Some new results on Borel irreducibility of equivalence relations *

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Abstract

We prove that orbit equivalence relations (ERs, for brevity) of generically turbulent Polish actions are not Borel reducible to ERs of a family which includes Polish actions of S_∞, the group of all permutations of $\mathbb{N}$, and is closed under the Fubini product modulo the ideal Fin of all finite sets, and some other operations. Our second main result shows that T_2, an equivalence relation called "the equality of countable sets of the reals", is not Borel reducible to another family of ERs which includes continuous actions of Polish CLI groups, Borel equivalence relations with $\mathbf{G}_{\delta\sigma}$ classes, some ideals, and is closed under the Fubini product over Fin. Both results and their corollaries extend some earlier irreducibility theorems by Hjorth and Kechris.

Introduction

Classification problems for different types of mathematical structures have been in the center of interests in descriptive set theory since the beginning of the 90s. Suppose that X is a class of mathematical structures, identified modulo an equivalence relation E. This can be, *e.g.*, countable groups modulo the isomorphism relation, or unitary operators over a fixed space $\mathbb{C}^n$ modulo conjugacy, or probability measures over a fixed Polish space modulo the identification of

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measures having the same null sets, or, for instance, reals modulo the Turing reducibility. (The examples are taken from Hjorth's book [6] and Kechris' survey paper [12], where many more examples are given.) Suppose that Y is another class of mathematical structures, identified modulo an equivalence relation F. The classification problem is then to find out whether there exists a *definable*, or *effective* injection $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$. Such a map Θ can be seen as a classification of objects in X in terms of objects in Y, in a way which respects quotients over E and F. Its existence can be a result of high importance, for instance when objects in Y are of mich simpler mathematical nature than those in X.

In many cases, it turns out that the classes of structures X and Y can be considered as Polish (*i.e.*, separable complete metric) spaces, so that E, F become Borel or, more generally, analytic (as sets of pairs) relations, while reduction maps are usually required to be Borel [1]. In this case the problem can be studied by methods of descriptive set theory, where it takes the form: if E, F are Borel or analytic equivalence relations on Polish spaces, resp., X, Y, does there exist a Borel *reduction* E to F, *i.e.*, a Borel map $\vartheta : X \to Y$ satisfying $x \mathrel{\mathsf{E}} x' \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$ for all $x, x' \in X$. If such a map ϑ exists then E is said to be *Borel reducible* to F. Studies on Borel and analytic equivalence relations (ERs, for brevity) under Borel reducibility by methods of descriptive set theory, revealed a remarkable structure of reducibility and irreducibility theorems between ERs of different types (we can cite [2, 5, 6, 7, 12] as a partial account of the results obtained). Our paper belongs to this research direction.

Our first main theorem establishes Borel irreducibility between two large classes of ERs. *Class 1* consists of ERs induced by *generically turbulent* Polish actions [2]. Hjorth [6] proved that no ER of this class is Borel reducible to an ER of *Class 2* which consists of orbit ERs of Polish actions of S_∞, the group of all permutations of $\mathbb{N}$. (This result is also known in the form: generically turbulent ERs are not classifiable by countable structures, see comments in 3.3.)

One of possible proofs of Hjorth's theorem is as follows. *First*, any ER, Borel reducible to an ER in Class 2, is then Borel reducible, at least on a comeager set, to an ER in *Class 3* [3], which consists of those ERs that can be obtained from equalities on Polish spaces using the operation of countable power E^∞. *Second* (this involves Hjorth's turbulence theory), no ER in Class 1 is Borel reducible to a ER in Class 3, even on a comeager set. Our Theorem 1 generalizes the

[1] That is, with Borel graphs. Baire measurable maps and reductions satisfying certain algebraic requirements are also considered [2], as well as $\mathbf{\Delta}^1_2$ and more complicated reductions [9, 10], however they are not in the scope of this paper.

[2] That is, comeager many orbits, and even local orbits, are somewhere dense, see Definition 5.

[3] Introduced essentially by H. Friedman [4, 3].

second part. We consider *Class 4*, containing all ERs which can be obtained from equalities on Polish spaces with the operations of 1) countable union (if it results in a ER) of ERs in the same space, 2) Fubini product $\prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ modulo the ideal $\mathtt{Fin}$ of all finite subsets of $\mathbb{N}$, and 3) the countable power E^∞ (see 1.2 for exact definitions). Class 4 includes Class 3, of course, but contains many more various ERs, especially those defined using Fubini products, for instance, all ERs induced by generalized Fréchet, indecomposable, and Weiss ideals (see 1.2).

Theorem 1. *ERs in Class 1 are not Borel reducible (even not reducible by Baire measurable functions) to ERs in Class 4.*

The proof (Section 2) involves the induction on the construction of ERs in Class 4 with the help of the operations indicated. The technique of the turbulence theory will play the key role in the proof, in particular, the key step will be to prove that any ER in Class 1 is generically ergodic w. r. t. any ER in Class 4 (Theorem 6). As an application of this result, we derive the abovementioned Hjorth's theorem in a few rather simple steps in Section 3.

Amongst the habitants of Class 1 we have ERs of the form $x \mathsf{E}_{\mathcal{I}} y$ iff $x \,\Delta\, y \in \mathcal{I}$, for all $x,\, y \in \mathcal{P}(\mathbb{N})$, where $\mathcal{I}$ is an ideal on $\mathbb{N}$. Any ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is obviously an Abelian group with the symmetric difference Δ as the group operation, and $\mathsf{E}_{\mathcal{I}}$ is induced by the shift action of $\mathcal{I}$ on $\mathcal{P}(\mathbb{N})$ by Δ. Kechris [11] proved that this action is turbulent provided $\mathcal{I}$ is a Borel P-ideal [4] , with few exceptions mentioned below. This allows us to prove the following theorem in 3.1 as a corollary of Theorem 1.

Theorem 2. *If $\mathcal{Z}$ is a non-trivial [5] Borel P-ideal on $\mathbb{N}$ then $\mathsf{E}_{\mathcal{Z}}$ is not Borel reducible to a ER in Class 4 unless $\mathcal{Z}$ is $\mathtt{Fin}$ or a trivial variation of $\mathtt{Fin}$, or $\mathcal{Z}$ is isomorphic to $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ via a bijection between the underlying sets.*

Borel P-ideals form a widely studied class, which includes, for instance, $\mathtt{Fin}$, the ideal $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ of all sets $x \subseteq \mathbb{N}^2$ such that every cross-section $(x)_n = \{k : \langle n, k\rangle \in x\}$ is finite, and *trivial variations of* $\mathtt{Fin}$, *i. e.*, ideals of the form $\mathcal{I}_W = \{x \subseteq \mathbb{N} : x \cap W \in \mathtt{Fin}\}$, where $W \subseteq \mathbb{N}$ is infinite (see 1.3), as well as summable ideals, density ideals, and many more (see [2, 15]). Easily the ERs $\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$ and $\mathsf{E}_3 = \mathsf{E}_{\mathcal{I}_3}$ induced by the ideals $\mathtt{Fin}$ and $\mathcal{I}_3$, and those induced by trivial variations of $\mathtt{Fin}$, belong to Class 4, thus, the exclusion of $\mathtt{Fin}$, $\mathcal{I}_3$, and trivial variations of $\mathtt{Fin}$ in Theorem 2 is necessary and fully motivated.

[4] An ideal $\mathcal{I}$ on $\mathbb{N}$ is a *P-ideal* if for any sequence of sets $x_n \in \mathcal{I}$ there is a set $x \in \mathcal{I}$ such that $x_n \setminus x$ is finite for any n. For Borel ideals, this is equivalent to *polishability*, *i. e.*, the existence of a Polish topology on $\mathcal{I}$ which converts $\langle \mathcal{I}\,; \Delta\rangle$ in a Polish group.

[5] Here, containing all singletons $\{n\}$, $n \in \mathbb{N}$, and different from $\mathcal{P}(\mathbb{N})$.

Note that a weaker form of Theorem 2, with Class 3 instead of Class 4, is implicitly contained in Kechris [11]. A very partial result, that $\mathsf{E}_{\mathscr{Z}_0}$, the ER associated with the density-0 ideal $\mathscr{Z}_0$, is not Baire reducible to any ER of Class 3, was announced in [4]. (Friedman gives a proof in [3].)

The final Section 4 is written in attempt to obtain results of the opposite character, *i.e.*, that ERs in Class 4 are not Borel reducible to, say, turbulent or some other ERs of different nature. This is a comparably less developed area, and perhaps only one special theorem of this sort is known: Hjorth [5] proved that T_2, the ER defined on countable sequences of the reals so that $\{x_n\}\,\mathsf{T}_2\,\{y_n\}$ iff $\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$ [6], is not Borel reducible to any ER induced by a continuous action of a Polish group which admits a compatible complete left-invariant metric (a *CLI* group; this includes, for instance, Polish *Abelian* groups). It can be expected that T_2 is not Borel reducible to any Borel action of a Borel Abelian group, but this is still open, even w.r.t. shift Δ-actions of Borel ideals.

A possible way to solve the problem is connected with the following condition of a ER E (implicitly in [5]): *for any forcing notion* $\mathbb{P}$ *and any* $\mathbb{P}$*-term* $\boldsymbol{\xi}$, *if* $\mathbb{P} \times \mathbb{P}$ *forces* $\boldsymbol{\xi}_{\mathtt{left}}\,\mathsf{E}\,\boldsymbol{\xi}_{\mathtt{right}}$ *then there is a real* x *in the ground universe such that* $\mathbb{P}$ *forces* $x\,\mathsf{E}\,\boldsymbol{\xi}$. We call *pinned* all ERs satisfying this condition. Note that T_2 is not pinnedand not Borel reducible to any analytic pinned ER. We prove in Section 4 that 1) ERs induced by Polish actions of CLI groups are pinned (our proof is a simplification of Hjorth's proof in [5]), 2) Borel ERs whose equivalence classes are $\mathbf{G}_{\delta\sigma}$ sets are pinned (based on an idea extended to us by Hjorth), 3) ERs associated with exhaustive ideals of sequences of submeasures on $\mathbb{N}$ (not all of them are Polishable) are pinned, 4) Fubini products of analytic pinned ERs modulo $\mathtt{Fin}$ are pinned. All of those ERs do not Borel reduce T_2. For instance all ERs induced by Fréchet ideals are pinned and do not Borel reduce T_2.

1 Preliminaries

This Section contains a review of basic notation involved in the formulations and proofs of our main results.

1.1 Descriptive set theory

Some degree of knowledge of the theory of Borel and analytic sets in Polish (*i.e.*, complete separable metric) spaces is assumed. Recall that *analytic* sets (also

[6] T_2, sometimes denoted F_2, as in [6], is often called "the equality of countable sets of reals"; it belongs to Class 3 and is one of the most important Borel ERs.

known as Suslin, A-sets, or $\mathbf{\Sigma}^1_1$) are continuous images of Borel sets. Any Borel set is analytic, but (in uncountable Polish spaces) not conversely.

A map f (between Borel sets in Polish spaces) is called *Borel* iff its graph is a Borel set, or, which is the same, if all f-preimages of open sets are Borel. *BM* always means *Baire measurable*. A map $f : \mathbb{X} \to \mathbb{Y}$ is BM iff all f-preimages of open sets in $\mathbb{Y}$ have the Baire property in $\mathbb{X}$ (*i.e.*, are equal to open sets modulo meager sets, that is, sets of the 1st category). Any such a map is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq \mathbb{X}$ ($\mathbb{X}$, $\mathbb{Y}$ are supposed to be Polish).

Superpositions of Borel maps are easily Borel. Generally speaking, this is not true for BM maps, however, we have a useful partial result.

Lemma 3. *If $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ are Polish spaces, $f : \mathbb{X} \to \mathbb{Y}$ is BM, and $g : \mathbb{Y} \to \mathbb{Z}$ is Borel, then the superposition $f \circ g : \mathbb{X} \to \mathbb{Z}$ is BM.*

Proof. By definition, g-preimages of open subsets of $\mathbb{Z}$ are Borel in $\mathbb{Y}$, whose f-preimages are Borel combinations of sets having the Baire property. $\square$

1.2 Equivalence relations

ER means: equivalence relation. By $\mathsf{D}(X)$ we denote the equality on X considered as a ER.

Let E be a ER on a set X. Then $[y]_{\mathsf{E}} = \{x \in X : y \mathrel{\mathsf{E}} x\}$ is *the* E*-class* of any element $y \in X$. A set $Y \subseteq X$ is *pairwise* E*-equivalent*, if $x \mathrel{\mathsf{E}} y$ holds for all $x, y \in Y$.

Suppose that E and F are ERs on Polish spaces resp. $\mathbb{X}$, $\mathbb{Y}$.

- $*$ $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ (*Borel reducibility*; sometimes they write $\mathbb{X}/\mathsf{E} \le_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$) means that there is a Borel map $\vartheta : \mathbb{X} \to \mathbb{Y}$ (called *reduction*) such that $x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;
- $*$ $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$ means that $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \le_{\mathrm{B}} \mathsf{E}$ (*Borel bi-reducibility*);
- $*$ $\mathsf{E} <_{\mathrm{B}} \mathsf{F}$ means that $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ but $\mathsf{F} \not\le_{\mathrm{B}} \mathsf{E}$ (*strict Borel reducibility*).

