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On the Uniform Projection Problem in Descriptive Set Theory

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Abstract: For every $\mathbb{n} \geq 1$, generic models of **ZFC** will be presented for either of the following two sentences: 1. There exists a linear $\Sigma^1_{\mathbb{n}+2}$ set not equal to the projection of any uniform planar $\mathbf{\Pi}^1_{\mathbb{n}+2}$ set. 2. There exists a linear $\Delta^1_{\mathbb{n}+2}$ set not equal to the projection of any uniform planar $\mathbf{\Pi}^1_{\mathbb{n}+1}$ set. Ensuing consistency and independence corollaries are discussed.

Keywords: constructibility; projective hierarchy; uniform sets; projections

MSC: 03E15; 03E45

1. Introduction

This article is devoted to one of the problems in descriptive set theory, posed in Luzin's monograph [1] (1930). Luzin indicates that, after constructing the projective hierarchy, "we immediately meet" with a number of questions, the general meaning of which is, can some properties of the first level of the hierarchy be transferred to the following levels? Luzin raised several concrete problems of this kind in [1], pp. 274–276, 285, related to different results on Borel ($\mathbf{\Delta}^1_1$), analytic ($\mathbf{\Sigma}^1_1$), and coanalytic ($\mathbf{\Pi}^1_1$) sets, already known by that time. In particular, Luzin asked a few questions in [1] aimed at solving the uniform projection problem. To explain the essence and content of this problem, let us recall several definitions and relevant classical results.

We use **boldface** letters $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ for **boldface** projective classes (corresponding to resp. A_n, CA_n, B_n in the classical notation adopted in [1]), and *slanted* letters $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ for *lightface* (or effective) classes, as is customary in modern descriptive set theory.

As usual, elements of *the Baire space* ω^ω will be called *reals*. By definition (Kechris [2], Moschovakis [3]), a set $X \subseteq \omega^\omega$ belongs to $\mathbf{\Sigma}^1_{n+1}$ iff it is equal to the *projection* $\operatorname{dom} P = \{x : \exists\, y\, P(x,y)\}$ of a planar $\mathbf{\Pi}^1_n$ set $P \subseteq (\omega^\omega)^2$, in symbol $\mathbf{\Sigma}^1_{n+1} = \textbf{proj}\, \mathbf{\Pi}^1_n$. (As is customary in texts on modern set theory, we use $\operatorname{dom} P$ for the *projection* $\operatorname{dom} P = \{x : \exists\, y\, P(x,y)\}$ of a planar set P to the first coordinate, and we use compact *relational expressions* like $P(x,y)$, $Q(x,y,z)$, etc., instead of $\langle x,y\rangle \in P$, $\langle x,y,z\rangle \in Q$, etc.)

The picture drastically changes if we consider only *uniform* sets $P \subseteq (\omega^\omega)^2$, i.e., those satisfying $P(x,y) \wedge P(x,z) \implies y = z$. Indeed, it was established in the early years of descriptive set theory that these three classes coincide:

- Class $\mathbf{\Delta}^1_1$ of all Borel sets in ω^ω;
- Class **proj unif** $\mathbf{\Delta}^1_1$ of projections of uniform $\mathbf{\Delta}^1_1$ (that is, Borel) sets in $(\omega^\omega)^2$;
- Class **proj unif** $\mathbf{\Pi}^1_0$ of projections of uniform $\mathbf{\Pi}^1_0$ (that is, closed) sets in $(\omega^\omega)^2$.

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(See Luzin [1,4], and also [2,3] for modern treatment.) Thus, symbolically,

$$\textbf{proj unif}\,\mathbf{\Pi}^1_0 = \textbf{proj unif}\,\mathbf{\Delta}^1_1 = \mathbf{\Delta}^1_1 \underset{\neq}{\subsetneq} \mathbf{\Sigma}^1_1 = \textbf{proj}\,\mathbf{\Pi}^1_0. \tag{1}$$

Now, the common content of Luzun's relevant problems can be formulated as follows:

Problem 1 (uniform projection problem, Luzin [1]). *For any given $n \geq 2$, figure out the relations between the classes $\mathbf{\Delta}^1_n \underset{\neq}{\subsetneq} \mathbf{\Sigma}^1_n = \textbf{proj}\,\mathbf{\Pi}^1_{n-1}$ and $\textbf{proj unif}\,\mathbf{\Pi}^1_{n-1} \subseteq \textbf{proj unif}\,\mathbf{\Delta}^1_n$.*

The following two theorems are **the main results** of this paper.

Theorem 1. *Let $n \geq 3$ and $\omega_2^{\mathbf{L}} < \omega_1$. Then, there exists a generic extension of $\mathbf{L}$, the constructible universe, in which it is true that there is a Σ^1_n set $X \subseteq \omega^\omega$ not equal to the projection of any uniform $\mathbf{\Pi}^1_n$ set $P \subseteq (\omega^\omega)^2$.*

Theorem 2. *Let $n \geq 3$ and $\omega_2^{\mathbf{L}} < \omega_1$. Then, there is a generic extension of $\mathbf{L}$ in which it is true that there is a Δ^1_n set $X \subseteq \omega^\omega$ not equal to the projection of any uniform $\mathbf{\Pi}^1_{n-1}$ set $P \subseteq (\omega^\omega)^2$.*

We may observe that in fact Theorem 2 for some n is a simple corollary of Theorem 1 for $n-1$ just because $\Sigma^1_n \subseteq \Delta^1_{n+1}$. This leaves only the case $n = 3$ of Theorem 2 not already covered by Theorem 1. However the particular case $n = 3$ of Theorem 2 is not essentially easier than the general case within the methods used in this paper. Therefore we will present the proof of Theorem 2 for an arbitrary value $n \geq 3$.

Comments. The results (1) handle case $n = 1$ of the problem, of course. Case $n = 2$ was solved by *the Novikov–Kondo uniformization theorem* (Luzin and Novikov [5] and Kondo [6]), which asserts that every $\mathbf{\Pi}^1_1$ set $P \subseteq (\omega^\omega)^2$ is *uniformizable* by a $\mathbf{\Pi}^1_1$ set Q, meaning that $Q \subseteq P$, Q is uniform, and $\operatorname{dom} Q = \operatorname{dom} P$. That is, formally,

$$\textbf{proj unif}\,\mathbf{\Pi}^1_1 = \textbf{proj unif}\,\mathbf{\Delta}^1_2 = \mathbf{\Sigma}^1_2. \tag{2}$$

Thus, we have pretty different state of affairs in cases $n = 1$ and $n = 2$.

Generally, as far as higher levels $n \geq 3$ of the projective hierarchy are concerned, it has been established in modern set theory that many classical problems are unsolvable for higher projective levels. In other words, we cannot give definite answers to the questions posed on the basis of the axioms of the **ZFC** set theory of Zermelo–Fraenkel (Z: Zermelo, F: Fraenkel, C: choice, since this theory includes the axiom of choice). In such cases, additional axioms are used to study the problem under consideration, such as Gödel's axiom of constructibility $\mathbf{V} = \mathbf{L}$ [7], as well as various generic models of set theory, i.e., those defined by Cohen's forcing method [8]. (The axiom of constructibility postulates that all sets are constructible, that is, admit a special transfinite construction starting from the empty set. $\mathbf{V}$ traditionally denotes the universe of all sets, $\mathbf{L}$ the class of all constructible sets, and hence $\mathbf{V} = \mathbf{L}$ is a standard abbreviation of this axiom. See also Jech [9] or Kunen [10] for modern treatment of forcing.) This usually leads to consistency/independence results.

The axiom of constructibility and consistency. We have recently succeeded to prove that under $\mathbf{V} = \mathbf{L}$, Luzin's problem is answered in such a way that the statement

$$\textbf{proj unif}\,\mathbf{\Pi}^1_{n-1} = \textbf{proj unif}\,\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n. \tag{3}$$

holds for all $n \geq 3$, i.e., pretty similar to the solution in case $n = 2$ given by (2) in **ZFC** alone. As the axiom of constructibility $\mathbf{V} = \mathbf{L}$ is consistent with **ZFC** by Gödel [7], all its consequences, including (3) for all $n \geq 3$, are consistent as well.

Generic models and independence. In this context, our Theorems 1 and 2 witness that the negations of (3), in the forms

$$\mathbf{\Sigma}^1_n \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_n \quad\text{and}\quad \mathbf{\Delta}^1_n \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{n-1}\,,$$

for any given $n \geq 3$, hold in suitable generic models of **ZFC**. (We recall that $\Sigma^1_n \subsetneqq \mathbf{\Sigma}^1_n$ and $\Delta^1_n \subsetneqq \mathbf{\Delta}^1_n$.)

Corollary 1. *If* $n \geq 3$*, then each of the following three statements is consistent with and independent of* **ZFC** : $\mathbf{\Sigma}^1_n = \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{n-1}$, $\mathbf{\Sigma}^1_n \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{n-1}$.

