Varia
Ideals and Equivalence Relations, beta-version

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Introduction

We present a selection of basic results on Borel reducibility of ideals and equivalence relations, especially those with comparably short proofs. The focal point are reducibility/irreducibility results related to some special equivalences like E_0, E_1, E_2, E_3, E_∞, $\mathscr{Z}_0$, and Banach-induced equivalences $\boldsymbol{\ell}^p$. The bulk of results included in the book were obtained in the 1990s, but some rather recent theorems are presented as well, like Rosendal's proof that Borel ideals are cofinal within Borel equivalences of general form. [1]

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General set-theoretic notation used.

- $\mathbb{N} = \{0, 1, 2, \ldots\}$: natural numbers. $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.
- $\mathbb{N}^{\mathbb{N}}$ is *the Baire space*. If $s \in \mathbb{N}^{<\omega}$ (a finite sequence of natural numbers) then $\mathscr{O}_s(\mathbb{N}^{\mathbb{N}}) = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}$, a basic clopen nbhd in $\mathbb{N}^{\mathbb{N}}$.
- $X \subseteq^* Y$ means that the difference $X \smallsetminus Y$ is finite.
- If a basic set A is fixed then $\complement X = X^{\complement} = A \smallsetminus X$ for any $X \subseteq A$.
- If $X \subseteq A \times B$ and $a \in A$ then $(X)_a = \{b : \langle a, b\rangle \in X\}$, a *cross-section*.
- $\#X = \#(X)$ is the number of elements of a finite set X.
- $f\text{"}X = \{f(x) : x \in X \cap \mathtt{dom}\, f\}$, the *f-image* of X.
- Δ is the symmetric difference.
- $\exists^\infty x \ \ldots$ means: "there exist infinitely many x such that ...",
  $\forall^\infty x \ \ldots$ means: "for all but finitely many x, ... holds".
- $\{x_a\}_{a \in A}$ is the map f defined on A by $f(a) = x_a \tilde{t} \forall\, a$.
- $\mathscr{P}(X) = \{x : x \subseteq X\}$.
- $\mathscr{P}_{\mathtt{fin}}(X) = \{x : x \subseteq X \text{ is finite}\}$.

Contents

Chapter 1

Descriptive set theoretic background

We assume that the reader of this book has a basic knowledge of descriptive set theory, both classical and effective, in Polish spaces (recursively presented, in the effective case), including Borel and projective hierarchy, Borel sets and functions, analytic and coanalytic sets, and the like.

A map f (between Borel sets in Polish spaces) is *Borel* iff its graph is a Borel set iff all f-preimages of open sets are Borel. A map f is *Baire measurable* (*BM*, for brevity) iff all f-preimages of open sets are Baire measurable sets.

Apart of the very common knowledge, the whole instrumentarium of "effective" descriptive set theory employed in the study of reducibility of ideals and ERs, can be summarized in a rather short list of key "principles". In those below, by a *recursively presented* Polish space one can understand any product space of the form $\mathbb{N}^m \times (\mathbb{N}^{\mathbb{N}})^n$ without any harm for applications below, yet in fact this notion is much wider.

Remark 1.1. For the sake of brevity, the results below are formulated only for the "lightface" parameter-free classes Σ^1_1, Π^1_1, Δ^1_1, but they remain true for $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$ for any fixed real parameter p, as well as for the "boldface" classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$ of resp. analytic, coanalytic, Borel sets. □

Theorem 1.2 (Reduction, Separation). *If X, Y are Π^1_1 sets (of a recursively presented Polish space) then there exist* **disjoint** *Π^1_1 sets $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' = X \cup Y$.* The sets X', Y' are said to *reduce* the pair X, Y.

If X, Y are disjoint Σ^1_1 sets then there is a Δ^1_1 set Z with $X \subseteq Z$ and $Y \cap Z = \varnothing$. Such a set Z is said to *separate* X from Y. □

Theorem 1.3 (Countable-to-1 Projection). *If P is a Δ^1_1 subset of the product $\mathbb{X} \times \mathbb{Y}$ of two recursively presented Polish spaces, and for any $x \in \mathbb{X}$ the cross-section $P_x = \{y : P(x, y)\}$ is at most countable then $\mathtt{dom}\, P$ is a Δ^1_1 set.* □

It follows that images of Δ^1_1 sets via countable-to-1, in particular, 1-to-1 Δ^1_1 maps are Δ^1_1 sets, while images via arbitrary Δ^1_1 maps are, generally, Σ^1_1.

Theorem 1.4 (Countable-to-1 Enumeration). *If $P, \mathbb{X}, \mathbb{Y}$ are as in Theorem 1.3 then there is a Δ^1_1 map $f : \mathtt{dom}\, P \times \mathbb{N} \to \mathbb{Y}$ such that $P_x = \{f(x,n) : n \in \mathbb{N}\}$ for all $x \in \mathtt{dom}\, P$.* □

Theorem 1.5 (Borel Extension). *If P is a Σ^1_1 subset of the product $\mathbb{X} \times \mathbb{Y}$ of two recursively presented Polish spaces, and for any $x \in \mathbb{X}$ the cross-section $P_x = \{y : P(x,y)\}$ is at most countable then there is a Δ^1_1 superset $Q \supseteq P$ with all cross-sections Q_x at most countable. Similarly, if $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a uniform Σ^1_1 set then there is a uniform Δ^1_1 superset $Q \supseteq P$.* □

Recall that a set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is *uniform* iff the cross-section P_x contains at most one point for any $x \in \mathbb{X}$. This is the same as a *partial function* $\mathbb{X} \to \mathbb{Y}$.

Theorem 1.6 (Countable-to-1 Uniformization). *If $P, \mathbb{X}, \mathbb{Y}$ are as in Theorem 1.3 then P can be uniformized by a Δ^1_1 set.* □

Theorem 1.7 (Kreisel Selection). *If $\mathbb{X}$ is a recursively presented Polish space, $P \subseteq \mathbb{X} \times \mathbb{N}$ is a Π^1_1 set, and $X \subseteq \mathtt{dom}\, P$ is a Δ^1_1 set then there is a Δ^1_1 function $f : X \to \mathbb{N}$ such that $\langle x, f(x)\rangle \in P$ for all $x \in X$.* □

Proof. Let $Q \subseteq P$ be a Π^1_1 set which uniformizes P. For any $x \in X$ let $f(x)$ be the only n with $\langle x, n\rangle \in Q$. Immediately, (the graph of) f is Π^1_1, however, as $\mathtt{ran}\, f \subseteq \mathbb{N}$, we have $f(x) = n \iff \forall m \neq n\, (f(x) \neq m)$ whenever $x \in X$, which demonstrates that f is Σ^1_1 as well. □

The next theorem provides a useful enumeration of Δ^1_1 sets.

Theorem 1.8 (Δ^1_1 Enumeration). *If $\mathbb{X}$ is a recursively presented Polish space then there exist Π^1_1 sets $C \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{N} \times \mathbb{X}$ such that $W_e = W'_e$ for all $e \in C$, and a set $X \subseteq \mathbb{X}$ is Δ^1_1 iff there is $e \in C$ such that $X = W_e = W'_e$.* (Here $W_e = \{x : W(e,x)\}$ and similarly W'_e.) □

There is a generalization useful for relativised classes $\Delta^1_1(y)$.

Theorem 1.9 (Relativized Δ^1_1 Enumeration). *If $\mathbb{X}, \mathbb{Y}$ are recursively presented Polish spaces then there exist Π^1_1 sets $C \subseteq \mathbb{Y} \times \mathbb{N}$ and $W \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ and a Σ^1_1 set $W' \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ such that $W_{ye} = W'_{ye}$ for all $\langle y, e\rangle \in C$ and, for any $y \in \mathbb{Y}$, a set $X \subseteq \mathbb{X}$ is $\Delta^1_1(y)$ iff there is e such that $\langle y, e\rangle \in C$ and $X = W_{ye} = W'_{ye}$.* (Here $W_{ye} = \{x : W(y,e,x)\}$ and similarly W'_{ye}.) □

Suppose that $\mathbb{X}$ is a recursively presented Polish space. A set $U \subseteq \mathbb{N} \times \mathbb{X}$, is a *a universal Π^1_1 set* if for any Π^1_1 set $X \subseteq \mathbb{X}$ there is an index $e \in \mathbb{N}$ with $X = U_e = \{x : \langle e, x\rangle \in U\}$, and a *a "good" universal Π^1_1 set* if in addition for any other Π^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$ there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $V_e = U_{f(e)}$ for all e.

The notions of universal and "good" universal Σ^1_1 sets are similar.

Theorem 1.10 (Universal Sets). *For any recursively presented Polish space $\mathbb{X}$ there exist a "good" universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$ and a "good" universal Σ^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$.* (In fact we can take $V = (\mathbb{N} \times \mathbb{X}) \smallsetminus U$.) □

If a "good" universal Π^1_1 set U is fixed then a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{e : U_e \in \mathscr{A}\}$ is a Π^1_1 set. Similarly, if a "good" universal Σ^1_1 set V is fixed then a collection $\mathscr{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{e : V_e \in \mathscr{A}\}$ is a Π^1_1 set. These notions quite obviously do not depend on the choice of "good" universal sets.

To show how "good" universal sets work, we prove:

Proposition 1.11. *Let $\mathbb{X}$ be a recursively presented Polish space and $U \subseteq \mathbb{N}\times\mathbb{X}$ a "good" universal Π^1_1 set. Then for any pair of Π^1_1 sets $V, W \subseteq \mathbb{N} \times \mathbb{X}$ there are recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$ the pair of cross-sections $U_{f(m,n)}, U_{g(m,n)}$ reduces the pair V_m, W_n.*

Proof. Consider the following Π^1_1 sets in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{X}$:

$$P = \{\langle m, n, x\rangle : \langle m, x\rangle \in V \wedge n \in \mathbb{N}\}, \ \ Q = \{\langle m, n, x\rangle : \langle n, x\rangle \in W \wedge m \in \mathbb{N}\}.$$

By Reduction, there is a pair of Π^1_1 sets $P' \subseteq P$ and $Q' \subseteq Q$ which reduce the given pair P, Q. Accordingly, the pair P'_{mn}, Q'_{mn} reduces P_{mn}, Q_{mn} for any m, n. Finally, by the "good" universality there are recursive functions f, g such that $P'_{mn} = U_{f(m,n)}$ and $Q'_{mn} = U_{g(m,n)}$ for all m, n. □

The following theorem is less elementary than the results cited above, but it is very useful because it allows to "compress" some sophisticated arguments with multiple applications of Separation and Kreisel selection.

Theorem 1.12 (Reflection). *Let $\mathbb{X}$ be a recursively presented Polish space.*

Π^1_1 **form.** *Suppose that a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. (In the sense of a fixed "good" universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$.) Then for any $X \in \mathscr{A}$ there is a Δ^1_1 set $Y \in \mathscr{A}$ with $Y \subseteq X$.*

Σ^1_1 **form.** *Suppose that a collection $\mathscr{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. Then for any $X \in \mathscr{A}$ there is a Δ^1_1 set $Y \in \mathscr{A}$ with $X \subseteq Y$.* □

One of (generally, irrelevant here) consequences of this principle is that the set of all codes of a properly Π^1_1 set or properly Σ^1_1 set is never Π^1_1.

Chapter 2

Borel ideals

This Chapter does not mean any broad introduction into Borel ideals; we rather consider some issues close to the content of the book, including P-ideals, polishable ideals, LSC submeasures, summable, density, and Fréchet ideals, and Rudin – Blass reducibility of ideals.

2a Introduction to Borel ideals

Recall that an *ideal* on a set A is any non-empty set $\mathscr{I} \subseteq \mathscr{P}(A)$ closed under $\cup$ and satisfying $x \in \mathscr{I} \implies y \in \mathscr{I}$ whenever $y \subseteq x \subseteq A$. Thus, any ideal contains the empty set $\varnothing$. Usually they consider only *nontrivial* ideals, *i.e.*, those which contain all singletons $\{a\}$, $a \in A$, and do not contain A, *i.e.*, $\mathscr{P}_{\mathtt{fin}}(A) \subseteq \mathscr{I} \subsetneqq \mathscr{P}(A)$. But sometimes the ideal $\{\varnothing\}$, whose only element is the empty set $\varnothing$ is considered and often denoted by 0.

If A is a countable set then identifying $\mathscr{P}(A)$ with 2^A via characterictic functions we equip $\mathscr{P}(A)$ with the Polish product topology. In this sense, a *Borel* ideal on A is any ideal which is a Borel subset of $\mathscr{P}(A)$ in this topology. Let us give several important examples of Borel ideals.

- $\mathsf{Fin} = \{x \subseteq \mathbb{N} : x \text{ is finite}\}$, the ideal of all finite sets;
- $\mathscr{I}_1 = \{x \subseteq \mathbb{N}^2 : \{k : (x)_k \neq \varnothing\} \in \mathsf{Fin}\}$, where $(x)_a = \{b : \langle a, b\rangle \in x\}$;
- $\mathscr{I}_2 = \{x \subseteq \mathbb{N} : \sum_{n \in x} \frac{1}{n+1}\} < +\infty$, the *summable ideal*;
- $\mathscr{I}_3 = \{x \subseteq \mathbb{N}^2 : \forall\, k\, ((x)_k \in \mathsf{Fin})\}$;
- $\mathscr{Z}_0 = \{x \subseteq \mathbb{N} : \mathtt{lim}_{n \to +\infty} \frac{\#(x \cap [0,n))}{n} = 0\}$, the *density ideal*.

For any ideal $\mathscr{I}$ on a set A, we define $\mathscr{I}^+ = \mathscr{P}(A) \smallsetminus \mathscr{I}$ ($\mathscr{I}$*-positive* sets) and $\mathscr{I}^{\complement} = \{X : \complement X \in \mathscr{I}\}$ (*the dual filter*). Clearly $\varnothing \neq \mathscr{I}^{\complement} \subseteq \mathscr{I}^+$.

If $B \subseteq A$, then we put $\mathscr{I} \restriction B = \{x \cap B : x \in \mathscr{I}\}$.

2b P-ideals, submeasures, polishable ideals

Many important Borel ideals belong to the class of P-ideals.

Definition 2.1. An ideal $\mathscr{I}$ on $\mathbb{N}$ is a *P-ideal* if for any sequence of sets $x_n \in \mathscr{I}$ there is a set $x \in \mathscr{I}$ such that $x_n \subseteq^* x$ (*i.e.*, $x_n \smallsetminus x \in \mathsf{Fin}$) for all n; □

For instance, the ideals Fin, $\mathscr{I}_2$, $\mathscr{I}_3$, $\mathscr{Z}_0$ (but not $\mathscr{I}_1$!) are P-ideals.

This class admits several apparently different but equivalent characterizations, one of which is connected with submeasures.

Definition 2.2. A *submeasure* on a set A is any map $\varphi : \mathscr{P}(A) \to [0, +\infty]$, satisfying $\varphi(\varnothing) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \le \varphi(x \cup y) \le \varphi(x) + \varphi(y)$.

A submeasure φ on $\mathbb{N}$ is *lover semicontinuous*, or LSC for brevity, if we have $\varphi(x) = \mathtt{sup}_n\, \varphi(x \cap [0, n))$ for all $x \in \mathscr{P}(\mathbb{N})$. □

To be a *measure*, a submeasure φ has to satisfy, in addition, that $\varphi(x \cup y) = \varphi(x) + \varphi(y)$ whenever x, y are disjoint. Note that any σ-additive measure is LSC, but if φ is LSC then φ_∞ is not necessarily LSC itself.