The following operations over ERs on Polish spaces are considered.

(e1) *the countable union* (if it results in a ER) and *the countable intersection* of ERs on one and the same space;

(e2) *the countable disjoint union* $\mathsf{F} = \bigvee_{k \in \mathbb{N}} \mathsf{F}_k$ of ERs F_k on Polish spaces $\mathbb{S}_k$ is a ER on $\mathbb{S} = \bigcup_k (\{k\} \times \mathbb{S}_k)$ (with the Polish topology generated by sets of the form $\{k\} \times U$, where $U \subseteq \mathbb{S}_k$ is open) defined as follows: $\langle k, x\rangle \mathrel{\mathsf{F}} \langle l, y\rangle$ iff $k = l$ and $x \mathrel{\mathsf{F}_k} y$ [7];

[7] If $\mathbb{S}_k$ are pairwise disjoint and open in $\mathbb{S}' = \bigcup_k \mathbb{S}_k$ then we can equivalently define $\mathsf{F} = \bigvee_k \mathsf{F}_k$ on $\mathbb{S}'$ so that $x \mathrel{\mathsf{F}} y$ iff x, y belong to the same $\mathbb{S}_k$ and $x \mathrel{\mathsf{F}_k} y$.

(e3) *the product* $\mathsf{P} = \prod_k \mathsf{F}_k$ of ERs F_k on spaces $\mathbb{S}_k$ is a ER on $\prod_k \mathbb{S}_k$ defined so that $x \,\mathsf{P}\, y$ iff $x_k \,\mathsf{F}_k\, y_k$ for all k, in particular, if E, F live in, resp., $\mathbb{X}$, $\mathbb{Y}$ then $\mathsf{P} = \mathsf{E} \times \mathsf{F}$ is defined on $\mathbb{X} \times \mathbb{Y}$ so that $\langle x, y\rangle \,\mathsf{P}\, \langle x', y'\rangle$ iff $x \,\mathsf{E}\, x'$ and $y \,\mathsf{F}\, y'$;

(e4) *the Fubini product* $\prod_{k\in\mathbb{N}} \mathsf{F}_k \,/\, \mathcal{I}$ of ERs F_k on spaces $\mathbb{S}_k$, modulo an ideal $\mathcal{I}$ on $\mathbb{N}$, is a ER on the product space $\prod_{k\in\mathbb{N}} \mathbb{S}_k$ defined as follows: $x \,\mathsf{F}\, y$ iff $\{k : x_k \not\mathsf{E}_k\, y_k\} \in \mathcal{I}$;

(e5) *the countable power* F^∞ of a ER F on a space $\mathbb{S}$ is a ER on $\mathbb{S}^\mathbb{N}$ defined as follows: $x \,\mathsf{F}^\infty\, y$ iff $\{[x_k]_\mathsf{F} : k \in \mathbb{N}\} = \{[y_k]_\mathsf{F} : k \in \mathbb{N}\}$, so that for any k there is l with $x_k \,\mathsf{F}\, y_l$ and for any l there is k with $x_k \,\mathsf{F}\, y_l$.

Note that operations (e1), (e2), (e3), (e5), and (e4) with $\mathcal{I} = \mathtt{Fin}$, always yield Borel, resp., analytic ERs provided given ERs are Borel, resp., analytic.

The operations are not independent. In particular, $\bigcap_{k\in\mathbb{N}} \mathsf{F}_k$ is Borel reducible to $\prod_k \mathsf{F}_k$ via the map $x \mapsto \langle x, x, x, ...\rangle$, while the disjoint union $\bigvee_{k\in\mathbb{N}} \mathsf{F}_k$ is reducible to $\mathsf{D}(\mathbb{N}) \times \prod_k \mathsf{F}_k$ via $\langle k, x\rangle \mapsto \langle k, x_0, ..., x_{k-1}, x, x_{k+1}, ...\rangle$, where $x_k \in \mathbb{S}_k$ are fixed once and for all. The product $\prod_k \mathsf{F}_k$ itself is expressible in terms of the Fubini product modulo $\mathtt{Fin}$. Indeed, let $f : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ be any map such that $f^{-1}(n)$ is infinite for any n. Put $\mathsf{E}_k = \mathsf{F}_{f(k)}$. For any $x = \langle x_0, x_1, x_2, ...\rangle \in \prod_k \mathbb{S}_k$ (where $\mathbb{S}_k$ is the domain of F_k) let $\vartheta(x) = \langle y_0, y_1, y_2, ...\rangle$, where $y_k = x_{f(k)}$. Then ϑ is a Borel reduction of $\prod_k \mathsf{F}_k$ to $\prod_k \mathsf{E}_k \,/\, \mathtt{Fin}$. Yet the Fubini product and countable power are surely not reducible to each other, and we know little on the countable union in (e1).

It follows that Class 4 of ERs, mentioned in our theorems 1 and 2, is the least class of ERs which contains equalities $\mathsf{D}(\mathbb{S})$ on Polish spaces $\mathbb{S}$ and is closed under the operations (e1) — (e5) (with $\mathcal{I} = \mathtt{Fin}$ in (e4)), and all ERs in Class 4 are Borel ERs on Polish spaces.

There are many interesting ERs in Class 4, for instance, the sequence of ERs T_α, $\alpha < \omega_1$, of H. Friedman [3] which begins with $\mathsf{T}_0 = \mathsf{D}(\mathbb{N})$, the equality relation on $\mathbb{N}$, then $\mathsf{T}_{\alpha+1} = {\mathsf{T}_\alpha}^\infty$ for any α, and $\mathsf{T}_\lambda = \bigvee_{\alpha<\lambda} \mathsf{T}_\gamma$ for limit ordinals λ. Thus, $\mathtt{dom}\,\mathsf{T}_1 = \mathbb{N}^\mathbb{N}$ and $x \,\mathsf{T}_1\, y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$, for $x, y \in \mathbb{N}^\mathbb{N}$. The map $\vartheta(x) =$ the characteristic function of $\mathtt{ran}\, x$ witnesses that $\mathsf{T}_1 \le_{\mathrm{B}} \mathsf{D}(2^\mathbb{N})$. To show the converse, define, for any $a \in 2^\mathbb{N}$, $\beta(a)$ to be the only increasing bijection $\mathbb{N} \xrightarrow{\text{onto}} |a| = \{k : a(k) = 1\}$ whenewer $|a|$ is infinite, while if $|a| = \{k_0, ..., k_n\}$ then put $\beta(a)(i) = k_i$ for $i < n$ and $\beta(a)(i) = k_n$ for $i \ge n$. The function β witnesses $\mathsf{D}(2^\mathbb{N}) \le_{\mathrm{B}} \mathsf{T}_1$, hence, $\mathsf{T}_1 \sim_{\mathrm{B}} \mathsf{D}(2^\mathbb{N})$. It easily follows that $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{D}(2^\mathbb{N})^\infty$; the right-hand side is often taken as the definition of the ER T_2, which by this reason is usually called “the equality of countable sets of reals”.

1.3 Ideals

An ideal on a set A is any set $\emptyset \neq \mathcal{I} \subseteq \mathcal{P}(A)$, closed under $\cup$ and satisfying $x \in \mathcal{I} \wedge y \subseteq x \Longrightarrow y \in \mathcal{I}$. In this case, $\mathsf{E}_{\mathcal{I}}$ is a ER on $\mathcal{P}(A)$ defined as follows: $X \mathrel{\mathsf{E}_{\mathcal{I}}} Y$ iff $X \mathbin{\Delta} Y \in \mathcal{I}$. Note that $\mathsf{E}_{\mathcal{I}}$ is Borel provided the ideal $\mathcal{I}$ is such. Many important ERs appear in the form $\mathsf{E}_{\mathcal{I}}$, among them

$$\begin{array}{lllll} \mathsf{E}_0 & = & \mathsf{E}_{\mathtt{Fin}} & , \quad \text{where} & \mathtt{Fin} = \{x \subseteq \mathbb{N} : x \text{ is finite}\}\,; \\ \mathsf{E}_1 & = & \mathsf{E}_{\mathcal{I}_1} & , \quad \text{where} & \mathcal{I}_1 = \mathtt{Fin} \times 0 = \{x \subseteq \mathbb{N}^2 : \{k : (x)_k \neq \emptyset\} \in \mathtt{Fin}\}\,; \\ \mathsf{E}_3 & = & \mathsf{E}_{\mathcal{I}_3} & , \quad \text{where} & \mathcal{I}_3 = 0 \times \mathtt{Fin} = \{x \subseteq \mathbb{N}^2 : \forall\, k\, ((x)_k \in \mathtt{Fin})\}\,; \end{array}$$

all of them belong to Class 4 [8]. Ideals of the form $\mathtt{Fin}_W = \{x \in \mathcal{P}(\mathbb{N}) : x \cap W \in \mathtt{Fin}\}$, where $W \subseteq \mathbb{N}$ is infinite and coinfinite, called *trivial variations of* $\mathtt{Fin}$, also produce ERs in Class 4.

We write $\mathcal{I} \leq_{\mathrm{B}} \mathcal{J}$, $\mathcal{I} \sim_{\mathrm{B}} \mathcal{J}$, *etc.* if resp. $\mathsf{E}_{\mathcal{I}} \leq_{\mathrm{B}} \mathsf{E}_{\mathcal{J}}$, $\mathsf{E}_{\mathcal{I}} \sim_{\mathrm{B}} \mathsf{E}_{\mathcal{J}}$, *etc.*

The *Fubini product* $\prod_{k \in \mathbb{N}} \mathcal{J}_k \,/\, \mathcal{I}$ of ideals $\mathcal{J}_k$ on sets B_k, over an ideal $\mathcal{I}$ on $\mathbb{N}$, is an ideal of all sets $y \subseteq B = \{\langle k, b\rangle : k \in \mathbb{N} \wedge b \in B_k\}$ such that the set $\{k : (y)_k \notin \mathcal{J}_k\}$ belongs to $\mathcal{I}$, where $(y)_k = \{b : \langle k, b\rangle \in Y\}$, a *cross-section* of Y. (Compare with the Fubini product of ERs !) In particular, if $\mathcal{I}$, $\mathcal{J}$ are ideals on resp. $\mathbb{N}$, B, then $\mathcal{I} \otimes \mathcal{J} = \prod_{k \in \mathbb{N}} \mathcal{J}_k \,/\, \mathcal{I}$, where each $\mathcal{J}_k = \mathcal{J}$ for all $k \in \mathbb{N}$. Thus, $\mathcal{I} \otimes \mathcal{J}$ is the ideal of all sets $y \subseteq \mathbb{N} \times B$ such that $\{k : (y)_k \notin \mathcal{J}\} \in \mathcal{I}$.

P-ideals. An ideal $\mathcal{I}$ on $\mathbb{N}$ is called *a P-ideal* if for any sequence of sets $x_n \in \mathcal{I}$ there is a set $x \in \mathcal{I}$ such that $x_n \setminus x \in \mathtt{Fin}$ for all n. For instance, $\mathtt{Fin}$ and $\mathcal{I}_3$ (but not $\mathcal{I}_1$!) are P-ideals.

This class admits different characterizations. A *submeasure* on a set A is any map $\varphi : \mathcal{P}(A) \to [0, +\infty]$, satisfying $\varphi(\emptyset) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \leq \varphi(x \cup y) \leq \varphi(x) + \varphi(y)$. A submeasure φ on $\mathbb{N}$ is *lover semicontinuous*, or l. s. c. for brevity, if we have $\varphi(x) = \mathtt{sup}_n\, \varphi(x \cap [0, n))$ for all $x \in \mathcal{P}(\mathbb{N})$. Solecki [15]) proved that Borel P-ideals are exactly those of the form $\mathtt{Exh}_\varphi = \{x \in \mathcal{P}(\mathbb{N}) : \varphi_\infty(x) = 0\}$, where φ is a l. s. c. submeasure on $\mathbb{N}$ and $\varphi_\infty(x) = \mathtt{inf}_n(x \cap [n, \infty))$, and Borel P-ideals are the same as *polishable* ideals, *i. e.*, those which admit a Polish group topology with Δ as the group operation.

Kechris [11] proved that the shift action of any Borel P-ideal $\mathcal{I}$, except for $\mathtt{Fin}$, $\mathcal{I}_3$, and trivial variations of $\mathtt{Fin}$, is generically turbulent, hence, the corresponding ER $\mathsf{E}_{\mathcal{I}}$ belongs to Class 1.

Fréchet family. This is the least family $\mathtt{Fr}$ of ideals containing $\mathtt{Fin}$ and closed under the Fubini products $\prod_{n \in \mathbb{N}} \mathcal{I}_n \,/\, \mathtt{Fin}$. For instance, the *iterated Fréchet ideals* $\mathcal{J}_\alpha$, defined by induction on $\alpha < \omega_1$ so that $\mathcal{J}_0 = \mathtt{Fin}$,

[8] To show that E_0 belongs to Class 4 let, for any k, F_k be the equality on a 2-element set in (e4). To see that E_3 belongs to Class 4 take each F_k to be E_0 in (e3).

