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TRANSFER THEOREMS AND THE ALGEBRA OF MODAL OPERATORS

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A set theory ZFI' which does not employ the Law of the Excluded Middle $\varphi \vee \neg\varphi$, for all φ, retains the stock of expressive capacities of the classical set theory ZF, on the one hand, and has many of the features of an effective theory on the other. In the article, a broad class of formulas ζ is constructed for which $ZF \vdash \zeta$ implies $ZFI' \vdash \zeta$. This result provides a generalization of Friedman's theorem on AE-arithmetic formulas. Besides, we prove transfer theorems of classical logic for the case of rings; in particular, Hilbert's theorem on zeros and Artin's theorem on ordered fields are extended to the case of regular f-rings, and we bring in appropriate upper bounds for them.

A set theory ZFI′ which does not make use of the Law of the Excluded Middle $\varphi \vee \neg\varphi$ (LEM) for all formulas φ, while keeping up expressive capacities of the classical set theory ZF (for describing schemes, images, relations, and the like), has a lot of traits of an effective theory. For instance, formulas of the form $\forall x \exists y\, \varphi_0$ inferable in ZF often define functions in a certain effective way.

In the article we construct a broad class of formulas ζ for which ZF $\vdash \zeta$ implies ZFI′ $\vdash \zeta$. This result provides a generalization of Friedman's known theorem on AE-arithmetic formulas [1], and is used to ground an algorithm designed in [2]. Besides, here we prove transfer theorems of classical logic for the case of rings, ordered rings included. As an illustration, Hilbert's theorem on zeros and Artin's theorem on ordered fields are extended to the case of regular f-rings, and we supply appropriate upper bounds.

As a classical set theory ZF we consider the Zermelo–Fraenkel set theory, with the ε-induction and collection axiom taken instead of foundation and replacement axioms, respectively, that is, we deal with the ordinary system of axioms for classical set theory. As a corresponding intuitionistic set theory ZFI′ we consider ZF that is freed of the axiom $\varphi \vee \neg\varphi$.

The symbol $\rightleftharpoons$ stands for "is equal by definition" or "is equivalent by definition." The reader is expected to be familiar with [3-5].

A phi-*formula* is one in which the premise of any one of its implications satisfies the following: it does not contain the quantifier $\forall$, while the quantifier $\exists$ does not enter into the domain of the connective $\Rightarrow$. Every formula is classically equivalent to a phi-formula, for instance, to a formula in the prenex form. Further, an AE-*formula* is one of the form $\forall x_1 \ldots x_n \exists y_1 \ldots y_n\, \varphi_0$, where φ_0 is quantifier-free. A *formula with tight negations* is one that contains an implication only in the form of negation of its atomic parts. The negation $\neg\varphi$ is always understood as $\varphi \Rightarrow \perp$.

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Fix an arbitrary language that includes the equality symbol $=$, and also some functional symbols $f, \dots$ and predicate symbols $P, \dots$. For instance, let f and P be binary. Assume that K is some interpretation of those symbols in a countable constructive set, say, in the set of all positive integers $\omega \rightleftharpoons \{0, 1, 2, \dots, \}$. Under this interpretation, $x = y$ is thought of as an identity, that is, two sets x and y coincide, and $f : \omega^2 \to \omega$, $P \subseteq \omega^2$. In essence, $\bar{K} \rightleftharpoons \langle \omega, f, \dots, P, \dots \rangle$ is an arbitrary countable structure (with functions and relations). The approach attempted below makes use of the countability of a support and aims at verifying relations $(*_1)$ and $(*_2)$, specified later. The relations are conditions on a support K of the structure $\bar{K}$, satisfied for $K = \omega$. If we assume that these are satisfied for K then for a structure of the form $\bar{K} \rightleftharpoons \langle K, f, \dots, P, \dots \rangle$, $f : K^2 \to K$, $P \subseteq K^2$, Theorem 1 (see below), too, will be valid. A question as to which sets K apart from ω satisfy the above conditions will be dealt with in some other work.

Thus, let φ be an arbitrary phi-formula and ψ an arbitrary AE-formula in our language and let $(\varphi \Rightarrow \psi)_{\bar{K}}$ express some property of the structure $\bar{K}$ and be a formula in the language of the set theory ZF; more specifically, it will be one of the following: either $\forall f, P, \dots (f : \omega^2 \to \omega \wedge P \subseteq \omega^2 \wedge \dots \Rightarrow (\forall \bar{x}\, (\varphi \Rightarrow \psi))_{\omega, f, P, \dots})$ or $\forall f, P, \dots\ (\kappa(f, P, \dots) \Rightarrow (\forall \bar{x}\ (\varphi \Rightarrow \psi))_{\omega, f, P, \dots})$, where $\bar{x}$ are all free variables in φ and ψ and κ is a formula in the language of ZF describing the structure $\bar{K}$. Here we limit ourselves to the first case. The second case is treated similarly (see Remark 2 below). Denote by ζ the first formula, and by ζ' a formula obtained from ζ by adding $\forall x, y \in \omega\ (P(x, y) \vee \neg P(x, y))$ in the premise for all predicate symbols $P, \dots$ occurring in ζ. If, for instance, the premise of ζ says that P is a recursive predicate then we can take ζ itself to be ζ'.

THEOREM 1. If $\mathrm{ZF} \vdash \zeta$ then $\mathrm{ZFI}' \vdash \zeta'$.

Proof. Suppose that the condition of the theorem holds. Further argument is an explicit metamathematical description of the inference in ZFI$'$, spoken of in the conclusion. From that, in particular, we see that the length of an intuitionistic inference is a linear function of length of a corresponding classical inference, with certain small coefficients, which it is easy to explicitly specify.

Put $\mathbb{Z}_2 \rightleftharpoons \{0, 1\}$, where $<$ is defined to be $0 < 1$. Then $u \le v \rightleftharpoons u < v \vee u = v$. This structure is a Boolean algebra. (Of course, its completeness is not maintained.) Let $\mathcal{J}_2$ be a set of all ideals in $\mathbb{Z}_2$; as usual, the *ideal* a is a subset of $\mathbb{Z}_2$, with the following properties: $0 \in a$, $\forall e_1 e_2 \in a\ (e_1 \vee e_2 \in a)$, and $\forall e \in \mathbb{Z}_2\ \forall e_1 \in a\ (e \le e_1 \Rightarrow e \in a)$. An order in $\mathcal{J}_2$ is naturally defined as follows: $a \le b \rightleftharpoons a \subseteq b$. That structure is a complete Heyting algebra. For instance, $(\bigvee_\alpha a_\alpha) \wedge b \le \bigvee_\alpha (a_\alpha \wedge b)$. In this case $a \wedge b \rightleftharpoons a \cap b$ and $\vee A \rightleftharpoons \{0\} \cup \bigcup A$ since $\{0\} \cup \bigcup A$ is an ideal. Fix an embedding of the Boolean algebra $\mathbb{Z}_2$ in $\mathcal{J}_2$ to be $0 \mapsto \{0\}$, $1 \mapsto \mathbb{Z}_2$.

Denote by A_2 a set of all modal operators, or, in other words, of all J-operators on $\mathcal{J}_2$; see [6]. As usual, a *J-operator* is the map $J : \mathcal{J}_2 \to \mathcal{J}_2$ for which $J(a) \ge a$, $J(a \wedge b) = J(a) \wedge J(b)$, $J(J(a)) = J(a)$, $\forall a, b \in \mathcal{J}_2$. An order in A_2 is defined thus: $J_1 \le J_2 \rightleftharpoons \forall a \in \mathcal{J}_2\ (J_1(a) \subseteq J_2(a))$. This structure is a complete Heyting algebra. In this case $\left(\bigwedge_\alpha J_\alpha\right)(a) \rightleftharpoons \bigcap_\alpha (J_\alpha(a))$ and $\left(\bigvee_\alpha J_\alpha\right)(a) \rightleftharpoons \cap\{b | a \subseteq b,\ J_\alpha(b) = b,\ \forall \alpha\}$. Define the embedding $\mathcal{J}_2 \to A_2$ as $a \mapsto J_a$, where $J_a(b) \rightleftharpoons a \vee b$. This is a cHa-embedding, that is, $\{0\} \mapsto J_0 \rightleftharpoons \mathrm{id}$, where $\mathrm{id}\,(a) \equiv a$, $\mathbb{Z}_2 \mapsto J_1$; here, by definition, $J_1(a) \equiv \mathbb{Z}_2$, $J_{a \wedge b} = J_a \wedge J_b$, and $J_{\bigvee_\alpha a_\alpha} = \bigvee_\alpha J_{a_\alpha}$. Note that $(\neg\neg_{A_2}) J_a = J_a$ since $(\neg_{A_2}) J_a = J^a$, where $J^a(b) \rightleftharpoons a \to b$. Indeed, $J_a \wedge J^a = \mathrm{id} = J_0$, $J_a \vee J^a = J_1$. Therefore, any $J_a \in \mathcal{B}_2 \rightleftharpoons \{J \in A_2 |\ (\neg\neg_{A_2}) J = J\}$, that is, $\mathcal{J}_2$ is cHa-embedded also in $\mathcal{B}_2$. The $\mathcal{B}_2$ is a complete Boolean algebra, tailored to this form — as any algebra — from the complete Heyting algebra.