2. Outline of the Proof

We will make use of a wide range of methods related to forcing. Section 3 contains a brief introduction to iterated perfect sets; it is written for the convenience of the reader. Section 4 introduces a natural set of permutations of iterated perfect sets and their actions. Section 5 briefly describes the forcing notion $\mathscr{X} \in \mathbf{L}$ used in this paper; it depends on the value of n (or rather on $\mathbb{n} = n-2$, so we can write $\mathscr{X} = \mathscr{X}[\mathbb{n}]$) in Theorems 1 and 2. The forcing notion $\mathscr{X}$ involves several significant ideas and constructions in forcing theory, including the following:

(I) Jensen's [11] construction of a forcing notion say $\mathbb{J} \in \mathbf{L}$ which adjoins a minimal non-constructible $\mathit{\Pi}^1_2$ real singleton $\{a\}$, and hence a Δ^1_3 real a;
(II) Generalized iterations of the Sacks forcing as in Groszek and Jech [12];
(III) Definable-generic construction of forcing notions by Harrington [13] that yields non-homogeneous forcing notions as elementary subforcings of homogeneous forcings;
(IV) A forcing notion $\mathbb{J}[\mathbb{n}] \in \mathbf{L}$ (for a given $\mathbb{n} \geq 2$) in [14], based on (I) and (III), which adjoins a minimal $\mathit{\Pi}^1_n$ real singleton but does not adjoin non-constructible Δ^1_n reals;
(V) Jensen's model of **ZF** in which countable **AC** holds but the principle of dependent choices **DC** fails (see [15] or pp. 155–159 in Felgner [16]), obtained by adjoining a $\boldsymbol{I}$-sequence of Cohen reals with subsequent suitable symmetrization — where $\boldsymbol{I}$ is the tree of all non-empty finite tuples of ordinals $\alpha < \omega_1$;
(VI) An ω_1-long iteration of Jensen's forcing by Abraham [17] and a generalized $\boldsymbol{I}$-iteration of Jensen's forcing by Gitman [18], based on (I), (II), (V), used to obtain models with various forms of the countable axiom of choice;
(VII) A generalized $\boldsymbol{I}$-iteration $\mathscr{X} = \mathscr{X}[\mathbb{n}]$ of the forcing notion $\mathbb{J}[\mathbb{n}]$ as in (IV) was defined in [19] following the ideas in (VI) — equivalently, it can be viewed as Harrington-style sub-forcing (as in (III)) of the generalized $\boldsymbol{I}$-iteration of the Sacks forcing.

Forcing $\mathscr{X} = \mathscr{X}[\mathbb{n}]$ is used in this paper as well.

The relevant generic extensions of $\mathbf{L}$ are considered in Sections 6 and 7. We will refer to [19] in matters of their key properties. We introduce the associated forcing relation in Section 8, consider its invariance in Section 9, and prove some related *isolation* results in Section 10.

Sections 11 and 12 contain proofs of Theorems 1 and 2. To prove the result, we consider certain subextensions of an $\mathscr{X}$-generic extension of $\mathbf{L}$.

Section 13 contains conclusive remarks and offers some problems for further study.

3. Preliminaries: Spaces, Projections, Iterated Perfect Sets

Arguing in L in this section, we define, in $\mathbf{L}$, the set $\boldsymbol{I} = \omega_1^{<\omega} \smallsetminus \{\Lambda\} \in \mathbf{L}$ of all non-empty tuples $\boldsymbol{i} = \langle \gamma_0, \ldots, \gamma_{n-1} \rangle$, $n \geq 1$, of ordinals $\gamma_k < \omega_1$. The set $\boldsymbol{I}$ is partially ordered by *the strict extension* $\subset$ of tuples. Then, $\boldsymbol{I}$ is a tree without a root because Λ, the empty tuple, is excluded. Characters $\boldsymbol{i}, \boldsymbol{j}$ are used to denote *elements* of $\boldsymbol{I}$.

If $\boldsymbol{i} \in \boldsymbol{I}$, then $\mathtt{lh}\,\boldsymbol{i}$ is the length of $\boldsymbol{i}$; $\mathtt{lh}\,\boldsymbol{i} \geq 1$ since Λ is excluded.

Our plan is to define a generic extension $\mathbf{L}[\mathbf{a}]$ of $\mathbf{L}$ by an array $\mathbf{a} = \langle \mathbf{a}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$ of reals $\mathbf{a}_{\boldsymbol{i}} \in 2^\omega$, where the structure of iterated genericity of the reals $\mathbf{a}_{\boldsymbol{i}}$ will be determined by $\boldsymbol{I}$.

Let Ξ be the set of all at most countable *initial segments* (in the sense of $\subset$) $\zeta \subseteq \boldsymbol{I}$. Greek letters ξ, η, ζ, ϑ, τ denote sets in Ξ. For any $\boldsymbol{i} \in \zeta \in \Xi$, we consider initial segments

$$[\subset\boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subset \boldsymbol{i}\} \;\subsetneqq\; [\subseteq\boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}$$

Let $\mathscr{D} = 2^\omega \subseteq \omega^\omega$ be the *Cantor space*.

For any set ξ, $\mathscr{D}^\xi$ is the topological product of ξ-many copies of $\mathscr{D}$, a compact space.

Projections. Assume that $\eta \subseteq \xi$ belong to Ξ. If $x \in \mathscr{D}^\xi$, then let $x{\downarrow}\eta := x \restriction \eta \in \mathscr{D}^\eta$, the usual restriction. If $X \subseteq \mathscr{D}^\xi$, then let $X{\downarrow}\eta = \{x{\downarrow}\eta : x \in X\}$.

If $Y \subseteq \mathscr{D}^\eta$, then let $Y{\uparrow}\xi = \{x \in \mathscr{D}^\xi : x{\downarrow}\eta \in Y\}$.

We define $X{\downarrow}_{\subseteq\boldsymbol{i}} = X{\downarrow}[\subseteq\boldsymbol{i}]$, and similarly $X{\downarrow}_{\subset\boldsymbol{i}}$, and $x{\downarrow}_{\subseteq\boldsymbol{i}}$ etc. for points x.

Definition 1 (iterated perfect sets)**.** *For any $\zeta \in \Xi$, let $\mathbf{IPS}_\zeta$ be the collection of all sets $X \subseteq \mathscr{D}^\zeta$ such that there is a homeomorphism $H : \mathscr{D}^\zeta \xrightarrow{\text{onto}} X$ satisfying*

$$x_0{\downarrow}\eta = x_1{\downarrow}\eta \iff H(x_0){\downarrow}\eta = H(x_1){\downarrow}\eta$$

for all elements x_0, $x_1 \in \operatorname{dom} H$ and all sets $\eta \in \Xi$, $\eta \subseteq \zeta$.

We put $\mathbf{IPS} = \bigcup_{\xi\in\Xi} \mathbf{IPS}_\xi$. Sets in $\mathbf{IPS}$ are called iterated perfect sets.

The set $\mathbf{IPS}$ is ordered *by the relation: $X \downarrow\subseteq Y$ iff $\eta = \dim Y \subseteq \|X\|$ and $X{\downarrow}\eta \subseteq Y$.*

If $X \in \mathbf{IPS}_\xi$, then let $\|X\| = \xi$ (the dimension *of X).*

For instance, the empty set $\varnothing$ belongs to Ξ, $\mathscr{D}^\varnothing = \{\varnothing\}$, $\mathbb{1} = \{\varnothing\} \in \mathbf{IPS}_\varnothing$.

Suppose that $\zeta \in \Xi$ in $\mathbf{L}$. The set $\mathbf{IPS}_\zeta$, defined in $\mathbf{L}$, can be considered as a forcing notion, ordered by $\subseteq$. It is proved in [20] (Theorem 1 and Section 6.1) that $\mathbf{IPS}_\zeta$ adjoins a generic array $\boldsymbol{v} \in \mathscr{D}^\zeta$ of reals $\boldsymbol{v}(\boldsymbol{i}) \in \mathscr{D} = 2^\omega$, $\boldsymbol{i} \in \zeta$, such that each real $\boldsymbol{v}(\boldsymbol{i})$ is Sacks-generic over $\mathbf{L}[\boldsymbol{v}{\downarrow}_{\subset\boldsymbol{i}}]$. Thus, $\mathbf{IPS}_\zeta$ works as a generalized ζ-long iteration of the Sacks (perfect set) forcing. This is why we call sets in $\mathbf{IPS}$ *iterated perfect sets*.

4. Permutations

Let **Perm** be the group of all bijections $\pi : \boldsymbol{I} \xrightarrow{\text{onto}} \boldsymbol{I}$, $\pi \in \mathbf{L}$, $\subset$*-invariant* in the sense that $\boldsymbol{i} \subset \boldsymbol{j} \iff \pi(\boldsymbol{i}) \subset \pi(\boldsymbol{j})$ for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$. Thus, $\mathbf{Perm} \in \mathbf{L}$. Bijections $\pi \in \mathbf{Perm}$ will be called *permutations*. Any $\pi \in \mathbf{Perm}$ is *length-preserving*, so that $\mathtt{lh}\,\boldsymbol{i} = \mathtt{lh}\,\pi(\boldsymbol{i})$ for all $\boldsymbol{i} \in \xi$,

The superposition $\circ$ is the group operation: $(\pi \circ \rho)(\boldsymbol{i}) = \pi(\rho(\boldsymbol{i}))$.

Any permutation $\pi \in \mathbf{Perm}$ induces transformations **left-acting** on several types of objects as follows.

- If $\xi \in \Xi$, or generally $\xi \subseteq \boldsymbol{I}$, then $\pi{\bullet}\xi := \pi''\xi = \{\pi(\boldsymbol{i}) : \boldsymbol{i} \in \xi\}$.
- If $\xi \subseteq \boldsymbol{I}$ and $x \in \mathscr{D}^\xi$, then $\pi{\bullet}x \in \mathscr{D}^{\pi\bullet\xi}$ is defined by $(\pi{\bullet}x)(\pi(\boldsymbol{i})) = x(\boldsymbol{i})$ for all $\boldsymbol{i} \in \xi$.