Suppose that φ is a submeasure on $\mathbb{N}$. Define the *tailsubmeasure* $\varphi_\infty(x) = ||x||_\varphi = \mathtt{inf}_n(\varphi(x \cap [n, \infty)))$. The following ideals are considered:

$$\begin{aligned} \mathtt{Fin}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) < +\infty\} & ; \\ \mathtt{Null}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) = 0\} & ; \\ \mathtt{Exh}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi_\infty(x) = 0\} = \mathtt{Null}_{\varphi_\infty} & . \end{aligned}$$

Example 2.3. $\mathsf{Fin} = \mathtt{Exh}_\varphi = \mathtt{Null}_\varphi$, where $\varphi(x) = 1$ for any $x \neq \varnothing$. We also have $0 \times \mathsf{Fin} = \mathtt{Exh}_\psi$, where $\psi(x) = \sum_k 2^{-k}\, \varphi(\{l : \langle k, l \rangle \in x\})$ is LSC. □

It turns out (Solecki, see Theorem 8.5 below) that analytic P-ideals are the same as ideals of the form $\mathtt{Exh}_\varphi$, where φ is a LSC submeasure on $\mathbb{N}$. This implies that any analytic P-ideal is $\mathbf{\Pi}^0_3$.

There is one more useful characterization of Borel P-ideals. Let T be the ordinary Polish product topology on $\mathscr{P}(\mathbb{N})$. Then $\mathscr{P}(\mathbb{N})$ is a Polish group in the sense of T and the symmetric difference as the operation, and any ideal $\mathscr{I}$ on $\mathbb{N}$ is a subgroup of $\mathscr{P}(\mathbb{N})$.

Definition 2.4. An ideal $\mathscr{I}$ on $\mathbb{N}$ is *polishable* if there is a Polish group topology τ on $\mathscr{I}$ which produces the same Borel subsets of $\mathscr{I}$ as $T \restriction \mathscr{I}$. □

The same Solecki's theorem (Theorem 8.5) proves that, for analytic ideals, to be a P-ideal is the same as to be polishable. It follows (see Example 2.3) that, for instance, Fin and $\mathscr{I}_3$ are polishable, but $\mathscr{I}_1$ is not. The latter will be shown directly after the next lemma.

Lemma 2.5. *Suppose that an ideal* $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ *is polishable. Then there is a unique Polish group topology* τ *on* $\mathscr{I}$*. This topology refines* $T \restriction \mathscr{I}$ *and is metrizable by a* Δ*-invariant metric. If* $Z \in \mathscr{I}$ *then* $\tau \restriction \mathscr{P}(Z)$ *coincides with* $T \restriction \mathscr{P}(Z)$*. In addition,* $\mathscr{I}$ *itself is* T*-Borel.*

Proof. Let τ witness that $\mathscr{I}$ is polishable. The identity map $f(x) = x\colon \langle \mathscr{I}\,; \tau\rangle \to \langle \mathscr{P}(\mathbb{N})\,; T\rangle$ is a Δ-homomorphism and is Borel-measurable because all $(T \restriction \mathscr{I})$-open sets are τ-Borel, hence, by the Pettis theorem (Kechris [34, ??]), f is continuous. It follows that all $(T \restriction \mathscr{I})$-open subsets of $\mathscr{I}$ are τ-open, and that $\mathscr{I}$ is T-Borel in $\mathscr{P}(\mathbb{N})$ because $1-1$ continuous images of Borel sets are Borel.

A similar "identity map" argument shows that τ is unique if exists.

It is known (Kechris [34, ??]) that any Polish group topology admits a left-invariant compatible metric, which, in this case, is right-invariant as well since Δ is an abelian operation.

Let $Z \in \mathscr{P}(\mathbb{N})$. Then $\mathscr{P}(Z)$ is T-closed, hence, τ-closed by the above, subgroup of $\mathscr{I}$, and $\tau \restriction \mathscr{P}(Z)$ is a Polish group topology on $\mathscr{P}(Z)$. Yet $T \restriction \mathscr{P}(Z)$ is another Polish group topology on $\mathscr{P}(Z)$, with the same Borel sets. The same "identity map" argument proves that T and τ coincide on $\mathscr{P}(Z)$. □

Example 2.6. The ideal $\mathscr{I}_1$ is not polishable. Indeed we have $\mathscr{I}_1 = \bigcup_n W_n$, where $W_n = \{x : x \subseteq \{0, 1, \dots, n\} \times \mathbb{N}\}$. Let, on the contrary, τ be a Polish group topology on $\mathscr{I}_1$. Then τ and the ordinary topology T coincide on each set W_n by the lemma, in particular, each W_n remains τ-nowhere dense in W_{n+1}, hence, in $\mathscr{I}_1$, a contradiction with the Baire category theorem for τ. □

2c Summable and density ideals

Any sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$\mathscr{S}_{\{r_n\}} = \{X \subseteq \mathbb{N} : \sum_{n\in X} r_n < +\infty\} = \{X : \mu_{\{r_n\}}(X) < +\infty\}\,,$$

where $\mu_{\{r_n\}}(X) = \sum_{n\in X} r_n$. These ideals are called *summable ideals*; all of them are $\mathbf{F}_\sigma$ in the product Polish topology on $\mathscr{P}(\mathbb{N})$. Any summable ideal is easily a P-ideal: indeed, $\mathscr{S}_{\{r_n\}} = \mathtt{Exh}_\varphi$, where $\varphi(X) = \sum_{n\in X} r_n$ is a σ-additive measure. Summable ideals are perhaps the easiest to study among all P-ideals. More on summable ideals see [46, 48, 7].

Farah [7, § 1.10] defines a non-summable $\mathbf{F}_\sigma$ P-ideal as follows. Let $I_k = [2^k, 2^{k+1})$ and $\psi_k(s) = k^{-2}\,\mathtt{min}\{k, \#s\}$ for all k and $s \subseteq I_k$, and then

$$\psi(X) = \sum_{k=0}^{\infty} \psi_k(X \cap I_k) \quad \text{and} \quad \mathscr{I} = \mathtt{Fin}_\psi\,;$$

it turns out that $\mathscr{I}$ is an $\mathbf{F}_\sigma$ P-ideal, but not summable. To show that $\mathscr{I}$ distincts from any $\mathscr{S}_{\{r_n\}}$, Farah notes that there is a set X (which depends on

$\{r_n\}$) such that the differences $|\mu_{\{r_n\}}(X \cap I_k) - \psi_k(X \cap I_k)|$, $k = 0, 1, 2, \ldots$, are unbounded.

There exist other important types of Borel P-ideals. Any sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$EU_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} : \lim_{n\to+\infty} \frac{\sum_{i\in x\cap[0,n)} r_i}{\sum_{i\in[0,n)} r_i} = 0 \right\}.$$

These ideals are called *Erdös – Ulam* (or: EU) *ideals*. Examples: $\mathscr{Z}_0 = EU_{\{1\}}$ and $\mathscr{Z}_{\mathtt{log}} = EU_{\{1/n\}}$.

This definition can be generalized. Let $\mathtt{supp}\,\mu = \{n : \mu(\{n\}) > 0\}$, for any measure μ on $\mathbb{N}$. Measures μ, ν are *orthogonal* if we have $\mathtt{supp}\,\mu \cap \mathtt{supp}\,\nu = \varnothing$. Now suppose that $\vec{\mu} = \{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal measures on $\mathbb{N}$, with finite sets $\mathtt{supp}\,\mu_i$. Define $\varphi_{\vec{\mu}}(X) = \mathtt{sup}_n\,\mu_n(X)$: this is a LSC submeasure on $\mathbb{N}$. Let finally $\mathscr{D}_{\vec{\mu}} = \mathtt{Exh}(\varphi_{\vec{\mu}}) = \{X : ||X||_{\varphi_\mu} = 0\}$. Ideals of this form are called *density ideals* by Farah [7, § 1.13]. This class includes all EU ideals (although this is not immediately transparent), and some other ideals: for instance, $\mathscr{I}_3$ is a density but non-EU ideal. Generally density ideals are more complicated than summables. We obtain an even wider class if the requirement, that the sets $\mathtt{supp}\,\mu_n$ are finite, is dropped: this wider family includes all summmable ideals, too.

See [27], [7, § 1.13] on density ideals.

2d Operations on ideals and Fréchet ideals

Suppose that A is any non-empty set, and $\mathscr{J}_a$ is an ideal on a set B_a for all $a \in A$. The following two operations on such a family of ideals are defined.

Disjoint sum $\sum_{a\in A} \mathscr{J}_a$ is the ideal on the set $B = \{\langle a, b\rangle : a \in A \wedge b \in B_a\}$ that consists of all sets $x \subseteq B$ such that $(x)_a \in \mathscr{J}_a$ for all $a \in A$, where $(x)_a = \{b : \langle a, b\rangle \in x\}$ (the cross-section). If the sets B_a are pairwise disjoint then $\sum_{a\in A} \mathscr{I}_a$ can be equivalently defined as the ideal on $B = \bigcup_{a\in A} B_a$ that consists of all sets of the form $\bigcup_{a\in A} x_a$, where $x_a \in \mathscr{I}_a$ for all a.

In the case of two or finitely many summands, the disjoint sum $\mathscr{I} \oplus \mathscr{J}$ of ideals $\mathscr{I}$, $\mathscr{J}$ on disjoint sets A, B is equal to $\{x \cup y : x \in \mathscr{I} \wedge y \in \mathscr{J}\}$.

Fubini sum and product Suppose in addition that $\mathscr{I}$ is an ideal on A. The *Fubini sum* $\sum_{a\in A} \mathscr{J}_a / \mathscr{I}$ of the ideals $\mathscr{J}_a$ modulo $\mathscr{I}$ is the ideal on the set B (defined as above) which consists of all sets $y \subseteq B$ such that the set $\{a : (x)_a \notin \mathscr{J}_a\}$ belongs to $\mathscr{I}$. This ideal obviously coincides with the plain disjoint sum $\sum_{a\in A} \mathscr{J}_a$ whenever $\mathscr{I} = \{\varnothing\}$.

In particular, the *Fubini product* $\mathscr{I} \otimes \mathscr{J}$ of two ideals $\mathscr{I}$, $\mathscr{J}$ on sets resp. A, B is equal to $\sum_{a\in A} \mathscr{J}_a / \mathscr{I}$, where $\mathscr{J}_a = \mathscr{J}$, $\forall\, a$. Thus $\mathscr{I} \otimes \mathscr{J}$ consists of all sets $y \subseteq A \times B$ such that $\{a : (y)_a \notin \mathscr{J}\} \in \mathscr{I}$.

Coming back to the ideals defined in Section 2a, $\mathscr{I}_1$ and $\mathscr{I}_3$ coincide with resp. $\mathtt{Fin} \times 0$ and $0 \times \mathtt{Fin}$, where, we recall, 0 denotes the least ideal $0 = \{\varnothing\}$.

The operations of Fubini sum and product allow us to define the following interesting family of Borel ideals (mainly, non-P-ideals) on countable sets.

Fréchet ideals. This family consists of ideals $\mathtt{Fr}_\xi$, $\xi < \omega_1$, defined by transfinite induction. We put $\mathtt{Fr}_1 = \mathtt{Fin}$ and $\mathtt{Fr}_{\xi+1} = \mathtt{Fin} \otimes \mathtt{Fr}_\xi$ for all ξ. Limit steps cause a certain problem. The most natural idea would be to define $\mathtt{Fr}_\lambda = \sum_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, \mathtt{Fin}_\lambda$ for any limit λ, where $\mathtt{Fin}_\lambda$ is the ideal of all finite subsets of λ, or perhaps $\mathtt{Fr}_\lambda = \sum_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, \mathtt{Bou}_\lambda$, where $\mathtt{Bou}_\lambda$ is the ideal of all bounded subsets of λ, or even $\mathtt{Fr}_\lambda = \sum_{\xi<\lambda} \mathtt{Fr}_\xi \,/\, \{\varnothing\}$. Yet this appears not to be fully satisfactory in [23], where they define $\mathtt{Fr}_\lambda = \sum_{n\in\mathbb{N}} \mathtt{Fr}_{\xi_n} \,/\, \mathtt{Fin}$, where $\{\xi_n\}$ is a once and for all fixed cofinal increasing sequence of ordinals below λ, with understanding that the result is independent of the choice of ξ_n, modulo a certain equivalence.

2e Some other ideals

We consider two interesting families of Borel ideals (mainly, non-P-ideals), united by their relation to countable ordinals. Note that the underlying sets of these ideals are countable sets different from $\mathbb{N}$.

Indecomposable ideals. Let $\mathtt{otp}\, X$ be the order type of $X \subseteq \mathtt{Ord}$. For any ordinals $\xi, \vartheta < \omega_1$ define:

$$\mathscr{I}^\xi_\vartheta = \{A \subseteq \vartheta : \mathtt{otp}\, A < \omega^\xi\} \quad (\text{nontrivial only if } \vartheta \ge \omega^\xi)\,.$$

To see that the sets $\mathscr{I}^\xi_\vartheta$ are really ideals note that ordinals of the form ω^ξ and only those ordinals are *indecomposable*, *i.e.*, are not sums of a pair of smaller ordinals, hence, the set $\{A \subseteq \vartheta : \mathtt{otp}\, A < \gamma\}$ is an ideal iff $\gamma = \omega^\xi$ for some ξ.

Weiss ideals. Let $|X|_{\mathrm{CB}}$ be the *Cantor-Bendixson rank* of $X \subseteq \mathtt{Ord}$, *i.e.*, the least ordinal α such that $X^{(\alpha)} = \varnothing$. Here $X^{(\alpha)}$ is defined by induction on α : $X^{(0)} = X$, $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ at limit steps λ, and finally $X^{(\alpha+1)} = (X^{(\alpha)})'$, where A', the Cantor-Bendixson derivative, is the set of all ordinals $\gamma \in x$ which are limit points of X in the interval topology. For any ordinals $\xi, \vartheta < \omega_1$ define:

$$\mathscr{W}^\xi_\vartheta = \{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \omega^\xi\} \quad (\text{nontrivial only if } \vartheta \ge \omega^{\omega^\xi})\,.$$

It is less transparent that all $\mathscr{W}^\xi_\vartheta$ are ideals (see Farah [7, § 1.14]) while $\{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \gamma\}$ is not an ideal if γ is not of the form ω^ξ.

Ideals on finite sequences. The set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers is countable, yet its own order structure is quite different from that of $\mathbb{N}$. We can exploit this in several ways, for instance, with ideals of sets $X \subseteq \mathbb{N}^{<\omega}$ which intersect every branch in $\mathbb{N}^{<\omega}$ by a set which belongs to a given ideal on $\mathbb{N}$.

Further entry: Farah [6, 8, 9] on *Tsirelson ideals*.

Nowhere dense ideal etc

2f Reducibility of ideals

There are different methods of reduction of an ideal $\mathscr{I}$ on a set A to an ideal $\mathscr{J}$ on a set B, where the reducibility means that $\mathscr{I}$ is in some sense simpler (in non-strict way) than $\mathscr{J}$.

Rudin–Keisler order: $\mathscr{I} \le_{\rm RK} \mathscr{J}$ iff there exists a function $\beta : B \to A$ (a *Rudin – Keisler* reduction) such that $x \in \mathscr{I} \iff b^{-1}(x) \in \mathscr{J}$.

Rudin–Blass order: $\mathscr{I} \le_{\rm RB} \mathscr{J}$ iff there is a finite-to-one function $\beta : B \to A$ (a *Rudin–Blass* reduction) with the same property.

A version: $\mathscr{I} \le^{+}_{\rm RB} \mathscr{J}$ allows β to be defined on a proper subset of B, in other words, we have pairwise disjoint finite non-empty sets $w_a = \beta^{-1}(\{a\})$, $a \in A$, such that $x \in \mathscr{I} \iff w_x = \bigcup_{a\in x} w_a \in \mathscr{J}$.

Another version $\mathscr{I} \le^{++}_{\rm RB} \mathscr{J}$, applicable in the case when $A = B = \mathbb{N}$, requires that in addition the sets w_a satisfy $\mathtt{max}\, w_a < \mathtt{min}\, w_{a+1}$.