Define the *evaluation*

$$[\![k = t]\!]_{\bar{K}} \rightleftharpoons \{0\} \cup \{x \mid x = 1,\ k = t\} \subseteq \mathbb{Z}_2,\ \ [\![\cdot = \cdot]\!]_{\bar{K}} : K^2 \to \mathcal{J}_2.$$

In fact, each of its values is an ideal. For any terms s_1 and s_2, define $[s_1(\bar{k}) = s_2(\bar{k})]_{\bar{K}}$ to be $[k = t]_{\bar{K}}$, where $s_1(\bar{k}) \rightleftharpoons k$, $s_2(\bar{k}) \rightleftharpoons t$ (s_1 and s_2 are computed in K). Similarly we define $[P(s_1, s_2)]_{\bar{K}} \rightleftharpoons [P(s_1^0, s_2^0)]_{\bar{K}}$ (where s_1^0 and s_2^0 are values of the terms s_1 and s_2 in $\bar{K}$) $\rightleftharpoons \{0\} \cup \{x \mid x = 1,\ P(s_1^0, s_2^0)\} \in \mathcal{J}_2$. To extend the map $[\cdot = \cdot]_{\bar{K}}$ from the set of all terms with parameters in K to the set of all formulas with parameters from K (without free variables), we use induction on the connectives. There are two ways. The first one is to use operations in $\mathcal{J}_2$, and the second — in B_2. These result in the maps, which we denote by, respectively, $[\![\cdot]\!]_{\mathcal{J}_2}$ and $[\![\cdot]\!]_{B_2}$. The condition

$$\forall k, t \in K \ (k = t \vee k \neq t) \tag{$*_1$}$$

is satisfied for the support $\bar{K} \rightleftharpoons \omega$. The outcome is that the situation becomes much simpler: $[\![s_1 = s_2]\!]_{\bar{K}} = \{0\}$ or $[\![s_1 = s_2]\!]_{\bar{K}} = \mathbb{Z}_2$. Similarly, $[\![P(s_1, s_2)]\!]_{\bar{K}} = \{0\}$ or $[\![P(s_1, s_2)]\!]_{\bar{K}} = \mathbb{Z}_2$, due to an extra premise $P \vee \neg P$ in ζ'. The property that we have specified will be called the *normality of a valuation*. Accordingly, $[\![s_1 = s_2]\!]_{B_2} = J_0$ or $[\![s_1 = s_2]\!]_{B_2} = J_1$ and $[\![P(s_1, s_2)]\!]_{B_2} = J_0$ or $[\![P(s_1, s_2)]\!]_{B_2} = J_1$. Here (and in the theorems below), we can directly assume the normality condition: $([\![k = t]\!]_{\bar{K}}, [\![P(k,t)]\!]_{\bar{K}} \in \mathbb{Z}_2)$, $\forall k, t \in K$ [resp., $\in B(K)$].

LEMMA 1. Any formula φ satisfies the following:

$$\varphi_{\bar{K}} \Leftrightarrow ([\![\varphi]\!]_{\mathcal{J}_2} = \mathbb{Z}_2).$$

The proof is by induction on the length of φ. We have $(s_1 = s_2)_{\bar{K}} \Leftrightarrow (k = t)_{\bar{K}} \Leftrightarrow ([\![k = t]\!]_{\bar{K}} = \mathbb{Z}_2) \Leftrightarrow ([\![s_1 = s_2]\!]_{\bar{K}} = \mathbb{Z}_2)$. Similarly $P(s_1, s_2)_{\bar{K}} \Leftrightarrow P(k, t) \Leftrightarrow ([\![P(s_1, s_2)]\!]_{\bar{K}} = \mathbb{Z}_2)$. If $[\![\varphi \vee \psi]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$, then $1 \in ([\![\varphi]\!] \cup [\![\psi]\!])$, $1 \in [\![\varphi]\!]$, or $1 \in [\![\psi]\!]$. If $(\varphi \Rightarrow \psi)_{\bar{K}}$ and $e \in [\![\varphi]\!]_{\mathcal{J}_2}$, then $e = 0$ or $e = 1$, and by induction, $e \in [\![\psi]\!]$. If $\varphi_{\bar{K}}$ and $[\![\varphi]\!]_{\mathcal{J}_2} \leq [\![\psi]\!]_{\mathcal{J}_2}$, then $[\![\psi]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$, $\psi_{\bar{K}}$. If $[\![\exists x \varphi]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$, then $1 \in (\{0\} \cup \bigcup_k [\![\varphi(k)]\!]_{\mathcal{J}_2}$ and $[\![\varphi(k)]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$, for some $k \in \bar{K}$.

Denote by V^{B_2} the Boolean-valued universe for the complete Boolean algebra B_2. The class V of all sets is then embedded in V^{B_2} in the usual way as follows: $x^\vee \rightleftharpoons \{y^\vee \mid y \in x\}_-$, where X_- stands for the identity function defined on X. Here $(\cdot)^\vee : V \to V^{B_2}$. By induction on the length of the argument, it is easy to infer that if ZF $\vdash \zeta$ then $[\![\zeta]\!]_{V^{B_2}} = J_1$ and, in particular, $[\![f^\vee : (\omega^\vee)^2 \to \omega^\vee,\ P^\vee \subseteq (\omega^\vee)^2 \wedge \ldots \Rightarrow (\forall \bar{x}(\varphi \Rightarrow \psi)_{\omega^\vee, f^\vee, P^\vee})]\!] = J_1$,

$$[\![\varphi_{\omega^\vee}]\!]_{V^{B_2}} \leq [\![\psi_{\omega^\vee}]\!]_{V^{B_2}}. \tag{1}$$

We obtain relation (1) in accordance with the premise of Theorem 1, and use it repeatedly below; our appeal to the premise of Theorem 1 will be limited to just this case. In deriving (1), we note, use is also made of the equality $[\![f^\vee : (\omega^\vee)^2 \to \omega^\vee]\!]_{V^{B_2}} = J_1$. The proof of single-valuedness is nontrivial: we need to show that

$$[\![k_1^\vee = t_1^\vee]\!]_{V^{B_2}} \wedge [\![k_2^\vee = t_2^\vee]\!]_{V^{B_2}} \leq [\![f(k_1, k_2)^\vee = f(t_1, t_2)^\vee]\!]_{V^{B_2}}.$$

This follows from Lemma 2; the case with $k_1, t_1, k_2, t_2 \in \omega$ is overt in virtue of ω being strongly decidable. Recall that X is called *strongly decidable* if the transitive closure X^+ of X, defined by ε-induction to be $X^+ \rightleftharpoons X \cup \bigcup \{Y^+ \mid Y \in X\}$, possesses the property that $\forall u, v \in X^+ \ (u = v \vee \exists w \in u \ (w \notin v) \vee \exists w \in v \ (v \notin u))$.

LEMMA 2. Any formula φ satisfies the following:

$$[\![\varphi(k_1, \ldots, k_n)]\!]_{B_2} = [\![\varphi(k_1^\vee, \ldots, k_n^\vee)_{\omega^\vee, f^\vee, P^\vee}]\!]_{V^{B_2}}.$$

The proof is by induction on the length of φ.

1st atomic case. Obviously, $[\![k = t]\!]_{\bar{K}} \leq [\![k^\vee = t^\vee]\!]_{V^{\mathcal{J}_2}} \leq [\![k^\vee = t^\vee]\!]_{V^{\mathcal{B}_2}}$ (alternatively, using normality, we immediately obtain $[\![k = t]\!]_{\bar{K}} \leq [\![k^\vee = t^\vee]\!]_{V^{\mathcal{B}_2}}$).

Assume the condition

$$[\![k^\vee = t^\vee]\!]_{V^{\mathcal{B}_2}} \leq [\![k = t]\!]_{\bar{K}}. \qquad (*_2)$$

For $k, t \in \omega$, this is satisfied trivially, as is the case with any other strongly decidable set. Thus, $[\![k = t]\!]_{\bar{K}} = [\![k^\vee = t^\vee]\!]_{V^{\mathcal{B}_2}}$.

Term case. For one functional symbol, we have $[\![f(k^\vee, t^\vee) = r^\vee]\!]_{\mathcal{B}_2} = [\![\langle k^\vee, t^\vee, r^\vee\rangle \in f^\vee]\!]_{\mathcal{B}_2} = \bigvee_{u,v\in K} [\![k^\vee = u^\vee]\!]_{\mathcal{B}_2} \wedge [\![t^\vee = v^\vee]\!]_{\mathcal{B}_2} \wedge [\![r^\vee = f(u,v)^\vee]\!]_{\mathcal{B}_2} = \bigvee_{u,v\in K} [\![k = u]\!]_{\bar{K}} \wedge [\![t = v]\!]_{\bar{K}} \wedge [\![r = f(u,v)]\!]_{\bar{K}} = \bigvee_{u,v\in K} [\![k = u]\!] \wedge [\![t = v]\!] \wedge [\![r = f(k,t)]\!] = [\![r = f(k,t)]\!]_{\bar{K}}$. The last but one equality follows from $[\![k = u]\!]_{\bar{K}} \wedge [\![t = v]\!]_{\bar{K}} \wedge [\![r = f(k,t)]\!]_{\bar{K}} \leq [\![r = f(u,v)]\!]_{\bar{K}}$. In the general case $[\![f(t_1,t_2) = s]\!]_{\mathcal{B}_2} = [\![(\exists x, y\ (f(x,y) = s \wedge t_1 = x \wedge t_2 = y))_{K^\vee}]\!]_{\mathcal{B}_2} = \bigvee_{x,y\in K} [\![f(x^\vee, y^\vee) = s]\!]_{\mathcal{B}_2} \wedge [\![t_1 = x^\vee]\!]_{\mathcal{B}_2} \wedge [\![t_2 = y^\vee]\!]_{\mathcal{B}_2} = \bigvee_{x,y\in K} [\![f(x,y) = s]\!]_{\bar{K}} \wedge [\![t_1 = x]\!]_{\bar{K}} \wedge [\![t_2 = y]\!]_{\bar{K}} = \bigvee_{x,y\in K} [\![t_1 = x]\!]_{\bar{K}} \wedge [\![t_2 = y]\!]_{\bar{K}} \wedge [\![f(t_1,t_2) = s]\!]_{\bar{K}} = [\![f(t_1,t_2) = s]\!]_{\bar{K}}$.