$\mathcal{J}_{\alpha+1} = \mathtt{Fin} \otimes \mathcal{J}_\alpha$ for all α, and $\mathcal{J}_\lambda = \prod_{\alpha<\lambda} \mathcal{J}_\alpha \,/\, \mathtt{Fin}_\lambda$ for any limit λ, where $\mathtt{Fin}_\lambda$ is the ideal of all finite subsets of λ, belong to Fr. (A modification of this construction in [8] involves a cofinal ω-sequece fixed in each limit λ.)

By definition, if $\mathcal{I} \in \mathsf{Fr}$ then $\mathsf{E}_\mathcal{I}$ is a ER of Class 4.

Let $\mathtt{otp}\, X$ be the order type of $X \subseteq \mathtt{Ord}$. For any γ, $\alpha < \omega_1$, the set

$$\mathcal{I}^\gamma_\alpha = \{A \subseteq \alpha : \mathtt{otp}\, A < \omega^\gamma\} \quad (\text{non-trivial only if } \alpha \geq \omega^\gamma)\,.$$

is an ideal (because ordinals of the form ω^γ are not sums of a pair of smaller ordinals); those ideals, especially in the case when $\alpha = \omega^\gamma$, are called *indecomposable*. We don't know whether each $\mathcal{I}^\gamma_\alpha$ is really (isomorphic to) an ideal in Fr, yet it can be shown that any $\mathcal{I}^\gamma_\alpha$ is Borel reducible to an ideal in Fr. Similarly,

$$\mathcal{W}^\gamma_\alpha = \{A \subseteq \gamma : |A|_{\mathrm{CB}} < \omega^\gamma\} \quad (\text{the } \textit{Weiss ideals}, \text{ non-trivial only if } \alpha \geq \omega^{\omega^\gamma}),$$

where $|X|_{\mathrm{CB}}$ is the Cantor-Bendixson rank of $X \subseteq \mathtt{Ord}$, see Farah [2, § 1.14], are Borel reducible to ideals in Fr.

2 Proof of Theorem 1

This section proves Theorem 1. Our method of demonstration of the non-reducibility employs the following auxiliary notions. Let E, F be ERs on Polish spaces, resp., $\mathbb{X}$, $\mathbb{Y}$. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is called

- E,F-*invariant* if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all x, $y \in \mathbb{X}$;
- *generically* [9] (or *gen.*, for brevity) E,F-*invariant* if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all x, y in a comeager set $X \subseteq \mathbb{X}$;
- *gen.* F-*constant* if $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ holds for all x, y in a comeager set $X \subseteq \mathbb{X}$.

Finally, E is *generically* F-*ergodic* (Hjorth [6, 3.1]) if every BM E,F-invariant map is gen. F-constant.

Proposition 4. (i) *If* E *is gen.* F-*ergodic and does not have a comeager equivalence class then* E *is not reducible to* F *by a BM map.*

(ii) *If* E *is gen.* F-*ergodic then even every BM gen.* E,F-*invariant map is gen.* F-*constant.* □

Accordingly, our plan will be to show that any orbit ER induced by a Polish turbulent action is gen. F-ergodic for any ER F in Class 4.

[9] In this research direction, "generically", or, in our abbreviation, "gen." (property) is understood so that the property holds on a comeager domain.

2.1 Local orbits and turbulence

An *action* of a group $\mathbb{G}$ on $\mathbb{X}$ is any map $a : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$, usually written as $a(g,x) = g{\cdot}x$, such that 1) $e{\cdot}x = x$, and 2) $g{\cdot}(h{\cdot}x) = (gh){\cdot}x$. In this case, $\langle \mathbb{X}\,; a\rangle$, as well as $\mathbb{X}$ itself, is called a $\mathbb{G}$*-space*. A continuous action of a Polish group [10] $\mathbb{G}$ on a Polish space $\mathbb{X}$ is called *Polish action* and $\mathbb{X}$ itself is called a *Polish* $\mathbb{G}$*-space*.

Any action a of $\mathbb{G}$ on $\mathbb{X}$ induces *the orbit ER*, $\mathsf{E}^{\mathbb{X}}_a = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$, defined on $\mathbb{X}$ so that $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} y$ iff there is $a \in \mathbb{G}$ with $y = a{\cdot}x$, whose equivalence classes

$$[x]_{\mathbb{G}} = [x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} = \{y : \exists\, g \in \mathbb{G}\ (g{\cdot}x = y)\}\,.$$

are $\mathbb{G}$-orbits. Induced ERs of Polish actions are analytic (as sets of pairs), sometimes even Borel [1, Chapter 7].

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$. If $G \subseteq \mathbb{G}$ and $X \subseteq \mathbb{X}$ then we define

$$\mathsf{R}^X_G = \{\langle x,y\rangle \in X^2 : \exists\, g \in G\ (x = g \cdot y)\}$$

and let $\sim^X_G$ denote the ER-hull of R^X_G, *i.e.*, the $\subseteq$-least ER on X such that $x \mathrel{\mathsf{R}^X_G} y \Longrightarrow x \sim^X_G y$. In particular $\sim^{\mathbb{X}}_{\mathbb{G}} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$, but generally we have $\sim^X_G \subsetneqq \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \restriction X$. Finally, define $\mathcal{O}(x,X,G) = [x]_{\sim^X_G} = \{y \in X : x \sim^X_G y\}$ for $x \in X$ – the *local orbit* of x. In particular, $[x]_{\mathbb{G}} = [x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} = \mathcal{O}(x,\mathbb{X},\mathbb{G})$, the full $\mathbb{G}$-orbit of $x \in \mathbb{X}$.

Definition 5 (This version is taken from Kechris [13, § 8]**).** Suppose that $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting on $\mathbb{X}$ continuously.

(t1) A point $x \in \mathbb{X}$ is *turbulent* if for any open non-empty set $X \subseteq \mathbb{X}$ containing x and any nbhd $G \subseteq \mathbb{G}$ (not necessarily a subgroup) of $1_{\mathbb{G}}$, the local orbit $\mathcal{O}(x,X,G)$ is somewhere dense (*i.e.*, not a nowhere dense set) in $\mathbb{X}$.

(t2) An orbit $[x]_{\mathbb{G}}$ is *turbulent* if x is such (then all $y \in [x]_{\mathbb{G}}$ are turbulent).

(t3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *gen. turbulent* and $\mathbb{X}$ is a *gen. turbulent* Polish $\mathbb{G}$-space, if all orbits are dense and meager. The action is *generically*, or *gen. turbulent* and $\mathbb{X}$ is a *gen. turbulent* Polish $\mathbb{G}$-space, if the union of all dense, turbulent, and meager orbits $[x]_{\mathbb{G}}$ is comeager. □

ERs induced by gen. turbulent Polish actions are what is called *Class 1* in Theorem 1. To prove Theorem 1, we are going to show that all ERs in Class 1 are gen. F-ergodic, for any F in Class 4. The method of proof will be by induction on the construction of ERs in Class 4. It is a slight inconvenience that we have to consider a somewhat stronger property through the induction scheme.

Suppose that F is an ER on a Polish space.

[10] That is, a topological group whose underlying set is a Polish space and the group operation and the inverse map are continuous.

- An action of $\mathbb{G}$ on $\mathbb{X}$ is *hereditarily generically* (*h. gen.*, for brevity) F-ergodic if ER $\sim_G^X$ is generically F-ergodic whenever $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a non-empty open set containing $1_{\mathbb{G}}$, and local orbits $\mathcal{O}(x, X, G)$ are dense in X for comeager (in X) many $x \in X$.

This obviously implies gen. F-ergodicity provided the action is gen. turbulent.

Theorem 6. *Suppose that $\mathbb{G}$ is a Polish group, $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is h. gen. F-ergodic, hence* (by Proposition 4)*, not reducible to any ER F of Class 4 by a BM map.*

2.2 Preliminaries to the proof of Theorem 6

We begin with two rather simple technical facts related to turbulence.

Lemma 7. *In the assumptions of the theorem, suppose that $\emptyset \neq X \subseteq \mathbb{X}$ is an open set, $G \subseteq \mathbb{G}$ is a nbhd of $1_{\mathbb{G}}$, and $\mathcal{O}(x, X, G)$ is dense in X for X-comeager many $x \in X$. Let $U, U' \subseteq X$ be non-empty open and $D \subseteq X$ be comeager in X. Then there exist points $x \in D \cap U$ and $x' \in D \cap U'$ with $x \sim_G^X x'$.*

Proof. Under our assumptions there exist points $x_0 \in U$ and $x_0' \in U'$ with $x_0 \sim_G^X x_0'$, *i.e.*, there are elements $g_1, ..., g_n \in G$ with $x_0' = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x_0$ and $g_k \cdot ... \cdot g_1 \cdot x_0 \in X$ for all $k \leq n$. Since the action is continuous, there is a nbhd $U_0 \subseteq U$ of x_0 such that $g_k \cdot ... \cdot g_1 \cdot x \in X$ and $g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x \in U_2$ for all $x \in U_0$. Since D is comeager, easily there is $x \in U_0 \cap D$ such that $x' = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x \in U' \cap D$. □

Lemma 8. *In the assumptions of the theorem, for any open non-empty $U \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ with $1_{\mathbb{G}} \in G$ there is an open set $\emptyset \neq U' \subseteq U$ such that the local orbit $\mathcal{O}(x, U', G)$ is dense in U' for U'-comeager many $x \in U'$.*

Proof. Let $\textsc{Int}\,\overline{X}$ be the interior of the closure of X. If $x \in U$ and $\mathcal{O}(x, U, G)$ is somewhere dense (in U) then the set $U_x = U \cap \textsc{Int}\,\overline{\mathcal{O}(x, U, G)} \subseteq U$ is open and $\sim_G^U$-invariant (an observation made, *e.g.*, in [13, proof of 8.4]), moreover, $\mathcal{O}(x, U, G) \subseteq U_x$, hence, $\mathcal{O}(x, U, G) = \mathcal{O}(x, U_x, G)$. It follows from the invariance that the sets U_x are pairwise disjoint, and it follows from the turbulence that the union of them is dense in U. Take any non-empty U_x as U'. □

Our proof of Theorem 6 goes on by induction on the construction of ERs in Class 4 with the help of the operations mentioned in Section 1.2, in several following subsections. We begin with the base of the induction: prove that, under the assumptions of the theorem, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is h. gen. $\mathsf{D}(\mathbb{N})$-ergodic. Suppose that $X \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ are non-empty open sets, $1_{\mathbb{G}} \in G$, and $\mathcal{O}(x, X, G)$ is dense in X for X-comeager many points $x \in X$. Prove that $\sim_G^X$ is generically $\mathsf{D}(\mathbb{N})$-ergodic.

Consider a BM gen. $\sim_G^X, \mathsf{D}(\mathbb{N})$-invariant map $\vartheta : \mathbb{X} \to \mathbb{N}$. Suppose, on the contrary, that ϑ is not gen. $\mathsf{D}(\mathbb{N})$-constant. Then there exist open non-empty sets $U_1, U_2 \subseteq X$, numbers $\ell_1 \neq \ell_2$, and a comeager set $D \subseteq X$ such that $\vartheta(x) = \ell_1$ for all $x \in D \cap U_1$ and $\vartheta(x) = \ell_2$ for all $x \in D \cap U_2$. Lemma 7 yields a pair of points $x_1 \in U_1 \cap D$ and $x_2 \in U_2 \cap D$ with $x_1 \sim_G^X x_2$, contradiction.

2.3 Inductive step of the countable power

Consider a gen. turbulent Polish $\mathbb{G}$-space $\mathbb{X}$ and a Borel ER F on a Polish space $\mathbb{Y}$. Assume that the action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F-ergodic, and prove that the action is h. gen. F^∞-ergodic. Fix a non-empty open set $X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that $\mathcal{O}(x, X_0, G_0)$ is dense in X_0 for X_0-comeager many $x \in X_0$. Prove that any given $\sim_{G_0}^{X_0}, \mathsf{F}^\infty$-invariant BM function $\vartheta : X_0 \to \mathbb{Y}^{\mathbb{N}}$ is gen. F^∞-constant. By definition, we have

(1) for $x, x' \in X_0$: $x \sim_{G_0}^{X_0} x' \implies \forall k \ \exists l \ \big(\vartheta_k(x) \ \mathsf{F} \ \vartheta_l(x')\big)$, where $\vartheta_k(x) = \vartheta(x)(k)$.

Note that ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq X_0$.

Lemma 9. *For each k and open $\emptyset \neq U \subseteq X_0$ there is an open set $\emptyset \neq W \subseteq U$ such that ϑ_k is gen. F-constanta on W.*

Proof. A simple category argument beginning with (1) yields a number l and open non-empty sets $W \subseteq U$ and $Q \subseteq G_0$, and a dense in $W \times Q$ set $P \subseteq W \times Q$ of class $\mathbf{G}_\delta$ such that $\vartheta_k(x) \ \mathsf{F} \ \vartheta_l(g \cdot x)$ holds for all pairs $\langle x, g \rangle \in P$. We can assume that $\langle x, g \rangle \in P \implies x \in D$. Since Q is open, there is an element $g_0 \in Q$ and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ with $G^{-1} = G$ such that $g_0 \, G \subseteq Q$.

The continuation of the proof involves forcing [11].

