3$, define a model of **ZF** in which the countable **AC** holds whereas **DC** fails for some $\mathit{\Pi}^1_{\mathtt{n}}$ relation but holds for $\mathbf{\Pi}^1_{\mathtt{n}-1}$. A recent paper [30], containing some consistency results related to different forms of the countable **AC**, is a step in this direction.

Problem 2. The method of definable generic forcing notions has been recently applied for some definability problems in modern set theory, including the following applications:

- A model of **ZFC**, in which the separation principle holds for a given effective projective type $\Sigma^1_{\mathtt{n}}$, $\mathtt{n} \geq 3$, is defined in [31];
- A model of **ZFC**, in which well-orderings of the reals first appear at a given projective level, is defined in [32];
- A model of **ZFC**, in which the full basis theorem holds in the absence of analytically definable well-orderings of the reals, is defined in [34].

It is a common problem related to all these results to establish their $\mathbf{PA}_2$-consistency versions similar to Theorem 2.

Problem 3. A somewhat modified forcing notion, say $\mathbb{P}'(\mathtt{n}) \subseteq \mathbf{PT}$, rather similar to $\mathbb{P}(\mathtt{n})$ of Theorem 4, is defined in [35]. It is invariant under some transformations so that, instead of a single generic real by $\mathbb{P}(\mathtt{n})$, it adjoins a E_0-*equivalence class* of $\mathbb{P}'(\mathtt{n})$-generic reals. (Recall that reals $a, b \in 2^\omega$ are E_0-equivalent if $a(n) = b(n)$ for all but finite n. See some generalizations in [36].) It turns out that this $\mathbb{P}'(\mathtt{n})$-generic E_0-class is a (countable) $\mathit{\Pi}^1_{\mathtt{n}}$ set containing no OD (ordinal-definable) elements in the extension, and, at the same time, every countable $\Sigma^1_{\mathtt{n}}$ set definitely contains OD elements.

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