2nd atomic case. $[\![P^\vee(k^\vee, t^\vee)]\!]_{V^{\mathcal{B}_2}} \rightleftharpoons [\![\langle k^\vee, t^\vee\rangle \in P^\vee]\!]_{V^{\mathcal{B}_2}} = \bigvee_{\langle u,v\rangle \in P} [\![k^\vee = u^\vee]\!]_{V^{\mathcal{B}_2}} \wedge [\![t^\vee = v^\vee]\!]_{V^{\mathcal{B}_2}} = \bigvee_{\langle u,v\rangle\in P} [\![k = u]\!]_{\bar{K}} \wedge [\![t = v]\!]_{\bar{K}} = [\![P(k,t)]\!]_{\bar{K}}$. The last equality is verified directly. Similarly $[\![P^\vee(s_1, s_2)]\!]_{V^{\mathcal{B}_2}} \rightleftharpoons [\![(\exists x, y\ (P^\vee(x,y) \wedge s_1 = x \wedge s_2 = y))_{K^\vee}]\!]_{V^{\mathcal{B}_2}} = \bigvee_{x,y\in K} [\![P^\vee(x^\vee, y^\vee)]\!]_{\mathcal{B}_2} \wedge [\![s_1 = x^\vee]\!]_{\mathcal{B}_2} \wedge [\![s_2 = y^\vee]\!]_{\mathcal{B}_2} = \bigvee_{x,y\in K} [\![P(x,y)]\!]_{\bar{K}} \wedge [\![s_1 = x]\!]_{\bar{K}} \wedge [\![s_2 = y]\!]_{\bar{K}} = [\![P(s_1,s_2)]\!]_{\bar{K}}$. The last equality is verified directly.

Cases with connectives are obvious.

Remark 1. In the above argument, we did not make use of the statement that $[\![k^\vee = t^\vee]\!]_{\mathcal{J}_2} = [\![k = t]\!]_{\bar{K}}$, where k and t are arbitrary sets, which is nevertheless useful for a better understanding of the idea behind it.

Indeed, let that equality hold for all $x \in k$ and $y \in t$. In one direction, the induction hypothesis is not needed: $e \in [\![k = t]\!]_{\bar{K}} \Rightarrow e = 0 \vee e = 1$ and $0 \in [\![k^\vee = t^\vee]\!]$ or $k^\vee = t^\vee$. If $e \in [\![k^\vee = t^\vee]\!]_{\mathcal{J}_2}$, then $e = 0 \vee e = 1$. The first case is trivial. In the second case $1 = e \in [\![k^\vee = t^\vee]\!]_{\mathcal{J}_2} = \bigwedge_{x\in k} \bigvee_{y\in t} [\![x^\vee = y^\vee]\!]_{\mathcal{J}_2} \wedge \ldots$, that is, for any $x \in k$, there exists a $y \in t$ such that $1 \in [\![x^\vee = y^\vee]\!]_{\mathcal{J}_2}$ and $1 \in [\![x = y]\!]_{\bar{K}}$, $x = y$, that is, $x \in t$, $k \subseteq t$, and similarly $t \subseteq k$. Thus $k = t$ and $[\![k = t]\!]_{\bar{K}} = \mathbb{Z}_2$, $1 \in [\![k = t]\!]_{\bar{K}}$.

LEMMA 3. (a) For any phi-formula φ with parameters $\bar{k} = \langle k_1, \ldots, k_n\rangle \in K$, $[\![\varphi(\bar{k})]\!]_{\mathcal{J}_2} \leq [\![\varphi(\bar{k})]\!]_{\mathcal{B}_2}$ holds.

(b) For any AE-formula ψ with parameters $\bar{k} = \langle k_1, \ldots, k_n\rangle \in K$, $(a \leq [\![\psi(\bar{k})]\!]_{\mathcal{B}_2}) \Rightarrow (a \leq [\![\psi(\bar{k}]\!]_{\mathcal{J}_2})$ holds for any $a \in \mathcal{J}_2$.

Proof. (a) Assume, first, that φ has no quantifier $\forall$ and that the quantifier $\exists$ does not enter into the domain of $\Rightarrow$. Use induction on the length of φ to verify that if φ does not contain $\exists$ then $[\![\varphi]\!]_{\mathcal{J}_2} = [\![\varphi]\!]_{\mathcal{B}_2} \in \mathbb{Z}_2$, and if φ does contain $\exists$ then $[\![\varphi]\!]_{\mathcal{J}_2} = [\![\varphi]\!]_{\mathcal{B}_2}$. In fact, for φ an atomic formula, the first statement follows by the normality property. For the cases with $\wedge$ and $\vee$, it is trivial. For $\Rightarrow$, the formula φ does not contain $\exists$ by assumption, and the result follows. The case with $\exists$ is trivial.

Now let φ be a phi-formula. If φ is atomic or is constructed via the connectives $\wedge$, $\vee$, $\exists$, and $\forall$, then statement (a) of Lemma 3 needs no elucidation. If φ is obtained through $\varphi_1 \Rightarrow \varphi_2$ then, for φ_1, we have $[\![\varphi_1]\!]_{\mathcal{J}_2} = [\![\varphi_1]\!]_{\mathcal{B}_2}$ by the previous paragraph. By the induction hypothesis, $[\![\varphi_2]\!]_{\mathcal{J}_2} \leq [\![\varphi_2]\!]_{\mathcal{B}_2}$, and so $([\![\varphi_1]\!]_{\mathcal{J}_2} \to_{\mathcal{J}_2} [\![\varphi_2]\!]_{\mathcal{J}_2}) \leq [\![\varphi_1]\!]_{\mathcal{J}_2} \to_{\mathcal{B}_2} [\![\varphi_2]\!]_{\mathcal{J}_2} \leq [\![\varphi_1]\!]_{\mathcal{B}_2} \to_{\mathcal{B}_2} [\![\varphi_2]\!]_{\mathcal{B}_2}$.

(b) If $\psi(\bar{k})$ is atomic, that is, of the form $s_1 = s_2$ or $P(s_1, s_2)$, then $[\![\psi(\bar{k})]\!]_{\mathcal{J}_2} = [\![\psi(\bar{k})]\!]_{\mathcal{B}_2}$ by definition, and $[\![\psi(\bar{k})]\!] \in \mathbb{Z}_2$ by the normality property. Propositional operations $\wedge$, $\vee$, $\to$ in $\mathcal{J}_2$ and in $\mathcal{B}_2$, if applied to elements of $\mathbb{Z}_2$, that is, to the ideals $\{0\}$ and $\mathbb{Z}_2$ or to the operators J_0 and J_1, yield the same result belonging to $\mathbb{Z}_2$. For $\psi(\bar{k})$ a quantifier-free formula, therefore, again we obtain $[\![\psi(\bar{k})]\!]_{\mathcal{J}_2} = [\![\psi(\bar{k})]\!]_{\mathcal{B}_2} \in \mathbb{Z}_2$ by induction on the length of φ. Here condition $(*_1)$ is essential. For the case with $\exists y\psi$, $[\![\exists y\psi(y,\bar{k})]\!]_{\mathcal{J}_2} = [\![\exists y\psi(y,\bar{k})]\!]_{\mathcal{B}_2} \in \mathcal{J}_2$ follows from the fact that $\mathcal{J}_2$ is cHa-embedded in $\mathcal{B}_2$. Lastly, for the case with $\forall x\exists y\psi$, the inequality that we are verifying is straightforward.

We finish the proof of Theorem 1. Assume that $f : \omega^2 \to \omega$, $P \subseteq \omega^2$, $\varphi_{\omega,f,P}(\bar{k})$, and that $(*_2)$ holds. By Lemma 1, then, we obtain $[\![\varphi]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$, and by (a) of Lemma 3, $[\![\varphi]\!]_{\mathcal{B}_2} = J_1$. By Lemma 2, $[\![\varphi(k_1^\vee,\ldots,k_n^\vee)_{K^\vee}]\!]_{V^{\mathcal{B}_2}} = J_1$. By relation (1), we have $[\![\psi(k_1^\vee,\ldots,k_n^\vee)_{K^\vee}]\!]_{V^{\mathcal{B}_2}} = J_1$, by Lemma 2, $[\![\psi(\bar{k})]\!]_{\mathcal{B}_2} = J_1$, and by (b) of Lemma 3, $[\![\psi(\bar{k})]\!]_{\mathcal{J}_2} = \mathbb{Z}_2$. In view of Lemma 1, $\psi_{\omega,f,P}(\bar{k})$.

Remark 2. In the second case envisaged before Theorem 1, we must first assume that the formula κ is *absolute*, that is,

$$\kappa(\omega, f, P, \ldots) \Rightarrow ([\![\kappa(\omega^\vee, f^\vee, P^\vee, \ldots)]\!]_{V^{\mathcal{B}_2}} = 1). \qquad (*_3)$$

Next assume that $\kappa(\omega, f, P, \ldots)$ to arrive at $[\![\kappa(\omega^\vee, f^\vee, P^\vee, \ldots)]\!]_{V^{\mathcal{B}_2}} = 1$, whence (1), and then proceed further as in the proof of Theorem 1. Of course, the typical structure $\langle\omega, +, -, \cdot, \leq, 0, 1\rangle$ is described by an absolute formula. This is an instance of Friedman's theorem. All recursive functions and relations on ω are also described by absolute formulas. If κ is positive, with bounded quantifier $\forall$, then it is absolute. If κ is with tight negations and relativized to the set U such that the transitive closure of $\{x, y\}$ (for any $x, y \in U$) is strongly decidable, then κ is absolute. If $\mathcal{B}_2$ is an apartness algebra, then any formula κ with bounded quantifiers is absolute. The proof of all these cases is by a straightforward induction. A statement similar to Theorem 1 will hold for the many-sorted language, which is the case, for instance, in our Theorem 3 (see below) where an extra sort of variables runs over an algebraically or really closed extension of the initial ring.

Instead of one formula φ we can consider a theory T consisting of the set of phi-formulas, in which case $\mathrm{T}_{\bar{K}}$ is understood as $\bar{K} \models \mathrm{T}$, for a suitable description of T in terms of a set of codes, which are natural numbers. Normally, T contains a countable set of axioms and can be described, for instance, as some $\alpha \subseteq \omega$. Therefore, if $\forall n \in \alpha\ (\bar{K} \models n)$, then $[\![\forall n \in \alpha^\vee(\bar{K} \models n)]\!] = \wedge\{[\![(\varphi_n)_K]\!] \mid n \in \alpha\} = J_1$.