  That is, formally $\pi{\bullet}x = x \circ \pi^{-1}$, the superposition.
- If $\xi \subseteq \boldsymbol{I}$ and $X \subseteq \mathscr{D}^\xi$, then $\pi{\bullet}X := \{\pi{\bullet}x : x \in X\} \subseteq \mathscr{D}^{\pi\bullet\xi}$.
- If $G \subseteq \mathbf{IPS}$, then $\pi{\bullet}G := \{\pi{\bullet}X : X \in G\}$.

If $\pi \in \mathbf{Perm}$ and $X \in \mathbf{IPS}_\xi$, then easily $\pi{\bullet}X \in \mathbf{IPS}_{\pi\bullet\xi}$. Moreover π is an $\|\ldots\|$-preserving and $\downarrow\subseteq$-preserving automorphism of $\mathbf{IPS}$.

Lemma 1 ([19], Lemma 14.3)**.** *Let $\pi, \rho \in \mathbf{Perm}$, $\eta \in \Xi$, and $v \in \mathscr{D}^{\boldsymbol{I}}$. Then*

(i) $\pi{\bullet}(\rho{\bullet}v) = (\pi \circ \rho){\bullet}v$ *— the group action property;*

(ii) $(\pi{\bullet}v){\downarrow}(\pi{\bullet}\eta) = \pi{\bullet}(v{\downarrow}\eta)$*, equivalently,* $(\pi{\bullet}v){\downarrow}\eta = \pi{\bullet}(v{\downarrow}(\pi^{-1}{\bullet}\eta))$.

Thus, in general $\pi{\bullet}(v{\downarrow}\eta) = (\pi{\bullet}v){\downarrow}(\pi{\bullet}\eta)$ is not equal to $(\pi{\bullet}v){\downarrow}\eta$!

To define an important subgroup of **Perm**, recall that every ordinal α can be represented in the form $\alpha = \lambda + m$, where $\lambda \in \mathbf{Ord}$ is a limit ordinal and $m < \omega$; then, α is called *odd*, resp., *even*, if the number m is odd, resp., even. A tuple $\boldsymbol{i} = \langle \alpha_0, \ldots, \alpha_k \rangle \in \boldsymbol{I}$ is *odd*, resp., *even*, if such is the last term α_k. If $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$, then $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ will mean that $\mathtt{lh}\,\boldsymbol{i} = \mathtt{lh}\,\boldsymbol{j}$ and if $k < \mathtt{lh}\,\boldsymbol{i}$, then the ordinals $\boldsymbol{i}(k)$ and $\boldsymbol{j}(k)$ have the same parity.

Odd and even tuples will play different roles in our forcing construction. Namely, even tuples will be involved in the coding procedures, whereas the role of odd tuples will be to obscure things enough so that the desired counterexamples will not be available at levels of the hierarchy lower than prescribed.

Let $\boldsymbol{\Pi}$ be the subgroup of all permutations $\pi \in \mathbf{Perm}$, such that $\boldsymbol{i} \approx_{\mathtt{par}} \pi(\boldsymbol{i})$ for every $\boldsymbol{i} \in \boldsymbol{I}$, that is, the subgroup of all *parity-preserving* permutations.

If $\eta \in \Xi$, then put $\boldsymbol{\Pi}(\eta) = \{\pi \in \boldsymbol{\Pi} : \forall\, \boldsymbol{i} \in \eta\, (\boldsymbol{i} = \pi(\boldsymbol{i}))\}$, a subgroup of $\boldsymbol{\Pi}$.

5. The Forcing Notion

It has taken considerable effort to actually define in [19] the forcing notion $\mathscr{X} = \mathscr{X}[\mathbb{n}] \in \mathbf{L}$, $\mathscr{X} \subseteq \mathbf{IPS}$ (for a given $\mathbb{n} \geq 1$), which we will use here for the proof of our main results. As the notion of iterated perfect set and many related notions are definitely non-absolute, we add the following warning.

Remark 1. *The definition of* **IPS** *in Section 3 and all other relevant definitions in Sections 3–5, are assumed to be relativized to* **L** *by default, and we will never bother to add the sign* $^{\mathbf{L}}$ *of relativization. In other words,* $\boldsymbol{I}$ *is* $(\boldsymbol{I})^{\mathbf{L}}$, Ξ *is* $(\Xi)^{\mathbf{L}}$, $\mathbf{IPS} = (\mathbf{IPS})^{\mathbf{L}}$, $\boldsymbol{\Pi} = (\boldsymbol{\Pi})^{\mathbf{L}}$, $\mathbf{NFo} = (\mathbf{NFo})^{\mathbf{L}}$ *(see below), etc.*

Theorem 3 ([19], Thm 36.1)**.** *If* $\mathbb{n} \geq 1$ *then there is a set* $\mathscr{X} \in \mathbf{L}$, $\mathscr{X} \subseteq \mathbf{IPS}$, *which is a* **normal forcing** *with the* **Fusion, Structure,** $\mathbb{n}$**-Definability,** *and* $\mathbb{n}$**-Odd Expansion** *properties.*

The four mentioned properties will be explained in Section 7. See below in this Section on the concept of normal forcing.

Definition 2. *We fix both* $\mathbb{n} \geq 1$ *and a set* $\mathscr{X} = \mathscr{X}[\mathbb{n}] \in \mathbf{L}$ *as in Theorem 3 for the remainder of this paper. We assume that* $\mathscr{X}$ *is ordered by the relation* ${\downarrow}{\subseteq}$ *(Definition 1), so that*

$$\text{if } X \mathbin{\downarrow\subseteq} Y \text{ then } X \text{ is a stronger condition.}$$

The inequality $\omega_2^{\mathbf{L}} < \omega_1$ *will be our* **blanket assumption***.*

We do not reproduce here quite a complicated construction of $\mathscr{X} = \mathscr{X}[\mathbb{n}]$ given in [19]. Yet, we will gradually explain all notions involved in Theorem 3. The first of them is the concept of **a normal forcing**. Recall that $\|X\| = \xi$ in case $X \subseteq \mathscr{D}^{\xi}$. If $\mathscr{X} \subseteq \mathbf{IPS}$, then put

$$\begin{aligned} \mathscr{X}{\downarrow}\eta &= \{X{\downarrow}\eta : X \in \mathscr{X} \wedge \eta \subseteq \|X\|\}, \\ \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} &= \mathscr{X}{\downarrow}\eta, \text{ where } \eta = [\subseteq \boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}. \end{aligned}$$

Arguing in L, say that a set $\mathscr{X} \subseteq \mathbf{IPS}$ in $\mathbf{L}$ is a *normal forcing*, $\mathscr{X} \in \mathbf{NFo}$ for brevity, iff the following conditions 1°–6° hold:

1°. $\mathscr{X} \subseteq \mathbf{IPS}$, and if $\tau \in \Xi$, then $\mathscr{D}^{\tau} \in \mathscr{X}$.

2°. If $\xi \subseteq \tau$ belong to Ξ and $X \in \mathscr{X} \cap \mathbf{IPS}_{\tau}$, then $X{\downarrow}\xi \in \mathscr{X}$. In particular, the set $\mathbb{1} = \{\varnothing\} = X{\downarrow}\varnothing$ belongs to $\mathscr{X}{\downarrow}\varnothing$, and $\mathbb{1} \mathbin{\downarrow\subseteq} X$ for any $X \in \mathscr{X}$.

3°. If $\xi \subseteq \tau$ belong to Ξ, $X \in \mathscr{X}{\downarrow}\tau$, $Y \in \mathscr{X}{\downarrow}\xi$, and $Y \subseteq X{\downarrow}\eta$, then $X \cap (Y{\uparrow}\tau) \in \mathscr{X}{\downarrow}\tau$. In particular, if $Y \in \mathscr{X}{\downarrow}\xi$, then $Y{\uparrow}\tau \in \mathscr{X}{\downarrow}\tau$.

4°. If $\tau \in \Xi$, $X \in \mathscr{X}{\downarrow}\tau$, $Y \in \mathbf{IPS}_\tau$, $Y \subseteq X$ is open in X, then $Y \in \mathscr{X}$.

5°. $\mathscr{X}$ is $\mathbf{\Pi}$-invariant: if $\pi \in \mathbf{\Pi}$ and $X \in \mathbf{IPS}$, then $X \in \mathscr{X} \iff \pi{\bullet}X \in \mathscr{X}$.

6°. If $\tau \in \Xi$, $X \in \mathbf{IPS}_\tau$, and $X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \tau$, then $X \in \mathscr{X}$.

For instance, **IPS** itself belongs to **NFo**; see [19], Section 21.

6. Generic Arrays and Generic Extensions

We are going to establish our main results (Theorems 1 and 2) by means of generic extensions of $\mathbf{L}$ via the forcing notion $\mathscr{X} = \mathscr{X}[\mathfrak{n}] \in \mathbf{L}$ fixed by Definition 2. It is clear that $\mathscr{X}$-generic extensions of $\mathbf{L}$ do exist under the *consistent* assumption that $\omega_2^{\mathbf{L}} < \omega_1$ in the universe, which we suppose in Theorems 1 and 2.

Under $\omega_2^{\mathbf{L}} < \omega_1$, if $\zeta \in \Xi$ (i.e., $\zeta \in \mathbf{L}$ and $\mathbf{L} \models \zeta \in \Xi$), then every set $X \in \mathbf{IPS}_\zeta$ is a countable subset of $\mathscr{D}^\zeta$ in the universe. However it transforms to a perfect set in the universe by the closure operation: *the topological closure* $X^\#$ of a set $X \in \mathbf{IPS}_\zeta$ is closed in $\mathscr{D}^\zeta$ in the universe. (And in fact $X^\#$ satisfies the definition of $\mathbf{IPS}_\zeta$ in the universe.)