Lemma 2.7. *Suppose that $r_n \ge 0$, $r_n \to 0$, and $\sum_n r_n = +\infty$. Then any summable ideal $\mathscr{I}$ satisfies $\mathscr{I} \le^{++}_{\rm RB} \mathscr{S}_{\{r_n\}}$.*

Proof. Let $I = \mathscr{S}_{\{p_n\}}$, where $p_n \ge 0$ (no other requirements !). Under the assumptions of the lemma we can associate a finite set $w_n \subseteq \mathbb{N}$ to any n so that $\mathtt{max}\, w_n < \mathtt{min}\, w_{n+1}$ and $|r_n - \sum_{j\in w_n} r_i| < 2^{-n}$. □

Another type of reducibility is connected with Δ-homomorphisms.

Suppose that $\mathscr{I}, \mathscr{J}$ are ideals on sets resp. A, B. The power sets $\mathscr{P}(A)$, $\mathscr{P}(B)$ can be considered as groups with Δ as the operation and $\varnothing$ as the neutral element. Then a Δ-*homomorphism* is any map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $\vartheta(x) \,\Delta\, \vartheta(y) = \vartheta(x \,\Delta\, y)$ for all $x, y \subseteq A$.

The quotient $\mathscr{P}(A)/\mathscr{I}$ consists of $\mathscr{I}$-classes $[x]_{\mathscr{I}} = \{x \,\Delta\, a : a \in \mathscr{I}\}$ of sets $x \subseteq A$; it is endowed by the group operation $[x]_{\mathscr{I}} \,\Delta\, [y]_{\mathscr{I}} = [x \,\Delta\, y]_{\mathscr{I}}$. Similarly $\mathscr{P}(B)/\mathscr{J}$. For a map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ to induce in obvious way a group homomorphism of $\mathscr{P}(A)/\mathscr{I}$ to $\mathscr{P}(B)/\mathscr{J}$, it is necessary and sufficient that

(1) $(\vartheta(x) \,\Delta\, \vartheta(y)) \,\Delta\, \vartheta(x \,\Delta\, y) \in \mathscr{J}$ for all $x, y \subseteq A$, and

(2) $x \in \mathscr{I} \iff \vartheta(x) \in \mathscr{J}$ for all $x \subseteq A$.

Let us call any such a map *an* $(\mathscr{I}, \mathscr{J})$*-approximate* Δ*-homomorphism*.

Borel Δ-reducibility: $\mathscr{I} \le^{\Delta}_{\rm B} \mathscr{J}$ iff there is a Borel $(\mathscr{I}, \mathscr{J})$-approximate Δ-homomorphism $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$.

Note that if a map $\beta : B \to A$ witnesses, say, $\mathscr{I} \le_{\rm RK} \mathscr{J}$ then the map $\vartheta(x) = \beta^{-1}(x)$ obviously witnesses $\mathscr{I} \le^{\Delta}_{\rm B} \mathscr{J}$.

Isomorphism $\mathscr{I} \cong \mathscr{J}$ of ideals $\mathscr{I}, \mathscr{J}$ on sets resp. A, B means that there is a bijection $\beta : A \xrightarrow{\text{onto}} B$ such that $x \in \mathscr{I} \iff \beta"x \in \mathscr{J}$ for all $x \subseteq A$.

The following notion belongs to a somewhat different category since it does not allow to really define $\mathscr{I}$ in terms of $\mathscr{J}$.

Reducibility via inclusion (see [23])**:** $\mathscr{I} \le_{\mathrm{I}} \mathscr{J}$ iff there is a map $\beta : B \to A$ such that $x \in \mathscr{I} \Longrightarrow \beta^{-1}(x) \in \mathscr{J}$. (Note $\Longrightarrow$ instead of $\Longleftrightarrow$!)

In particular if $\mathscr{I} \subseteq \mathscr{J}$ (and $B = A$) then $\mathscr{I} \le_{\mathrm{I}} \mathscr{J}$ via $\beta(a) = a$. It follows that this order is not fully compatible with the Borel reducibility $\le_{\mathrm{B}}$.

Chapter 3

Introduction to equivalence relations

Recall that an *equivalence relation* (ER, for brevity) on a set A is any reflexive, transitive, and symmetric binary relation on A.

- If E is an ER on a set X then
  $[y]_{\mathsf{E}} = \{x \in X : y \mathrel{\mathsf{E}} x\}$ for any $y \in X$ (the E-*class* of x) and
  $[Y]_{\mathsf{E}} = \bigcup_{y \in Y} [y]_{\mathsf{E}}$ (the E-*saturation* of Y) for $Y \subseteq X$.
  A set $Y \subseteq X$ is E-*invariant* if $[Y]_{\mathsf{E}} = Y$.

- If E is an ER on a set X then a set $Y \subseteq X$ is *pairwise* E-*equivalent*, resp., *pairwise* E-*inequivalent*, if $x \mathrel{\mathsf{E}} y$, resp., $x \not\mathrel{\mathsf{E}} y$ holds for all $x \neq y$ in Y.

- If X, Y are sets and E any binary relation then $X \mathrel{\mathsf{E}} Y$ means that we have both $\forall x \in X\, \exists y \in Y\, (x \mathrel{\mathsf{E}} y)$ and $\forall y \in Y\, \exists x \in X\, (x \mathrel{\mathsf{E}} y)$.

3a Some examples of Borel equivalence relations

Let EQ_X denote the equality on a set X, considered as an equivalence relation on X. This is the most elementary type of ERs. A much more diverse family consists of equivalence relations $\mathsf{E}_{\mathscr{I}}$ generated by Borel ideals.

- If $\mathscr{I}$ is an ideal on a set A then $\mathsf{E}_{\mathscr{I}}$ denotes an equivalence relation on $\mathscr{P}(A)$, defined so that $x \mathrel{\mathsf{E}_{\mathscr{I}}} y$ iff $x \,\Delta\, y \in \mathscr{I}$.

Equivalently, $\mathsf{E}_{\mathscr{I}}$ can be considered as an equivalence relation on 2^A defined so that $f \mathrel{\mathsf{E}_{\mathscr{I}}} g$ iff $f \,\Delta\, g \in \mathscr{I}$, where $f \,\Delta\, g = \{a \in A : f(a) \neq g(a)\}$. Note that $\mathsf{E}_{\mathscr{I}}$ is Borel provided so is $\mathscr{I}$. We obtain the following important equivalence relations: [1]

[1] The notational system we follow is not the only one used in modern texts. For instance $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$ are sometimes denoted differently, see *e.g.* [14].

$\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$ is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathsf{E}_0 y$ iff $x \Delta y \in \mathtt{Fin}$.

$\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$ is an ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$, and $x \mathsf{E}_1 y$ iff $(x)_k = (y)_k$ for all but finite k, where, we recall, $(x)_k = \{n : \langle k, n \rangle \in x\}$ for $x \subseteq \mathbb{N} \times \mathbb{N}$.

$\mathsf{E}_2 = \mathsf{E}_{\mathscr{I}_2}$ is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathsf{E}_2 y$ iff $\sum_{k \in x \Delta y} k^{-1} < \infty$.

$\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$ is an ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$, and $x \mathsf{E}_3 y$ iff $(x)_k \mathsf{E}_0 (y)_k$, $\forall k$.

$\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$ is an ER on $\mathscr{P}(\mathbb{N})$, and $x \mathsf{Z}_0 y$ iff $\mathtt{lim}_{n \to \infty} \frac{\#((x \Delta y) \cap [0,n))}{n} = 0$.

Alternatively, E_0 can be viewed as an equivalence relation on $2^{\mathbb{N}}$ defined as $a \mathsf{E}_1 b$ iff $a(k) = b(k)$ for all but finite k. Similarly, E_1 can be viewed as a ER on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or even on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x \mathsf{E}_1 y$ iff $x(k) = y(k)$ for all but finite k, for all $x, y \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, while E_3 can be viewed as a ER on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x \mathsf{E}_3 y$ iff $x(k) \mathsf{E}_0 y(k)$ for all k.

The next group includes equivalence relations induced by actions of (the additive groups of) some Banach spaces — see below on group actions. The following Banach spaces are well known from textbooks:

$$
\begin{array}{rcllcl}
\boldsymbol{\ell}^p & = & \{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x(n)|^p < \infty\} \ (p \ge 1); & \|x\|_p & = & (\sum_n |x(n)|^p)^{\frac{1}{p}}; \\
\boldsymbol{\ell}^\infty & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{sup}_n |x(n)| < \infty\}; & \|x\|_\infty & = & \mathtt{sup}_n |x(n)|; \\
\mathbf{c} & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n x(n) < \infty \text{ exists}\}; & \|x\| & = & \mathtt{sup}_n |x(n)|; \\
\mathbf{c}_0 & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n x(n) = 0\}; & \|x\| & = & \mathtt{sup}_n |x(n)|.
\end{array}
$$

Note that $\boldsymbol{\ell}^p$, $\mathbf{c}$, $\mathbf{c}_0$ are separable spaces while $\boldsymbol{\ell}^\infty$ is non-separable. The domain of each of these spaces consists of infinite sequences $x = \{x(n)\}_{n \in \mathbb{N}}$ of reals, and is a subgroup of the group $\mathbb{R}^{\mathbb{N}}$ (with the componentwise addition). The latter can be naturally equipped with the Polish product topology, so that $\boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty$, $\mathbf{c}$, $\mathbf{c}_0$ are Borel subgroups of $\mathbb{R}^{\mathbb{N}}$. (But not topological subgroups since the distances are different. The metric definitions as in $\boldsymbol{\ell}^p$ or $\boldsymbol{\ell}^\infty$ do not work for $\mathbb{R}^{\mathbb{N}}$.)

Each of the four mentioned Banach spaces induces an *orbit equivalence* — a Borel equivalence relation on $\mathbb{R}^{\mathbb{N}}$ also denoted by, resp., $\boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty$, $\mathbf{c}$, $\mathbf{c}_0$. For instance, $x \,\boldsymbol{\ell}^p\, y$ if and only if $\sum_k |x(k) - y(k)|^p < +\infty$ (for all $x, y \in \mathbb{R}^{\mathbb{N}}$).

Another important equivalence relation is

T_2, often called *the equality of countable sets of reals*, is an ER defined on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ so that $g \mathsf{T}_2 h$ iff $\mathtt{ran}\, g = \mathtt{ran}\, h$ (for $g, h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$).

There is no reasonable way to turn $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$, the set of all at most countable subsets of $\mathbb{N}^{\mathbb{N}}$, into a Polish space, in order to directly define the equality of countable sets of reals in terms of EQ.. However, nonempty members of $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$ can be identified with equivalence classes in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} / \mathsf{T}_2$. (See Chapter 10 on the whole series of equivalence relations T_α, $\alpha < \omega_1$.)

We finish with another important equivalence relation,

E_∞, *the universal countable Borel ER*. The countability here means that all E-equivalence classes $[x]_\mathsf{E}$ are at most countable sets. The notion of universality will be explained below.

See Example 3.9 on an exact definition of E_∞.

3b Operations on equivalence relations

The following operations over equivalence relations are in part parallel to the operations on ideals in Section 2d. Suppose that A is any non-empty and *at most countable* set, and F_a is an equivalence relation on a set X_a for all $a \in A$. The following operations on such a family of ERs are defined.

(o1) **Union** $\bigcup_{a\in A} \mathsf{F}_a$ (if it results in an equivalence relation) and **intersection** $\bigcap_{a\in A} \mathsf{F}_a$ (it always results in an equivalence relation) — in the case when all F_a are ERs on one and the same set $X = X_a$, $\forall\, a$.

(o2) **Countable disjoint union** $\bigvee_{a\in A} \mathsf{F}_a$ is an ER E on the set $X = \bigcup_a (\{a\} \times X_a)$ defined as follows: $\langle a, x\rangle \mathrel{\mathsf{E}} \langle b, y\rangle$ iff $a = b$ and $x \mathrel{\mathsf{E}_a} y$.

If the sets X_a are pairwise disjoint then we can equivalently define an equivalence relation $\mathsf{E} = \bigvee_a \mathsf{F}_a$ on the set $Y = \bigcup_a X_a$ so that $x \mathrel{\mathsf{E}} y$ iff x, y belong to the same X_a and $x \mathrel{\mathsf{F}_a} y$.

(o3) **Product** $\prod_{a\in A} \mathsf{F}_a$ is an ER E on the cartesian product $\prod_{a\in A} X_a$ defined so that $x \mathrel{\mathsf{E}} y$ iff $x(a) \mathrel{\mathsf{F}_a} y(a)$ for all $a \in A$.

In particular the product $\mathsf{F}_1 \times \mathsf{F}_2$ of ERs E, F on sets resp. X_1, X_2 is an ER E on $X_1 \times X_2$ defined so that $\langle x_1, x_2\rangle \mathrel{\mathsf{E}} \langle y_1, y_2\rangle$ iff $x_1 \mathrel{\mathsf{F}_1} y_1$ and $x_2 \mathrel{\mathsf{F}_2} y_2$.

If $X_a = X$ and $\mathsf{F}_a = \mathsf{F}$ for all a then the power notation F^A can be used instead of $\prod_{a\in A} \mathsf{F}_a$.

(o4) The **Fubini product** (ultraproduct) $\prod_{a\in A} \mathsf{F}_a \,/\, \mathscr{I}$ modulo an ideal $\mathscr{I}$ on A is the ER on the product space $\prod_a X_a$ defined as follows: $x \mathrel{\mathsf{E}} y$ iff the set $\{a : x(a) \not\mathrel{\mathsf{F}_a} y(a)\}$ belongs to $\mathscr{I}$.

If $X_a = X$ and $\mathsf{F}_a = \mathsf{F}$ for all a then the ultrapower notation $\mathsf{F}^A/\mathscr{I}$ can be used instead of $\prod_{a\in A} \mathsf{F}_a \,/\, \mathscr{I}$.

(o5) **Countable power** of an equivalence relation F on a set X is an ER F^+ defined on the set $X^\mathbb{N}$ as follows: $x \mathrel{\mathsf{F}^+} y$ iff $\{[x(k)]_\mathsf{E} : k \in \mathbb{N}\} = \{[y(k)]_\mathsf{E} : k \in \mathbb{N}\}$, so that for any k there is l with $x(k) \mathrel{\mathsf{F}} y(l)$ and for any l there is k with $x(k) \mathrel{\mathsf{F}} y(l)$.

Example 3.2. In these terms, the equivalence relations E_1 and E_3 coincide with resp. $(\mathsf{EQ}_{2^\mathbb{N}})^\mathbb{N}/\mathtt{Fin}$ and ${\mathsf{E}_0}^\mathbb{N}$ modulo obvious bijections between the spaces considered. Generally, the operations on ideals introduced in Section 2d transform in some regular way into operations on the corresponding equivalence relations. For instance $\mathsf{E}_{\sum_{a\in A}\mathscr{J}_a/\mathscr{I}}$ is equal to $\prod_{a\in A}\mathsf{E}_{\mathscr{J}_a}/\mathscr{I}$, while $\mathsf{E}_{\mathscr{I}\otimes\mathscr{J}}$ is equal to $(\mathsf{E}_{\mathscr{J}})^A/\mathscr{I}$, where A is the domain of $\mathscr{I}$.

Accordingly, $\mathsf{E}_{\sum_a \mathscr{J}_a}$ is equal to $\prod_a \mathsf{E}_{\mathscr{J}_a}$. In particular if $\mathscr{I}, \mathscr{J}$ are ideals on disjoint sets A, B then $\mathsf{E}_{\mathscr{I}\oplus\mathscr{J}}$ is equal to $\mathsf{E}_{\mathscr{I}} \times \mathsf{E}_{\mathscr{J}}$. □

Example 3.3. The equivalence relation T_2 defined in Section 3a coincides with $\mathsf{EQ}_{\mathbb{N}^\mathbb{N}}{}^+$. □

Iterating these operations, we obtain a lot of interesting equivalence relations starting just with very primitive ones.