Below we give Theorem 2, according to which to some axioms in the inference one can apply elimination procedures such as cut-elimination and the elimination of LEM in Theorem 1. Statements concerning the possibility of such eliminations are sometimes referred to as *transfer theorems*.

A formula φ is called *weakly positive* if it is constructed from atomic formulas inductively via the connectives $\wedge$, $\vee$, $\exists$, $\forall$, and by the special rule for the implication: if φ_1 is a P-formula and φ_2 is weakly positive then $(\exists\bar{x}\,\varphi_1)\wedge(\forall\bar{x}\ (\varphi_1 \Rightarrow \varphi_2))$ is weakly positive. A *weakly Horn* formula ψ is determined inductively as one constructed from atomic formulas via the connectives $\wedge$, $\exists$, $\forall$, and by the special rule for the implication: if φ_1 is weakly positive and φ_2 is weakly Horn then $(\varphi_1 \Rightarrow \varphi_2)$ is weakly Horn. Recall that a *P-formula* is defined as atomic or as one that obtains via $\wedge$, $\exists$, $\forall$, and by the special rule for the implication: $(\exists\bar{x}\,\varphi_1) \wedge (\forall\bar{x}\ (\varphi_1 \Rightarrow \varphi_2))$, where φ_1 and φ_2 are P-formulas.

Let φ and ψ be such formulas in the language of rings.

THEOREM 2. (a) If $\mathrm{ZFI}' \vdash \forall +, -, \cdot, 0, 1\ (+ : \omega^2 \to \omega, - : \omega \to \omega, \cdot : \omega^2 \to \omega, 0, 1 \in \omega \Rightarrow [\Phi_3 \Rightarrow \forall\bar{x}\ (\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))]_{\omega,+,-,\cdot,0,1})$, then $\mathrm{ZFI}' \vdash \forall +, -, \cdot, 0, 1\ (\ldots \Rightarrow [K(\omega, +, -, \cdot, 0, 1) \Rightarrow \forall\bar{x}\ (\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))]_{\omega,+,-,\cdot,0,1})$, where ... stands for the corresponding expression in the premise, $K(\omega, +, -, \cdot, 0, 1)$ says that the structure $\langle\omega, +, -, \cdot, 0, 1\rangle$ is a ring, and Φ_3 says that the ring $\bar{K} \rightleftharpoons \langle\omega, +, -, \cdot, 0, 1\rangle$ is indecomposable, that is, each of its decompositions into a direct sum of ideals is trivial (of the form $K \oplus \{0\}$).

(b) The claim of (a) remains valid if we omit the assumption on the countability of $\bar{K}$, that is, assume that the support K of the ring $\bar{K}$ is an arbitrary set, and add, instead, a condition on the decidability of the set $B(\bar{K})$ of all central idempotents of $\bar{K}$, that is, put $\forall e_1, e_2 \in B(K)$ $(e_1 = e_2 \vee e_1 \neq e_2)$.

(c) If ZF $\vdash \forall +, -, \cdot, 0, 1$ $[i \Rightarrow \forall \bar{x}$ $(\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))]_{\omega,+,-,\cdot,0,1}$, then ZFI$' \vdash \forall +, -, \cdot, 0, 1$ $[i' \Rightarrow \forall \bar{x}$ $(\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))]_{\omega,+,-,\cdot,0,1}$. Here, in addition, φ is a phi-formula, ψ is an AE-formula, and $\langle i, i' \rangle$, for instance, are the following pairs of properties (including that of being a ring): $\langle$biregular, quasisimple$\rangle$, $\langle$strongly regular, a division ring$\rangle$. In the hypotheses of (a) and (b), we can add in the premise and the conclusion the property $\Phi_1 \rightleftharpoons$ "is a normal ring."

Thus, in clauses (a) and (b), the property Φ_3 is eliminated; in (c), the Law of the Excluded Middle is eliminated, and i is replaced by a much weaker property i'. Recall, for instance, that strongly regular rings are specified by the condition $\forall x \in K$ $\exists y \in K$ $(x^2 \cdot y = x)$. Those form a class of rings which is much broader than the class of division rings.

The proof is as in Theorem 1. We point out the differences. Given a $\bar{K}$, form a Boolean algebra $B(\bar{K})$ (not $\mathbb{Z}_2$), and then proceed as above to form $\mathcal{J}(K)$ and $\mathcal{B}(K)$. Define the evaluation $[k = t]_K \rightleftharpoons \{e \in B(\bar{K}) \mid e \cdot k = e \cdot t\} \in \mathcal{J}(K)$ and extend it to $[\cdot]_{\mathcal{J}(K)}$ and $[\![\cdot]\!]_{\mathcal{B}(K)}$. Now, the conclusion of Lemma 1 fails and the following holds instead:

LEMMA 4. (a) If φ is weakly positive then $\varphi_K \Rightarrow ([\![\varphi]\!]_{\mathcal{J}(K)} = B)$.

(b) If φ is weakly Horn then $([\![\varphi]\!]_{\mathcal{J}(K)} = B) \Rightarrow \varphi_K$.

Proof. Both clauses are proved by a simultaneous induction on the length of φ. If φ is atomic, then $(s_1 = s_2)_K \Leftrightarrow ([s_1 = s_2]_K = B)$. We handle case (a). For the connectives $\wedge$, $\vee$, $\exists$, $\forall$, the argument is trivial, and for $\Rightarrow$, we have the following: if $(\exists x \varphi_1 \wedge (\forall x$ $(\varphi_1 \Rightarrow \varphi_2)))_K$, then $\exists k_0 \in K([\varphi_1(k_0)]_{\mathcal{J}} = B)$, whence $[\forall x\,(\varphi_1 \Rightarrow \varphi_2)]_{\mathcal{J}} = \cap\{[\![\varphi_2(k)]\!]_{\mathcal{J}} \mid k \in K,$ $[\varphi_1(k)]_{\mathcal{J}} = B\}$, from which it follows that the last expression is equal to B. We check the first equality. It suffices to show that $[\forall x\,(\varphi_1 \Rightarrow \varphi_2)]_{\mathcal{J}} \geq$ the "right part," that is, $([\![\varphi_1(k)]\!] \to [\varphi_2(k)]\!]) \geq$ the "right part," $\forall k$. This follows from the inequality $[\varphi_1(k)]\!] \wedge$ the "right part" $\leq [\varphi_2(k)]\!]$, that is, from $\forall e \in [\![\varphi_1(k)]\!]$, $\forall k \exists k_1$ $[\langle e \rangle \wedge [\varphi_2(k_1)]\!] \leq [\![\varphi_2(k)]\!]]$, where $[\![\varphi_1(k)]\!] = B$. We claim that the latter is satisfied if we put $k_1 \rightleftharpoons e \cdot k + (1 - e) \cdot k_0$. Indeed, let e' be any element on the left-hand side. Then $e' \leq e$, and $e' \in [k = k_1] \wedge [\varphi_2(k_1)]\!] \leq [\varphi_2(k)]\!]$, and $e \in [k = k_1] \wedge [\![\varphi_1(k)]\!] \leq [\![\varphi_1(k_1)]\!]$, $(1-e) \in [\![k_1 = k_0]\!] \wedge [\![\varphi_1(k_0)]\!] \leq [\varphi_1(k_1)]\!]$, $[\varphi_1(k_1)]\!]_{\mathcal{J}} = B$. In what follows, $\langle e \rangle$ is a principal ideal generated by e.

We turn to case (b). For $\wedge$, $\forall$, the argument is trivial, and for $\exists$ we obtain the following: if $[\![\exists x \varphi]\!]_{\mathcal{J}} = B$, then $1 = e_1 \vee \ldots \vee e_n = e'_1 \vee \ldots \vee e'_n$, where the $\{e'_i\}$ are pairwise disjoint and $e'_i \in [\![\varphi(k_i)]\!]$, and then put $k_0 \rightleftharpoons \sum_i e'_i \cdot k_i$. It follows that $e'_i \cdot k_0 = e_i \cdot k_0$, $e'_i \in [\![k_0 = k_i]\!]_{\mathcal{J}} \wedge [\![\varphi(k_i)]\!]_{\mathcal{J}} \leq [\varphi(k_0)]\!]_{\mathcal{J}}$, $B = [\varphi(k_0)]\!]_{\mathcal{J}}$, and by the induction hypothesis, $\varphi(k_0)_K$. For the connective $\Rightarrow$, we have: if $[\varphi_1 \Rightarrow \varphi_2]_{\mathcal{J}} = B$ and $(\varphi_1)_K$, then by (a), $[\varphi_1]\!]_{\mathcal{J}} = B$, $[\varphi_2]\!]_{\mathcal{J}} = B$, and $(\varphi_2)_K$.