Recall that $\mathscr{X} \subseteq \mathbf{IPS}$, $\mathscr{X} \in \mathbf{L}$ is a normal forcing by Definition 2, that is, 1°–6° above hold (in $\mathbf{L}$). Let $G \subseteq \mathscr{X}$ be a filter $\mathscr{X}$-generic over $\mathbf{L}$. It easily follows from 4°, that there is a unique array $\boldsymbol{v} = \boldsymbol{v}[G] = \langle \boldsymbol{v}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}} \in \mathscr{D}^{\boldsymbol{I}}$, called *$\mathscr{X}$-generic array* (over $\mathbf{L}$), all terms $\boldsymbol{v}_{\boldsymbol{i}} = \boldsymbol{v}_{\boldsymbol{i}}[G] = \boldsymbol{v}(\boldsymbol{i})$ being reals (i.e., elements of $\mathscr{D} = 2^\omega$), such that the equivalence

$$\boldsymbol{v}{\downarrow}\zeta \in X^\# \iff X \in G$$

holds for all $X \in \mathscr{X}$ and $\zeta = \|X\| \in \Xi$. Then, the model $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{v}[G]] = \mathbf{L}[\langle \boldsymbol{v}_{\boldsymbol{i}}[G] \rangle_{\boldsymbol{i} \in \boldsymbol{I}}]$ is an *$\mathscr{X}$-generic extension of* $\mathbf{L}$.

7. Four Key Properties of the Forcing Notion

Now, we actually define those four key properties of the normal forcing notion $\mathscr{X}$ mentioned in Theorem 3. We follow [19]. We'll have to introduce some preliminary notions.

Suppose that $X \in \mathbf{IPS}$ and $\mathscr{Y} \subseteq \mathbf{IPS}$. We define $X \subseteq^{\mathtt{fd}} \bigcup \mathscr{Y}$, iff there is a finite set $\mathscr{Y}' \subseteq \mathscr{Y}$ such that (1) $\|Y\| \subseteq \xi = \|X\|$ for all $Y \in \mathscr{Y}'$, (2) $X \subseteq \bigcup_{Y \in \mathscr{Y}'} (Y{\uparrow}\xi)$, and (3) $(Y{\uparrow}\xi) \cap (Z{\uparrow}\xi) = \varnothing$ for all $Y \neq Z$ in $\mathscr{Y}'$.

Definition 3 (odd expansions and saturation)**.** *If $\eta \subseteq \tau$ belong to Ξ, then τ is an* odd expansion *of η, in symbol $\eta \subseteq_{\mathsf{odd}} \tau$, iff every tuple $\boldsymbol{i} \in \tau \smallsetminus \eta$ is odd.*

A set $J \subseteq \boldsymbol{I}$ is odd-saturated *iff we have $\eta \in J \implies \tau \in J$ whenever $\eta \subseteq_{\mathsf{odd}} \tau$.*

***Fusion* property of $\mathscr{X}$:** for any sequence $\langle \mathscr{Y}_k \rangle_{k<\omega} \in \mathbf{L}$ of dense sets $\mathscr{Y}_k \subseteq \mathscr{X}$, the set $\mathscr{Y} = \{X \in \mathscr{X} : \forall k\, (X \subseteq^{\mathtt{fd}} \bigcup \mathscr{Y}_k)\}$ is dense in $\mathscr{X}$ as well.

***Structure* property of $\mathscr{X}$:** if $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is an $\mathscr{X}$-generic array over $\mathbf{L}$, then, for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$, we have: $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}[\boldsymbol{v}(\boldsymbol{j})]$ iff $\boldsymbol{i} \subseteq \boldsymbol{j}$;

$\mathfrak{n}$-*Definability* property of $\mathscr{X}$: if $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is an $\mathscr{X}$-generic array over $\mathbf{L}$, $U \in \mathbf{L}[\boldsymbol{v}]$ and $\mathfrak{M} = \mathbf{L}[U]$, then the set $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$ belongs to $\mathfrak{M}$ and is a $\Pi^1_{\mathfrak{n}+1}$ set in $\mathfrak{M}$, where

$$\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) = \{\langle k, \boldsymbol{v}(\boldsymbol{i}) \rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is even} \wedge \mathtt{lh}\, \boldsymbol{i} = k\}.$$

(This formulation of the $\mathfrak{n}$-Definability property is somewhat weaker than the original formulation in [19], where a similarly defined set $\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v})$ was involved along with $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v})$, and we had to consider some cases when $\mathfrak{M}$ is not necessarily even a model of **ZF**. In this paper, there is no need in such a generalization.)

$\mathfrak{n}$-*Odd-Expansion*, or $\mathfrak{n}$-*OE*, *property* of $\mathscr{X}$: if $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic over $\mathbf{L}$, then for every $\eta \in \Xi$ and every $\Pi^1_{\mathfrak{n}}$ formula $\varphi(\cdot)$, with parameters in $\mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$, if

$\mathbf{L}[\boldsymbol{v}] \models \exists x\, \varphi(x)$, then there is an *odd expansion* $\tau \in \Xi$ of η and some $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}\!\downarrow\!\tau]$ such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(x)$ — *this definitely holds in case* $\mathbb{n} = 1$ *by the Shoenfield absoluteness* [21].

We may note that, for example, **IPS** as a forcing notion does have the Fusion, Structure and $\mathbb{n}$-OE properties, but does not have the $\mathbb{n}$-Definability property for any $\mathbb{n}$.

The Fusion property is another formalization of some features of the Sacks forcing. It somewhat differs from a more commonly used *Axiom A* (see Jech [9]), Def. 31.10, but it fits better to applications in this paper. On the other hand, the Fusion property is a weaker form of ω-distributivity as in [9], Def. 15.5. The next lemma presents several applications of the Fusion property of $\mathscr{X}$, including continuous reading of real names by (iv).

Lemma 2 (Theorem 27.1 in [19]). *Assume that* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is* $\mathscr{X}$*-generic over* $\mathbf{L}$*. Then:*

(i) *If* $h \in \mathbf{L}[\boldsymbol{v}]$, $h : \omega \to \mathbf{L}$, *then there is a map* $H \in \mathbf{L}$ *such that* $\operatorname{dom} H = \omega$, *and, for each* $k < \omega$, $h(k) \in H(k)$ *and* $H(k)$ *is finite;*

(ii) *every* $\mathbf{L}$*-cardinal remains a cardinal in* $\mathbf{L}[\boldsymbol{v}]$*;*

(iii) *If* $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}]$, *then* $x \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\xi]$ *for some* $\xi \in \Xi$, *— and more generally, If* $J \in \mathbf{L}$, $J \subseteq \boldsymbol{I}$ *is an initial segment, and* $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v} \restriction J]$ *then* $x \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\xi]$ *for some* $\xi \in \Xi$, $\xi \subseteq J$*;*

(iv) *If* $\xi \in \Xi$ *and* $a \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}\!\downarrow\!\xi]$, *then there is a continuous map* $F : \mathscr{D}^\xi \to \omega^\omega$ *such that* $a = F(\boldsymbol{v}\!\downarrow\!\xi)$, *and* F is coded in $\mathbf{L}$ *in the sense that the restriction* $F_{\mathbf{L}} = F \restriction (\mathbf{L} \cap \mathscr{D}^\xi)$ *belongs to* $\mathbf{L}$.

Note that if $F_{\mathbf{L}} = F \restriction (\mathbf{L} \cap \mathscr{D}^\xi) \in \mathbf{L}$ in (iv), then $\mathbf{L} \models$ "$F_{\mathbf{L}} : \mathscr{D}^\xi \to \omega^\omega$ is continuous" and $F = F_{\mathbf{L}}^{\#}$ (the topological closure of $F_{\mathbf{L}}$ in $\mathscr{D}^\xi \times \omega^\omega$).

The two following corollaries are based on resp. the $\mathbb{n}$-Definability and the $\mathbb{n}$-Odd-Expansion properties of the forcing notion $\mathscr{X}$.

Corollary 2. *Let* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *be* $\mathscr{X}$*-generic over* $\mathbf{L}$, $J \in \mathbf{L}$, $J \subseteq \boldsymbol{I}$ *be an initial segment. Then,*

(i) *if* $\boldsymbol{i} \in \boldsymbol{I} \smallsetminus J$, *then* $\boldsymbol{v}(\boldsymbol{i}) \notin \mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$*;*

(ii) *the set* $\mathbf{E}_J^{\mathtt{evn}}(\boldsymbol{v}) = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in J \text{ is even} \wedge \mathtt{lh}\, \boldsymbol{i} = k\}$ *is equal to the set* $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$, *and hence* $\mathbf{E}_J^{\mathtt{evn}}(\boldsymbol{v})$ *belongs to* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$ *and is a* $\Pi^1_{\mathbb{n}+1}$ *set in* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$.

Proof. (i) By (iii) of Lemma 2, if $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$, then $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\xi]$ for some $\xi \in \Xi$, $\xi \subseteq J$. This contradicts Corollary 26.4 in [19], saying that $\boldsymbol{v}(\boldsymbol{i}) \notin \mathbf{L}[\boldsymbol{v}\!\downarrow\!\xi]$ provided $\boldsymbol{i} \notin \xi$.