Example 3.4. Iterating the operation of countable power, H. Friedman [12] defines the sequence of ERs T_ξ, $1 \le \xi < \omega_1$, as follows [2]. Let $\mathsf{T}_1 = \mathsf{EQ}_{\mathbb{N}^\mathbb{N}}$, the equality relation on $\mathbb{N}^\mathbb{N}$. Put $\mathsf{T}_{\xi+1} = \mathsf{T}_\xi{}^+$ for all $\xi < \omega_1$. If $\lambda < \omega_1$ is a limit ordinal, then put $\mathsf{T}_\lambda = \bigvee_{\xi<\lambda}\mathsf{T}_\xi$. The definition for the second term T_2 is equivalent with the separate definition of T_2 in Section 3a by 3.3. □

3c Orbit equivalence relations of group actions

An *action* of a group $\mathbb{G}$ on a space $\mathbb{X}$ is any map $\mathfrak{a} : \mathbb{G}\times\mathbb{X}\to\mathbb{X}$, usually written as $\mathfrak{a}(g,x) = g\cdot x$, such that 1) $e\cdot x = x$, and 2) $g\cdot(h\cdot x) = (gh)\cdot x$. Then, for any $g\in\mathbb{G}$, the map $x\mapsto g\cdot x$ is a bijection $\mathbb{X}$ onto $\mathbb{X}$ with $x\mapsto g^{-1}\cdot x$ being the inverse map. A $\mathbb{G}$-*space* is a pair $\langle\mathbb{X};\mathfrak{a}\rangle$, where $\mathfrak{a}$ is an action of $\mathbb{G}$ on $\mathbb{X}$; in this case $\mathbb{X}$ itself is also called a $\mathbb{G}$-space, and the *orbit ER*, or *ER induced by the action*, $\mathsf{E}^\mathbb{X}_\mathfrak{a} = \mathsf{E}^\mathbb{X}_\mathbb{G}$ is defined on $\mathbb{X}$ so that $x\,\mathsf{E}^\mathbb{X}_\mathbb{G}\,y$ iff there is $g\in\mathbb{G}$ with $y = g\cdot x$. $\mathsf{E}^\mathbb{X}_\mathbb{G}$-classes are the same as $\mathbb{G}$-orbits, *i.e.*,

$$[x]_\mathbb{G} = [x]_{\mathsf{E}^\mathbb{X}_\mathbb{G}} = \{y : \exists\, g\in\mathbb{G}\ (g\cdot x = y)\}\,.$$

Recall that a *Polish group* is a group whose underlying set is a Polish space and the operations are continuous. A *Borel group* is a group whose underlying set is a Borel set (in a Polish space) and the operations are Borel maps. A Borel group is *Polishable* if there is a Polish topology on the underlying set which 1) produces the same Borel sets as the original topology and 2) makes the group Polish.

- If both $\mathbb{X}$ and $\mathbb{G}$ are Polish and the action continuous, then $\langle\mathbb{X};\mathfrak{a}\rangle$ (and also $\mathbb{X}$) is called a *Polish* $\mathbb{G}$-space. If both $\mathbb{X}$ and $\mathbb{G}$ are Borel and the action is a Borel map, then $\langle\mathbb{X};\mathfrak{a}\rangle$ (and also $\mathbb{X}$) is called a *Borel* $\mathbb{G}$-space.

Example 3.6. Any ideal $\mathscr{I}\subseteq\mathscr{P}(\mathbb{N})$ is a group with Δ as the operation. We cannot expect this group to be Polish in the product topology inherited from

[2] Hjorth [19] uses F_ξ instead of T_ξ.

$\mathscr{P}(\mathbb{N})$ (indeed, $\mathscr{I}$ would have to be $\mathbf{G}_\delta$). However if $\mathscr{I}$ is a P-ideal then it is Polishable (see Section 2b), in other words, $\langle \mathscr{I}\,;\Delta\rangle$ is a Polish group in an appropriate Polish topology compatible with the Borel structure of $\mathscr{I}$. Given such a topology, the Δ-action of (a P-ideal) $\mathscr{I}$ on $\mathscr{P}(\mathbb{N})$ is Polish, too. □

Example 3.7. $\mathbb{G} = \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ is a countable subgroup of $\langle \mathscr{P}(\mathbb{N})\,;\Delta\rangle$. Define an action of $\mathbb{G}$ on $2^{\mathbb{N}}$ as follows: $(w\cdot x)(n) = x(n)$ whenever $n \notin w$ and $(w\cdot x)(n) = 1 - x(n)$ otherwise. The orbit equivalence relation $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ of this action is obviously E_0. This action is Polish (given $\mathbb{G} = \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ the discrete topology) and *free*: $x = w\cdot x$ implies $w = \varnothing$ (the neutral element of $\mathbb{G}$) for any $x \in 2^{\mathbb{N}}$. □

Consider any Borel pairwise E_0-inequivalent set $T \subseteq 2^{\mathbb{N}}$. Then $w\cdot T \cap T = \varnothing$ for any $\varnothing \neq w \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ by the above. It easily follows that T is meager in $2^{\mathbb{N}}$. (Otherwise T is co-meager on a basic clopen set $\mathscr{O}_s(2^{\mathbb{N}}) = \{x \in 2^{\mathbb{N}} : s \subset x\}$, where $s \in 2^{<\omega}$. Put $w = \{n\}$, where $n = \mathtt{lh}\, s$. Then $w \in \mathbb{G}$ maps $T \cap \mathscr{O}_{s^\wedge 0}(2^{\mathbb{N}})$ onto $T \cap \mathscr{O}_{s^\wedge 1}(2^{\mathbb{N}})$. Thus $w\cdot T \cap T \neq \varnothing$ – contradiction.) We conclude that $\mathbb{G}\cdot T = \bigcup_{w\in\mathbb{G}} w\cdot T$ is still a meager subset of $2^{\mathbb{N}}$ in this case, and hence T cannot be a full (Borel) transversal for E_0.

Example 3.8. The *canonical* (or *shift*) *action* of a group $\mathbb{G}$ on a set of the form $X^{\mathbb{G}}$ (X any set) is defined as follows: $g\cdot\{x_f\}_{f\in\mathbb{G}} = \{x_{g^{-1}f}\}_{f\in\mathbb{G}}$ for any element $\{x_f\}_{f\in\mathbb{G}} \in X^{\mathbb{G}}$ and any $g \in \mathbb{G}$. This is easily a Polish action provided $\mathbb{G}$ is countable, X a Polish space, and $X^{\mathbb{G}}$ given the product topology. The equivalence relation on $X^{\mathbb{G}}$ induced by this action is denoted by $\mathsf{E}(\mathbb{G}, X)$. □

Example 3.9. The free group of two generators F_2 consists of finite irreducible words composed of the symbols a, b, a^{-1}, b^{-1}, including the empty word (the neutral element). The group operation is the concatenation of words (followed by reduction, if necessary, *e.g.* $ab \cdot b^{-1}a = aa$).

The shift action of F_2 on the compact space 2^{F_2} is defined in accordance with the general scheme of Example 3.8, so that if $x \in 2^{F_2}$ and $w \in F_2$ then $(w\cdot x)(u) = x(w^{-1}u)$ for all $u \in F_2$. Put, for $x, y \in 2^{F_2}$, $x \,\mathsf{E}_\infty\, y$ iff $x = w\cdot y$ for some $w \in F_2$. Thus E_∞ is $\mathsf{E}(F_2, 2)$ in the sense of 3.8. □

Example 3.10. Come back to Banach spaces $\boldsymbol{\ell}^\infty, \boldsymbol{\ell}^p, \mathbf{c}, \mathbf{c}_0$ discussed in Section 3a. Each of them can be considered as a Polish group in the sense of componentwise addition in $\mathbb{R}^{\mathbb{N}}$. Each of them canonically acts on $\mathbb{R}^{\mathbb{N}}$ also by componentwise addition. For the sake of brevity, the orbit equivalence relations of these actions, *i.e.* $\mathsf{E}^{\mathbb{R}^{\mathbb{N}}}_{\boldsymbol{\ell}^\infty}, \mathsf{E}^{\mathbb{R}^{\mathbb{N}}}_{\boldsymbol{\ell}^p}, \mathsf{E}^{\mathbb{R}^{\mathbb{N}}}_{\mathbf{c}}, \mathsf{E}^{\mathbb{R}^{\mathbb{N}}}_{\mathbf{c}_0}$, are denoted by the same symbols resp. $\boldsymbol{\ell}^\infty, \boldsymbol{\ell}^p, \mathbf{c}, \mathbf{c}_0$. □

Example 3.11. The group S_∞ of all permutations of $\mathbb{N}$ (that is, all bijections $f : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$, with the superposition as the group operation) is a Polish group in the Polish product topology of $\mathbb{N}^{\mathbb{N}}$. It acts on any set of the form $X^{\mathbb{N}}$ as follows: for any $g \in S_\infty$ and $x \in X^{\mathbb{N}}$, $(g\cdot x)(k) = x(g^{-1}(k))$ for all k, or equivalently $(g{\cdot}x)(g(k)) = x(k)$ for all k. Formally, $g{\cdot}x = xg^{-1}$ the the sense of the superposition in the right-hand side.

Take $X = \mathbb{N}^{\mathbb{N}}$. Note that $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ with the product topolody is a Polish space and the above action is Polish. Its orbit equivalence $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ is quite similar to T_2, but in fact not equal. Indeed if $x, y \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ satisfy $x(0) = x(1) = y(0) = u$ and $x(k) = u(l) = v$ for all $k \ge 2,\ l \ge 1$, where $u \ne v \in \mathbb{N}^{\mathbb{N}}$, then $x \mathrel{\mathsf{T}_2} y$ holds while $x\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ fails. Still Lemma 4.2 will prove that T_2 and $\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ are Borel equivalent. □

3d Borel and Polish actions

The next theorem (too difficult to be proved here) shows that the type of the group is the essential component in the difference between Polish and Borel actions: roughly, any Borel action of a Polish group $\mathbb{G}$ is a Polish action of $\mathbb{G}$.

Theorem 3.12 ([3, 5.2.1])**.** *Suppose that $\mathbb{G}$ is a Polish group and $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space. Then $\mathbb{X}$ admits a Polish topology which 1) produces the same Borel sets as the original topology, and 2) makes the action to be Polish.* □

If $\langle \mathbb{X}\,; \mathfrak{a}\rangle$ is a Borel $\mathbb{G}$-space (and $\mathbb{G}$ is a Borel group) then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is easily a $\boldsymbol{\Sigma}^1_1$ equivalence relation on $\mathbb{X}$. Sometimes $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is even Borel: for instance, when $\mathbb{G}$ is a countable group and the action is Borel, or if $\mathbb{G} = \mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a Borel ideal, considered as a group with Δ as the operation, which acts on $\mathbb{X} = \mathscr{P}(\mathbb{N})$ by Δ — thus $\mathsf{E}_{\mathbb{G}}^{\mathscr{P}(\mathbb{N})} = \mathsf{E}_{\mathscr{I}}$ is a Borel relation because $x \mathrel{\mathsf{E}_{\mathbb{G}}^{\mathscr{P}(\mathbb{N})}} y$ is equivalent to $x \,\Delta\, y \in \mathscr{I}$. Several much less trivial cases when $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel are described in [3, Chapter 7], for instance, if all $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$-classes are Borel sets of bounded rank then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel [3, 7.1.1]. Yet rather surprisingly equivalence classes generated by Borel actions are always Borel.

Theorem 3.13 (see [34, 15.14])**.** *If $\mathbb{G}$ is a Polish group and $\langle \mathbb{X}\,; \mathfrak{a}\rangle$ is a Borel $\mathbb{G}$-space then every equivalence class of $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel.*

The first notable case of this theorem was established by Scott [53] in the course of the proof that for any countable order type t (not necessarily well-ordered) the set of all sets $x \subseteq \mathbb{Q}$ of order type t is Borel in $\mathscr{P}(\mathbb{Q})$.

Proof. It can be assumed, by Theorem 3.12, that the action is continuous. Then for any $x \in \mathbb{X}$ the *stabilizer* $\mathbb{G}_x = \{g : g{\cdot}x = x\}$ is a closed subgroup of $\mathbb{G}$. [3] We

[3] Kechris [34, 9.17] gives an independent proof. Both $\mathbb{G}_x$ and its topological closure, say, G' are subgroups, moreover, G' is a closed subgroup, hence, we can assume that $G' = \mathbb{G}$, in other words, that $\mathbb{G}_x$ is dense in $\mathbb{G}$, and the goal is to prove that $\mathbb{G}_x = \mathbb{G}$. By a simple argument, $\mathbb{G}_x$ is either comeager or meager in $\mathbb{G}$. But a comeager subgroup easily coincides with the whole group, hence, assume that $\mathbb{G}_x$ is meager (and dense) in $\mathbb{G}$ and draw a contradiction.

Let $\{V_n\}_{n\in\mathbb{N}}$ be a basis of the topology of $\mathbb{X}$, and $A_n = \{g \in \mathbb{G} : g{\cdot}x \in V_n\}$. Easily $A_n h = A_n$

can consider $\mathbb{G}_x$ as continuously acting on $\mathbb{G}$ by $g \cdot h = gh$ for all $g, h \in \mathbb{G}$. Let F denote the associated orbit ER. Then every F-class $[g]_{\mathsf{F}} = g\,\mathbb{G}_x$ is a shift of $\mathbb{G}_x$, hence, $[g]_{\mathsf{F}}$ is closed. On the other hand, the saturation $[\mathscr{O}]_{\mathsf{F}}$ of any open set $\mathscr{O} \subseteq \mathbb{G}$ is obviously open. Therefore, by Lemma 6.2(iv) below, F admits a Borel transversal $S \subseteq \mathbb{G}$. Yet $g \longmapsto g \cdot x$ is a Borel $1-1$ map of a Borel set S onto $[x]_{\mathsf{E}}$, hence, $[x]_{\mathsf{E}}$ is Borel by Countable-to-1 Projection. □

It follows that not all $\mathbf{\Sigma}^1_1$ ERs are orbit ERs of Borel actions of Polish groups: indeed, take a non-Borel $\mathbf{\Sigma}^1_1$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$, define $x \mathrel{\mathsf{E}} y$ if either $x = y$ or x, $y \in X$, this is a $\mathbf{\Sigma}^1_1$ ER with a non-Borel class X.

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for any $h \in \mathbb{G}_x$. It follows, because $\mathbb{G}_x$ is dense, that every A_n is either meager or comeager. Now, if $g \in \mathbb{G}$ then $\{g\} = \bigcap_{n \in N(g)} A_n$, where $N(g) = \{n : g \cdot x \in V_n\}$, thus, at least one of sets A_n containing g is meager. It follows that $\mathbb{G}$ is meager, contradiction.

Chapter 4

Borel reducibility of equivalence relations

There are several reasonable ways to compare equivalence relations in terms of existence of a *reduction*, that is, a map of certain kind which allows to derive one of the ERs from the other one. The Borel reducibility $\le_{\rm B}$ is the key one. The plan of this Chapter is to define $\le_{\rm B}$ and present a diargam which displays mutual $\le_{\rm B}$-reducibility of the equivalence relations introduced in Section 3a (the key equivalence relations). The proof of related reducibility/irreducibility claims will be the mail content of the remainder of the book.