As in Lemma 2, in the Boolean-valued universe $V^{\mathcal{B}(K)}$ we choose a nonstandard representation of the structure $\bar{K}$. (The index K will often be omitted.) In this case, this is not $\langle K^{\vee}, f^{\vee}, P^{\vee} \rangle$ but $\langle K', +', -', \cdot', 0', 1' \rangle$, where $K' \rightleftharpoons \{P_k \mid k \in K\}_-$, $P_k(t^{\vee}) \rightleftharpoons [k = t]_K$, t runs through K, and $+' \rightleftharpoons \{\langle P_k, P_t, P_{k+t} \rangle \mid k, t \in K\}_-$, and similarly for $-'$, $\cdot'$; lastly, $0' \rightleftharpoons P_0$, $1' \rightleftharpoons P_1$. It is worth noting that $[\![+' : (K')^2 \to K']\!]_{V^{\mathcal{B}}} = J_1$ (and similarly for all other operations including $[\![0', 1' \in K']\!]_{V^{\mathcal{B}}} = J_1$). The verification of single-valuedness is nontrivial: we have to arrive at $[P_{k_1} = P_{t_1}]_{\mathcal{B}} \wedge [P_{k_2} = P_{t_2}]_{\mathcal{B}} \leq [P_{k_1+k_2} = P_{t_1+t_2}]_{\mathcal{B}}$. By Lemma 2, which in this case does also hold under the same condition $(*_2)$ (see a proof below), we need first check that $[\![k_1 = t_1]\!]_{\mathcal{B}} \wedge [\![k_2 = t_2]\!]_{\mathcal{B}} \leq [k_1 + k_2 = t_1 + t_2]_{\mathcal{B}}$, that is, $[\![k_1 = t_1]\!]_K \wedge [k_2 = t_2]_K \leq [k_1 + k_2 = t_1 + t_2]_K$. The latter is true for any function f (e.g., of two arguments) for which $f(e \cdot k_1, e \cdot k_2) = e \cdot f(k_1, k_2)$, $\forall e \in B(K)$,

$\forall k_1, k_2 \in K$, in particular, for $+, -, \cdot$. More specifically, condition $(*_2)$ now has the form

$$[\![k^\vee = t^\vee]\!]_{V^{B(K)}} \le [\![k = t]\!]_K, \ \forall k, t \in K. \qquad (*_4)$$

In this case Lemma 2 follows from the condition $[\![P_k = P_t]\!]_{V^{B(K)}} \le [\![k = t]\!]_K$, that is, from

$$\bigcap_{x \in K} ([\![k = x]\!]_K \to \bigcup_{y \in K} [\![t = y]\!]_K \wedge [\![x^\vee = y^\vee]\!]_B) \wedge \ldots \le [\![k = t]\!]_K. \qquad (*_5)$$

Here ... stands for the reverse inclusion. Obviously, $(*_4) \Rightarrow (*_5)$.

The proof of Lemma 2 (for Theorem 2). We verify that

$$[\![\varphi(k_1, \ldots, k_n)]\!]_{B(K)} = [\![\varphi(P_{k_1}, \ldots, P_{k_n})]\!]_{V^{B(K)}}, \ \forall \bar{k} \in K. \qquad (2)$$

Atomic case. Condition $(*_5)$ immediately implies that

$$[\![k = t]\!]_K = [\![P_k = P_t]\!]_B, \ \forall k, t \in K. \qquad (3)$$

Condition $(*_4)$ is satisfied for any strongly decidable set.

Term case (with one functional symbol). We have $[\![f(P_k, P_t) = P_r]\!]_B = [\![\langle P_k, P_t, P_r\rangle \in f']\!]_B = \bigcup_{u,v \in K} [\![P_k = P_u]\!]_B \wedge [\![P_t = P_v]\!]_B \wedge [\![P_r = P_{f(u,v)}]\!] = \bigcup_{u,v} [\![k = u]\!]_K \wedge [\![t = v]\!]_K \wedge [\![r = f(u,v)]\!]_K = \bigcup_{u,v} [\![k = u]\!]_K \wedge [\![t = v]\!]_K \wedge [\![r = f(k,t)]\!]_K = [\![r = f(k,t)]\!]_K$. The last but one equality uses the relation $[\![k = u]\!]_K \wedge [\![t = v]\!]_K \wedge [\![r = f(u,v)]\!]_K \le [\![r = f(k,t)]\!]_K$, where f is any function with the property $e \cdot f(u,v) = f(e \cdot u, e \cdot v)$.

Term case (with a number of functional symbols). We have $[\![f(t_1, t_2) = s]\!]_B = [\![(\exists x, y\ (f(x,y) = s \wedge (t_1 = x) \wedge (t_2 = y)))_{K'}]\!]_B = \bigcup_{x,y \in K} [\![f(P_x, P_y) = s]\!]_B \wedge [\![t_1 = P_x]\!]_B \wedge [\![t_2 = P_y]\!]_B = \bigcup_{x,y} [\![f(x,y) = s]\!]_K \wedge [\![t_1 = x]\!]_K \wedge [\![t_2 = y]\!]_K = \bigcup [\![t_1 = x]\!] \wedge [\![t_2 = y]\!] \wedge [\![f(t_1, t_2) = s]\!]$.

Lemma 3 is carried over to this case without changes (the normality of a valuation follows from property i'). Theorem 2 is thus proved.

A *positively* AE-*Horn formula* is one of the form $\varphi \Rightarrow \psi$, where φ is a weakly positive phi-formula and ψ is an AE-weakly Horn formula. The set of all such formulas true in some structure or some class of structures is called a *positively* AE-*Horn theory* of that structure or that class of structures.

COROLLARY. A positively AE-Horn theory of the class of strongly regular rings coincides with a positively AE-Horn theory of the class of division rings, and this is also true for all pairs $\langle i', i\rangle$ of classes of rings described in (c) of Theorem 2.

Remark 3. In Theorem 2, the formula φ may also include any formulas of the form ζ', where ζ is arbitrary in (a) and (b), and ζ is a phi-formula in (c). In (a) and (b), the ψ can be arbitrary, and then in the conclusion we should write ψ' instead of ψ. In (c) and the corollary, the ψ can be an arbitrary AE-formula, and then in the conclusion we must write ψ' again.

The results presented above remain valid if, instead of φ and ψ, we consider theories consisting of formulas of the same types.

The language of rings can also be enriched by any predicate symbols P (as in Lemma 2 for Thm. 1), subject to the requirement that $P(x,y) \Rightarrow P(e \cdot x, e \cdot y)$, $\forall e \in B(K)$, $\forall x, y \in K$. For instance, if $\forall e \in B(K)$ $(e \ge 0)$, the requirement holds for a relation $\le$.

An illustration to Remark 3 is Theorem 3 below. We start by giving a number of general statements needed in its proof.

Let A be a strongly regular ordered f-ring considered in a language of rings with the extra relation $\leq$. That is, $\leq$ is the lattice order and $(x \geq 0,\ a \wedge b = 0) \Rightarrow (a \wedge (b \cdot x) = a \wedge (x \cdot b) = 0)$. We will make use of the following elementary properties of f-rings: if $c \geq 0$, then $(a \vee b) \cdot c = a \cdot c \vee b \cdot c$ (and similarly for $\wedge$ and from the left), $|a \cdot b| = |a| \cdot |b|$, $a^2 \geq 0$, and $a \wedge b = 0 \Rightarrow a \cdot b = 0$.

Let $B(A)$ be a Boolean algebra of all central idempotents of the ring A and let $B \subseteq \mathcal{J} \subseteq \mathcal{B}$ be ordinary extensions such as those in Theorem 2. The evaluations $[\![\cdot]\!]_A$, $[\![\cdot]\!]_{\mathcal{J}}$, and $[\![\cdot]\!]_{\mathcal{B}}$ are determined as above, and $[\![s_1 \leq s_2]\!]_A \rightleftharpoons [\![k \leq t]\!]_A$ (where k and t are values of the terms s_1 and s_2 in A) $\rightleftharpoons \{e \in B \mid e \cdot k \leq e \cdot t\}$. The order relation in B, defined above as $(e_1 \leq e_2) \rightleftharpoons e_1 \cdot e_2 = e_1$, coincides with an order relation induced by A. Indeed, if $e_1 \leq_B e_2$, that is, $e_1 \cdot e_2 = e_1$, then $(e_2 - e_1)^2 = (e_2 - e_1)$. Hence $e_2 - e_1 \geq_A 0$. If $e_1 \leq_A e_2$, then $(1 - e_2) \cdot e_1 \leq 0$ and $(1 - e_2) \cdot e_1 \geq 0$. Therefore $(1 - e_2) \cdot e_1 = 0$. Moreover, $e_1 \wedge_B e_2 = e_1 \wedge_A e_2$ and $e_1 \vee_B e_2 = e_1 \vee_A e_2$. In fact, $e_1 \cdot e_2 \leq e_1$, $e_1 \cdot e_2 \leq e_2$, and if $a \leq e_1$, $a \leq e_2$, then $(1 - e_1) \cdot a \leq 0$, $a \leq e_1 \cdot a$, $e_1 \cdot a \leq e_1 \cdot e_2$, $a \leq e_1 \cdot e_2$.

We handle the case with $\vee_B$. Here $(e_1 \vee_A e_2) \cdot (e_1 \vee_A e_2) = (e_1 \vee_A e_2)$, that is, $(e_1 \vee_A e_2) \in B$ and $e_1, e_2 \leq (e_1 \vee_A e_2)$, so $e_1 \vee_B e_2 \leq e_1 \vee_A e_2$ (in B) and $e_1 \vee_A e_2 \leq e_1 \vee_B e_2$ (in A).

Thus, the order relation and lattice operations in A are extensions of the corresponding relation and operations in B. It is also worth noting that

$$[\![0 \leq k]\!]_A = [\![k^- = 0]\!]_A. \tag{4}$$

Indeed, the condition $ek \geq 0$ implies $e(k \wedge 0) = ek \wedge 0 = 0$ and the condition $e(k \wedge 0) = 0$ implies $ek \wedge 0 = 0$, $ek \geq 0$. Hence, the normality of $[\![\cdot]\!]_A$ follows from its being normal for the equality, and the latter in turn is stipulated by the strong regularity. So, for any quantifier-free formula φ we have $[\![\varphi]\!]_{\mathcal{J},\mathcal{B}} \in B(A)$.

It is not hard to obtain $[\![\forall x \exists y\,(x = 0 \vee x \cdot y = y \cdot x = 1)]\!]_{\mathcal{J}} = B$ and $[\![\forall x, y\,(x \leq y \vee y \leq x)]\!]_{\mathcal{J}} = B$. Indeed, $[\![0 \leq x]\!]_{\mathcal{J}} \vee_{\mathcal{J}} [\![x \leq 0]\!]_{\mathcal{J}} = [\![x^- = 0]\!] \vee [\![x^+ = 0]\!]$. Let $x^+ = x^+ \cdot y \cdot x^+$ and $e \rightleftharpoons y \cdot x^+ \in B$; then $x^+ \cdot (1 - e) = 0$, that is, $1 - e$ belongs to the second summand and e belongs to the first, since $(x^+) \wedge (-x^-) = 0$, $(x^+) \cdot (-x^-) = 0$, $x^+ \cdot x^- = 0$, $yx^+ \cdot x^- = 0$, and $e \cdot x^- = 0$. Therefore, the union contains 1.