Claim (ii) follows from (i) and the $\mathbb{n}$-Definability property in the "hence" part. $\square$

Corollary 3. *Assume that* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is an array* $\mathscr{X}$*-generic over* $\mathbf{L}$*. Let* $J \subseteq J'$ *be odd-saturated initial segment of* $\boldsymbol{I}$ *in* $\mathbf{L}$*. Then,*

(i) *the classes* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$ *and* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J']$ *are elementary submodels of* $\mathbf{L}[\boldsymbol{v}]$ *w.r.t.* $\Sigma^1_{\mathbb{n}+1}$ *formulas with parameters in resp.* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$, $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J']$*;*

(ii) *every* $\Sigma^1_{\mathbb{n}+2}$ *formula with parameters in* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$ *true in* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$ *remains true in* $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J']$.

Proof. (i) Consider any $\Pi^1_{\mathbb{n}}$ formula $\varphi(x)$, with parameters in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$. By Lemma 2(iii), there is $\eta \in \Xi$ in $\mathbf{L}$, such that $\eta \subseteq J$ and each real parameter in φ belongs to $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\eta]$. By the $\mathbb{n}$-OE property of $\mathscr{X}$, if $\mathbf{L}[\boldsymbol{v}] \models \exists x\, \varphi(x)$, then there is an *odd expansion* $\tau \in \Xi$ of η and some $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}\!\downarrow\!\tau]$ such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(x)$. Now, we have $\tau \subseteq J$ by the odd-saturation of J, and hence $x \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!J]$, as required.

(ii) is a simple consequence of (i). $\square$

To conclude this Section, we may note that the $\mathbb{n}$-Definability property weakens, whereas the $\mathbb{n}$-Odd-Expansion strengthens with $\mathbb{n} \to \infty$. In particular, it occurs that $(\mathbb{n}+1)$-

Odd-Expansion is already incompatible with the $\mathbb{n}$-Definability property, and hence the combination of $\mathbb{n}$-Definability and $\mathbb{n}$-Odd-Expansion in Theorem 3 is well balanced.

To prove the incompatibility claim, let $v \in \mathscr{D}^{I}$ be an $\mathscr{X}$-generic array, and $\mathfrak{M} = \mathbf{L}[v]$. Let, by $\mathbb{n}$-Definability, $\varphi(k,x)$ be a $\Pi^1_{\mathbb{n}+1}$ formula which defines the set $\mathbf{E}^{\mathtt{evn}}(v)$ in $\mathbf{L}[v]$. Corollary 2(ii) with $J = I$ implies that φ defines the set $\mathbf{E}^{\mathtt{evn}}_{I}(v)$ in $\mathbf{L}[v]$. In other words,

$$\{\langle k, v(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in I \text{ is even } \wedge \mathtt{lh}\,\boldsymbol{i} = k\} = \{\langle k, x\rangle \in \mathbf{L}[v] : \mathbf{L}[v] \models \varphi(k,x)\}. \quad (4)$$

It follows that $\mathbf{L}[v] \models \exists\, x\, \varphi(1,x)$ (take $\boldsymbol{i} = \langle 0\rangle$ and $x = v(\boldsymbol{i})$). Applying $(\mathbb{n}+1)$-Odd-Expansion with $\eta = \varnothing$, we obtain an odd expansion $\tau \in \Xi$ of $\varnothing$ and some $x \in \omega^\omega \cap \mathbf{L}[v{\downarrow}\tau]$ satisfying $\mathbf{L}[v] \models \varphi(1,x)$, hence, by (4), $x = v(\boldsymbol{i})$, where $\boldsymbol{i} \in I$ is even and $\mathtt{lh}\,\boldsymbol{i} = 1$. Then, $\boldsymbol{i} \in \tau$ by Corollary 2(i). Yet τ contains only odd tuples by construction, a contradiction.

8. Forcing Relation

Recall that $\mathscr{X} \in \mathbf{NFo}$ is a fixed normal forcing, i.e., $\mathscr{X} \in \mathbf{L}$ and it holds in $\mathbf{L}$ that $\mathscr{X} \in \mathbf{NFo}$, see Remark 1 and Definition 2. To study $\mathscr{X}$-generic extensions of $\mathbf{L}$, we make use of a *forcing language* $\mathscr{L}$, containing the following $\mathbf{L}$-class $\mathbf{N}(\mathscr{L})$ of basic names:

- $\dot{x}$ for any $x \in \mathbf{L}$ — we will typically *identify* $\dot{x}$ with x itself, as usual;
- $\underline{\sigma v}$ for any $\sigma \in \mathbf{\Pi}$ — names of this form will be called *unlimited*;
- *Derived* names $\underline{\sigma v}{\downarrow}\eta$ for any $\sigma \in \mathbf{\Pi}$ and $\eta \in \Xi$;
- In particular names $\underline{v}$ and $\underline{v}{\downarrow}\eta$ will be shorthands for resp. $\underline{\varepsilon v}$ and $\underline{\varepsilon v}{\downarrow}\eta$, where $\varepsilon \in \mathbf{\Pi}$ is the identity.

All those names belong to $\mathbf{L}$ as $\mathbf{\Pi}, \Xi \in \mathbf{L}$; see Remark 1.

This definition of $\mathscr{L}$ does not include names of the form $\underline{\mathbf{W}_\Omega}$, very instrumental in [19], because we do not consider symmetric subextensions in this paper. Generally, using a suitable ramified language of this type is quite common in forcing theory; see e.g., [22] of recent papers.

An $\mathscr{L}$-formula is **limited** iff it contains unlimited names $\underline{\pi v}$ only via derived names $\underline{\sigma v}{\downarrow}\eta$, $\sigma \in \mathbf{\Pi}$ and $\eta \in \Xi$.

Given $v \in \mathscr{D}^{I}$ in the universe and an $\mathscr{L}$-formula φ, we define the *valuation* $\varphi[v]$ by the substitution of the valuations resp. $\dot{x}[v] := x$, $(\underline{\sigma v})[v] := \sigma{\bullet}v$, $(\underline{\sigma v}{\downarrow}\eta)[v] := (\sigma{\bullet}v){\downarrow}\eta$ for any basic names resp. $\dot{x}$, $\underline{\pi v}$, $\underline{\sigma v}{\downarrow}\eta$ in $\mathbf{N}(\mathscr{L})$ that occur in φ. All those sets belong to the extension $\mathbf{L}[v] = \mathbf{L}[\mathscr{G}_v]$, of course. In other words, $\underline{v}$ is a canonical name for a generic array $v \in \mathscr{D}^{I}$, each $\underline{\sigma v}$ is a name for $\sigma{\bullet}\underline{v}$, each $\underline{\sigma v}{\downarrow}\eta$ is a name for $(\sigma{\bullet}\underline{v}){\downarrow}\eta = \sigma{\bullet}(\underline{v}{\downarrow}\eta')$, where $\eta' = \sigma^{-1}{\bullet}\eta$ (we refer to Lemma 1).

Definition 4 (forcing relation)**.** *Assume that φ is a closed $\mathscr{L}$-*formula *(with names in $\mathbf{N}(\mathscr{L})$ as parameters). Let $X \in \mathscr{X}$, $\zeta = \|X\|$. As usual, we define $X \Vdash_{\mathscr{X}} \varphi$, iff $\varphi[v]$ holds in $\mathbf{L}[v]$ whenever v is an $\mathscr{X}$-generic array over $\mathbf{L}$, satisfying $v{\downarrow}\zeta \in X^\#$.*

In addition, in the set universe, if $\tau \in \Xi$, $v \in \mathscr{D}^\tau$, and there is $X \in \mathscr{X}$ such that $\zeta = \|X\| \subseteq \tau$, $v \in X^\#$, and $X \Vdash_{\mathscr{X}} \varphi$, then say that v forces *φ.*

9. Forcing and Permutations

Automorphisms of forcing notions have been widely used to define models with various set theoretic effects, basically since Cohen's times. Define the left action of permutations $\pi \in \mathbf{\Pi}$ (see Section 4) on names, as follows:

$$
\begin{aligned}
\pi{\bullet}\overset{\bullet}{x} &= \overset{\bullet}{x} \\
\pi{\bullet}\underline{\sigma\boldsymbol{v}} &= \underline{(\sigma\circ\pi^{-1})\boldsymbol{v}}, \text{ in particular, } \pi{\bullet}\underline{\boldsymbol{v}} = \underline{(\pi^{-1})\boldsymbol{v}}; \\
\pi{\bullet}(\underline{\sigma\boldsymbol{v}}{\downarrow}\eta) &= \underline{(\sigma\circ\pi^{-1})\boldsymbol{v}}{\downarrow}\eta, \text{ in particular, } \pi{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\eta) = \underline{(\pi^{-1})\boldsymbol{v}}{\downarrow}\eta.
\end{aligned}
$$

The group action property holds, for instance:

$$\rho{\bullet}(\pi{\bullet}\underline{\sigma\boldsymbol{v}}) = \rho{\bullet}\underline{(\sigma\circ\pi^{-1})\boldsymbol{v}} = \underline{(\sigma\circ\pi^{-1}\circ\rho^{-1})\boldsymbol{v}} = \underline{(\sigma\circ(\rho\circ\pi)^{-1})\boldsymbol{v}} = (\rho\circ\pi){\bullet}\underline{\sigma\boldsymbol{v}}.$$

If $\pi \in \mathbf{\Pi}$ and φ is an $\mathscr{L}$-formula, then we let $\pi\varphi$ be obtained by the substitution of $\pi{\bullet}\nu$ for any name ν in φ.

Recall that $\mathbf{\Pi}(\eta) = \{\pi \in \mathbf{\Pi} : \forall\, \boldsymbol{i} \in \eta\ (\boldsymbol{i} = \pi(\boldsymbol{i}))\}$, a subgroup of $\mathbf{\Pi}$, for any $\eta \in \Xi$.