Let us fix a countable transitive model $\mathfrak{M}$ of **ZFHC**, *i.e.*, **ZFC** minus the Power Set axiom but plus the axiom: “every set is hereditarily countable”. We can assume that $\mathbb{X}$ is coded in $\mathfrak{M}$ in the sense that there is a set $D_{\mathbb{X}} \in \mathfrak{M}$ which is a dense (countable) subset of $\mathbb{X}$, and $d_{\mathbb{X}} \restriction D_{\mathbb{X}}$ (the distance function of $\mathbb{X}$ restricted to $D_{\mathbb{X}}$) also belongs to $\mathfrak{M}$. Further, $\mathbb{G}$, $\mathbb{Y}$, the action of $\mathbb{G}$ on

[11] Some degree of the reader's acquaintance with forcing is assumed. The lemma could have been proved by purely topological arguments, yet then the reasoning then would not be so transparent.

$\mathbb{X}$, sets G, D, P and the map $\vartheta \restriction D$ are also assumed to be coded in $\mathfrak{M}$ in a similar sense.

Below, let $\mathbf{C}_{\mathbb{X}}$ be the Cohen forcing for $\mathbb{X}$, which consists of rational balls with centers in a fixed dense countable subset of $\mathbb{X}$, and let $\mathbf{C}_{\mathbb{G}}$ be the Cohen forcing for $\mathbb{G}$ defined similarly As usual, $U \subseteq V$ means that U is a stronger forcing condition. In these assumptions, the notions of Cohen generic, over $\mathfrak{M}$, elements of $\mathbb{X}$ and $\mathbb{G}$ makes sense, and the set of all Cohen generic, over $\mathfrak{M}$ points of $\mathbb{X}$ is a dense $\mathbf{G}_\delta$ subset of $\mathbb{X}$ included in D.

Claim 10 (The key point of the turbulence)**.** *If $x, x' \in W$ are $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ and $x \sim_G^W x'$ then $\vartheta_k(x) \,\mathsf{F}\, \vartheta_k(x')$.*

Proof. We argue by induction on the number $n(x,x')$ equal to the least n such that there exist $g_1, ..., g_n \in G$ satisfying

(2) $x' = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x$, and $g_k \cdot ... \cdot g_1 \cdot x \in W$ for all $k \le n$.

Suppose that $n(x,x') = 1$, thus, $x = h \cdot x'$ for some $h \in G \cap \mathfrak{M}[x,x']$ [12] Take any $\mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}[x,x']$, element $g \in Q$, close enough to g_0 for $g' = gh^{-1}$ to belong to Q. Then $\langle x, g\rangle$ is $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}$ by the product forcing theorem, thus, $\langle x, g\rangle \in P$ (because P is a dense $\mathbf{G}_\delta$ coded in $\mathfrak{M}$) and $\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(g \cdot x)$ by the choice of P. Moreover, g' also is $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}[x']$, so that $\vartheta_k(x') \,\mathsf{F}\, \vartheta_l(g' \cdot x')$ by the same argument. Yet we have $g' \cdot x' = gh^{-1} \cdot (h \cdot x) = g \cdot x$.

As for the inductive step, suppose that (2) holds for some $n \ge 2$. Take a $\mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}[x]$, element $g_1' \in G$ close enough to g_1 for $g_2' = g_2\, g_1\, {g_1'}^{-1}$ to belong to G and for $x^* = g_1' \cdot x$ to belong to W. Note that x^* is $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ (product forcing) and $n(x^*, x') \le n-1$ because $g_2' \cdot x^* = g_2 \cdot g_1 \cdot x$.

$\square$ (*Claim 10*)

To summarize, we have shown that ϑ_k is gen. $\sim_G^W$, F-invariant on W, *i.e.*, invariant on a comeager subset of W. We can also assume that the orbit $\mathcal{O}(x, W, G)$ is dense in W for W-comeager many points $x \in W$, by Lemma 8. Then, by the h. gen. F-ergodicity, ϑ_k is gen. F-constant on W, as required.

$\square$ (*Lemma 9*)

According to the lemma, there exist: an X_0-comeager set $Z \subseteq X_0$, and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{Y}$ such that, for any k and for any $x \in Z$ there is j with $\vartheta_k(x) \,\mathsf{F}\, y_j$. Let $\eta(x) = \bigcup_{k \in \mathbb{N}} \{j : \vartheta_k(x) \,\mathsf{F}\, y_j\}$. Then, for any pair

[12] Here $\mathfrak{M}[x,x']$ is defined as any (countable transitive) model of **ZFHC** containing x, x', and all sets in $\mathfrak{M}$, rather than a generic extension of $\mathfrak{M}$. The model $\mathfrak{M}[x,x']$ can contain more ordinals than $\mathfrak{M}$, but this is not essential here.

$x,\ x' \in Z,\ \ \vartheta(x)\,\mathsf{F}^\infty\,\vartheta(x')$ iff $\eta(x)=\eta(x')$, so that, by the invariance of ϑ, we have:

(3) $x \sim_{G_0}^{X_0} x' \implies \eta(x)=\eta(x')$ for all $x,\ x' \in Z$.

It remains to show that η is a constant on a comeager subset of Z.

Suppose, on the contrary, that there exist two non-empty open sets $U_1, U_2 \subseteq X_0$, a number $j \in \mathbb{N}$, and a comeager set $Z' \subseteq Z$ such that $j \in \eta(x_1)$ and $j \notin \eta(x_2)$ for all $x_1 \in Z' \cap U_1$ and $x_2 \in Z' \cap U_2$. Lemma 7 yields a contradiction to (3) as in the end of Section 2.2.

2.4 Inductive step of the Fubini product

Suppose that $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Prove that the action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F-ergodic, where $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$, F_k is a Borel ER on a Polish space $\mathbb{Y}_k$, and the action is h. gen. F_k-ergodic for any k. Fix an open set $\emptyset \neq X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that X_0-comeager many orbits $\mathcal{O}(x, X_0, G_0)$ with $x \in X_0$ are dense in X_0. Prove that any $\sim_{G_0}^{X_0}, \mathsf{F}$-invariant BM function $\vartheta : U_0 \to \mathbb{Y}$ is gen. F-constant on X_0. By definition

(4) for $x,\ y \in X_0:\ x \sim_{G_0}^{X_0} y \implies \exists\, k_0\ \forall\, k \ge k_0\ \big(\vartheta_k(x)\ \mathsf{F}_k\ \vartheta_k(y)\big)$,

where $\vartheta_k(x) = \vartheta(x)(k)$. Note that ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq X_0$.

Lemma 11. *For any open set $\emptyset \neq U \subseteq X_0$ there exist a number k_0 and open $\emptyset \neq W \subseteq U$ such that ϑ_k is gen. F-constant on W for all $k \ge k_0$.*

Proof. Applying (4), we can easily find a number k_0, open non-empty sets $W \subseteq U$ and $Q \subseteq G_0$, and a dense in $W \times Q$ set $P \subseteq W \times Q$ of class $\mathbf{G}_\delta$, such that $\vartheta_k(x)\,\mathsf{F}\,\vartheta_k(g \cdot x)$ holds for all $k \ge k_0$ and all pairs $\langle x, g\rangle \in P$. We can assume that $\langle x, g\rangle \in P \Longrightarrow x \in D$. Since Q is open, there exist an element $g_0 \in Q$ and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ with $G^{-1} = G$, such that $g_0\, G \subseteq Q$.

Let a model $\mathfrak{M}$ be as in the proof of Lemma 9. Similarly to Claim 10, we can prove that if points $x,\ x' \in W$ are $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$, $k \ge k_0$, and $x \sim_G^W x'$, then $\vartheta_k(x)\,\mathsf{F}_k\,\vartheta_k(x')$, in other words, each function ϑ_k with $k \ge k_0$ is gen. $\sim_G^W, \mathsf{F}_k$-invariant on W. We can assume, by Lemma 8, that W-comeager many orbits $\mathcal{O}(x, W, G)$ are dense in W. Now, by the h. gen. F_k-ergodicity, any ϑ_k with $k \ge k_0$ is gen. F_k-constant on such a set W, as required. □

It is clear that if W is chosen as in the lemma then ϑ itself is gen. F-constant on W. It remains to show that these constants are F-equivalent to each other. Suppose, on the contrary, that there exist two non-empty open sets $W_1,\ W_2 \subseteq X_0$ and a pair of points $y \not\mathsf{F}\, y'$ in $\mathbb{Y}$ such that $\vartheta(x)\,\mathsf{F}\,y$ and $\vartheta(x')\,\mathsf{F}\,y'$ for comeager many $x \in W_1$ and $x' \in W_2$. Contradiction follows as in the end of Section 2.3.

2.5 Other inductive steps

We carry out the induction steps related to operations (e1), (e2), (e3) of 1.2.

Countable union. Suppose that F_1, F_2, F_3, ... are Borel ERs on a Polish space $\mathbb{Y}$, and $\mathsf{F} = \bigcup_k \mathsf{F}_k$ is still a ER, and the Polish and gen. turbulent action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F_k-ergodic for any k, and prove that it remains h. gen. F-ergodic.

Consider a non-empty open set $X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that X_0-comeager many orbits $\mathcal{O}(x, X_0, G_0)$ with $x \in X_0$ are dense in X_0. Consider a $\sim_{G_0}^{X_0}, \mathsf{F}$-invariant BM function $\vartheta : X_0 \to \mathbb{Y}$, continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq X_0$. It follows from the invariance that for any open set $\emptyset \neq U \subseteq X_0$ there exist: a number k and open non-empty sets $Q \subseteq U$ and $Q \subseteq G_0$ such that $\vartheta(x)\ \mathsf{F}_k\ \vartheta(g \cdot x)$ holds for any $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}$, pair $\langle x, g \rangle \in W \times Q$. We can find, as above, $g_0 \in Q \cap \mathfrak{M}$ and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $g_0\, G \subseteq Q$. Similarly to Claim 10, we have $\vartheta(x)\ \mathsf{F}_k\ \vartheta(x')$ for any pair of $\mathbf{C}_{\mathbb{X}}$-generic, over $\mathfrak{M}$, elements x, $x' \in W$, satisfying $x \sim_G^W x'$. It follows, by the ergodicity, that ϑ is gen. F_k-constant, hence, F-constant, on W. That these F-constants are F-equivalent to each other, can be demonstrated exactly as in the end of Section 2.3. The operation of countable intersection is considered similarly.

Countable product. It is shown in 1.2 that this operation is reducible to the Fubini product, yet there is a simple independent argument. If F_k be ERs on spaces $\mathbb{Y}_k$ then $\mathsf{F} = \prod_k \mathsf{F}_k$ is a ER on the space $\mathbb{Y} = \prod_k \mathbb{Y}_k$. For any map $\vartheta : \mathbb{X} \to \mathbb{Y}$, to be E, F-invariant (where E is an arbitrary ER on $\mathbb{X}$) it is necessary and sufficient that every co-ordinate map $\vartheta_k(x) = \vartheta(x)(k)$ is E, F_k-invariant, which immediately yields the result required.

Disjoint union. It is shown in 1.2 that this operation is reducible to the product.

$\square$ (*Theorems 6 and 1*)

3 Applications

This section contains two applications of Theorem 6. One of them is Theorem 2. The other one shows how Hjort's theorem mentioned in the Introduction (that "turbulent" ERs are not Borel reducible to Polish actions of S_∞, the group of all permutations of $\mathbb{N}$) can be derived from Theorem 6 by rather simple arguments.

3.1 Proof of Theorem 2

Let us fix a non-trivial, as in Theorem 2, Borel P-ideal $\mathcal{Z} \subseteq \mathcal{P}(\mathbb{N})$. By a theorem of Solecki (see Section 1.3) there exists a l. s. c. submeasure φ on $\mathbb{N}$ such that $\mathcal{Z} = \{x \subseteq \mathbb{N} : \varphi_\infty(x) = 0\}$. Put $r_k = \varphi(\{k\})$.

Lemma 12 (Kechris [11]). *If $\mathcal{Z}$ is not equal to $\mathtt{Fin}$, is not a trivial variation of $\mathtt{Fin}$, and is not isomorphic to $\mathcal{I}_3 = 0 \times \mathtt{Fin}$, then there is a set $W \notin \mathcal{Z}$ such that $\{r_k\}_{k \in W} \to 0$.*

Proof. Put $U_n = \{k : r_k \le \frac{1}{n}\}$, separately $U_0 = \mathbb{N}$, thus, $U_{n+1} \subseteq U_n$ for all n. We claim that $\mathtt{inf}_{m \in \mathbb{N}} \varphi(U_m) > 0$. Otherwise a set $x \subseteq \mathbb{N}$ belongs to $\mathcal{Z}$ iff $x \setminus U_n$ is finite for any n. If the set $N = \{n : U_n \setminus U_{n+1} \text{ is infinite}\}$ is empty then easily $\mathcal{Z} = \mathcal{P}(\mathbb{N})$. If $N \ne \emptyset$ is finite then $\mathcal{Z}$ is either $\mathtt{Fin}$ (if eventually $U_n = \emptyset$) or a trivial variation of $\mathtt{Fin}$ (if U_n is non-empty for all n). If finally N is infinite then $\mathcal{Z}$ is isomorphic to $0 \times \mathtt{Fin}$. (For instance, if all sets $D_n = U_n \setminus U_{n+1}$ are infinite then $x \in \mathcal{Z}$ iff $x \cap D_n$ is finite for all n.) Thus we always have a contradiction to the assumptions of the lemma.