1. For $\mathcal{J}$- and $\mathcal{B}$-global truths, we can state that

$$A \text{ is a (linearly) ordered division ring.} \tag{5}$$

The second statement follows from the fact that the notion of linearity and the concept of a division ring are defined in terms of phi-formulas, and clause (a) of Lemma 3 does also hold — in the form $[\![\varphi]\!]_{\mathcal{J}(A)} \leq [\![\varphi]\!]_{\mathcal{B}(A)}$.

Now, define some extension $\bar{A}$ of the ring A. To do this, we follow Theorem 2 to define $A' \in V^{\mathcal{J}} \subseteq V^{\mathcal{B}}$ such that $[\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{B}} = [\![\varphi(P_{k_1}, \ldots, P_{k_n})_{A'}]\!]_{V^{\mathcal{B}}}$, for all formulas φ such as in Lemma 2. Then $[\![A'$ is a linearly ordered division ring$]\!]_{V^{\mathcal{B}}} = J_1$. Let $[\![A''$ is a really closed ordered division ring, $A' \subseteq A'']\!]_{V^{\mathcal{B}}} = J_1$. Put $\bar{A} \rightleftharpoons (A'')^{\wedge_{\mathcal{B}(A)}} \rightleftharpoons \{g \in V^{\mathcal{B}(A)} \mid [\![g \in A'']\!]_{V^{\mathcal{B}}} = J_1\}$. We have

$$[\![\bar{A}_- = A'']\!]_{V^{\mathcal{B}}} = \top.$$

Define $[\![\cdot]\!]_{\mathcal{B}(A),\bar{A}}$ in the usual way by setting $[\![f = g]\!]_{\mathcal{B}(A),\bar{A}} \rightleftharpoons [\![f = g]\!]_{V^{\mathcal{B}}}$ for any $f, g \in \bar{A}$, and similarly for $\leq$. Operations in $\bar{A}$ are induced by those in $(\bar{A})_-$ via the predicate $([\![\cdot]\!]_{V^{\mathcal{B}}} = \top)$. There are two sorts of variables: x, y, z, running through A, and α, β, γ running through $\bar{A}$; moreover, $[\![\forall x \varphi(x)]\!]_{\mathcal{B}(A),\bar{A}} \rightleftharpoons \wedge\{[\![\varphi(P_x)]\!]_{V^{\mathcal{B}}} \mid x \in A\}$, etc.

LEMMA 5. (a) A ring $\bar{A}$ is an extension of the ring A under the embedding $k \mapsto P_k$ including operations $\wedge$ and $\vee$; $[\![(\bar{A})_- = A'']\!]V^{\mathcal{B}} = \mathrm{T}$; the valuations $[\![\varphi]\!]_{\mathcal{B}(A),\bar{A}}$ and $[\![\varphi_{A''}]\!]_{V^{\mathcal{B}}}$ coincide, the $\wedge$ and $\vee$ included; $[\![\cdot]\!]_{\mathcal{B}(A),\bar{A}}$ coincides with $[\![\cdot]\!]_{\mathcal{J}(A),\bar{A}}$ for all atomic formulas $s_1 = s_2$ and $s_1 \leq s_2$, the $\wedge$ and $\vee$ included.

(b) The structure $\langle \bar{A}, [\![\cdot]\!]_{\mathcal{B}(A),\bar{A}} \rangle$ is a $\mathcal{B}$-orthocomplete, really closed, strongly regular f-ring.

(c) For every $\mathcal{B}$-orthocomplete, really closed, strongly regular f-ring A_1 extending A, there exists an $A'' \in V^{\mathcal{B}}$ such that $[\![A''$ is a really closed division ring and an extension of $A']\!]_{V^{\mathcal{B}}} = \top$ and $A_1 = (A'')^{\wedge \mathcal{B}(A)}$.

Proof. (a) If $k +_A t = r$, then $[\![\langle P_k, P_t, P_{k+t}\rangle \in +']\!]_{V^{\mathcal{B}}} = \top$ and $[\![P_{k+t} = P_r]\!]_{V^{\mathcal{B}}} = \top$, whence $P_k +_{\bar{A}} P_t = P_r$. If $k \leq_A t$, then $[\![\langle P_k, P_t\rangle \in \leq']\!]_{V^{\mathcal{B}}} = \top$ and $P_k \leq_{\bar{A}} P_t$. If $k \wedge_A t = r$, then $r \leq k, t \wedge \forall u$ $(u \leq k, t \Rightarrow u \leq r)$, $P_r \leq P_k$, P_t, and $[\![P_k \leq P_t \vee P_t \leq P_k]\!]_{V^{\mathcal{B}}} = \top$, where $[\![P_k \leq P_t]\!]_{V^{\mathcal{B}}} =^{(1)} [\![k \leq t]\!]_A \rightleftharpoons J_a$ and $[\![P_t \leq P_k]\!]_{V^{\mathcal{B}}} = [\![t \leq k]\!]_A \rightleftharpoons J_b$. In other words, $J_a \vee_{\mathcal{B}} J_b = J_{a \vee_{\mathcal{J}} b} = J_1$, $a \vee_{\mathcal{J}} b = B$, $\exists e_1 \in a$, $e_2 \in b$ $(e_1 \vee e_2 = 1)$, where $e_1 k \leq e_1 t$ and $e_2 t \leq e_2 k$. Then $(e_1 k) \wedge_A e_1 t = e_1 t = e_1 r$, $e_1 k = e_1 r$. Since $[\![P_k = P_r]\!]_{V^{\mathcal{B}}} =^{(2)} [\![k = r]\!]_A$, we have $J_{(e_1)} \leq [\![P_k = P_r]\!]_{\mathcal{B}} \wedge [\![P_k \leq P_t]\!]_{\mathcal{B}} \leq [\![P_k \wedge_{A''} P_t = P_r]\!]_{\mathcal{B}}$. Similarly, $J_{(e_2)} \leq [\![P_k \wedge_{A''} P_t = P_r]\!]_{\mathcal{B}}$. Therefore, $[\![P_k \wedge_{A''} P_t = P_r]\!]_{\mathcal{B}} = \top$ and $P_k \wedge_{\bar{A}} P_t = P_r$.

It remains to verify equalities (1) and (2). Equality (2) is formula (3), which was checked earlier, and (1) follows immediately from (2): $[\![P_k \leq P_t]\!]_{\mathcal{B}} \rightleftharpoons [\![\langle P_k, P_t\rangle \in \leq_{A''}]\!]_{\mathcal{B}} = [\![\langle P_k, P_t\rangle \in \leq_{A'}]\!]_{\mathcal{B}} = \mathbb{V}_{\mathcal{B}}\{[\![P_k = P_u]\!]_{\mathcal{B}} \wedge [\![P_t = P_v]\!]_{\mathcal{B}} \mid u \leq v\} = \mathbb{V}_{\mathcal{J}}\{[\![k = u]\!]_A \wedge [\![t = v]\!]_A \mid u \leq v\} = [\![k \leq t]\!]_A$.

The next statement of (a) is obvious and so omitted. We proceed further to first check that

$$\begin{gathered}[\![(s_1 + s_2)^\circ = s_1^\circ + s_2^\circ]\!]_{V^{\mathcal{B}}} = \top, \ [\![s = s^\circ]\!]_{V^{\mathcal{B}}} = \top, \\ ([\![s_1 + s_2 = s_3]\!]_{V^{\mathcal{B}_2}} = \top) \Leftrightarrow s_1 + s_2 = s_3 \text{ in } \bar{A}.\end{gathered} \tag{6}$$

Here s and s° are, respectively, a term and its value in $\bar{A}$. It follows immediately that $[\![s_1 = s_2]\!]_{\bar{A}} \rightleftharpoons [\![s_1^\circ = s_2^\circ]\!]_{\bar{A}} \rightleftharpoons [\![s_1^\circ =_{A''} s_2^\circ]\!]_{V^{\mathcal{B}}} = [\![s_1 = s_2]\!]_{V^{\mathcal{B}}}$ and $[\![s_1 \leq s_2]\!]_{\bar{A}} \rightleftharpoons [\![s_1^\circ \leq s_2^\circ]\!]_{\bar{A}} \rightleftharpoons [\![s_1^\circ \leq_{A''} s_2^\circ]\!]_{V^{\mathcal{B}}} = [\![s_1 \leq s_2]\!]_{V^{\mathcal{B}}}$, and we use induction on the number of connectives to eventually see that values of all the formulas coincide. The first relation in (6) is, in essence, a definition, and the second is verified inductively as follows: $[\![s_1 + s_2 = (s_1 + s_2)^\circ]\!]_{\mathcal{B}}$ (where $[\![s_1 = s_1^\circ]\!] = [\![s_2 = s_2^\circ]\!] = \top$) $= \mathbb{V}\{[\![s_1 = x]\!] \wedge [\![s_2 = y]\!] \wedge [\![\langle x, y, (s_1 + s_2)^\circ\rangle \in +]\!]\} \geq \mathbb{T}$.