If φ is an $\mathscr{L}$-formula, then let $\|\varphi\| = \bigcup\{\sigma^{-1}{\bullet}\eta : \underline{\sigma\boldsymbol{v}}{\downarrow}\eta \text{ occurs in } \varphi\}$; $\|\varphi\| \in \Xi$.

Theorem 4 (Theorem 25.2 in [19])**.** *Assume that, in* $\mathbf{L}$, φ *is a closed* $\mathscr{L}$*-formula, and* $\pi \in \mathbf{\Pi}$*. Let* $X \in \mathscr{X}$*. Then,* $X \Vdash_{\mathscr{X}} \varphi$ *iff* $\pi{\bullet}X \Vdash_{\mathscr{X}} \pi\varphi$*.*

Lemma 3. *Assume that* $J \in \mathbf{L}$, $J \subseteq \boldsymbol{I}$ *is an initial segment;* $\eta, \zeta \in \Xi$, $\boldsymbol{i} \in J$, $\varphi(\cdot)$ *is a closed limited formula,* $\|\varphi\| \subseteq \eta \subseteq J$, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is an* $\mathscr{X}$*-generic array over* $\mathbf{L}$, $\boldsymbol{v}{\downarrow}\zeta$ *forces "*$\mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}J] \models \varphi(\underline{\boldsymbol{v}}(\boldsymbol{i}))$*". Let* $\pi \in \mathbf{\Pi}(\eta \cup \zeta)$*, and let* $\boldsymbol{i}' = \pi(\boldsymbol{i})$, $J' = \pi''J$*. Then,* $\mathbf{L}[\boldsymbol{v}{\downarrow}J'] \models \varphi(\boldsymbol{v}(\boldsymbol{i}'))$*.*

Proof. By definition, there is a condition $X \in \mathscr{X}$ such that $\xi = \|X\| \subseteq \zeta$, $\boldsymbol{v}{\downarrow}\xi \in X^{\#}$, and $X \Vdash_{\mathscr{X}}$ "$\mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}J] \models \varphi(\underline{\boldsymbol{v}}(\boldsymbol{i}))$". Acting by π, we obtain by Theorem 4 that

$$\pi{\bullet}X \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{(\pi^{-1})\boldsymbol{v}}{\downarrow}J] \models \pi\varphi(\underline{(\pi^{-1})\boldsymbol{v}}(\boldsymbol{i}))\text{”}.$$

However, $\pi{\bullet}X = X$ because $\|X\| \subseteq \zeta$ and π is the identity on ζ. Moreover, $\pi\varphi(\cdot)$ is identical to $\varphi(\cdot)$ since $\|\varphi\| \subseteq \eta$ and π is the identity on η. Thus, we have

$$X \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{(\pi^{-1})\boldsymbol{v}}{\downarrow}J] \models \varphi(\underline{(\pi^{-1})\boldsymbol{v}}(\boldsymbol{i}))\text{”}.$$

However $\boldsymbol{v}{\downarrow}\xi \in X^{\#}$ and $\boldsymbol{v}$ is generic. It follows that

$$\mathbf{L}[\underline{(\pi^{-1})\boldsymbol{v}}[\boldsymbol{v}]{\downarrow}J] \models \varphi(\underline{(\pi^{-1})\boldsymbol{v}}[\boldsymbol{v}](\boldsymbol{i})).$$

Now, we compute the valuation $\underline{(\pi^{-1})\boldsymbol{v}}[\boldsymbol{v}] = \pi^{-1}{\bullet}\boldsymbol{v}$, and hence, by Lemma 1(ii),

$$\mathbf{L}[\underline{(\pi^{-1})\boldsymbol{v}}[\boldsymbol{v}]{\downarrow}J] = \mathbf{L}[(\pi^{-1}{\bullet}\boldsymbol{v}){\downarrow}J] = \mathbf{L}[\pi^{-1}{\bullet}(\boldsymbol{v}{\downarrow}J)] = \mathbf{L}[\boldsymbol{v}{\downarrow}J']$$

because $\pi \in \mathbf{L}$. On the other hand, $\underline{(\pi^{-1})\boldsymbol{v}}[\boldsymbol{v}](\boldsymbol{i}) = ((\pi^{-1}){\bullet}\boldsymbol{v})(\boldsymbol{i}) = (\boldsymbol{v}\circ\pi)(\boldsymbol{i}) = \boldsymbol{v}(\boldsymbol{i}')$.

Thus, finally $\mathbf{L}[\boldsymbol{v}{\downarrow}J'] \models \varphi(\boldsymbol{v}(\boldsymbol{i}'))$, as required. □

10. Permutations and Isolation

The next lemma involves the notion of isolation. Let $J \subseteq \boldsymbol{I}$ be an initial segment. Say that a set $\tau \in \Xi$, $\tau \subseteq J$ is *isolated in* J, iff for each set $\vartheta \in \Xi$ with $\tau \subseteq \vartheta$ (not necessarily $\vartheta \subseteq J$) there is a permutation $\pi \in \mathbf{\Pi}(\tau)$ satisfying (A) $\pi''J = J$, and (B) $\vartheta \cap (\pi''\vartheta) = \tau$.

Lemma 4. *Assume that* $J \in \mathbf{L}$, $J \subseteq \boldsymbol{I}$ *is an initial segment,* $\tau \in \Xi$, $\tau \subseteq J$ *is isolated in* J, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is an* $\mathscr{X}$*-generic array over* $\mathbf{L}$*, and a set* $x \in \mathbf{L}[\boldsymbol{v}{\downarrow}J]$, $x \subseteq \mathbf{L}$*, is definable in* $\mathbf{L}[\boldsymbol{v}{\downarrow}J]$ *by a formula with sets in* $\mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$ *as parameters. Then,* $x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$*.*

Proof. We have $x = \{y \in \mathbf{L} : \mathbf{L}[v{\downarrow}J] \models \varphi(y)[v]\}$, where φ is a limited $\mathscr{L}$-formula containing only $\underline{v}{\downarrow}\tau$ and sets in $\mathbf{L}$ as names. We claim that

$$x = \{y \in \mathbf{L} : \exists W \in \mathscr{X}\,(\|W\| = \tau \wedge v{\downarrow}\tau \in W^{\#} \wedge W \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{v}{\downarrow}J] \models \varphi(y)\text{”}\}). \tag{5}$$

(Recall that we identify any $\mathscr{L}$-name $\mathring{y}$ with y itself.) The direction $\supseteq$ in (5) easily follows from the genericity of v. This allows us to concentrate on the direction $\subseteq$. Thus, let $y \in x$.

By the genericity of v, there is a condition $X \in \mathscr{X}$ such that $v \restriction \xi \in X^{\#}$, where $\xi = \|X\|$, and

$$X \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{v}{\downarrow}J] \models \varphi(y)\text{”}. \tag{6}$$

We can assume that $\tau \subseteq \xi$. (Otherwise take $X' = X{\uparrow}(\tau \cup \xi)$ in $\mathbf{L}$ instead of X. Then, $X' \in \mathscr{X}$ by 3° of Section 5, whereas $X' {\downarrow}{\subseteq} X$ is obvious.) Then, $W = X{\downarrow}\tau \in \mathscr{X}$ by 2°, and $v{\downarrow}\tau \in W^{\#}$ by construction. It remains to prove that

$$W \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{v}{\downarrow}J] \models \varphi(y)\text{”}. \tag{7}$$

Suppose towards the contrary that (7) fails. Then, there is a condition $U \in \mathscr{X}$ such that $U {\downarrow}{\subseteq} W$ and

$$U \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{v}{\downarrow}J] \models \neg\,\varphi(y)\text{”}. \tag{8}$$

We put $\eta = \|U\|$ (then $\tau = \|W\| \subseteq \eta$) and $\vartheta = \eta \cup \xi$. By the isolation assumption, there is a permutation $\pi \in \mathbf{\Pi}(\tau)$ satisfying (A) $\pi''J = J$, and (B) $\vartheta \cap (\pi''\vartheta) = \tau$. Acting on (8), we obtain

$$Y \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{(\pi^{-1})v}{\downarrow}J] \models \neg\,\varphi(y)\text{”} \tag{9}$$

by Theorem 4, where $Y = \pi{\bullet}U$. However, if $v \in \mathscr{D}^{\boldsymbol{I}}$, then the valuation $(\underline{(\pi^{-1})v})[v] = \pi^{-1}{\bullet}v$ satisfies $(\pi^{-1}{\bullet}v){\downarrow}J = \pi^{-1}{\bullet}(v{\downarrow}J')$ by Lemma 1(ii), where $J' = \pi{\bullet}J = \pi''J = J$ by the choice of π. Thus, $(\pi^{-1}{\bullet}v){\downarrow}J = \pi^{-1}{\bullet}(v{\downarrow}J)$. It follows that $\mathbf{L}[(\pi^{-1}{\bullet}v){\downarrow}J] = \mathbf{L}[v{\downarrow}J]$. Therefore, (9) implies

$$Y \Vdash_{\mathscr{X}} \text{“}\mathbf{L}[\underline{v}{\downarrow}J] \models \neg\,\varphi(y)\text{”}. \tag{10}$$

It suffices now to prove that *conditions Y and X (see above) are compatible in $\mathscr{X}$*, so that (10) contradicts (6), and this completes the proof of (7) and the lemma.