4a Borel reducibility

Suppose that E and F are equivalence relations on Borel sets X, Y in some Polish spaces. We define

$\mathsf{E} \le_{\rm B} \mathsf{F}$ (*Borel reducibility* of E to F) iff there is a Borel map $\vartheta : X \to Y$ (called *reduction*) such that $x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in X$;

$\mathsf{E} \sim_{\rm B} \mathsf{F}$ iff $\mathsf{E} \le_{\rm B} \mathsf{F}$ and $\mathsf{F} \le_{\rm B} \mathsf{E}$ (*Borel bi-reducibility*, or *Borel equivalence*);

$\mathsf{E} <_{\rm B} \mathsf{F}$ iff $\mathsf{E} \le_{\rm B} \mathsf{F}$ but not $\mathsf{F} \le_{\rm B} \mathsf{E}$ (*strict Borel reducibility*).

If $\mathsf{E} \le_{\rm B} \mathsf{F}$ (resp. $\mathsf{E} <_{\rm B} \mathsf{F}$, $\mathsf{E} \sim_{\rm B} \mathsf{F}$) then E is said to be *Borel reducible* (resp. *Borel strictly reducible*, *Borel equivalent* or *bi-reducible*) to F.

$\mathsf{E} \sqsubseteq_{\rm B} \mathsf{F}$ iff there is a Borel *embedding*, *i.e.*, a $1-1$ reduction;

$\mathsf{E} \approx_{\rm B} \mathsf{F}$ iff $\mathsf{E} \sqsubseteq_{\rm B} \mathsf{F}$ and $\mathsf{F} \sqsubseteq_{\rm B} \mathsf{E}$ (a rare form, [22, § 0]);

$\mathsf{E} \sqsubseteq_{\rm B}^{\rm i} \mathsf{F}$ iff there is a Borel *invariant* embedding, *i.e.*, an embedding ϑ such that $\mathtt{ran}\,\vartheta = \{\vartheta(x) : x \in \mathbb{X}\}$ is an F-*invariant* set (meaning that the F-*saturation* $[\mathtt{ran}\,\vartheta]_{\mathsf{F}} = \{y' : \exists\, x\, (y \mathrel{\mathsf{F}} \vartheta(x))\}$ equals $\mathtt{ran}\,\vartheta$);

Sometimes they write $\mathbb{X}/\mathsf{E} \le_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$ instead of $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$.

Borel reducibility of ideals: $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{J}}$. Thus it is required that there is a Borel map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $x \,\Delta\, y \in \mathscr{I}$ iff $\vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{J}$. (Here $\mathscr{I}, \mathscr{J}$ are ideals on countable sets A, B.)

In the domain of ideals, $\le_{\mathrm{B}}$ is weaker than all reducibilities of more special nature discussed in Section 2f — in the sense that, for instance, each of $\mathscr{I} \le_{\mathrm{RB}} \mathscr{J}$ and $\mathscr{I} \le_{\mathrm{B}}^{\Delta} \mathscr{J}$ implies $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$. The only exception is the reducibility via inclusion $\le_{\mathrm{I}}$ — it does not imply $\le_{\mathrm{B}}$. Indeed we have $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$ while the summable ideal $\mathscr{S}_{\{1/n\}}$ and the density-0 ideal $\mathscr{Z}_0$ are known to be $\le_{\mathrm{B}}$-incomparable, see below. [1]

It would be interesting to figure out exact relationship between $\le_{\mathrm{B}}$ and the Δ-reducibility $\le_{\mathrm{B}}^{\Delta}$. If the next questions answers in the negative then the whole theory of Borel reducibility for Borel ideals can be greatly simplified because reduction maps which are Δ-homomorphisms are much easier to deal with.

Question 4.1. Is there a pair of Borel ideals $\mathscr{I}, \mathscr{J}$ such that $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ but not $\mathscr{I} \le_{\mathrm{B}}^{\Delta} \mathscr{J}$? □

The remainder of the book will be concentrated on the Borel reducibility/irreducibility theorems. The following rather elementary lemma gives a couple of examples.

Lemma 4.2. (i) $\mathsf{EQ}_{\mathbb{N}^{\mathbb{N}}} \sim_{\mathrm{B}} \mathsf{EQ}_{\mathbb{N}}{}^{+}$. (ii) $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}$ (see Example 3.11).

Proof. (i) By definition $\mathsf{EQ}_{\mathbb{N}}{}^{+}$ is an ER on $\mathbb{N}^{\mathbb{N}}$ and $x \,\mathsf{EQ}_{\mathbb{N}}{}^{+}\, y$ holds iff $\mathtt{ran}\, x = \mathtt{ran}\, y$. Thus the map $\vartheta(x) = \chi_{\mathtt{ran}\, x}$ (the characterictic function) witnesses that $\mathsf{EQ}_{\mathbb{N}}{}^{+} \le_{\mathrm{B}} \mathsf{EQ}_{\mathbb{N}^{\mathbb{N}}}$. To prove the converse put, for $x \in \mathbb{N}^{\mathbb{N}}$,

$$r(x) = \{x(0)\,,\ x(0) + x(1) + 1\,,\ x(0) + x(1) + x(2) + 2\,,\ \dots\}\,;$$

then $\vartheta(x) = \chi_{r(x)}$ witnesses $\mathsf{EQ}_{\mathbb{N}^{\mathbb{N}}} \le_{\mathrm{B}} \mathsf{EQ}_{\mathbb{N}}{}^{+}$.

(ii) Suppose that $x, y \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. Then $x \,\mathsf{T}_2\, y$ means that

$$\forall\, k\, \exists\, l\, (x(k) = y(l)) \quad \text{and} \quad \forall\, l\, \exists\, k\, (x(k) = y(l)),$$

while $x \,\mathsf{E}_{S_\infty}^{(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}}\, y$ means that there is a bijection $f : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ such that $x(k) = y(f(k))$ for all k. The latter condition is, generally speaking, stronger, but the two are equivalent provided for any k there exist infinitely many indices l such that $x(k) = x(l)$ and the same for y. It follows that the map $\vartheta : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$,

[1] Some examples of this kind were recently found in the class of Borel countable equivalence relations, see [1, 61].

defined so that $\vartheta(x) = x'$ iff $x'(2^n(2k+1)-1) = x(k)$ for all n, k, is a Borel reduction of T_2 to $\mathsf{E}_{S_\infty}^{(\mathbb{N}^\mathbb{N})^\mathbb{N}}$.

A Borel reduction ϑ of $\mathsf{E}_{S_\infty}^{(\mathbb{N}^\mathbb{N})^\mathbb{N}}$ to T_2 can be defined as follows: $\vartheta(x) = x'$, where $x'(k) = n_x(k)^\wedge x(k)$ for all k, $n_x(k)$ is the number of all l satisfying $x(l) = x(k)$ (including $l = k$) or 0 if there exist infinitely many of such l, and $n^\wedge a$ for $a \in \mathbb{N}^\mathbb{N}$ is defined as the only element of $\mathbb{N}^\mathbb{N}$ such that $(n^\wedge a)(0) = n$ and $(n^\wedge a)(j+1) = a(j)$ for all j. □

4b Borel, continuous, Baire measurable, additive reductions

The Borel reducibility and related notions in Section 4a admit weaker Baire measurable (BM, for brevity) versions, which claims that the reduction postulated to exist is only a BM, not necessarily Borel, map. (Recall that a map is *Baire measurable* if the preimages of open sets are sets with the Baire property.) Those versions will be denoted with a subscript BM instead of B. Also there are stronger continuous versions, that will be denoted with a subscript C. Thus

$\mathsf{E} \le_{\rm BM} \mathsf{F}$, $\mathsf{E} \sim_{\rm BM} \mathsf{F}$, $\mathsf{E} <_{\rm BM} \mathsf{F}$ mean the reducibility, resp., bi-reducibility, strict reducibility by **Baire measurable** maps.

$\mathsf{E} \le_{\rm C} \mathsf{F}$, $\mathsf{E} \sim_{\rm C} \mathsf{F}$, $\mathsf{E} <_{\rm C} \mathsf{F}$ mean the reducibility, resp., bi-reducibility, strict reducibility by **continuous** maps.

It is known that a Baire measurable map defined on a Polish space is continuous on a comeager set. Thus BM reducibility is the same as a Borel, or even continuous reducibility on a comeager set. On the other hand, according to the following result of Just [25] and Louveau [41], continuous reducibility on the full domain can sometimes be derived from Borel reducibility.

Lemma 4.3. *If* $\mathscr{I}$ *is a Borel ideal on a countable* A, E *an equivalence relation on a Polish space* $\mathbb{X}$, *and* $\mathsf{E}_\mathscr{I} \le_{\rm BM} \mathsf{E}$, *then* $\mathsf{E}_\mathscr{I} \le_{\rm C} \mathsf{E} \times \mathsf{E}$ (via a continuous reduction). *In addition there is a set* $X \subseteq A$, $X \notin \mathscr{I}$ *such that* $\mathsf{E}_{\mathscr{I} \restriction X} \le_{\rm C} \mathsf{E}$, *where* $\mathscr{I} \restriction X = \mathscr{I} \cap \mathscr{P}(X)$.

Here $\mathsf{E} \times \mathsf{E}$ is an equivalence relation on $\mathbb{X} \times \mathbb{X}$ defined so that $\langle x, y \rangle$ and $\langle x', y' \rangle$ are equivalent iff both $x \mathrel{\mathsf{E}} x'$ and $y \mathrel{\mathsf{E}} y'$. Note that $\mathsf{E} \times \mathsf{E} \le_{\rm C} \mathsf{E}$ holds for various equivalence relations E, and in such a case the condition $\mathsf{E}_\mathscr{I} \le_{\rm C} \mathsf{E} \times \mathsf{E}$ in the theorem can be replaced by $\mathsf{E}_\mathscr{I} \le_{\rm C} \mathsf{E}$.

Proof. We have to define continuous maps $\vartheta_0, \vartheta_1 : \mathscr{P}(A) \to \mathbb{X}$ such that, for any $x, y \in \mathscr{P}(\mathbb{N})$, $x \mathbin{\Delta} y \in \mathscr{I}$ iff both $\vartheta_0(x) \mathrel{\mathsf{E}} \vartheta_0(y)$ and $\vartheta_1(x) \mathrel{\mathsf{E}} \vartheta_1(y)$. Suppose *w.l.o.g.* that $A = \mathbb{N}$. Let $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathbb{X}$ witness that $\mathsf{E}_\mathscr{I} \le_{\rm BM} \mathsf{E}$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D = \bigcap_i D_i \subseteq \mathscr{P}(\mathbb{N})$, all D_i being dense open, and $D_{i+1} \subseteq D_i$. A sequence $0 = n_0 < n_1 < n_2 < \dots$ and, for any i, a set $u_i \subseteq [n_i, n_{i+1})$ can be easily defined, by induction on i, so that $x \cap [n_i, n_{i+1}) = u_i \Longrightarrow x \in D_i$. [2] Let

$$N_1 = \bigcup_i [n_{2i}, n_{2i+1}), \quad N_2 = \bigcup_i [n_{2i+1}, n_{2i+2}), \quad U_1 = \bigcup_i u_{2i}, \quad U_2 = \bigcup_i u_{2i+1}.$$

Now set $\vartheta_1(x) = \vartheta((x \cap N_1) \cup U_2)$ and $\vartheta_2(x) = \vartheta((x \cap N_2) \cup U_1)$ for $x \subseteq \mathbb{N}$.

To prove the second claim let X be that one of the sets N_1, N_2 which does not belong to $\mathscr{I}$. (Or any of them if neither belongs to $\mathscr{I}$.) Let say $X = N_1 \notin \mathscr{I}$. Then the map ϑ_1 proves $\mathsf{E}_{\mathscr{I} \upharpoonright X} \le_{\mathrm{C}} \mathsf{E}$. □

The following question should perhaps be answered in the negative in general and be open for some particular cases.

Question 4.4. Suppose that $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ are Borel ERs. Does there always exist a *continuous* reduction ? □

There is a special useful type of continuous reducibility, actually a "clone" of the Rudin–Blass order of ideals considered in Section 2f.

Suppose that $X = \prod_{k \in \mathbb{N}} X_k$ and $Y = \prod_{k \in \mathbb{N}} Y_k$, the sets X_i, Y_i are finite, $0 = n_0 < n_1 < n_2 < \dots$, and $H_i : X_i \to \prod_{n_i \le k < n_{i+1}} Y_k$ for any i. Define

$$\Psi(x) = H_0(x(0)) \cup H_1(x(1)) \cup H_2(x(2)) \cup \dots \in Y$$

for each $x \in X$. Maps Ψ of this kind are called *additive* (Farah [9]). More generally, if, in addition, $0 = m_0 < m_1 < m_2 < \dots$, and $H_i : \prod_{m_i \le j < m_{i+1}} X_j \to \prod_{n_i \le k < n_{i+1}} Y_k$ for any i, then define

$$\Psi(x) = H_0(x \upharpoonright [m_0, m_1)) \cup H_1(x \upharpoonright [m_1, m_2)) \cup H_2(x \upharpoonright [m_2, m_3)) \cup \dots \in Y$$

for each $x \in X$. Farah [9] calls maps Ψ of this kind *asymptotically additive*. All of them are continuous functions $X \to Y$ in the sense of the product Polish topology. (Recall that X_i, Y_i are finite.)

Suppose now that E and F are ERs on resp. $X = \prod_k X_k$ and $Y = \prod_k Y_k$.

Additive reducibility: $\mathsf{E} \le_{\mathrm{A}} \mathsf{F}$ if there is an additive reduction of E to F. As usual $\mathsf{E} \sim_{\mathrm{A}} \mathsf{F}$ means that simultaneously $\mathsf{E} \le_{\mathrm{A}} \mathsf{F}$ and $\mathsf{F} \le_{\mathrm{A}} \mathsf{E}$, while $\mathsf{E} <_{\mathrm{A}} \mathsf{F}$ means that $\mathsf{E} \le_{\mathrm{A}} \mathsf{F}$ but not $\mathsf{F} \le_{\mathrm{A}} \mathsf{E}$.

A version: $\mathsf{E} \le_{\mathrm{AA}} \mathsf{F}$ if there exists an asymptotically additive reduction of E to F.

The additive reducibility coincides with $\le_{\mathrm{RB}}^{++}$ on the domain of Borel ideals:

Lemma 4.5 (Farah [9]**).** *Assume that $\mathscr{I}$ and $\mathscr{J}$ are Borel ideals on $\mathbb{N}$. Then $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.*

[2] Sets like u_i are called *stabilizers*, they are of much help in study of Borel ideals.

By definition $\mathsf{E}_{\mathscr{I}}$ and $\mathsf{E}_{\mathscr{J}}$ are equivalence relations on $\mathscr{P}(\mathbb{N})$, however we can consider them as ERs on $2^{\mathbb{N}} = \prod_{k\in\mathbb{N}}\{0,1\}$, as usual, which yields the intended meaning for the relation $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.

Proof. If $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ via a sequence of finite sets w_i with $\mathtt{max}\, w_i < \mathtt{min}\, w_{i+1}$ then we put $n_0 = 0$ and $n_i = \mathtt{min}\, w_i$ for $k \ge 1$, so that $w_i \subseteq [n_i, n_{i+1})$, and, for any i, put $H_i(0) = [n_i, n_{i+1}) \times \{0\}$ and let $H_i(1)$ be the characteristic function of w_i within $[n_i, n_{i+1})$. Conversely, if $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$ via a sequence $0 = n_0 < n_1 < n_2 < \dots$ and a family of maps $H_i : \{0,1\} \to 2^{[n_i, n_{i+1})}$ then $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ via the sequence of sets $w_i = \{k \in [n_i, n_{i+1}) : H_i(0)(k) \ne H_i(1)(k)\}$. □

4c Diargam of Borel reducibility of key equivalence relations

The diagram on page 26 begins, at the low end, with cardinals $1 \le n \in \mathbb{N}$, $\aleph_0$, $\mathfrak{c}$, naturally identified with the equivalence relation of equality on resp. finite (of a certain number n of elements), countable, uncountable Polish spaces. As all uncountable Polish spaces are Borel isomorphic, the equivalence relations $\mathsf{EQ}_{\mathbb{X}}$, $\mathbb{X}$ a Polish space, are characterized, modulo $\le_{\mathrm{B}}$, or even modulo Borel isomorphism between the domains, by the cardinality of the domain, which can be any finite $1 \le n < \omega$, or $\aleph_0$, or $\mathfrak{c} = 2^{\aleph_0}$.