The third relation in (6) is straightforward: if $[\![\exists x, y, z \in A''\ (s_1 = x \wedge s_2 = y \wedge s_3 = z \wedge x + y = z)]\!] = \mathbb{T}$, then $\exists f, g, h \in \bar{A}([\![s_1 = f \wedge s_2 = g \wedge s_3 = h \wedge f + g = h]\!] = \mathbb{T}$, $s_1^\circ = f$, $s_2^\circ = g$, $s_3^\circ = h)$, and vice versa. It is worth mentioning that the operations $\wedge$ and $\vee$ in $\bar{A}$ have ordinary meanings:

> if $f \wedge_{\bar{A}} g = h$, that is, $[\![f \wedge_{A''} g = h]\!]_{\mathcal{B}} = \mathbb{T}$, then $h \leq_{\bar{A}} f, g$, and if $u \leq_{\bar{A}} f, g$ then $u \leq_{\bar{A}} h$. Conversely, if h is the greatest lower bound for f and g in $\bar{A}$ then $[\![h \leq f, g]\!]_{\mathcal{B}} \wedge [\![\forall u \leq f, g\, (u \leq h)]\!]_{\mathcal{B}} = \mathbb{T}$. (7)

Finally, we verify the last relation stated in Lemma 5. For atomic cases, we have $[\![s_1 = s_2]\!]_A$ (as was checked in the proof of Lemma 2 for Theorem 2) $= [\![s_1 =_{A''} s_2]\!]_{V^{\mathcal{B}}} \rightleftharpoons [\![s_1 = s_2]\!]_{\bar{A}}$ and $[\![s_1 \leq s_2]\!]_A \rightleftharpoons [\![s_1^\circ \leq s_2^\circ]\!]_A$ [by equality (1)]$= [\![s_1^\circ \leq s_2^\circ]\!]_{V^{\mathcal{B}}} = [\![s_1 \leq s_2]\!]_{\bar{A}}$. Here we use the relation $[\![P_{(s(k_1,\ldots,k_n))^\circ} = (s(P_{k_1},\ldots,P_{k_n}))^\circ]\!]_{V^{\mathcal{B}}} = \mathbb{T}$, which may be verified by using induction on the length of a term s since $[\![P_{(s_1+s_2)^\circ} = P_{s_1^\circ + s_2^\circ} = P_{s_1^\circ} + P_{s_2^\circ} = s_1 + s_2]\!]_{\mathbb{T}}$.

(In the derivation above, use was made of just one fact — that A'' is a linearly ordered extension of A'.)

(b) The property of being $\mathcal{B}$-*orthocomplete* means that, for any family $\{\langle b_\alpha, f_\alpha\rangle\}$ of "conforming" pairs (i.e., $b_\alpha \wedge b_\beta \leq [\![f_\alpha = f_\beta]\!]_{\bar{A}}$, $\forall \alpha$, β, where $b_\alpha \in \mathcal{B}$ and $f_\alpha \in \bar{A}$), there exists an $f_0 \in \bar{A}$ for which $b_\alpha \leq [\![f_0 = f_\alpha]\!]$, $\forall \alpha$. In our case this property is obvious. The properties of being really closed and strongly regular, as well as the f-property, are expressed via Horn formulas, whence the result.

(c) The proof of this statement can be found in [4, p. 119].

Thus, in the class $\mathcal{K}_A \rightleftharpoons \{K \supseteq A|\ K$ is a strongly regular f-ring$\}$, β-orthocomplete, really closed elements correspond — in a nonstandard sense — to really closed division rings extending the nonstandard image A' of the ring A.

Suppose that the language of (ordered) rings is extended by adding a new sort of variables α, β, $\gamma\ldots$, running through $\bar{A}$, $\bar{A} \subseteq \mathcal{B}(A)$, an ordered, strongly regular f-ring, which now may or may not be chosen in the same way as above.

A *P-formula in the extended language* is defined as atomic or as one that obtains by means of $\wedge$, $\forall x$, $\exists\alpha$, $\forall\alpha$, and also as $(\exists\alpha\varphi_1) \wedge (\forall\alpha(\varphi_1 \Rightarrow \varphi_2))$, where φ_1 and φ_2 are P-formulas. A *weakly positive formula* in the same language is defined as atomic or as one that obtains by means of $\wedge$, $\vee$, $\exists x$, $\forall x$, $\exists\alpha$, $\forall\alpha$, and also as $(\exists\alpha\varphi_1) \wedge (\forall\alpha(\varphi_1 \Rightarrow \varphi_2))$, where φ_1 is a P-formula and φ_2 is weakly positive.

An *input formula* in the language in question is defined as weakly positive of the form φ', where φ is a phi-formula in the initial language of rings, or as a weakly positive phi-formula in the initial language of rings, or as one that obtains via the connectives $\wedge$, $\vee$, $\exists x$, $\forall x$ and $\exists\alpha$, $\forall\alpha$. Recall that φ' is a formula in the initial language of rings, equivalent to $[\varphi]_{\mathcal{J}} = \top$; see [4, p. 115]. A *normal formula* is one of the form $\varphi \Rightarrow \psi$, where φ is an input formula and ψ is an AE-formula in the initial language of rings.

LEMMA 6. (a) For any input formula φ, if $\varphi_{A,\bar{A}}$, then $[(\varphi^0)_{A',(\bar{A})_-}]_{V^{\mathcal{B}}} = J_1$, where φ^0 is constructed from φ by changing each part of the form u' by u, that is, by deleting the sign $'$.

The proof is by induction that follows the definition of an input formula. For φ', we have $[\varphi]_{\mathcal{J}(A)} = B$ by the definition of φ', $[\varphi]_{\mathcal{B}(A)} = J_1$ by the condition of being a phi-formula, and $[\varphi_{A'}]_{V^{\mathcal{B}(A)}} = J_1$ by Lemma 2. For a weakly positive formula, we proceed by induction on its length. There are two atomic cases to consider — in A and in $\bar{A}$. For $s_1 = s_2$ and $s_1 \leq s_2$, where s_1 and s_2 are terms over $\bar{A}$, as in (a) of Lemma 3 we use induction on the length of terms, applying $[\![s_1 = s_2]\!]_A = [(s_1 = s_2)_{A'}]_{V^{\mathcal{B}}}$ and $[s_1 \leq s_2]_A = [(s_1 \leq s_2)_{A'}]_{V^{\mathcal{B}}}$. For the case where s_1 and s_2 are terms over $\bar{A}$, appeal to the definition of $\bar{A}$. The case with connectives is obvious.

For simplicity, from this point on we assume that A is a commutative regular ordered f-ring. Then $[\![A'$ is an ordered field$]\!]_{V^{\mathcal{B}}} = J_1$. There exists a really closed extension of A', and we let $[A''$ is a really closed field, $A' \subseteq A'']_{V^{\mathcal{B}}} = J_1$. Put $\bar{A} \rightleftharpoons (A'')^{\wedge \mathcal{B}(A)}$.

Let $\varphi \Rightarrow \psi$ be a normal formula. By (c) of Lemma 4, $[\varphi^0_{A',(\bar{A})_-}]\!]_{V^{\mathcal{B}}} = J_1$. If ZFC $\vdash (\varphi^0 \Rightarrow \psi)_{A,\bar{A}}$ for any ordered field A and really closed extension $\bar{A}$, then $[\psi_{A'}]_{V^{\mathcal{B}}} = J_1$. As above, we obtain ψ'_{A}, and hence also ψ_A, provided that ψ is weakly Horn.

An *extension* is always taken in that class of rings in which the initial ring is taken. Below, if we say that something is valid we mean that a corresponding statement is inferable in ZFC.

We have thus proved the following:

THEOREM 3. Let $\varphi \Rightarrow \psi$ be a normal formula. If, in the class of ordered fields A and their really closed extensions $\bar{A}$, the formula $\varphi^0 \Rightarrow \psi$ is valid, then $\varphi \Rightarrow \psi'$ is valid in the class of regular commutative f-rings A and their $\mathcal{B}$-orthocomplete really closed extensions $\bar{A}$.

As above, a *positively* AE-*Horn* formula is one of the form $\varphi \Rightarrow \psi$, where φ is a weakly positive formula in the extended language of rings or a weakly positive phi-formula in the initial language of rings, and ψ is an AE-weakly Horn formula in the initial language. The set of all such formulas true in some structure or some class of structures is referred to as a *positively* AE-*Horn theory* of that structure or that class of structures.

COROLLARY. A positively AE-Horn theory for the class of ordered fields and their really closed extensions coincides with a positively AE-Horn theory for the class of regular commutative f-rings and their $\mathcal{B}$-orthocomplete really closed extensions.

Remark 4. In the above corollary and in Theorem 3, as $\bar{A}$ we can take, respectively, only one real closure of a field A and only one $\mathcal{B}$-orthocomplete closure of a ring A, which is $(A'^{-})^{\wedge_{\mathcal{B}(A)}}$ by definition.

Let $\mathcal{K}$ be a class of all regular commutative f-rings. An example to the corollary may be furnished by Hilbert's theorem on zeros (including a bound for the degree and degrees of polynomials) and by Artin's theorem, which were stated for the class $\mathcal{K}$. We cite the second of them.

For every ring A in $\mathcal{K}$, there exists an (above-described) class of really closed extensions $\bar{A}$ in $\mathcal{K}$ such that for any polynomial f over A, if $f \geq 0$ over $\bar{A}$, then f is represented as a sum of squares of functions f_i rational over A, that is, $f = \sum_{i=1}^{m} c_i \cdot (f_i)^2$, where c_i is in A and $c_i \geq 0$. The bound for the number m and degrees of polynomials occurring in f_i is the same as is the case with fields.

This statement, as many others, gives an affirmative answer to Hilbert's 17th problem, for the class of rings $\mathcal{K}$.

Remark 5. In theorem 3, instead of paired properties such as ⟨a strongly regular f-ring, an ordered division ring⟩ and ⟨a regular commutative f-ring, an ordered field⟩, we can take all typical pairs of ring properties (like in [3, 7, 8]), or take a pair $\varphi' \leftrightarrow \varphi$ in the general form.

Proof. We bring out only those parts of proofs that relate to the passage from one member of a pair to the other, keeping the rest unchanged.

(1) A is a projective f-ring iff $[\![A'$ is linearly ordered$]\!]_{V^{\mathcal{B}}} = \top$;

(2) A is a quasiregular f-ring iff $[\![A'$ is l-simple, linearly ordered$]\!]_{V^{\mathcal{B}}} = \top$;

(3) A is a projective f-ring without nilpotent elements iff $[\![A'$ is linearly ordered, without zero divisors$]\!]_{V^{\mathcal{B}}} = \top$.

The same relations are true also for $[\![\cdot]\!]_{V^{\mathcal{J}}}$- and $[\![\cdot]\!]_{\mathcal{B}(A),\bar{A}}$-evaluations.