We argue in L. To prove the compatibility, note that $X, Y \in \mathscr{X}$, $\|X\| = \xi$, $\|Y\| = \eta' := \pi''\eta$. It holds by construction and the choice of π that $\eta' \cap \eta = \tau$ and $Y{\downarrow}\tau = U{\downarrow}\tau$. On the other hand, $\xi \cap \eta' = \tau$ as well, and $Y{\downarrow}\tau = U{\downarrow}\tau \subseteq W = X{\downarrow}\tau$. To conclude,

$$\|X\| = \xi, \quad \|Y\| = \eta', \quad \xi \cap \eta' = \tau, \quad Y{\downarrow}\tau \subseteq X{\downarrow}\tau. \tag{11}$$

Now, consider the set $P = X{\uparrow}(\xi \cup \eta')$; $P \in \mathscr{X}$ by 3° of Section 5, and $P {\downarrow}{\subseteq} X$. Moreover, easily $Y \subseteq P{\downarrow}\eta'$ by the last claim of (11). It follows by still 3° that the set $Z = P \cap (Y{\uparrow}(\xi \cup \eta'))$ belongs to $\mathscr{X}$. Finally, $Z {\downarrow}{\subseteq} X$ and $Z {\downarrow}{\subseteq} Y$ by construction. □

11. Proof of the First Main Theorem

Here, we prove Theorem 1. We work under the assumptions of Definition 2. If $s \subseteq \omega_1^{\mathbf{L}}$ and $v \in \mathscr{D}^{\boldsymbol{I}}$ then we put

$$H_s[v] = \{v(\langle\alpha\rangle) : \alpha \in s\}.$$

Here, $\langle\alpha\rangle$ is the “tuple” in $\boldsymbol{I}$ containing a single term α. A set of the form $H_s[v]$ will be the desired counterexample for Theorem 1. We define

$$s_1 = \{\lambda + 4k : \lambda < \omega_1^{\mathbf{L}} \text{ is limit} \wedge k < \omega\}, \quad t_1 = \{\lambda + 4k + 2 : \lambda < \omega_1^{\mathbf{L}} \text{ is limit} \wedge k < \omega\},$$

so that $s_1 \cap t_1 = \varnothing$ and $s_1 \cup t_1 =$ all even ordinals $\alpha < \omega_1^{\mathbf{L}}$.

Let $\boldsymbol{J}_1$ be the set of all tuples $\boldsymbol{i} \in \boldsymbol{I}$ such that if $\mathtt{lh}\,\boldsymbol{i} \geq 2$ and $\boldsymbol{i}(0) \notin s_1$, then $\boldsymbol{i}(1)$ is an odd ordinal.

Lemma 5 (in $\mathbf{L}$)**.** *$\boldsymbol{J}_1 \subseteq \boldsymbol{I}$ is an odd-saturated initial segment in the sense of $\subseteq$.*

In addition, every set $\tau \in \Xi$ is isolated in $\boldsymbol{J}_1$.

Proof. We argue in $\mathbf{L}$. The saturation claim is obvious. To prove the isolation claim, assume that $\tau \subsetneq \vartheta$ belong to Ξ and $\tau \subseteq \boldsymbol{J}_1$. As ϑ is countable, there is a limit ordinal $\lambda < \omega_1$ such that $\boldsymbol{i} \in \vartheta \implies \operatorname{ran} \boldsymbol{i} \subseteq \lambda$. Define a bijection $b : \omega_1 \xrightarrow{\text{onto}} \omega_1$ as follows:

$$b(\alpha) = \begin{cases} \lambda + \alpha\,, & \text{in case} \quad \alpha < \lambda\,, \\ \gamma\,, & \text{in case} \quad \alpha = \lambda + \gamma \wedge \gamma < \lambda\,, \\ \alpha\,, & \text{in case} \quad \alpha \geq \lambda\,. \end{cases}$$

Now, suppose that $\boldsymbol{i} \in \boldsymbol{I}$, $m = \mathtt{lh}\,\boldsymbol{i}$. Let k me the largest of the numbers $1 \leq k \leq m$ such that $\boldsymbol{i} \restriction k \in \tau$ — if such k do exist, and otherwise just $k = 0$. Define $\boldsymbol{j} = \pi(\boldsymbol{i}) \in \boldsymbol{I}$ so that still $\mathtt{lh}\,\boldsymbol{j} = m$, $\boldsymbol{j} \restriction k = \boldsymbol{i} \restriction k$ (void in case $k = 0$), and if $k \leq \ell < m$, then $\boldsymbol{j}(\ell) = b(\boldsymbol{i}(\ell))$. This permutation π belongs to $\mathbf{\Pi}(\tau)$, and satisfies $\pi''\boldsymbol{J}_1 = \boldsymbol{J}_1$ and $\vartheta \cap (\pi''\vartheta) = \tau$, and hence witnesses that τ is $\boldsymbol{J}_1$-isolated. □

The next theorem implies Theorem 1 (with the shift $n = \mathfrak{n} + 2$).

Theorem 5. *Assume that $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is an array $\mathscr{X}$-generic over $\mathbf{L}$. Then, it holds in $\mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$ that*

(i) *the set $H_{s_1}[\boldsymbol{v}] \subseteq 2^\omega$* (belongs to $\mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$ and) *is $\Sigma^1_{\mathfrak{n}+2}$;*

(ii) *$H_{s_1}[\boldsymbol{v}]$ is not equal to the projection of a uniform $\mathbf{\Pi}^1_{\mathfrak{n}+2}$ set $P \subseteq (\omega^\omega)^2$.*

Proof. (i) By definition, $\boldsymbol{J}_1 \in \mathbf{L}$ is an odd-saturated (Definition 3) initial segment in $\boldsymbol{I}$ (in the sense of $\subset$), containing all tuples of length 1. Moreover, we have

$$\alpha \in s_1 \iff \exists\, \boldsymbol{j} \in \boldsymbol{J}_1 \big(\langle\alpha\rangle \subset \boldsymbol{j} \wedge \mathtt{lh}\,\boldsymbol{j} = 2 \wedge \boldsymbol{j} \text{ is even}\big) \;\text{— for all } \alpha < \omega_1^{\mathbf{L}}. \tag{12}$$

It follows by Corollary 2(ii) and the Structure property of $\mathscr{X}$ that

$$H_{s_1}[\boldsymbol{v}] = \{x \in 2^\omega : \exists\, y \in 2^\omega\, (\langle 2, y\rangle \in \mathbf{E}^{\mathtt{evn}}_{\boldsymbol{J}_1}(\boldsymbol{v}) \wedge x \in \mathbf{L}[y])\}$$

in $\mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$. This implies (i) by the $\mathfrak{n}$-Definability property of $\mathscr{X}$.

(ii) Suppose towards the contrary that, in $\mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$, $P \subseteq (\omega^\omega)^2$ is a *uniform* $\mathbf{\Pi}^1_{\mathfrak{n}+2}$ set satisfying $\operatorname{dom} P = H_{s_1}[\boldsymbol{v}]$. The set $P = \{\langle x, y\rangle : \varphi(a, x, y)\}$ is defined in $\mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$ by a $\mathit{\Pi}^1_{\mathfrak{n}+2}$ formula $\varphi(a, x, y)$ with a single real $a \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{J}_1]$ as a parameter. It follows from Lemma 2(iii),(iv) that there is $\eta \in \Xi$, $\eta \subseteq \boldsymbol{J}_1$ in $\mathbf{L}$ such that $a \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ $a = F(\boldsymbol{v}{\downarrow}\eta)$, where $F : \omega^\omega \to \omega^\omega$ is a continuous map, *coded in* $\mathbf{L}$ in the sense that the restriction $f = F \restriction (\mathbf{L} \cap \mathscr{D}^\xi)$ belongs to $\mathbf{L}$. Note that then $F = f^\#$ (the topological closure of f in $(\omega^\omega)^2$), and hence $\varphi(a, x, y)$ is $\varphi(f^\#(\boldsymbol{v}{\downarrow}\eta), x, y)$.

Then, there is $\eta_1 \in \Xi$ such that $\boldsymbol{v}{\downarrow}\eta_1$ forces (in the sense of Definition 4) that

(A) $\quad \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\boldsymbol{J}_1] \models H_{s_1}[\boldsymbol{v}] = \{x : \exists\, y\, \varphi(f^\#(\underline{\boldsymbol{v}}{\downarrow}\eta), x, y)\}$.

The set $D = \{\boldsymbol{i}(0) : \boldsymbol{i} \in \eta \cup \eta_1\}$ is at most countable in $\mathbf{L}$. Therefore, there exist ordinals $\alpha \in s_1 \smallsetminus d$ and $\beta \in t_1 \smallsetminus d$. In particular $\beta \notin s_1$, and hence $\boldsymbol{v}{\downarrow}\eta_1$ forces that

(B) $\quad \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\boldsymbol{J}_1] \models \neg\, \exists\, y\, \varphi(f^\#(\underline{\boldsymbol{v}}{\downarrow}\eta), \underline{\boldsymbol{v}}(\langle\beta\rangle), y)$,

along with (A). Now, we set up for an application of Lemma 3.

Arguing in L, we easily define a bijection $b\colon \omega_1^{\mathbf{L}} \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$ such that $b(\gamma) = \gamma$ for all odd γ and all $\gamma \in d$, $b(\beta) = \alpha$, and $b''s_1 = s_1 \smallsetminus \{\alpha\}$ (equivalently, $b''t_1 = t_1 \cup \{\alpha\}$). This bijection b induces a permutation $\pi : \boldsymbol{I} \xrightarrow{\text{onto}} \boldsymbol{I}$ acting so that if $\boldsymbol{i} \in \boldsymbol{I}$ and $\mathtt{lh}\,\boldsymbol{i} = m$, then $\boldsymbol{j} = \pi(\boldsymbol{i}) \in \boldsymbol{I}$ satisfies $\mathtt{lh}\,\boldsymbol{j} = m$, $\boldsymbol{j}(0) = b(\boldsymbol{i}(0))$, but $\boldsymbol{j}(\ell) = \boldsymbol{i}(\ell)$ whenever $1 \le \ell < m$.