The linearity breaks above E_0 : each one of the four equivalence relations E_1, E_2, E_3, E_∞ of the next level is strictly $<_{\mathrm{B}}$-bigger than E_0, and they are pairwise $\le_{\mathrm{B}}$-incomparable with each other.

The framebox [?] points on an interesting open problem (Question 4.8 below). The framebox [$\mathbf{c}_0$-eqs] denotes $\mathbf{c}_0$-*equalities*, a family of Borel ERs introduced by Farah [9], all of them are $\le_{\mathrm{B}}$-between E_3 and $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, and there is continuum-many $\le_{\mathrm{B}}$-incomparable among them.

The "non-P domain" denotes the family of all Borel ERs that cannot be induced by a Polish action. E_1 belongs to this family, and it is conjectured that E_1 is a $\le_{\mathrm{B}}$-least ER in this family. Solecki [57, 58] proved this conjecture for ERs generated by Borel ideals: for instance for a Borel ideal $\mathscr{I}$ to be not a P-ideal it is necessary and sufficient that $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. See Chapter 8 for more details.

Finally, the framebox [ctble] denotes the family of all Borel countable ERs (meaning that equivalence classes are at most countable); all of them are Borel reducible to E_∞. The following theorem of Adams – Kechris [2] shows that this is quite a rich family.

Theorem 4.6 (not to be proved here). *There is continuum many pairwise $\le_{\mathrm{B}}$-incomparable countable Borel equivalence relations.*

A somewhat weaker result that implies the existence of continuum many pairwise $\le_{\mathrm{B}}$-incomparable (not necessarily countable) Borel equivalence relations will be established by Theorem 14.12.

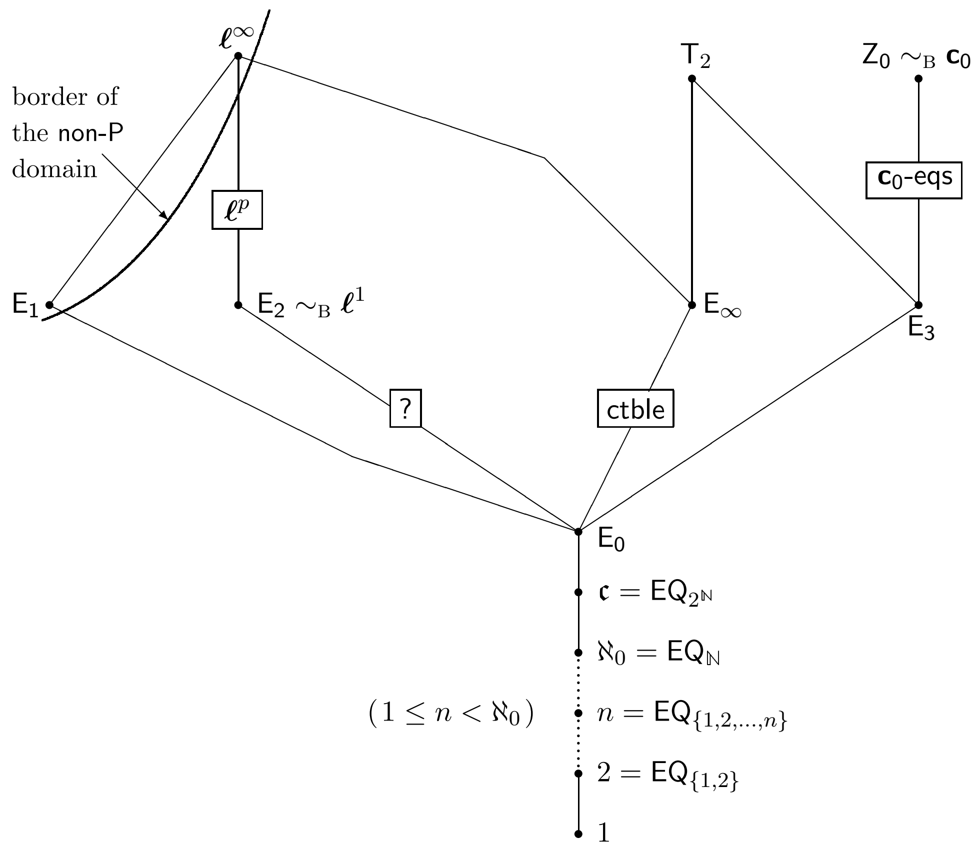

Figure 1: Reducibility between the key equivalence relations. Connecting lines here indicate Borel reducibility of lower ERs to upper ones.

4d Reducibility and irreducibility on the diagram

Here we discuss, without going into technicalities, the structure of the diagram on page 26 and related theorems.

Recall that any straight line on the diagram indicates Borel reducibility of the ER at the lower end to the ERat the upper end. Some of these reducibility claims are witnessed by a simple and obvious reductions. Slightly less obvious are reductions of E_∞ and E_3 to T_2 and E_3 to $\mathbf{c}_0$, see lemmas 5.2, 5.3. Finally, to prove that $\mathsf{E}_1, \mathsf{E}_\infty$, and all of $\boldsymbol{\ell}^p$ (including $\boldsymbol{\ell}^1 \sim_{\mathrm{B}} \mathsf{E}_2$), are Borel reducible to $\boldsymbol{\ell}^\infty$, we apply Rosendal's theorem in [52] saying that $\boldsymbol{\ell}^\infty$ is a $\le_{\mathrm{B}}$-largest $\mathbf{F}_\sigma$.

That $\mathsf{E}_2 \sim_{\mathrm{B}} \boldsymbol{\ell}^1$ and $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$ see lemmas 5.6 and 5.7.

See Theorem 5.11 on the equivalence $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^q \iff p \le q$.

It is the most interesting question whether the diagram on page 26 is complete in the sense that there is no Borel reducibility interrelations between the ERs mentioned in the diagram except for those explicitly indicated by straight lines.

Some of these irreducibility claims are trivial by a simple cardinality argument: clearly an ER E having strictly more equivalence classes than F is not Borel reducible to F.

However this argument is not applicable in more complicated cases, beginning with the irreducibility claim $\mathsf{E}_0 \not\leq_{\rm B} \mathsf{EQ}_{2^{\mathbb{N}}}$: each of the two relations has exactly continuum-many classes. Here we have to employ the borelness. Suppose towards the contrary that $\vartheta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel reduction of E_0 to $\mathsf{EQ}_{2^{\mathbb{N}}}$. Then the pre-image $\{x : \vartheta(x) = y\}$ of any $y \in 2^{\mathbb{N}}$ is countable (or empty). We conclude, using some classical theorems of descriptive set theory (theorems 1.6 and 1.3) that there is a Borel set $T \subseteq 2^{\mathbb{N}}$ having exactly one element with each E_0-class. But this contradicts the borelness of T — see a short argument after Example 3.7. [3]

As for the rest of the diagram, to establish its completeness one has to prove the following irreducibility claims:

(1) $\mathsf{E}_1 \not\leq_{\rm B}$: E_2, T_2, $\mathbf{c}_0$;

(2) $\boldsymbol{\ell}^\infty \not\leq_{\rm B}$: E_1, E_2, T_2, $\mathbf{c}_0$;

(3) $\mathsf{E}_2 \not\leq_{\rm B}$: E_1, T_2, $\mathbf{c}_0$;

(4) $\mathsf{E}_\infty \not\leq_{\rm B}$: E_1, E_2, $\mathbf{c}_0$ — **this group contains open problems**;

(5) $\mathsf{E}_3 \not\leq_{\rm B}$: $\boldsymbol{\ell}^\infty$;

(6) $\mathsf{T}_2 \not\leq_{\rm B}$: $\boldsymbol{\ell}^\infty$, $\mathbf{c}_0$;

(7) $\mathbf{c}_0 \not\leq_{\rm B}$: $\boldsymbol{\ell}^\infty$, T_2.

Beginning with (1), we note that E_1 is not Borel reducible to any equivalence relation induced by a Polish action by Theorem 9.9 (Kechris – Louveau). On the other hand, $\mathsf{E}_2, \mathsf{T}_2, \mathbf{c}_0$ obviously belong to this category of ERs.

(2) follows from (1) and (3) since $\mathsf{E}_1 \leq_{\rm B} \boldsymbol{\ell}^\infty$ and $\mathsf{E}_2 \leq_{\rm B} \boldsymbol{\ell}^\infty$.

The result $\mathsf{E}_2 \not\leq_{\rm B} \mathbf{c}_0$ in (3) is Hjorth's Theorem 5.8(ii). The result $\mathsf{E}_2 \not\leq_{\rm B} \mathsf{E}_1$ (Corollary 8.4) will be established by reference to Kechris' Theorem 8.1 on the structure of ideals Borel reducible to E_1.

The results $\mathsf{E}_2 \not\leq_{\rm B} \mathsf{T}_2$ and $\mathbf{c}_0 \not\leq_{\rm B} \mathsf{T}_2$ in (3) and (7) are proved below in Chapter 11 (Corollary 11.17); this will involve turbulence theory by Hjorth and Kechris.

The result of (5) is Lemma 5.1. It also implies $\mathbf{c}_0 \not\leq_{\rm B} \boldsymbol{\ell}^\infty$ in (7).

(6) was obtained by Hjorth, see Chapter 15.

This leaves us with (4). We don't know how to prove $\mathsf{E}_\infty \not\leq_{\rm B} \mathsf{E}_1$ easily and directly. There are two indirect ways. The first one is to apply some results in the theory of countable and hyperfinite equivalence relations — see Corollary 9.2. The second one is based on theorems 9.4 (3rd dichotomy) and 9.9 —

[3] Alternatively, one can derive $\mathsf{EQ}_{2^{\mathbb{N}}} \leq_{\rm B} \mathsf{E}_0$ from an old result of Sierpiński [54]: any linear ordering of all E_0-classes yields a Lebesgue non-measurable set of the same descriptive complexity as the given ordering.

see Corollary 9.11.

Question 4.7. Is E_∞ Borel reducible to $\mathbf{c}_0$? to $\boldsymbol{\ell}^1$ or any other $\boldsymbol{\ell}^p$? □

A related question whether E_∞ is Borel reducible to E_3 answers in the negative on the base of 6th dichotomy theorem by Corollary 12.2.

The irreducibility results in (1) – (7) can be partitioned into two rather distinct categories. The first group consists of those having proofs that involve only common methods of descriptive set theory, as the proof of $\mathsf{E}_0 \not\leq_{\rm B} \mathsf{EQ}_{2^{\mathbb{N}}}$ outlined above. This includes such results as $\mathsf{E}_2 \not\leq_{\rm B} \mathbf{c}_0$, $\boldsymbol{\ell}^\infty \not\leq_{\rm B} \mathbf{c}_0$, $\mathsf{E}_3 \not\leq_{\rm B} \boldsymbol{\ell}^\infty$, $\mathbf{c}_0 \not\leq_{\rm B} \boldsymbol{\ell}^\infty$, and also $\mathsf{E}_2 \not\leq_{\rm B} \mathsf{E}_1$ as a transitional claim between the first and second group: it refers to Theorem 8.1, a special result on the $\leq_{\rm B}$-structure of ideals below $\mathscr{I}_1$, rather complicated but still based on classics of descriptive set theory.

Note that some results in this group belong to the earliest of this type. For instance Just proved that E_2 is mutually $\leq_{\rm B}$-irreducible with Z_0 [26] and with $\mathsf{E}_{\mathtt{Fin}\otimes\mathtt{Fin}}$ [25]. According to [37, 1.4] the irreducibility claim $\mathsf{E}_1 \not\leq_{\rm B} \mathsf{E}_\infty$ goes back to even earlier paper [10].

The other group consists of irreducibility results that involve (as far as we know) methods that definitely go beyond common tools of descriptive set theory. This includes such resuts as $\mathsf{E}_1 \not\leq_{\rm B} \mathsf{E}_2$, $\mathsf{E}_1 \not\leq_{\rm B} \mathsf{T}_2$, $\mathsf{E}_1 \not\leq_{\rm B} \mathbf{c}_0$, based on the fact that E_1 is not reducible to a Polish action (Theorem 9.9), $\mathsf{E}_2 \not\leq_{\rm B} \mathsf{T}_2$ and $\mathbf{c}_0 \not\leq_{\rm B} \mathsf{T}_2$ based on the turbulence theory, $\mathsf{E}_\infty \not\leq_{\rm B} \mathsf{E}_1$ and $\mathsf{E}_\infty \not\leq_{\rm B} \mathsf{E}_3$ based on resp. 3th and 6th dichotomy theorems (see the next Section), and finally $\mathsf{T}_2 \not\leq_{\rm B} \boldsymbol{\ell}^\infty$ and $\mathsf{T}_2 \not\leq_{\rm B} \mathbf{c}_0$ based on the theory of *pinned* equivalence relations (Chapter 15).

4e Dichotomy theorems

Another general problem related to the diagram is the $\leq_{\rm B}$-structure of certain domains, for instance, $\leq_{\rm B}$-intervals between adjacent equivalence relations. Some results in this direction are known as dichotomy theorems because of their distinguished dichotomical form.

1st dichotomy (Theorem 7.1 below). *Any Borel, even any* $\mathbf{\Pi}^1_1$ *equivalence relation* E **either** *has at most countably many equivalence classes, formally,* $\mathsf{E} \leq_{\rm B} \mathsf{EQ}_{\mathbb{N}}$, **or** *satisfies* $\mathsf{EQ}_{2^{\mathbb{N}}} \leq_{\rm B} \mathsf{E}$.

Thus not only the strict $<_{\rm B}$-interval between the ERs $\aleph_0 = \mathsf{EQ}_{\mathbb{N}}$ and $\mathfrak{c} = \mathsf{EQ}_{2^{\mathbb{N}}}$ is empty, but the union of the lower $\leq_{\rm B}$-cone of the former and the upper $\leq_{\rm B}$-cone of the latter cover the whole family of Borel equivalence relations! The same is true for the next $<_{\rm B}$-interval:

2nd dichotomy (Thm 7.2). *Any Borel ER* E *satisfies* **either** $\mathsf{E} \leq_{\rm B} \mathfrak{c}$ **or** $\mathsf{E}_0 \leq_{\rm B} \mathsf{E}$.

What is going on in the $<_{\mathrm{B}}$-intervals between E_0 and the equivalence relations E_1, E_2, E_3? The following dichotomy theorems provide some answers.

3rd dichotomy (Theorem 9.4). *Any equivalence relation* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$ *satisfies* **either** $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ **or** $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_1$.

4th dichotomy (Theorem 13.1). *Any equivalence relation* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ **either** *is essentially countable* **or** *satisfies* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_2$.

An equivalence relation E is *essentially countable* iff it is Borel reducible to a Borel countable (*i.e.*, with at most countable equivalence classes) ER. The **either** case in 4th dichotomy remains mysterious. This is marked by the framebox [?] on the diagram.

Question 4.8. In 4th dichotomy, can the **either** case be improved to $\le_{\mathrm{B}} \mathsf{E}_0$? □

The fifth dichotomy theorem is a bit more special, it will be addressed below.

6th dichotomy (Theorem 12.1). *Any equivalence relation* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_3$ *satisfies* **either** $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ **or** $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

On the other hand, the interval between E_0 and E_∞ contains all countable Borel ERs and among them plenty of pairvise $\sim_{\mathrm{B}}$-inequivalent ERs by Theorem 4.6.