It follows that there is $\varepsilon > 0$ such that $\varphi(U_m) > \varepsilon$ for all m. As φ is l. s. c., we can define an increasing sequence of numbers $n_1 < n_2 < n_3 < \ldots$ and for any l a finite set $w_l \subseteq U_{n_l} \setminus U_{n_{l+1}}$ with $\varphi(w_l) > \varepsilon$. Then $W = \bigcup_l w_l \notin \mathcal{Z}$ and obviously $\{r_k\}_{k \in W} \to 0$. □

Since obviously $\mathsf{E}_{\mathcal{Z} \restriction W} \le_{\mathrm{B}} \mathsf{E}_{\mathcal{Z}}$, the following lemma is sufficient for Theorem 2:

Lemma 13. *If $\mathcal{Z}$, φ, r_k are as above, and $\{r_k\} \to 0$, then the shift action of $\mathcal{Z}$ on $\mathcal{P}(\mathbb{N})$ is gen. turbulent.*

Proof. $\mathcal{Z}$ is a Polish group (with the operation Δ) in the topology τ induced by the metric $r(x, y) = \varphi(x \,\Delta\, y)$. The action of $\mathcal{Z}$ by Δ on the space $\mathcal{P}(\mathbb{N})$ (considered in the product topology; $\mathcal{P}(\mathbb{N})$ is here identified with $2^{\mathbb{N}}$) by Δ is then continuous. It remains to verify the turbulence.

Let $x \in \mathcal{P}(\mathbb{N})$. The orbit $[x]_{\mathcal{Z}} = \mathcal{Z} \,\Delta\, x$ is easily dense and meager, hence, it suffices to prove that x is a turbulent point of the action. Consider an open set $X \subseteq \mathcal{P}(\mathbb{N})$ containing x, and a τ-hbhd G of $\emptyset$ (the neutral element of $\mathcal{Z}$); we may assume that, for some k, $X = \{y \in \mathcal{P}(\mathbb{N}) : y \cap [0, k) = u\}$, where $u = x \cap [0, k)$, and $G = \{g \in \mathcal{Z} : \varphi(g) < \varepsilon\}$ for some $\varepsilon > 0$. Prove that the local orbit $\mathcal{O}(x, X, G)$ is somewhere dense (*i.e.*, not a nowhere dense set) in X.

Let $l \ge k$ be big enough for $r_n < \varepsilon$ for all $n \ge l$. Put $v = x \cap [0, l)$ and prove that $\mathcal{O}(x, X, G)$ is dense in $Y = \{y \in \mathcal{P}(\mathbb{N}) : y \cap [0, l) = v\}$. Consider an open set $Z = \{z \in Y : z \cap [l, j) = w\}$, where $j \ge l$, $w \subseteq [l, j)$. Let z be the only element of Z with $z \cap [j, +\infty) = x \cap [j, +\infty)$, thus, $x \,\Delta\, z = \{l_1, \ldots, l_m\} \subseteq [l, j)$.

Each $g_i = \{l_i\}$ belongs to G by the choice of l (indeed, $l_i \geq l$). Moreover, easily $x_i = g_i \,\Delta\, g_{i-1} \,\Delta \dots \Delta\, g_1 \,\Delta\, x = \{l_1, \dots, l_i\} \,\Delta\, x$ belongs to X for any $i = 1, \dots, m$, and $x_m = z$, thus, $z \in \mathcal{O}(x, X, G)$, as required.

□ (*Lemma and Theorem 2*)

3.2 Irreducibility to actions of the group of all permutations of $\mathbb{N}$

Recall that S_∞ is the group of all permutations of $\mathbb{N}$, *i.e.*, 1–1 maps $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$, with the superposition as the group operation. A compatible Polish metric on S_∞ can be defined by $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, where d is the ordinary Polish metric of $\mathbb{N}^\mathbb{N}$, *i.e.*, $d(x,y) = 2^{-m-1}$, where m is the least number such that $x(m) \neq y(m)$.

Hjorth proved in mid-90s that turbulent ERs are not reducible to those induced by Polish actions of S_∞. The proof (as, *e.g.*, in [6, 13]) is quite complicated, in particular, containing references to some model theoretic facts and methods like Scott's analysis. We decided to include a simplified proof, based on the following theorem. This will be still a lengthy argument, because, to make the exposition friendly to a reader not experienced in special topics related to group actions and model theory, we outline proofs of some auxiliary results involved.

Theorem 14. *Any ER E, induced by a Polish action and reducible to an orbit ER of a Polish action of S_∞ by a BM map, is reducible to one of ERs T_γ by a BM map* [13] *— hence, by Theorem 1, such an ER E cannot be induced by a gen. turbulent Polish action.*

3.3 Classifiability by countable structures

Isomorphism relations of various classes of countable structures are amongst those induced by Polish actions of S_∞. Indeed, suppose that $\mathcal{L} = \{R_i\}_{i \in I}$ is a countable relational language, *i.e.*, $\mathtt{card}\, I \leq \aleph_0$ and each R_i is an m_i-ary relational symbol. Put [14] $\mathtt{Mod}_\mathcal{L} = \prod_{i \in I} \mathcal{P}(\mathbb{N}^{m_i})$, the space of *$\mathcal{L}$-structures* on $\mathbb{N}$ as the underlying set. *The logic action* $j_\mathcal{L}$ of S_∞ on $\mathtt{Mod}_\mathcal{L}$ is defined as follows: if $x = \{x_i\}_{i \in I} \in \mathtt{Mod}_\mathcal{L}$ and $g \in S_\infty$ then $y = j_\mathcal{L}(g, x) = g \cdot x = \{y_i\}_{i \in I} \in \mathtt{Mod}_\mathcal{L}$, where

$$\langle k_1, \dots, k_{m_i} \rangle \in x_i \iff \langle g(k_1), \dots, g(k_{m_i}) \rangle \in y_i$$

[13] We cannot claim Borel reducibility here, because, as all ERs T_γ are easily Borel, any ER Borel reducible to some T_γ is Borel itself, while on the other hand even ERs of the form $\cong_\mathcal{L}$ are, generally, non-Borel (but analytic).

[14] $X_\mathcal{L}$ is often used to denote $\mathtt{Mod}_\mathcal{L}$.

for all $i \in I$ and $\langle k_1, ..., k_{m_i} \rangle \in \mathbb{N}^{m_i}$. Then $\langle \mathtt{Mod}_{\mathcal{L}} ; j_{\mathcal{L}} \rangle$ is a Polish S_∞-space, while $j_{\mathcal{L}}$-orbits in $\mathtt{Mod}_{\mathcal{L}}$ are exactly the isomorphism classes of $\mathcal{L}$-structures, which is a reason to denote the associated equivalence relation $\mathsf{E}_{j_{\mathcal{L}}}^{\mathtt{Mod}_{\mathcal{L}}}$ by $\cong_{\mathcal{L}}$. All ERs of the form $\cong_{\mathcal{L}}$ are analytic, of course.

Hjorth [6, 2.38] defined an ER E to be *classifiable by countable structures* if there is a countable relational language $\mathcal{L}$ such that $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathcal{L}}$.

Theorem 15 (Becker and Kechris [1]**).** *Any ER induced by a Polish action of S_∞ is classifiable by countable structures.*

Thus all ERs induced by Polish actions of S_∞ (in fact also of any closed subgroup of S_∞) are Borel reducible to a very special kind of actions of S_∞.

Proof (by Hjorth [6, 6.19]**).** Let $\mathbb{X}$ be a Polish S_∞-space with basis $\{U_l\}_{l\in\mathbb{N}}$, and let $\mathcal{L}$ be the language with relations R_{lk} of arity k. If $x \in \mathbb{X}$ then define $\vartheta(x) \in \mathtt{Mod}_{\mathcal{L}}$ by stipulation that $\vartheta(x) \models R_{lk}(s_0, ..., s_{k-1})$ if and only if $s_i \ne s_j$ whenever $i < j < k$, and $g^{-1}\cdot x \in U_l$ whenever $g \in S_\infty$ satisfies $\langle s_0, ..., s_{k-1} \rangle \subset g$. Then ϑ reduces $\mathsf{E}_{S_\infty}^{\mathbb{X}}$ to $\cong_{\mathcal{L}}$. □

3.4 Reduction to countable graphs

It could be expected that more complicated languages $\mathcal{L}$ produce more complicated ER $\cong_{\mathcal{L}}$. However this is not the case: it turns out that a single binary relation can code structures of any countable language. Let $\mathcal{G}$ be the language of (oriented binary) graphs, *i.e.*, $\mathcal{G}$ contains a single binary predicate, say $R(\cdot,\cdot)$.

Theorem 16. *If $\mathcal{L}$ is a countable relational language then $\cong_{\mathcal{L}} \le_{\mathrm{B}} \cong_{\mathcal{G}}$.*

Becker and Kechris [1, 6.1.4] outline a proof based on coding in terms of lattices, unlike the following argument, yet it may in fact involve the same idea.

Proof. Let $\mathrm{HF}(\mathbb{N})$ be the set of all hereditarily finite sets over the set $\mathbb{N}$ considered as the set of atoms, and ε be the associated "membership" (no $n \in \mathbb{N}$ has ε-elements, $\{0,1\}$ is different from 2, *etc.*). Let $\simeq_{\mathrm{HF}(\mathbb{N})}$ be the $\mathrm{HF}(\mathbb{N})$ version of $\cong_{\mathcal{G}}$, *i.e.*, if $P, Q \subseteq \mathrm{HF}(\mathbb{N})^2$ then $P \simeq_{\mathrm{HF}(\mathbb{N})} Q$ means that there is a bijection b of $\mathrm{HF}(\mathbb{N})$ on itself such that $Q = b\cdot P = \{\langle b(s), b(t)\rangle : \langle s,t\rangle \in P\}$. Obviously $(\cong_{\mathcal{G}}) \sim_{\mathrm{B}} (\simeq_{\mathrm{HF}(\mathbb{N})})$, thus, we have to prove that $\cong_{\mathcal{L}} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $\mathcal{L}$.

An action $\circ$ of S_∞ on $\mathrm{HF}(\mathbb{N})$ is defined as follows: $g \circ n = g(n)$ for any $n \in \mathbb{N}$, and, by ε-induction, $f \circ \{a_1, ..., a_n\}) = \{f \circ a_1, ..., f \circ a_n\}$ for all $a_1, ..., a_n \in \mathrm{HF}(\mathbb{N})$. If $g \in S_\infty$ then $a \longmapsto g \circ a$ is a ε-isomorfism of $\mathrm{HF}(\mathbb{N})$.

Lemma 17. *Suppose that* $X, Y \subseteq \mathrm{HF}(\mathbb{N})$ *are* ε*-transitive subsets of* $\mathrm{HF}(\mathbb{N})$, *the sets* $\mathbb{N} \setminus X$ *and* $\mathbb{N} \setminus Y$ *are infinite, and* $\varepsilon \restriction X \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$. *Then there is a permutation* $f \in S_\infty$ *such that* $Y = f \circ X = \{f \circ s : s \in X\}$.

Proof. It follows from the assumption $\varepsilon \restriction X \cong_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$ that there is an ε-isomorphism $\pi : X \xrightarrow{\text{onto}} Y$. Easily $\pi \restriction (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto $Y_0 = Y \cap \mathbb{N}$, hence, there is $f \in S_\infty$ such that $f \restriction X_0 = \pi \restriction X_0$, and then we have $f \circ s = \pi(s)$ for any $s \in X$. □ (*Lemma*)

Coming back to the proof of Theorem 16, we first show that $\cong_{\mathcal{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $m \ge 3$, where $\mathcal{G}(m)$ is the language with a single m-ary predicate. We observe that $\langle i_1, ..., i_m \rangle \in \mathrm{HF}(\mathbb{N})$ whenever $i_1, ..., i_m \in \mathbb{N}$. Put $\Theta(x) = \{\vartheta(s) : s \in x\}$ for every $x \in \mathtt{Mod}_{\mathcal{G}(m)} = \mathcal{P}(\mathbb{N}^m)$, where $\vartheta(s) = \mathrm{TC}_\varepsilon(\{\langle 2i_1, ..., 2i_m \rangle\})$ for each $s = \langle i_1, ..., i_m \rangle \in \mathbb{N}^m$, and finally, for $X \subseteq \mathrm{HF}(\mathbb{N})$, $\mathrm{TC}_\varepsilon(X)$ is the least ε-transitive set $T \subseteq \mathrm{HF}(\mathbb{N})$ with $X \subseteq T$. It easily follows from Lemma 16 that $x \cong_{\mathcal{G}(m)} y$ is equivalent to $\varepsilon \restriction \Theta(x) \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction \Theta(y)$, which ends the proof of $\cong_{\mathcal{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathcal{L}'} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$, where $\mathcal{L}'$ is the language with infinitely many binary predicates. In this case $\mathtt{Mod}_{\mathcal{L}'} = \mathcal{P}(\mathbb{N}^2)^{\mathbb{N}}$, so that we can assume that every $x \in \mathtt{Mod}_{\mathcal{L}'}$ has the form $x = \{x_n\}_{n \ge 1}$, with $x_n \subseteq (\mathbb{N} \setminus \{0\})^2$ for all n. Let $\Theta(x) = \{s_n(k, l) : n \ge 1 \wedge \langle k, l \rangle \in x_n\}$ for any such x, where

$$s_n(k, l) = \mathrm{TC}_\varepsilon(\{\{...\{\langle k, l \rangle\}...\}\,, 0\})\,, \text{ with } n+2 \text{ pairs of brackets } \{\,,\,\}\,.$$

Then Θ is a continuous reduction of $\cong_{\mathcal{L}'}$ to $\simeq_{\mathrm{HF}(\mathbb{N})}$. □ (*Theorem*)

3.5 Proof of Theorem 14

The proof (a version of the proof in [3]) is based on Scott's analysis.