Then, $\pi \in \mathbf{\Pi}$, π is the identity on $\eta \cup \eta_1$, $\pi(\langle\beta\rangle) = \langle\alpha\rangle$, and $\pi''\boldsymbol{J}_1 \subsetneqq \boldsymbol{J}_1$ (as $b''s_1 \subsetneqq s_1$).

Now, we can apply Lemma 3 to the statement that $\boldsymbol{v}\!\downarrow\!\eta_1$ forces (B), obtaining

(C) $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_1{}'] \models \neg\,\exists\, y\, \varphi(f^{\#}(\boldsymbol{v}\!\downarrow\!\eta), \boldsymbol{v}(\langle\alpha\rangle), y)$, where $\boldsymbol{J}_1{}' = \pi''\boldsymbol{J}_1 \subsetneqq \boldsymbol{J}_1$ by construction.

On the other hand, $\alpha \in s_1$, hence $\boldsymbol{v}(\langle\alpha\rangle) \in H_{s_1}[\boldsymbol{v}]$. Thus, by the uniformity of P there exists unique $y \in \omega^{\omega}$, such that $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_1] \models \varphi(f^{\#}(\boldsymbol{v}\!\downarrow\!\eta), \boldsymbol{v}(\langle\alpha\rangle), y)$. This real y belongs to $\mathbf{L}[\boldsymbol{v}\!\downarrow\!(\eta \cup \{\langle\alpha\rangle\})]$ by Lemma 4 with $\tau = \eta \cup \{\langle\alpha\rangle\}$ (applicable by Lemma 5!).

However $\tau \subseteq \boldsymbol{J}_1{}'$ by construction, and hence $y \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_1{}']$. It follows that the $\mathbf{\Pi}^1_{\mathbb{n}+2}$ formula $\varphi(f^{\#}(\boldsymbol{v}\!\downarrow\!\eta), \boldsymbol{v}(\langle\alpha\rangle), y)$, true in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_1]$, has to be true in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_1{}']$ as well by Corollary 3(ii) since the odd-saturation of the sets $\boldsymbol{J}_1{}' \subseteq \boldsymbol{J}_1$ is clear by construction. But, this contradicts (C). □

12. Proof of the Second Main Theorem

Here, we prove Theorem 2. We utilize the same disjoint sets $s_1, t_1 \subseteq \omega_1^{\mathbf{L}}$ as in Section 11. Yet, we make use of another set $\boldsymbol{J}_2$ instead of $\boldsymbol{J}_1$. Namely, let $\boldsymbol{J}_2$ consist of all tuples $\boldsymbol{i} \in \boldsymbol{I}$ such that (1) if $\mathtt{lh}\,\boldsymbol{i} \ge 2$ and $\boldsymbol{i}(0) \notin s_1$ then $\boldsymbol{i}(1)$ is odd, and (2) if $\mathtt{lh}\,\boldsymbol{i} \ge 3$ and $\boldsymbol{i}(0) \notin t_1$ then $\boldsymbol{i}(2)$ is odd.

Lemma 6 (in $\mathbf{L}$)**.** *$\boldsymbol{J}_2 \subseteq \boldsymbol{I}$ is an odd-saturated initial segment in the sense of $\subseteq$.*

In addition, every set $\tau \in \Xi$ is isolated in $\boldsymbol{J}_2$.

Proof. Pretty similar to the proof of Lemma 5. □

The next theorem implies Theorem 2 (with the shift $n = \mathbb{n} + 2$).

Theorem 6. *Assume that $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is an array $\mathscr{X}$-generic over $\mathbf{L}$. Then, it holds in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_2]$ that*

(i) *The set $H_{s_1}[\boldsymbol{v}] \subseteq 2^{\omega}$ (belongs to $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_2]$ and) is $\Delta^1_{\mathbb{n}+2}$;*

(ii) *$H_{s_1}[\boldsymbol{v}]$ is not equal to the projection of a uniform $\mathbf{\Pi}^1_{\mathbb{n}+1}$ set $P \subseteq (\omega^{\omega})^2$.*

Proof. (i) Similar to the proof of Theorem 5(i), the equations

$$
\begin{aligned}
H_{s_1}[\boldsymbol{v}] &= \{x \in 2^{\omega} : \exists\, y \in 2^{\omega}\, (\langle 2, y\rangle \in \mathbf{E}^{\mathbf{evn}}_{\boldsymbol{J}_1}(\boldsymbol{v}) \wedge x \in \mathbf{L}[y])\} \\
H_{t_1}[\boldsymbol{v}] &= \{x \in 2^{\omega} : \exists\, y \in 2^{\omega}\, (\langle 3, y\rangle \in \mathbf{E}^{\mathbf{evn}}_{\boldsymbol{J}_1}(\boldsymbol{v}) \wedge x \in \mathbf{L}[y])\}
\end{aligned}
$$

hold in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_2]$ and imply that both $H_{s_1}[\boldsymbol{v}]$ and $H_{t_1}[\boldsymbol{v}]$ belong to $\Sigma^1_{\mathbb{n}+2}$ in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_2]$. Moreover, $H := H_{s_1}[\boldsymbol{v}] \cup H_{t_1}[\boldsymbol{v}] = \{\langle\alpha\rangle : \alpha < \omega_1^{\mathbf{L}} \text{ is even}\}$, and hence

$$
H = \{x : \langle 0, x\rangle \in \mathbf{E}^{\mathbf{evn}}_{\boldsymbol{J}_2}(\boldsymbol{v})\} \in \Pi^1_{\mathbb{n}+1} \text{ — in } \mathbf{L}[\boldsymbol{v}\!\downarrow\!\boldsymbol{J}_2]
$$

by Corollary 2(ii). We conclude that $H_{s_1}[\boldsymbol{v}] = H \smallsetminus H_{t_1}[\boldsymbol{v}] \in \Pi^1_{\mathbb{n}+2}$, so that even $H_{s_1}[\boldsymbol{v}] \in \Delta^1_{\mathbb{n}+2}$, as required.

(ii) The proof goes exactly the same way as the proof of (ii) in Theorem 5 above. All the arguments go through with the only difference being that the inclusion $\boldsymbol{J}_2{}' = \pi''\boldsymbol{J}_2 \subseteq \boldsymbol{J}_2$ does not take place. But, this inclusion can be circumvented here, because now P is a set in $\mathbf{\Pi}^1_{\mathbb{n}+1}$ rather than $\mathbf{\Pi}^1_{\mathbb{n}+2}$, and therefore it is possible to use Corollary 3(i) instead of Corollary 3(ii). □

13. Conclusions and Problems

In this study, methods of forcing theory are employed in the solution of an old problem of classical descriptive set theory raised by Luzin in 1930 and related to uniform projections of projective sets. (Theorems 1 and 2). In addition, we established (Corollary 1) an ensuing consistency and independence result. These are new results, and they make a significant contribution to descriptive set theory in generic universes. The technique developed in this paper may lead to further progress in studies of different aspects of the projective hierarchy under the axiom of constructibility.

The following problems arise from our study. (Our short list of problems begins with Problem 2 since Problem 1 already appears in Section 1.)

Problem 2. *Find a model of* **ZFC** *in which the conclusion of Theorem 1 holds for all* $n \geq 3$ *rather than for a chosen value of* n.

Problem 3. *Coming back to the model* $\mathbf{L}[v \downarrow J_1]$ *of Theorem 5, suppose that* $m \neq \mathtt{n}$*. Is it true in* $\mathbf{L}[v \downarrow J_1]$ *that there is a* $\mathbf{\Sigma}^1_{m+2}$ *set not equal to the projection of a uniform* $\mathbf{\Pi}^1_{\mathtt{n}+1}$ *set? The answer may depend on whether* $m < \mathtt{n}$ *or* $m > \mathtt{n}$.

We hope that these problems can be solved by further development of the method of definable generic forcing notions, introduced by Harrington [13,23]. This method has been recently applied for some definability problems in modern set theory, including the following applications:

- A generic model of **ZFC** in [24], with a Groszek–Laver pair (see [25]) that consists of two OD-indistinguishable E_0-classes $X \neq Y$, whose union $X \cup Y$ is a Π^1_2 set;
- A generic model of **ZFC** in [26], in which, for a given $\mathtt{n} \geq 3$, there is a $\Delta^1_{\mathtt{n}}$ real coding the collapse of $\omega_1^{\mathbf{L}}$, whereas all $\Delta^1_{\mathtt{n}-1}$ reals are constructible, that generalizes a result by Abraham in [27];
- A generic model of **ZFC** in [28], which solves the Alfred Tarski [29] 'definability of definable' problem.

We hope that this study of generic models will eventually contribute to a solution of the following well-known key problem by S. D. Friedman; see [30], p. 209 and [31], p. 602:

Problem 4. *Find a model of* **ZFC**, *for a given* $\mathtt{n}$, *in which all* $\mathbf{\Sigma}^1_{\mathtt{n}}$ *sets of reals are Lebesgue measurable and have the Baire and perfect set properties, and in the same time there exists a* $\mathbf{\Delta}^1_{\mathtt{n}+1}$ *well-ordering of the reals.*

We also hope that this research can be useful in creating algorithms or computational algorithmic models that represent the evolution of cell types and are related to the storage and processing of genomic information.

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