(1) Recall that a projective ring A is specified by the condition

$$\forall a_1, a_2 \in A\, \exists b_1, b_2 \in A\, (a_1 = b_1 + b_2 \wedge |b_1| \wedge |a_2| = 0 \wedge \forall b \in A\, (|b| \wedge |a_2| = 0 \Rightarrow |b_2| \wedge |b| = 0)), \qquad (8)$$

from which it follows that $A = a_2^{\perp} + a_2^{\perp\perp}$, $\forall a_2 \in A$, where $a_2^{\perp} \rightleftharpoons \{b \in A |\ |b| \wedge |a_2| = 0\}$ is a *polar* of a_2 (i.e., "every polar" is a "direct summand"). Each polar $M^{\perp}$ is an l-ideal, where $M \subseteq A$. Assuming that the left-hand side is satisfied, we verify that $[\![0 \leq x]\!]_A \vee [\![x \leq 0]\!]_A = [\![x^- = 0]\!]_A \vee [\![x^+ = 0]\!]_A = \mathbb{T}$. By assumption, $A = (x^+)^{\perp} + (x^+)^{\perp\perp}$. Choose e so that $1 = e + y$, $e \in (x^+)^{\perp}$, $y \in (x^+)^{\perp\perp}$. Then $\forall u \in (x^+)^{\perp}$ $(eu = ue = u)$ since $u = ue + uy = ue$, and $\forall v \in (x^+)^{\perp\perp}$ $(ev = ve = 0)$. It follows that $\forall a \in A$ $(a = u + v$, $ae = ea)$, that is, e is a central element. Since e is an idempotent $(1 = e + y = e^2 + y^2$, $e - e^2 = y^2 - y = 0)$, we have $e \in B(A)$ and $e \geq 0$. Thus $x^+ \wedge (-x^-) = 0$, $x^+ \cdot (-x^-) = 0$, $x^+ \cdot x^- = 0$, $x^- \in (x^+)^{\perp}$, $e \cdot x^- = x^-$, $(1 - e) \cdot x^- = 0$, and $1 - e$ belongs to the first summand. Because $x^+ \in (x^+)^{\perp\perp}$, we have $e \cdot x^+ = 0$ and e belongs to the second summand.

The converse statement for a $\mathcal{J}$-evaluation holds by reason of the fact that formula (8) follows from the linearity condition and is Horn. For a $\mathcal{B}$-evaluation, if we assume that $[\![A'$ is linear$]\!]_{V^{\mathcal{B}}} = \mathbb{T}$, we obtain $[\![P_x \leq P_y \vee P_y \leq P_x]\!]_{V^{\mathcal{B}}} = \mathbb{T}$. In view of equality (1), $[\![x \leq y]\!]_A \vee_{\mathcal{B}} [\![y \leq x]\!]_A = [\![x \leq y]\!]_A \vee_{\mathcal{J}} [\![y \leq x]\!]_A = \mathbb{T}$ and $[\![\forall x, y \in A'\ (x \leq y \vee y \leq x)]\!]_{V^{\mathcal{J}}} = \mathbb{T}$; for a $[\![\cdot]\!]_{\mathcal{J}}$-evaluation, there is nothing to prove.

(2) A quasisimple ring A is defined by the condition

$$A = \langle a_2 \rangle + a_2^{\perp},\ \forall a_2 \in A, \text{ that is, } \forall a_1, a_2 \in A\, \exists b_1, b_2 \in A$$
$$(a_1 = b_1 + b_2 \wedge \exists n \in \mathbb{N}\, \exists d_1, \ldots, d_n, f_1, \ldots, f_n \in A\, (|b_1| \leq \qquad (9)$$

$$d_1 \cdot |a_2| \cdot f_1 + \ldots + d_n \cdot |a_2| \cdot f_n \wedge |b_2| \wedge |a_2| = 0)).$$

An l-simple ring A is specified by the condition of having no proper l-ideals, that is,

$$\forall a \in A\ (a = 0 \wedge \forall b \in A\ \exists n \in \mathbb{N} \exists d_1, \ldots, d_n, f_1, \ldots, f_n \in A\ (|b| \leq d_1 \cdot |a| \cdot f_1 + \ldots + d_n \cdot |a| \cdot f_n)). \quad (10)$$

We have $a^{\perp\perp} \supseteq \langle a \rangle$ and $a^{\perp\perp} = \langle a \rangle$ (if $z \in a^{\perp\perp} \wedge a^{\perp}$ then $|z| \wedge |z| = |z| = 0$), from which it follows that a quasisimple ring A is projective, and by clause (1), $[\![A'$ is linear$]\!] = \mathbb{T}$. Let us check that $[\![A'$ is l-simple$]\!] = \mathbb{T}$; see (10). Let e be a central idempotent corresponding to an l-ideal $\langle a \rangle$, that is, $(1-e) \cdot e = 0$. Then $[\![a = 0]\!]_A \ni (1-e)$. Now if we show that e is contained in the second summand of (10) we obtain the desired statement. Take an arbitrary factor corresponding to b and choose b_1 and b_2 for which $b = b_1 + b_2$, $|b_1| \leq d_1 \cdot |a| \cdot f_1 + \ldots + d_n \cdot |a| \cdot f_n$, and $|b_2| \wedge |a| = 0$. We have $[\![|b| \leq d_1 \cdot |a| \cdot f_1 + \ldots + d_n |a| f_n]\!]_A \ni e$ since $[\![|b| \leq |b_1| + |b_2|]\!]_A = \mathbb{T}$, $[\![|b_1| + |b_2| \leq d_1 \cdot |a| \cdot f_1 + \ldots + d_n \cdot |a| \cdot f_n + |b_2|]\!] = \mathbb{T}$, and $e \cdot |b_2| = 0$ in view of $|b_2| \in a^{\perp}$, $(1-e) \cdot |b_2| = |b_2|$.

The converse statement for a $\mathcal{J}$-evaluation follows by observing that l-simplicity and linearity imply quasiregularity and projectivity, which are expressed via Horn formulas; here we also use the fact that an algebra $\mathcal{J}$ is compact. For a $\mathcal{B}$-evaluation, the passage from the linearity in $V^{\mathcal{B}}$ to that in $V^{\mathcal{J}}$ is as in clause (1). To express the condition of being l-simple, write $\forall a_2 \in A\ \forall a_1 \in A\ \exists n \in \mathbb{N} \exists d_1, \ldots, d_n, f_1, \ldots, f_n \in A$ $(a = 0 \vee (|a_1| \leq d_1 \cdot |a_2| \cdot f_1 + \ldots + d_n \cdot |a_2| \cdot f_n))$. This formula is $\mathcal{B}$-globally true and so $\mathcal{J}$-globally true, in view of equalities (1) and (2). This, by the above, implies quasiregularity.

(3) Let A be a projective f-ring without nilpotent elements. We check that $[\![\forall x, y\ (x \cdot y = 0 \Rightarrow x = 0 \vee y = 0)]\!] = \mathbb{T}$. If $e \in [\![x \cdot y = 0]\!]$, then $exy = 0$, $e|x| \cdot |y| = 0$. Further, $0 \leq (e|x| \wedge |y|)^2 = e|x|^2 \wedge e|x| \cdot |y| \wedge e|y| \cdot |x| \wedge |y|^2 \leq e|x| \cdot |y| = 0$, whence $e|x| \wedge |y| = 0$, $ex \in y^{\perp}$, by one of the conditions. By the other condition, $y^{\perp} + y^{\perp\perp} = A$. Let e' be a central idempotent corresponding to the summand $y^{\perp}$. Then $(1-e') \cdot ex = 0$, $(1-e') \cdot e \in [\![x = 0]\!]$. On the other hand, $e' \in y^{\perp}$, $e' \wedge |y| = 0$, $e' \cdot |y| = 0$, $|e'y| = 0$, $e'y = 0$, $ee' \cdot y = 0$, $ee' \in [\![y = 0]\!]$. Therefore $(1-e') \cdot e \vee e'e = e \in [\![x = 0]\!] \vee_{\mathcal{J}} [\![y = 0]\!]$.

Argument for the converse statement is as in clauses (1) and (2).

A number of statements, which are true for rings on the right-hand sides of the above-envisaged equivalences (viz., for linearly ordered rings, l-simple rings, rings without zero divisors, division rings or fields, and the like), or for algebras over such rings, have the above-specified form $\varphi \Rightarrow \psi$, or we are able to reduce them to a series of statements in this form. The statements can then be carried over to rings or to algebras over rings occurring on the left-hand sides.*

Remark 6. Theorems 2 and 3 can be formulated for arbitrary structures like Theorem 1. Let φ be a set of functions defined on the set K. Elements of K can be represented as constants, and we may — in this sense — confine our account to functions. A *basis* in φ is the part $B \subseteq \varphi$ such that $e \in B \rightleftharpoons e \circ e = e, \ldots$. Let $\varphi_0 \rightleftharpoons \{f \in \varphi |\ f \circ e = e \circ f\}$ (on the left, e is applied to all arguments of the function f). Putting $[\![k = t]\!]_K \rightleftharpoons \{e \in B |\ e \circ k = e \circ t\}$, then, we can develop a theory close to the one above. The set φ_0 may also include relations P such that $P(x) \Rightarrow P(ex)$. This, we think, will make it possible to define a semantics for some language of functional programming.

The classes of input and output formulas can be extended as follows. Let a formula φ be of the form that guarantees that $\varphi_K \Rightarrow ([\![\varphi]\!] \in j_0)$, where j_0 is some filter. By set-theoretic considerations, then, we

*Currently, I am preparing a summary of these results. Among them are Ritt's theorem on zeros for differential polynomials, the Lean–Zeidenberg theorem on critical points of a polynomial mapping $\mathbb{C} \to \mathbb{C}^2$, the Gelfand–Ponomaryov theorem on the representation of free modular lattices, the classification of Henselian fields, etc.

obtain $[\![\psi]\!] \in j_1$, where j_1 is, generally speaking, another filter with the property $([\![\psi]\!] \in j_1) \Rightarrow \psi_K$ for formulas ψ from a certain class. In the end, as was shown above, $j_0 = j_1 = \{\mathbb{T}\}$.

Note: Parts of the theorems presented in this article are contained in [3]; for the language of rings, they are given with a proof in [7, p. 111] and without a proof in [8].

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