Define a family of Borel binary relations $\equiv^\alpha_{st}$ on $\mathcal{P}(\mathbb{N}^2)$, where $\alpha < \omega_1$ and $s, t \in \mathbb{N}^{<\omega}$, as follows:

- $A \equiv^0_{st} B$ iff $A(s_i, s_j) \Longleftrightarrow B(t_i, t_j)$ for all $i, j < \mathtt{lh}\, s = \mathtt{lh}\, t$;
- $A \equiv^{\alpha+1}_{st} B$ iff $\forall\, k\, \exists\, l\, (A \equiv^\alpha_{s^\wedge k\,,\, t^\wedge l} B)$ and $\forall\, l\, \exists\, k\, (A \equiv^\alpha_{s^\wedge k\,,\, t^\wedge l} B)$;
- if $\lambda < \omega_1$ is limit then: $A \equiv^\lambda_{st} B$ iff $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$.

We define $\langle s, A \rangle \equiv^\alpha \langle t, B \rangle$ iff $A \equiv^\alpha_{st} B$; then, by induction on α, each $\equiv^\alpha$ is easily a Borel ER on $\mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2)$, and $\equiv^\beta \subseteq \equiv^\alpha$ whenever $\alpha < \beta$.

Let $\mathsf{E} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ be a ER induced by a Polish action of a Polish group $\mathbb{G}$ on a Polish space $\mathbb{X}$. Suppose that E is reducible to a Polish action of S_∞ by a BM map. According to Theorems 15, 16, and Proposition 3, there is a BM reduction

$\vartheta : \mathbb{X} \to \mathcal{P}(\mathbb{N}^2)$ of E to $\cong_{\mathcal{G}}$. The reduction is continuous on a dense $\mathbf{G}_\delta$ set $D_0 \subseteq \mathbb{X}$. Recall that, for $A, B \subseteq \mathbb{N}^2$, $A \cong_{\mathcal{G}} B$ means that there is $f \in S_\infty$ with $A(k,l) \iff B(f(k), f(l))$ for all k, l. We easily prove $\cong_{\mathcal{G}} \subseteq \equiv^\alpha_{st}$, where $t = f \circ s$, by induction on α, in particular, $\cong_{\mathcal{G}} \subseteq \equiv^\alpha_{\Lambda\Lambda}$, where Λ is the empty sequence. Since ϑ is a reduction, the equivalence $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \cong_{\mathcal{G}} \vartheta(y)$ holds for all x, y. Our goal is to find a $\mathbf{G}_\delta$ dense set $D \subseteq D_0$ and an ordinal $\alpha < \omega_1$ such that

($*$) the implication $x \not\mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ holds for all $x, y \in D$.

To find D fix a countable transitive model $\mathfrak{M}$ of **ZFHC** (see above). We assume that $\mathbb{X}$, the group $\mathbb{G}$, the action, D_0, $\vartheta \restriction D_0$ are assumed to be coded in $\mathfrak{M}$ in the same sense as in the proof of Lemma 9. We assert that the set D of all Cohen generic, over $\mathfrak{M}$, points of $\mathbb{X}$ (a dense $\mathbf{G}_\delta$ subset of $\mathbb{X}$ included in D_0) satisfies ($*$).

Suppose that $x, y \in D$. First consider the case when $\langle x, y\rangle$ is a Cohen generic pair over $\mathfrak{M}$. If $x \not\mathrel{\mathsf{E}} y$ then, by the choice of ϑ, we have $\vartheta(x) \not\cong_{\mathcal{G}} \vartheta(y)$, hence, this fact holds in $\mathfrak{M}[x, y]$ by the Mostowski absoluteness. Therefore, arguing in $\mathfrak{M}[x, y]$ (which is still a model of **ZFHC**, see Footnote 12), we find an ordinal $\alpha \in \mathtt{Ord}^{\mathfrak{M}} = \mathtt{Ord}^{\mathfrak{M}[x,y]}$ with $\vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$. Moreover, since the Cohen forcing satisfies CCC, there is an ordinal $\alpha \in \mathfrak{M}$ such that $\vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ for <u>all</u> Cohen generic, over $\mathfrak{M}$, pairs $\langle x, y\rangle \in D^2$ with $x \not\mathrel{\mathsf{E}} y$. It remains to show that this also holds when $x, y \in D$ with $x \not\mathrel{\mathsf{E}} y$ do not form a Cohen generic pair.

Let $g \in \mathbb{G}$ be Cohen generic over $\mathfrak{M}[x, y]$. Then $z = g \cdot x \in \mathbb{X}$ is easily Cohen generic over $\mathfrak{M}[x, y]$ (because the action is continuous), furthermore, $x \mathrel{\mathsf{E}} z$, hence, $y \not\mathrel{\mathsf{E}} z$. However y is generic over $\mathfrak{M}$ and z is generic over $\mathfrak{M}[y]$, thus, $\langle y, z\rangle$ is Cohen generic over $\mathfrak{M}$, hence, we have $\vartheta(z) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ by the above. On the other hand, $\vartheta(x) \equiv^\alpha_{\Lambda\Lambda} \vartheta(z)$ holds because $x \mathrel{\mathsf{E}} z$, thus, we finally obtain $\vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$, as required by ($*$).

To conclude, we have $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ for all $x, y \in D$. In this case we can easily redefine ϑ on the complement of D in $\mathbb{X}$ so that the equivalence holds for all $x, y \in \mathbb{X}$, in other words, the improved ϑ is a BM (because $\vartheta \restriction D$ is continuous and D is a dense $\mathbf{G}_\delta$) reduction of E to $\equiv^\alpha_{\Lambda\Lambda}$.

The following result completes the proof of the theorem.

Proposition 18. *Any ER $\equiv^\alpha$ is Borel reducible to some T_γ .*

Proof. We have $\equiv^0 \le_{\mathrm{B}} \mathsf{T}_0$ since $\equiv^0$ has countably many equivalence classes, all of which are open-and-closed sets. To carry out the step $\alpha \mapsto \alpha+1$ note that the map $\langle s, A\rangle \mapsto \{\langle s^\wedge k, A\rangle\}_{k\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^\alpha)^\infty$. As for the limit step, let $\lambda = \{\alpha_n : n \in \mathbb{N}\}$ be a limit ordinal, and $\mathsf{R} = \bigvee_{n\in\mathbb{N}} \equiv^{\alpha_n}$, *i.e.*, R is a ER on $\mathbb{N} \times \mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2)$ defined so that $\langle m, s, A\rangle \mathrel{\mathsf{R}} \langle n, t, B\rangle$ iff $m = n$

and $A \equiv_{st}^{\alpha_m} B$. However the map $\langle s, A\rangle \mapsto \{\langle m, s, A\rangle\}_{m\in\mathbb{N}}$ is a Borel reduction of $\equiv^\lambda$ to R^∞. $\square$

$\square$ (*Theorem 14*)

4 Pinned ERs and irreducibility of T_2

This section contains a theorem saying that the ER T_2 of equality of countable sets of the reals is not Borel reducible to ERs which belong to a family of *pinned* ERs, including, for instance, continuous actions of CLI groups, some ideals, not only Polishable, together with ERs having $\mathbf{G}_{\delta\sigma}$ equivalence classes, and is closed under the Fubini product modulo $\mathtt{Fin}$. The definition of the family is based on a rather metamathematical property which we extracted from Hjorth [5].

4.1 Pinned ERs

First of all, if X is an analytic set in the universe $\mathbb{V}$ of all sets (in particular, this applies when X is Borel), and $\mathbb{V}^+$ is a generic extension $\mathbb{V}$, then $X^\#$ will denote the result of the sequence of operations contained in the definition of X but applied in $\mathbb{V}^+$. The correctness of this definition follows from the Shoenfield absoluteness theorem, and easily $X = X^\# \cap \mathbb{V}$.

For instance, if, in $\mathbb{V}$, E is an analytic ER on a polish space $\mathbb{X}$, then, still by the Shoenfield absoluteness, $\mathsf{E}^\#$ is an analytic ER on $\mathbb{X}^\#$. If now $x \in \mathbb{X}$ (hence, $x \in \mathbb{V}$) then the E-class $[x]_{\mathsf{E}} \subseteq \mathbb{X}$ of x (defined in $\mathbb{V}$) is included in a unique $\mathsf{E}^\#$-class $[x]_{\mathsf{E}^\#} \subseteq \mathbb{X}^\#$ (in $\mathbb{V}^+$). Classes of the form $[x]_{\mathsf{E}^\#}$, $x \in \mathbb{X}$, belong to a wider category of $\mathsf{E}^\#$-classes which admit a description from the $\mathbb{V}$-th point of view.

Definition 19. Assume that $\mathbb{P}$ is a notion of forcing in $\mathbb{V}$. A *virtual* E*-class* is any $\mathbb{P}$-term $\boldsymbol{\xi}$ such that $\mathbb{P}$ forces $\boldsymbol{\xi} \in \mathbb{X}^\#$ and $\mathbb{P}\times\mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \,\mathsf{E}^\#\, \boldsymbol{\xi}_{\mathtt{right}}$. [15] A virtual class is *pinned* if there is, in $\mathbb{V}$, a point $x \in \mathbb{X}$ which pins it in the sense that $\mathbb{P}$ forces $x \,\mathsf{E}^\#\, \boldsymbol{\xi}$. Finally, an analytic ER E is *pinned* if, for any forcing notion $\mathbb{P} \in \mathbb{V}$, all virtual E-classes are pinned. $\square$

If $\boldsymbol{\xi}$ is a virtual E-class then, in any extension $\mathbb{V}^+$ of $\mathbb{V}$, if U and V are generic subsets of $\mathbb{P}$ then $x = \boldsymbol{\xi}[U]$ and $y = \boldsymbol{\xi}[V]$ belong to $\mathbb{X}^\#$ and satisfy $x \,\mathsf{E}^\#\, y$, hence, $\boldsymbol{\xi}$ induces a $\mathsf{E}^\#$-class in the extension. If $\boldsymbol{\xi}$ is pinned then this

[15] $\boldsymbol{\xi}_{\mathtt{left}}$ and $\boldsymbol{\xi}_{\mathtt{right}}$ are $\mathbb{P}\times\mathbb{P}$-terms meaning $\boldsymbol{\xi}$ associated with the resp. left and right factors $\mathbb{P}$ in the product forcing, formally, $\boldsymbol{\xi}_{\mathtt{left}}[U\times V] = \boldsymbol{\xi}[U]$ and $\boldsymbol{\xi}_{\mathtt{right}}[U\times V] = \boldsymbol{\xi}[V]$ for any $\mathbb{P}\times\mathbb{P}$-generic set $U\times V$, where, say, $\boldsymbol{\xi}[U]$ is the interpretation of a term $\boldsymbol{\xi}$ via a generic set U.

class contains an element in the ground universe $\mathbb{V}$ — in other words, pinned virtual classes induce $\mathsf{E}^{\#}$-equivalence classes of the form $[x]_{\mathsf{E}^{\#}}$, $x \in \mathbb{V}$ in the extensions of the universe $\mathbb{V}$.

We prove below that T_2 is not pinned, moreover, T_2 is not Borel reducible to any pinned analytic ER. In addition, we give a simplified proof of Hjorth's theorem that continuous actions of Polish CLI groups never induce pinned orbit ERs, introduce a family of pinned ERs associated with $\mathbf{F}_{\sigma\delta}$ ideals, show that any Borel ER whose all equivalence classes are $\mathbf{G}_{\delta\sigma}$ is pinned, and prove that the class of all pinned analytic ERs is closed under the Fubini product over $\mathtt{Fin}$.

4.2 Pinned ERs do not reduce T_2

Recall that, modulo $\sim_{\mathrm{B}}$, T_2 is a ER on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $x \mathrel{\mathsf{T}_2} y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$.

Lemma 20. T_2 *is not pinned. If* E, F *are analytic ERs,* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$, *and* F *is pinned, then so is* E. *Hence,* T_2 *is not Borel reducible to a pinned analytic ER.*

Proof. To prove that T_2 is not pinned, consider $\mathbb{P} = \textsc{Coll}(\mathbb{N}, 2^{\mathbb{N}})$, a forcing to produce a generic map $f : \mathbb{N} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. ($\mathbb{P}$ consists of all functions $p : u \to 2^{\mathbb{N}}$ where $u \subseteq \mathbb{N}$ is finite.) Let $\boldsymbol{\xi}$ be a $\mathbb{P}$-term for the set $\mathtt{ran}\, f = \{f(n) : n \in \mathbb{N}\}$. Then $\boldsymbol{\xi}$ is obviously a virtual T_2-class, but it is not pinned because $\mathbb{N}^{\mathbb{N}}$ is uncountable in the ground universe $\mathbb{V}$.