It was once considered [20] as a plausible hypothesis that any Borel ER which is not $\le_{\mathrm{B}} \mathsf{E}_\infty$, *i.e.*, not essentially countable, satisfies $\mathsf{E}_i \le_{\mathrm{B}} \mathsf{E}$ for at least one $i = 1, 2, 3$. This turns out to be not the case: Farah [8, 6] and Velickovic [63] found an independent family of uncountable Borel ERs, based on *Tsirelson ideals*, $\le_{\mathrm{B}}$-incomparable with $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$.

Question 4.9. It there any reasonable "basis" of Borel ERs above E_0? □

4f Borel ideals in the structure of Borel reducibility

Some of equivalence relations on the diagram are obviously generated by Borel ideals, for some other ones this is not clear. This leads to the question what is the place of Borel ideals in the whole structure od Borel equivalence relations. The answer obtained in the studies of last years can be formulated as follows: Borel ideals are $\le_{\mathrm{B}}$-cofinal, but rather rare, in the $\le_{\mathrm{B}}$-structure of Borel ERs. We prove the following theorem, the cofinality claim of which is due to Rosental [52] (Theorem 16.1 in Chapter 16) while on the other claim see Corollary 11.19.

Theorem 4.10. *For any Borel equivalence relation* E *there exists a Borel ideal* $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ *such that* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. *On the other hand there is no Borel ideal* $\mathscr{I}$ *such that* $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.

Chapter 5

"Elementary" stuff

This Chapter gathers proofs of some reducibility/irreducibility results related to the diagram on page 26, elementary in the sense that they do not involve any special concepts. Some of them are really simple, as *e.g.* some lemmas on E_3 and T_2 in Section 5a or the equivalences $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$ and $\mathsf{E}_2 \sim_{\mathrm{B}} \boldsymbol{\ell}^1$ in Section 5b, while some other quite tricky. The latter category includes Hjorth's theorem on the irreducibility of nontrivial summable ideals to $\mathbf{c}_0$ in Section 5c, interrelations in the family of equivalence relations $\boldsymbol{\ell}^b$ in Section 5d, and the $\le_{\mathrm{B}}$-universality of $\boldsymbol{\ell}^\infty$ in the class of all $\mathbf{F}_\sigma$ equivalence relations in Section 5e.

5a E_3 and T_2

These equivalence relations, together with $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, are the only non-$\boldsymbol{\Sigma}^0_2$ equivalences explicitly mentioned on the diagram.

Lemma 5.1. E_3 *is Borel irreducible to* $\boldsymbol{\ell}^\infty$.

Proof. Suppose towards the contrary that $\vartheta : 2^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a Borel reduction of E_3 to $\boldsymbol{\ell}^\infty$. [1] Since obviously $\boldsymbol{\ell}^\infty \sim_{\mathrm{B}} \boldsymbol{\ell}^\infty \times \boldsymbol{\ell}^\infty$, Lemma 4.3 reduces the general case to the case of continuous ϑ. Define $\mathbf{0}, \mathbf{1} \in 2^{\mathbb{N}}$ by $\mathbf{0}(n) = 0$, $\mathbf{1}(n) = 1$, $\forall\, n$. Define $\mathbb{0} \in 2^{\mathbb{N}\times\mathbb{N}}$ by $\mathbf{0}(k,n) = 0$ for all k,n, thus $(\mathbb{0})_k = \mathbf{0}$, $\forall\, k$. Finally, for any k define $\mathbf{z}_k \in 2^{\mathbb{N}}$ by $\mathbf{z}_k(n) = 1$ for $n < k$ and $\mathbf{z}_k(n) = 0$ for $n \ge k$.

We claim that there are increasing sequences of natural numbers $\{k_m\}$ and $\{j_m\}$ such that $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for any m and any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ satisfying

$$(x)_k = \begin{cases} \mathbf{z}_{k_i} & \text{whenever } i < m \text{ and } k = k_i \\ \mathbf{0} & \text{for all } k < k_m \text{ not of the form } k_i. \end{cases}$$

[1] Recall that, for $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$, $x \mathrel{\mathsf{E}_3} y$ means $(x)_k \mathrel{\mathsf{E}_0} (y)_k$, $\forall\, k$, where $(x)_k \in 2^{\mathbb{N}}$ is defined by $(x)_k(n) = x(k,n)$ for all n while $a \mathrel{\mathsf{E}_0} b$ means that $a \,\Delta\, b = \{m : a(m) \ne b(m)\}$ is finite.

To see that this implies contradiction define $x \in 2^{\mathbb{N}\times\mathbb{N}}$ so that $(x)_{k_i} = \mathbf{z}_{k_i}$, $\forall i$ and $(x)_k = \mathbf{0}$ whenever k does not have the form k_i. Then obviously $x \mathsf{E}_3 \mathbb{0}$, but $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for all m, hence $\vartheta(x) \boldsymbol{\ell}^\infty \vartheta(\mathbb{0})$ fails, as required.

We put $k_0 = 0$. To define j_0 and k_1, consider $x_0 \in 2^{\mathbb{N}\times\mathbb{N}}$ defined by $(x_0)_0 = \mathbf{1}$ but $(x_0)_k = \mathbf{0}$ for all $k \geq 1$. Then $x_0 \mathsf{E}_3 \mathbb{0}$ fails, and hence $\vartheta(x_0) \boldsymbol{\ell}^\infty \vartheta(\mathbb{0})$ fails either. Take any j_0 with $|\vartheta(x_0)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$. As ϑ is continuous, there is a number $k_1 > 0$ such that $|\vartheta(x)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$ holds for any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ with $(x)_0 = \mathbf{z}_{k_1}$ and $(x)_k = \mathbf{0}$ for all $0 < k < k_1$.

To define j_1 and k_2, consider $x_1 \in 2^{\mathbb{N}\times\mathbb{N}}$ defined so that $(x_1)_0 = \mathbf{z}_{k_1}$, $(x_1)_k = \mathbf{0}$ whenever $0 < k < k_1$, and $(x_1)_{k_1} = \mathbf{1}$. Once again there is a number j_1 with $|\vartheta(x_1)(j_1) - \vartheta(\mathbb{0})(j_1)| > 1$, and a number $k_2 > k_1$ such that $|\vartheta(x)(j_1) - \vartheta(\mathbb{0})(j_1)| > 1$ for any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ with $(x)_0 = \mathbf{z}_{k_1}$, $(x)_{k_1} = \mathbf{z}_{k_1}$, and $(x)_k = \mathbf{0}$ for all $0 < k < k_1$ and $k_1 < k < k_2$.

Et cetera. □

Lemma 5.2. E_3 *is Borel reducible to both* T_2 *and* $\mathbf{c}_0$.

Proof. (1) If $a \in 2^{\mathbb{N}}$ and $s \in 2^{<\omega}$ then define $sx \in 2^{\mathbb{N}}$ by $(sx)(k) = x(k) +_2 s(k)$ for $k < \mathtt{lh}\, s$ and $(sx)(k) = x(k)$ for $k \geq \mathtt{lh}\, s$. If $m \in \mathbb{N}$ then $m^\wedge x \in 2^{\mathbb{N}}$ denotes the concatenation. In these terms, if $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$ then obviously

$$x \mathsf{E}_3 y \iff \{m^\wedge(s(x)_m) : s \in 2^{<\omega},\ m \in \mathbb{N}\} = \{m^\wedge(s(y)_m) : s \in 2^{<\omega},\ m \in \mathbb{N}\}.$$

Now any bijection $2^{<\omega} \times \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ yields a Borel reduction of E_3 to T_2.

(2) To reduce E_3 to $\mathbf{c}_0$ consider a Borel map $\vartheta : 2^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $\vartheta(x)(2^n(2k+1) - 1) = n^{-1}(x)_n(k)$. □

Lemma 5.3. *Any countable Borel ER is Borel reducible to* T_2.

Proof. Let E be a countable Borel ER on $2^{\mathbb{N}}$. It follows from Countable-to-1 Enumeration that there is a Borel map $f : 2^{\mathbb{N}} \times \mathbb{N} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f(a,n) : n \in \mathbb{N}\}$ for all $a \in 2^{\mathbb{N}}$. The map ϑ sending any $a \in 2^{\mathbb{N}}$ to $x = \vartheta(a) \in 2^{\mathbb{N}\times\mathbb{N}}$ such that $(x)_n = f(a,n)$, $\forall n$, is a reduction required. □

See further study on T_2 in Chapter 15, where it will be shown that T_2 is not Borel reducible to a big family of equivalence relations that includes $\mathbf{c}_0, \boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty, \mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \mathsf{E}_\infty$. On the other hand, the equivalence relations in this list, with the exception of $\mathsf{E}_3, \mathsf{E}_\infty$, are not Borel reducible to T_2 — this follows from the turbulence theory presented in Chapter 11.

5b Discretization and generation by ideals

Some equivalence relations on the diagram on page 26 are explicitly generated by ideals, like E_i, $i = 0, 1, 2, 3$. Some other ERs are defined differently. It will be shown below (Chapter 16) that **any** Borel ER E is Borel reducible to a ER of the form $\mathsf{E}_{\mathscr{I}}$, $\mathscr{I}$ a Borel ideal. On the other hand, $\mathbf{c}_0, \boldsymbol{\ell}^1, \boldsymbol{\ell}^\infty$ turn out to be Borel equivalent to some meaningful Borel ideals. Moreover, these equivalence relations admit "discretization" by means of restriction to certain subsets of $\mathbb{R}^{\mathbb{N}}$.

Definition 5.4. We define $\mathbb{X} = \prod_{n\in\mathbb{N}} X_n = \{x \in \mathbb{R}^{\mathbb{N}} : \forall\, n\,(x(n) \in X_n)\}$, where $X_n = \{\frac{0}{2^n}, \frac{1}{2^n}, \dots, \frac{2^n}{2^n}\}$. □

Lemma 5.5. *$\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$ and $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction \mathbb{X}$ for any $1 \le p < \infty$.*

On the other hand, $\boldsymbol{\ell}^\infty \le_{\mathrm{B}} \boldsymbol{\ell}^\infty \restriction \mathbb{Z}^{\mathbb{N}}$.

Proof. We first show that $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Let π be any bijection of $\mathbb{N} \times \mathbb{Z}$ onto $\mathbb{N}$. For $x \in \mathbb{R}^{\mathbb{N}}$, define $\vartheta(x) \in [0,1]^{\mathbb{N}}$ as follows. Suppose that $k = \pi(n,\eta)$ $(\eta \in \mathbb{Z})$. If $\eta \le x(n) < \eta+1$ then let $\vartheta(x)(k) = x(n)$. If $x(n) \ge \eta+1$ then put $\vartheta(x)(k) = 1$. If $x(n) < \eta$ then put $\vartheta(x)(k) = 0$. Then ϑ is a Borel reduction of $\mathbf{c}_0$ to $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Now we prove that $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}} \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$. For $x \in [0,1]^{\mathbb{N}}$ define $\psi(x) \in \mathbb{X}$ so that $\psi(x)(n)$ the largest number of the form $\frac{i}{2^n}$, $0 \le i \le 2^n$ smaller than $x(n)$. Then obviously $x\,\mathbf{c}_0\,\psi(x)$ holds for any $x \in [0,1]^{\mathbb{N}}$, and hence ψ is a Borel reduction of $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$ to $\mathbf{c}_0 \restriction \mathbb{X}$.

Thus $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$, and hence in fact $\mathbf{c}_0 \sim_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$.

The argument for $\boldsymbol{\ell}^1$ is pretty similar. The result for $\boldsymbol{\ell}^\infty$ is obvious: given $x \in \mathbb{R}^{\mathbb{N}}$, replace any $x(n)$ by the largest integer value $\le x(n)$.

The version for $\boldsymbol{\ell}^p, 1 < p < \infty$, needs some comments in the first part (reduction to $[0,1]^{\mathbb{N}}$). Note that if $\eta \in \mathbb{Z}$ and $\eta - 1 \le x(n) < \eta < \zeta \le y(n) < \zeta+1$ then the value $(y(n)-x(n))^p$ in the distance $\|y-x\|_p = (\sum_n |y(n)-x(n)|^p)^{\frac{1}{p}}$ is replaced by $(\zeta-\eta) + (\eta - x(n))^p + (y(n)-\zeta)^p$ in $\|\vartheta(y)-\vartheta(x)\|_p$. Thus if this happens infinitely many times then both distances are infinite, while otherwise this case can be neglected. Further, if $\eta - 1 \le x(n) < \eta \le y(n) < \eta+1$ then $(y(n)-x(n))^p$ in $\|y-x\|_p$ is replaced by $(\eta-x(n))^p + (y(n)-\eta)^p$ in $\|\vartheta(y)-\vartheta(x)\|_p$. However $(\eta - x(n))^p + (y(n)-\eta)^p \le (y(n)-x(n))^p \le 2^{p-1}((\eta-x(n))^p + (y(n)-\eta)^p)$, and hence these parts of the sums in $\|y-x\|_p$ and $\|\vartheta(y)-\vartheta(x)\|_p$ differ from each other by a factor between 1 and 2^{p-1}. Finally, if $\eta \le x(n)$, $y(n) < \eta+1$ for one and the same $\eta \in \mathbb{Z}$ then the term $(y(n)-x(n))^p$ in $\|y-x\|_p$ appears unchanged in $\|\vartheta(y)-\vartheta(x)\|_p$. Thus totally $\|y-x\|_p$ is finite iff so is $\|\vartheta(y)-\vartheta(x)\|_p$. □

Lemma 5.6 (Oliver [51]**).** $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$. (Recall that $\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$.)

Proof. Prove that $\mathbf{c}_0 \le_{\mathrm{B}} \mathsf{Z}_0$. It suffices, by Lemma 5.5, to define a Borel reduction $\mathbf{c}_0 \restriction \mathbb{X} \to \mathsf{Z}_0$, *i.e.*, a Borel map $\vartheta : \mathbb{X} \to \mathscr{P}(\mathbb{N})$ such that $x\,\mathbf{c}_0\,y \iff \vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{Z}_0$ for all $x, y \in \mathbb{X}$. Let $x \in \mathbb{X}$. Then, for any n, we have $x(n) = \frac{k(n)}{2^n}$ for some natural $k(n) \le 2^n$. The value of $k(n)$ determines the intersection $\vartheta(x) \cap [2^n, 2^{n+1})$: for each $j < 2^n$, we define $2^n + j \in \vartheta(x)$ iff $j < k(n)$. Then $x(n) = \frac{\#(\vartheta(x)\cap[2^n,2^{n+1}))}{2^n}$ for any n, and moreover $|y(n)-x(n)| =$

$\frac{\#([\vartheta(x)\,\Delta\,\vartheta(y)]\cap[2^n,2^{n+1}))}{2^n}$ for all $x,y\in\mathbb{X}$ and n. This easily implies that ϑ is as required.

To prove $\mathsf{Z}_0\le_{\mathrm{B}}\mathbf{c}_0$, we have to define a Borel map $\vartheta:\mathscr{P}(\mathbb{N})\to\mathbb{R}^{\mathbb{N}}$ such that $x\,\Delta\,x\in\mathscr{Z}_0\iff\vartheta(x)\;\mathbf{c}_0\;\vartheta(x)$. Most elementary ideas like $\vartheta(x)(n)=\frac{\#(x\cap[0,n))}{n}$ do not work, the right way is based on the following observation: for any sets $s,t\subseteq[0,n)$ to satisfy $\#(s\,\Delta\,t)\le k$ it is necessary and sufficient that $|\#(s\,\Delta\,z)-\#(t\,\Delta\,z)|\le k$ for any $z\subseteq[0,n)$. To make use of this fact, let us fix an enumeration (with repetitions) $\{z_j\}_{j\in\mathbb{N}}$ of all finite subsets of $\mathbb{N}$ such that

$$\{z_j:2^n\le j<2^{n+1}\}\quad=\quad\text{all subsets of }[0,n)$$

for every n. Put, for any $x\in\mathscr{P}(\mathbb{N})$ and $2^n\le j<2^{n+1}$, $\vartheta(x)(j)=\frac{\#(x\cap z_j)}{n}$. Then $\vartheta:\mathscr{P}(\mathbb{N})\to[0,1]^{\mathbb{N}}$ is a required reduction. □

Recall that for any sequence of reals $r_n\ge 0$, $\mathsf{S}_{\{r_n\}}$ is an equivalence relation on $\mathscr{P}(\mathbb{N})$ generated by the ideal $\mathscr{S}_{\{r_n\}}=\{x\subseteq\mathbb{N}:\sum_{n\in x}r_n<+\infty\}$.