Suppose that, in $\mathbb{V}$, $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel reduction of E to F, where $\mathbb{X} = \mathtt{dom}\, \mathsf{E}$ and $\mathbb{Y} = \mathtt{dom}\, \mathsf{F}$. We can assume that $\mathbb{X}$ and $\mathbb{Y}$ are just two copies of $2^{\mathbb{N}}$. Let $\mathbb{P}$ be a forcing notion and a $\mathbb{P}$-term $\boldsymbol{\xi}$ be a virtual E-class. By the Shoenfield absoluteness, $\vartheta^{\#}$ is a reduction of $\mathsf{E}^{\#}$ to $\mathsf{F}^{\#}$ in any extension of $\mathbb{V}$, hence, $\boldsymbol{\sigma}$, a $\mathbb{P}$-name for $\vartheta^{\#}(\boldsymbol{\xi})$, is also a virtual F-class. Since F is pinned, there is $y \in \mathbb{Y}$ such that $\mathbb{P}$ forces $y \mathrel{\mathsf{F}^{\#}} \boldsymbol{\sigma}$. Note that it is true in the $\mathbb{P}$-extension that $y \mathrel{\mathsf{F}^{\#}} \vartheta^{\#}(x)$ for some $x \in \mathbb{X}^{\#}$, hence, by Shoenfield, in the ground universe there is $x \in \mathbb{X}$ with $y \mathrel{\mathsf{F}} \vartheta(x)$. Clearly $\mathbb{P}$ forces $x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$. □

4.3 Fubini product of pinned ERs is pinned

Recall that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ of ERs E_k on $\mathbb{X}_k$ modulo $\mathtt{Fin}$ is a ER on $\mathbb{X} = \prod_k \mathbb{X}_k$ defined as follows: $x \mathrel{\mathsf{E}} y$ if $x(k) \mathrel{\mathsf{E}_k} y(k)$ for all but finite k.

Lemma 21. *The family of all analytic pinned ERs is closed under Fubini products modulo* $\mathtt{Fin}$.

Proof. Suppose that analytic ERs E_k on Polish spaces $\mathbb{X}_k$ are pinned; prove that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ is a pinned ER on $\mathbb{X} = \prod_k \mathbb{X}_k$. Consider a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-term $\boldsymbol{\xi}$ which is a virtual E-class. There is a number k_0 and conditions $p,\, q \in \mathbb{P}$ such that $\langle p, q\rangle$ $\mathbb{P}\times\mathbb{P}$-forces $\boldsymbol{\xi}_{\mathtt{left}}(k)\, {\mathsf{E}_k}^{\#}\, \boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. As all E_k are ERs, we conclude that the condition $\langle p, p\rangle$ also forces $\boldsymbol{\xi}_{\mathtt{left}}(k)\, {\mathsf{E}_k}^{\#}\, \boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. Therefore, since E_k are pinned, there exists, in $\mathbb{V}$, a sequence of points $x_k \in \mathbb{X}_k$ such that p $\mathbb{P}$-forces $x_k\, {\mathsf{E}_k}^{\#}\, \boldsymbol{\xi}(k)$ for any $k \ge k_0$. Let $x \in \mathbb{X}$ satisfy $x(k) = x_k$ for all $k \ge k_0$. (The values $x(k) \in \mathbb{X}_k$ for $k < k_0$ can be arbitrary.) Then p obviously $\mathbb{P}$-forces $x\, \mathsf{E}^{\#}\, \boldsymbol{\xi}$.

It remains to show that just every $q \in \mathbb{P}$ also forces $x\, \mathsf{E}^{\#}\, \boldsymbol{\xi}$. Suppose otherwise, *i.e.*, some $q \in \mathbb{P}$ forces that $x\, \mathsf{E}^{\#}\, \boldsymbol{\xi}$ fails. Consider the pair $\langle p, q\rangle$ as a condition in $\mathbb{P}\times\mathbb{P}$: it forces $x\, \mathsf{E}^{\#}\, \boldsymbol{\xi}_{\mathtt{left}}$ and $\neg\ x\, \mathsf{E}^{\#}\, \boldsymbol{\xi}_{\mathtt{right}}$, as well as $\boldsymbol{\xi}_{\mathtt{left}}\, \mathsf{E}^{\#}\, \boldsymbol{\xi}_{\mathtt{right}}$ by the choice of E and $\boldsymbol{\xi}$, which is a contradiction. $\square$

4.4 Left-invariant actions induce pinned ERs

Recall that a Polish group $\mathbb{G}$ is *complete left-invariant*, CLI for brevity, if $\mathbb{G}$ admits a compatible left-invariant complete metric. Then easily $\mathbb{G}$ also admits a compatible right-invariant complete metric, which will be practically used.

Theorem 22 (Hjorth [5])**.** *Any ER $\mathsf{E} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ induced by a Polish action of a CLI group $\mathbb{G}$ on a Polish space $\mathbb{X}$ is pinned, hence, T_2 is not Borel reducible to E.*

Proof. Let $\mathbb{P}$ be a forcing notion and $\boldsymbol{\xi}$ be a virtual E-class. Let $\le$ denote the partial order of $\mathbb{P}$; we assume that $p \le q$ means that p is a stronger condition. Let us fix a compatible complete right-invariant metric ρ on $\mathbb{G}$. For any $\varepsilon > 0$, put $G_\varepsilon = \{g \in \mathbb{G} : \rho(g, 1_{\mathbb{G}}) < \varepsilon\}$. Say that $q \in \mathbb{P}$ *is of size* $\le \varepsilon$ if $\langle q, q\rangle$ $\mathbb{P}\times\mathbb{P}$-forces the existence of $g \in {G_\varepsilon}^{\#}$ such that $\boldsymbol{\xi}_{\mathtt{left}} = g\cdot\boldsymbol{\xi}_{\mathtt{right}}$.

Lemma 23. *If $q \in \mathbb{P}$ and $\varepsilon > 0$, then there exists a condition $r \in \mathbb{P}$, $r \le q$, of size $\le \varepsilon$.*

Proof. Otherwise for any $r \in \mathbb{P}$, $r \le q$, there is a pair of conditions $r',\, r'' \in \mathbb{P}$ stronger than r and such that $\langle r', r''\rangle$ $\mathbb{P}\times\mathbb{P}$-forces that there is no $g \in {G_\varepsilon}^{\#}$ with $\boldsymbol{\xi}_{\mathtt{left}} = g\cdot\boldsymbol{\xi}_{\mathtt{right}}$. Applying an ordinary splitting construction in such a generic extension $\mathbb{V}^+$ of $\mathbb{V}$ where $\mathcal{P}(\mathbb{P})\cap\mathbb{V}$ is countable, we find an uncountable set $\mathcal{U}$ of generic sets $U \subseteq \mathbb{P}$ with $q \in U$ such that any pair $\langle U, V\rangle$ of $U \ne V$ in $\mathcal{U}$ is $\mathbb{P}\times\mathbb{P}$-generic (over $\mathbb{V}$), hence, there is no $g \in {G_\varepsilon}^{\#}$ with $\boldsymbol{\xi}[U] = g\cdot\boldsymbol{\xi}[V]$. [16]

[16] $\boldsymbol{\xi}[U]$ is the interpretation of the $\mathbb{P}$-term $\boldsymbol{\xi}$ obtained by taking U as the generic set.

Fix $U_0 \in \mathcal{U}$. We can associate, in $\mathbb{V}^+$, with each $U \in \mathcal{U}$, an element $g_U \in G^\#$ such that $\boldsymbol{\xi}[U] = g_U \cdot \boldsymbol{\xi}[U_0]$; then $g_U \notin {G_\varepsilon}^\#$ by the above. Moreover, we have $g_V g_U^{-1} \cdot \boldsymbol{\xi}[U] = \boldsymbol{\xi}[V]$ for all $U, V \in \mathcal{U}$, hence $g_V g_U^{-1} \notin {G_\varepsilon}^\#$ whenever $U \neq V$, which implies $\rho(g_U, g_V) \geq \varepsilon$ by the right invariance. But this contradicts the separability of G. □ (*Lemma*)

Coming back to the theorem, suppose on the contrary that a condition $p \in \mathbb{P}$ forces that there is no $x \in \mathbb{X}$ (in the ground universe $\mathbb{V}$) satisfying $x \mathsf{E}^\# \boldsymbol{\xi}$. According to the lemma, there is, in $\mathbb{V}$, a sequence of conditions $p_n \in \mathbb{P}$ of size $\leq 2^{-n}$, and closed sets $X_n \subseteq \mathbb{X}$ with $\mathbb{X}$-diameter $\leq 2^{-n}$, such that $p_0 \leq p$, $p_{n+1} \leq p_n$, $X_{n+1} \subseteq X_n$, and p_n forces $\boldsymbol{\xi} \in {X_n}^\#$ for any n. Let x be the common point of the sets X_n in $\mathbb{V}$. We claim that p_0 forces $x \mathsf{E}^\# \boldsymbol{\xi}$.

Indeed, otherwise there is $q \in \mathbb{P}$, $q \leq p_0$, which forces $\neg\, x \mathsf{E}^\# \boldsymbol{\xi}$. Consider an extension $\mathbb{V}^+$ of $\mathbb{V}$ rich enough to contain, for any n, a generic set $U_n \subseteq \mathbb{P}$ with $p_n \in U_n$ such that each pair $\langle U_n, U_{n+1} \rangle$ is $\mathbb{P} \times \mathbb{P}$-generic (over $\mathbb{V}$), and, in addition, $q \in U_0$. Let $x_n = \boldsymbol{\xi}[U_n]$ (an element of $\mathbb{X}^\#$). Then $\{x_n\}_{\to} x$. Moreover, for any n, both U_n and U_{n+1} contain p_n, hence, as p_n has size $\leq 2^{-n-1}$, there is $g_{n+1} \in {\mathbb{G}_\varepsilon}^\#$ with $x_{n+1} = g_{n+1} \cdot x_n$. Thus, $x_n = h_n \cdot x_0$, where $h_n = g_n ... g_1$. Note that $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \leq 2^{-n+1}$ by the right-invariance of the metric, thus, $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^\#$. Let $h = \mathtt{lim}_{n \to \infty} h_n \in \mathbb{G}^\#$ be its limit. As the action is continuous, we have $x = \mathtt{lim}_n x_n = h \cdot x_0$. It follows that $x \mathsf{E}^\# x_0$ holds in $\mathbb{V}^+$, hence, also in $\mathbb{V}[U_0]$. However $x_0 = \boldsymbol{\xi}[U_0]$ while $q \in U_0$ forces $\neg\, x \mathsf{E}^\# \boldsymbol{\xi}$, which is a contradiction.

Thus p_0 $\mathbb{P}$-forces $x \mathsf{E}^\# \boldsymbol{\xi}$. Then any $r \in \mathbb{P}$ also forces $x \mathsf{E}^\# \boldsymbol{\xi}$: indeed, if some $r \in \mathbb{P}$ forces $\neg\, x \mathsf{E}^\# \boldsymbol{\xi}$ then the pair $\langle p_0, r \rangle$ forces, in $\mathbb{P} \times \mathbb{P}$, that $x \mathsf{E}^\# \boldsymbol{\xi}_{\mathtt{left}}$ and $\neg\, x \mathsf{E}^\# \boldsymbol{\xi}_{\mathtt{right}}$, which contradicts the fact that $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \mathsf{E}^\# \boldsymbol{\xi}_{\mathtt{right}}$.

□ (*Theorem 22*)

4.5 All ERs with $\mathbf{G}_{\delta\sigma}$ classes are pinned

We have a non-pinned ER T_2, obviously of class $\mathbf{F}_{\sigma\delta}$; the following theorem shows that this is the simplest possible case of non-pinned ERs.

Theorem 24. *Any Borel ER* E *whose all equivalence classes are* $\mathbf{G}_{\delta\sigma}$ *is pinned.*

Proof (Based on an idea communicated by Hjorth). We can assume that $\mathtt{dom}\,\mathsf{E} = \mathbb{N}^\mathbb{N}$. It follows from a theorem of Louveau [14], that there is a Borel map γ, defined on $\mathbb{N}^\mathbb{N}$ so that $\gamma(x)$ is a $\mathbf{G}_{\delta\sigma}$-code of $[x]_\mathsf{E}$ for any $x \in \mathbb{N}^\mathbb{N}$, that is, for instance, $\gamma(x) \subseteq \mathbb{N}^2 \times \mathbb{N}^{<\omega}$ and $[x]_\mathsf{E} = \bigcup_i \bigcap_j \bigcup_{\langle i,j,s \rangle \in \gamma(x)} B_s$, where $B_s = \{a \in \mathbb{N}^\mathbb{N} : s \subset a\}$ for all $s \in \mathbb{N}^{<\omega}$.

Let $\mathbb{P} = \langle \mathbb{P}\,; \le \rangle$ be a forcing notion, and $\boldsymbol{\xi}$ be a virtual E-class, thus, $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\texttt{left}} \, \mathsf{E}^{\#} \boldsymbol{\xi}_{\texttt{right}}$, hence, there is a number i_0 and a condition $\langle p_0, q_0 \rangle \in \mathbb{P} \times \mathbb{P}$ which forces $\boldsymbol{\xi}_{\texttt{left}} \in \vartheta^{\#}(\boldsymbol{\xi}_{\texttt{right}})$, where $\vartheta(x) = \bigcap_j \bigcup_{\langle i_0, j, s \rangle \in \gamma(x)} B_s$ for all $x \in \mathbb{N}^{\mathbb{N}}$.

The key idea of the proof is to substitute $\mathbb{P}$ by the Cohen forcing. Let $\mathbb{S}$ denote the set of all $s \in \mathbb{N}^{<\omega}$ such that p_0 does <u>not</u> $\mathbb{P}$-force that $s \not\subset \boldsymbol{\xi}$. We consider $\mathbb{S}$ as a forcing, and $s \subseteq t$ (*i.e.*, t is an extension of s) means that t is a stronger condition; Λ, the empty sequence, is the weakest condition in $\mathbb{S}$. If $s \in \mathbb{S}$ then obviously there is at least one n such that $s^\wedge n \in \mathbb{S}$, hence, $\mathbb{S}$ forces an element of $\mathbb{N}^{\mathbb{N}}$, whose $\mathbb{S}$-name will be $\mathbf{a}$.

Lemma 25. *The pair* $\langle \Lambda, q_0 \rangle$ $\mathbb{S} \times \mathbb{P}$*-forces* $\mathbf{a} \in \vartheta^{\#}(\boldsymbol{\xi})$.

Proof. Otherwise some condition $\langle s_0, q \rangle \in \mathbb{S} \times \mathbb{P}$ with $q \le q_0$ forces $\mathbf{a} \notin \vartheta^{\#}(\boldsymbol{\xi})$. By the definition of ϑ we can assume that there is j_0 satisfying

$$\langle s_0, q \rangle \quad \mathbb{S} \times \mathbb{P}\text{-forces} \quad \neg\, \exists\, s \left(\langle i_0, j_0, s \rangle \in \gamma(\boldsymbol{\xi}) \wedge s \subset \mathbf{a} \right). \tag{$*$}$$

Since $s_0 \in \mathbb{S}$, there is a condition $p' \in \mathbb{P}$, $p' \le p_0$, which $\mathbb{P}$-forces $s_0 \subset \boldsymbol{\xi}$. By the choice of $\langle p_0, q_0 \rangle$ we can assume that, for some $s \in \mathbb{S}$ and $q' \in \mathbb{P}$, $q' \le q$,

$$\langle p', q' \rangle \quad \mathbb{P} \times \mathbb{P}\text{-forces} \quad \langle i_0, j_0, s \rangle \in \gamma(\boldsymbol{\xi}_{\texttt{right}}) \wedge s \subset \boldsymbol{\xi}_{\texttt{left}}\,.$$

This means that 1) p' $\mathbb{P}$-forces $s \subset \boldsymbol{\xi}$ and 2) q' $\mathbb{P}$-forces $\langle i_0, j_0, s \rangle \in \gamma(\boldsymbol{\xi})$. In particular, by the above, p' forces both $s_0 \subset \boldsymbol{\xi}$ and $s \subset \boldsymbol{\xi}$, therefore, either $s \subseteq s_0$ – then let $s' = s_0$, or $s_0 \subset s$ – then let $s' = s$. In both cases, $\langle s', q' \rangle$ $\mathbb{S} \times \mathbb{P}$-forces $\langle i_0, j_0, s \rangle \in \gamma(\boldsymbol{\xi})$ and $s \subset \mathbf{a}$, contradiction to $(*)$. $\square$ (*Lemma*)

Note that $\mathbb{S}$ is a subforcing of the Cohen forcing $\mathbb{C} = \mathbb{N}^{<\omega}$, therefore, by the lemma, there is a $\mathbb{C}$-term $\boldsymbol{\sigma}$ such that $\langle \Lambda, q_0 \rangle$ $\mathbb{C} \times \mathbb{P}$-forces $\boldsymbol{\sigma} \in \vartheta^{\#}(\boldsymbol{\xi})$, hence, forces $\boldsymbol{\sigma} \, \mathsf{E}^{\#} \boldsymbol{\xi}$. It follows, by consideration of the forcing $\mathbb{C} \times \mathbb{P} \times \mathbb{P}$, that generally $\mathbb{C} \times \mathbb{P}$ forces $\boldsymbol{\sigma} \, \mathsf{E}^{\#} \boldsymbol{\xi}$. Therefore, by ordinary arguments, first, $\mathbb{C} \times \mathbb{C}$ forces $\boldsymbol{\sigma}_{\texttt{left}} \, \mathsf{E}^{\#} \boldsymbol{\sigma}_{\texttt{right}}$, and second, to prove the theorem it suffices now to find $x \in \mathbb{N}^{\mathbb{N}}$ in $\mathbb{V}$ such that $\mathbb{C}$ forces $x \, \mathsf{E}^{\#} \boldsymbol{\sigma}$. This is our next goal.

Let $\mathbf{a}$ be the $\mathbb{C}$-name of the Cohen generic element of $\mathbb{N}^{\mathbb{N}}$. The term $\boldsymbol{\sigma}$ can be of arbitrary nature, but we can substitute it by a term of the form $f^{\#}(\mathbf{a})$, where $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel map in the ground universe $\mathbb{V}$. It follows from the above that $f^{\#}(\mathbf{a}) \, \mathsf{E}^{\#} f^{\#}(\mathbf{b})$ for any $\mathbb{C} \times \mathbb{C}$-generic, over $\mathbb{V}$, pair $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. We conclude that $f^{\#}(\mathbf{a}) \, \mathsf{E}^{\#} f^{\#}(\mathbf{b})$ also holds even for any pair of separately Cohen generic $\mathbf{a}$, $\mathbf{b} \in \mathbb{N}^{\mathbb{N}}$. Thus, in a generic extension of $\mathbb{V}$, where there exist comeager-many Cohen generic reals, there is a comeager $\mathbf{G}_\delta$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that $f^{\#}(a) \, \mathsf{E}^{\#} f^{\#}(b)$ for all a, $b \in X$. By the Shoelfield absoluteness, the statement of existence of such a set X is true also in $\mathbb{V}$, hence, in $\mathbb{V}$, there is $x \in \mathbb{N}^{\mathbb{N}}$ such that we have $x \mathsf{E} f(a)$ for comeager-many $a \in \mathbb{N}^{\mathbb{N}}$. This is again a Shoenfield absolute property of x, hence, $\mathbb{C}$ forces $x \mathsf{E}^{\#} f^{\#}(\mathbf{a})$, as required.

$\square$ (*Theorem 24*)

4.6 A family of pinned ideals

Let us say that a Borel ideal $\mathcal{I}$ is *pinned* if so is the induced ER $\mathsf{E}_{\mathcal{I}}$. It follows from Theorem 22 that any P-ideal is pinned because Borel P-ideals are polishable [15] while all Polish Abelian groups are CLI. Yet there exist non-P pinned ideals.

We introduce here a family of such ideals. Suppose that $\{\varphi_i\}_{i\in\mathbb{N}}$ is a sequence of lower semicontinuous (l. s. c.) submeasures on $\mathbb{N}$. Define

$$\mathtt{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}\,, \quad \text{where} \quad \varphi_\infty(X) = \limsup_{i\to\infty} \varphi_i(X)\,.$$

the exhaustive ideal of the sequence of submeasures. By Solecki's Theorem [15] for any Borel P-ideal $\mathcal{I}$ there is an l. s. c. submeasure φ such that $\mathcal{I} = \mathtt{Exh}_{\{\varphi_i\}} = \mathtt{Exh}_{\varphi}$, where $\varphi_i(x) = \varphi(x \cap [i,\infty))$, however, for example, the non-polishable ideal $\mathcal{I}_1 = \mathtt{Fin} \times 0$ also is of the form $\mathtt{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define $\varphi_i(x) = 0$ or 1 if resp. $x \subseteq$ or $\not\subseteq \{0,...,n-1\} \times \mathbb{N}$.

Theorem 26. *All ideals of the form* $\mathtt{Exh}_{\{\varphi_i\}}$ *are pinned.*

Proof. Let $\mathcal{I} = \mathtt{Exh}_{\{\varphi_i\}}$, where all φ_i are l. s. c. submeasures on $\mathbb{N}$. We can assume that the submeasures φ_i decrease, *i. e.*, $\varphi_{i+1}(x) \le \varphi_i(x)$ for any x, for if not consider the l. s. c. submeasures $\varphi'_i(x) = \sup_{j\ge i} \varphi_j(x)$.

Suppose that $\mathsf{E} = \mathsf{E}_{\mathcal{I}}$ is not pinned. Then there is a forsing notion $\mathbb{P}$, a virtual E-klass $\boldsymbol{\xi}$, and a condition $p \in \mathbb{P}$ which $\mathbb{P}$-forces $\neg\, x \mathsf{E}^{\#} \boldsymbol{\xi}$ for any $x \in \mathcal{P}(\mathbb{N})$ in $\mathbb{V}$. By definition, for any $p' \in \mathbb{P}$ i $n \in \mathbb{N}$ there are $i \ge n$ and conditions $q, r \in \mathbb{P}$ with $q, r \le p'$, such that $\langle q, r\rangle$ $\mathbb{P}\times\mathbb{P}$-forces the inequality $\varphi_i(\boldsymbol{\xi}_{\mathtt{left}} \,\Delta\, \boldsymbol{\xi}_{\mathtt{right}}) \le 2^{-n-1}$, hence, $\langle q, q\rangle$ $\mathbb{P}\times\mathbb{P}$-forces $\varphi_i(\boldsymbol{\xi}_{\mathtt{left}} \,\Delta\, \boldsymbol{\xi}_{\mathtt{right}}) \le 2^{-n}$. It follows that, in $\mathbb{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < ...$, and a sequence $p_0 \ge p_1 \ge p_2 \ge ...$ of conditions in $\mathbb{P}$, and, for any n, a set $u_n \subseteq [0,n)$, such that $p_0 \le p$ i

(1) each p_n $\mathbb{P}$-forces $\boldsymbol{\xi} \cap [0,n) = u_n$;

(2) each $\langle p_n, p_n\rangle$ $\mathbb{P}\times\mathbb{P}$-forces $\varphi_{i_n}(\boldsymbol{\xi}_{\mathtt{left}} \,\Delta\, \boldsymbol{\xi}_{\mathtt{right}}) \le 2^{-n}$.

Arguing in $\mathbb{V}$, put $a = \bigcup_n u_n$; then $a \cap [0,n) = u_n$ for all n. We claim that p_0 forces $a \mathsf{E}^{\#} \boldsymbol{\xi}$, contrary to the assumption above, which proves the theorem.

Indeed, otherwise there is a condition $q_0 \le p_0$ which forces $\neg\, a\mathsf{E}^{\#}\boldsymbol{\xi}$. Consider a generic extension $\mathbb{V}^+$ of the universe, where there is a sequence of $\mathbb{P}$-generic sets $U_n \subseteq \mathbb{P}$ such that, for any n, the pair $\langle U_n, U_{n+1}\rangle$ is $\mathbb{P}\times\mathbb{P}$-generic, $p_n \in U_n$, and in addition $q_0 \in U_0$. Then, in $\mathbb{V}^+$, the sets $x_n = \boldsymbol{\xi}[U_n] \in \mathcal{P}(\mathbb{N})$ satisfy $\varphi_{i_n}(x_n \,\Delta\, x_m) \le 2^{-n}$ by (2), whenever $n \le m$. It follows that $\varphi_{i_n}(x_n \,\Delta\, a) \le 2^{-n}$, because $a = \mathtt{lim}_m\, x_m$ by (1). However we assume that the submeasures φ_j decrease, hence, $\varphi_\infty(x_n \,\Delta\, a) \le 2^{-n}$. On the other hand, $\varphi_\infty(x_n \,\Delta\, x_0) = 0$ because $\boldsymbol{\xi}$ is a virtual E-class. We conclude that $\varphi_\infty(x_0 \,\Delta\, a) \le 2^{-n}$ for any n, in other words, $\varphi_\infty(x_0 \,\Delta\, a) = 0$, that is, $x_0 \mathrel{\mathsf{E}^\#} a$, which is a contradiction with the choice of U_0 because $x_0 = \boldsymbol{\xi}[U_0]$ and $q_0 \in U_0$. □

Questions

Question 1. Are all Borel ideals pinned ? The expected answer "yes" would show that T_2 is not Borel reducible to any Borel ideal. Moreover, is any orbit ER of a Borel action of a Borel CLI group pinned ? □

Question 2 (Kechris**).** Is there a $\le_{\mathrm{B}}$-least non-pinned Borel ER ? It was once expected that T_2 is such, but Hjorth informed us that there is a strictly $\le_{\mathrm{B}}$-smaller non-pinned Borel ER of a rather complicated nature. □

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