Lemma 5.7 (Attributed to Kechris in [17, 2.4])**.** *If* $r_n\ge 0$, $r_n\to 0$, $\sum_n r_n=+\infty$ *then* $\mathsf{S}_{\{r_n\}}\sim_{\mathrm{B}}\boldsymbol{\ell}^1$. *In particular,* $\mathsf{E}_2=\mathsf{S}_{\{1/n\}}$ *satisfies* $\mathsf{E}_2\sim_{\mathrm{B}}\boldsymbol{\ell}^1$.

Proof. To prove $\mathsf{S}_{\{r_n\}}\le_{\mathrm{B}}\boldsymbol{\ell}^1$, define $\vartheta(x)\in\mathbb{R}^{\mathbb{N}}$ for any $x\in\mathscr{P}(\mathbb{N})$ as follows: $\vartheta(x)(n)=r_n$ for any $n\in x$, and $\vartheta(x)(n)=0$ for any other n. Then $x\,\Delta\,y\in\mathscr{S}_{\{r_n\}}\iff\vartheta(x)\;\boldsymbol{\ell}^1\;\vartheta(y)$, as required.

To prove the other direction, it suffices to define a Borel reduction of $\boldsymbol{\ell}^1\restriction\mathbb{X}$ to $\mathsf{S}_{\{r_n\}}$. We can associate a (generally, infinite) set $s_{nk}\subseteq\mathbb{N}$ with any pair of n and $k<2^n$, so that the sets s_{nk} are pairwise disjoint and $\sum_{j\in s_{nk}}r_j=2^{-n}$. The map $\vartheta(x)=\bigcup_n\bigcup_{k<2^n x(n)}s_{nk}$, $x\in\mathbb{X}$, is the reduction required. □

5c Summables irreducible to density-0

The $\le_{\mathrm{B}}$-independence of $\boldsymbol{\ell}^1$ and $\mathbf{c}_0$, two best known “Banach” equivalence relations, is quite important. In one direction it is provided by (ii) of the next theorem. As for the other direction, Lemma 5.1 contains an even stronger irreducibility claim.

Is there any example of Borel ideals $\mathscr{I}\le_{\mathrm{B}}\mathscr{J}$ which do not satisfy $\mathscr{I}\le_{\mathrm{B}}^{\Delta}\mathscr{J}$? Typically the reductions found to witness $\mathscr{I}\le_{\mathrm{B}}\mathscr{J}$ are Δ-homomorphisms, and even better maps. The following lemma proves that Borel reduction yields $\le_{\mathrm{RB}}^{++}$-reduction in quite a representative case. Suppose that $\mathscr{I},\mathscr{J}$ are ideals over $\mathbb{N}$. Let us say that $\mathscr{I}\le_{\mathrm{RB}}^{++}\mathscr{J}$ *holds exponentially* if there exist a sequence of natural numbers k_i with and $k_{i+1}\ge 2k_i$ and a sequence of sets $w_i\subseteq[k_i,k_{i+1})$ that withesses $\mathscr{I}\le_{\mathrm{RB}}^{++}\mathscr{J}$ — in other words, the equivalence $A\in\mathscr{I}\iff w_A=\bigcup_{k\in A}w_k\in\mathscr{J}$ holds for any $A\subseteq\mathbb{N}$.

Theorem 5.8. *Suppose that* $r_n\ge 0$, $r_n\to 0$, $\sum_n r_n=+\infty$. *Then*

(i) (Farah [6, 2.1]) *If* $\mathscr{J}$ *is a Borel P-ideal and* $\mathscr{S}_{\{r_n\}} \le_{\rm B} \mathscr{J}$ *then we have* $\mathscr{S}_{\{r_n\}} \le_{\rm RB}^{++} \mathscr{J}$ *exponentially;*

(ii) (Hjorth [17]) $\mathscr{S}_{\{r_n\}}$ *is not Borel-reducible to* $\mathscr{Z}_0$.

Proof. (i) Let a Borel map $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ witness $\mathscr{S}_{\{r_n\}} \le_{\rm B} \mathscr{J}$. Let, according to Theorem 8.5, ν be a LSC submeasure on $\mathbb{N}$ with $\mathscr{J} = \mathtt{Exh}_\nu$. The construction makes use of stabilizers. Suppose that $n \in \mathbb{N}$. If $u, v \subseteq [0, n)$ then $(u \cup x) \,\Delta\, (v \cup x) = u \,\Delta\, v \in \mathscr{S}_{\{r_n\}}$ for any $x \subseteq [n, +\infty)$, therefore $\vartheta(u \cup x) \,\Delta\, \vartheta(v \cup x) \in \mathscr{J}$. It follows, by the choice of the submeasure ν, that for any $\varepsilon > 0$ there are numbers $n' > k > n$ and a set $s \subseteq [n, n')$ such that

$$\nu\big((\vartheta(u \cup s \cup x) \,\Delta\, \vartheta(v \cup s \cup x)) \cap [k, \infty)\big) \,<\, \varepsilon$$

holds for all $u, v \subseteq [0, n)$ and all generic $x \subseteq [n', \infty)$.

Remark 5.9. In the course of the proof, "generic" means Cohen-generic over a fixed countable transitive model $\mathfrak{M}$ of **ZFC**$^-$, the theory containing all of **ZFC** minus the Power Set axiom but plus the axiom: "for every set X, the countable power set $\mathscr{P}_{\mathtt{ctbl}}(X) = \{y \subseteq X : \mathtt{card}\, y \le \aleph_0\}$ exists".[2]

Note that Cohen-generic extensions of such a model are still models of **ZFC**$^-$.

We require that in addition $\mathfrak{M}$ contains all relevant real-type objects, together with codes of all relevant Borel sets. In particular, in the case considered, $\mathfrak{M}$ contains the sequence $\{r_n\}_{n\in\mathbb{N}}$ and also contains Borel codes of the ideal $\mathscr{I}$ and of the map ϑ. □

This allows us to define an increasing sequence of natural numbers $0 = k_0 = a_0 < b_0 < k_1 < a_1 < b_1 < k_2 < \dots$ and, for any i, a set $s_i \subseteq [b_i, a_{i+1})$ such that, for all generic $x, x \subseteq [a_{i+1}, \infty)$ and all $u, v \subseteq [0, b_i)$, we have

(1) $\nu\big((\vartheta(u \cup s_i \cup x) \,\Delta\, \vartheta(v \cup s_i \cup x)) \cap [k_{i+1}, \infty)\big) < 2^{-i}$;

(2) $\big(\vartheta(u \cup s_i \cup x) \,\Delta\, \vartheta(u \cup s_i \cup y)\big) \cap [0, k_{i+1}) = \varnothing$;

(3) any $z \subseteq \mathbb{N}$, satisfying $z \cap [b_i, a_{i+1}) = s_i$ for infinitely many i, is generic;

(4) $k_{i+1} \ge 2k_i$ for all i;

and in addition, under the assumptions on $\{r_n\}$,

(5) there is a set $g_i \subseteq [a_i, b_i)$ such that $|r_i - \sum_{n\in g_i} r_n| < 2^{-i}$.

It follows from (5) that $a \mapsto g_a = \bigcup_{i\in a} g_i$ is a reduction of $\mathscr{S}_{\{r_n\}}$ to $\mathscr{S}_{\{r_n\}} \restriction N$, where $N = \bigcup_i [a_i, b_i)$. Let $S = \bigcup_i s_i$; note that $S \cap N = \varnothing$.

[2] In fact generic points are precisely those which avoid certain meager $\mathbf{F}_\sigma$ sets, but the genericity assumption allows us not to specify those sets explicitly, giving instead a reference to $\mathfrak{M}$ where all relevant meager $\mathbf{F}_\sigma$ sets have to be coded.

Put $\xi(z) = \vartheta(z \cup S) \mathbin{\Delta} \vartheta(S)$ for any $z \subseteq N$. Then, for any sets $x, y \subseteq N$,

$$x \mathbin{\Delta} y \in \mathscr{S}_{\{r_n\}} \iff \vartheta(x \cup S) \mathbin{\Delta} \vartheta(y \cup S) \in \mathscr{J} \iff \xi(x) \mathbin{\Delta} \xi(y) \in \mathscr{J},$$

thus ξ reduces $\mathscr{S}_{\{r_n\}} \restriction N$ to $\mathscr{J}$. Now put $w_i = \xi(g_i) \cap [k_i, k_{i+1})$ and $w_a = \bigcup_{i \in a} w_i$ for $a \in \mathscr{P}(\mathbb{N})$. We assert that the map $i \mapsto w_i$ proves $\mathscr{S}_{\{r_n\}} \le_{\rm RB}^{++} \mathscr{J}$. In view of the above, it remains to show that $\xi(g_a) \mathbin{\Delta} w_a \in \mathscr{J}$ for any $a \in \mathscr{P}(\mathbb{N})$.

As $\mathscr{J} = \mathtt{Exh}_\nu$, it suffices to demonstrate that $\nu\big(w_i \mathbin{\Delta} (\xi(g_a) \cap [k_i, k_{i+1}))\big) < 2^{-i}$ for all $i \in a$ while $\nu(\xi(g_a) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin a$. After dropping the common term $\vartheta(S)$, it suffices to check that

(a) $\nu\big((\vartheta(g_i \cup S) \mathbin{\Delta} \vartheta(g_a \cup S)) \cap [k_i, k_{i+1})\big) < 2^{-i}$ for all $i \in a$ while

(b) $\nu\big((\vartheta(S) \mathbin{\Delta} \vartheta(g_a \cup S)) \cap [k_i, k_{i+1})\big) < 2^{-i}$ for $i \notin a$.

Note that any set of the form $x \cup S$, where $x \subseteq N$, is generic by (3). It follows, by (2), that we can assume, in (a) and (b), that $a \subseteq [0, i]$, *i.e.*, resp. $\mathtt{max}\, a = i$ and $\mathtt{max}\, a < i$. We can finally apply (1), with $u = a \cup \bigcup_{j<i} s_j$, $x = \bigcup_{j>i} s_j$, and $v = u_i \cup \bigcup_{j<i} s_j$ if $i \in a$ while $v = \bigcup_{j<i} s_j$ if $i \notin a$.

(ii) Otherwise $\mathscr{S}_{\{r_n\}} \le_{\rm RB}^{++} \mathscr{Z}_0$ exponentially by (i). Let this be witnessed by $i \mapsto w_i$ and a sequence of numbers k_i, so that $k_{i+1} \ge 2k_i$ and $w_i \subseteq [k_i, k_{i+1})$. If $d_i = \frac{\#(w_i)}{k_{i+1}} \to 0$ then easily $\bigcup_i w_i \in \mathscr{Z}_0$ by the choice of $\{k_i\}$. Otherwise there is a set $a \in \mathscr{S}_{\{r_n\}}$ such that $d_i > \varepsilon$ for all $i \in a$ and one and the same $\varepsilon > 0$, so that $w_a = \bigcup_{i \in a} w_i \notin \mathscr{Z}_0$. In both cases we have a contradiction with the assumption that the map $i \mapsto w_i$ witnesses $\mathscr{S}_{\{r_n\}} \le_{\rm RB}^{++} \mathscr{Z}_0$. □

Question 5.10. Farah [6] points out that Theorem 5.8(i) also holds for $0 \times \mathtt{Fin}$ (instead of $\mathscr{S}_{\{r_n\}}$) and asks for which other ideals it is true. □

5d The family ℓ^p

It follows from the next theorem that Borel reducibility between equivalence relations ℓ^p, $1 \le p < \infty$, is fully determined by the value of p.

Theorem 5.11 (Dougherty – Hjorth [5])**.** *If* $1 \le p < q < \infty$ *then* $\ell^p <_{\rm B} \ell^q$.

Proof. Part 1: show that $\ell^q \not\le_{\rm B} \ell^p$.

By Lemma 5.5, it suffices to prove that $\ell^q \restriction \mathbb{X} \not\le_{\rm B} \ell^p \restriction \mathbb{X}$. Suppose, on the contrary, that $\vartheta : \mathbb{X} \to \mathbb{X}$ is a Borel reduction of $\ell^q \restriction \mathbb{X}$ to $\ell^p \restriction \mathbb{X}$. Arguing as in the proof of Theorem 5.8, we can reduce the general case to the case when there exist increasing sequences of numbers $0 = j(0) < j(1) < j(2) < \dots$ and $0 = a_0 < a_1 < a_2 < \dots$ and a map $\tau : \mathbb{Y} \to \mathbb{X}$, where $\mathbb{Y} = \prod_{n=0}^\infty X_{j(n)}$, which reduces $\ell^q \restriction \mathbb{Y}$ to $\ell^p \restriction \mathbb{X}$ and has the form $\tau(x) = \bigcup_{n \in \mathbb{N}} t_n^{x(n)}$, where $t_n^r \in \prod_{k=a_n}^{a_{n+1}-1} X_k$ for any $r \in X_{j_n}$. (See Definition 5.4.)

Case 1: there are $c > 0$ and a number N such that $\|\tau_n^1 - \tau_n^0\|_p \ge c$ for all $n \ge N$. Since $p < q$, there is a non-decreasing sequence of natural numbers $i_n \le$

j_n, $n = 0, 1, 2, \ldots$, such that $\sum_n 2^{p(i_n - j_n)}$ diverges but $\sum_n 2^{q(i_n - j_n)}$ converges. (*Hint*: $i_n \approx j_n - p^{-1} \log_2 n$.)

Now consider any $n \ge N$. As $\|\tau_n^1 - \tau_n^0\|_p \ge c$ and because $\|\ldots\|_p$ is a norm, there exists a pair of rationals $u(n) < v(n)$ in X_{j_n} with $v(n) - u(n) = 2^{i_n - j_n}$ and $\|\tau_n^{v(n)} - \tau_n^{u(n)}\|_p \ge c\, 2^{i_n - j_n}$. In addition, put $u(n) = v(n) = 0$ for $n < N$. Then the $\boldsymbol{\ell}^q$-distance between the infinite sequences $u = \{u(n)\}_{n \in \mathbb{N}}$ and $v = \{v(n)\}_{n \in \mathbb{N}}$ is equal to $\sum_{n=N}^{\infty} 2^{q(i_n - j_n)} < +\infty$, while the $\boldsymbol{\ell}^p$-distance between $\tau(u)$ and $\tau(v)$ is non-smaller than $\sum_{n=N}^{\infty} c^p\, 2^{p(i_n - j_n)} = \infty$. But this contradicts the assumption that τ is a reduction.

Case 2: otherwise. Then there is a strictly increasing sequence $n_0 < n_1 < n_2 < \ldots$ with $\|\tau_{n_k}^1 - \tau_{n_k}^0\|_p \le 2^{-k}$ for all k. Let now $x \in \mathbb{Y}$ be the constant 0 while $y \in \mathbb{Y}$ be defined by $y(n_k) = 1$, $\forall k$ and $y(n) = 0$ for all other n. Then $x \; \boldsymbol{\ell}^q \; y$ fails ($|y(n) - x(n)| \not\to 0$) but $\tau(x) \; \boldsymbol{\ell}^p \; \tau(y)$ holds, contradiction.

Part 2: show that $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^q$.

It suffices to prove that $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}} \le_{\mathrm{B}} \boldsymbol{\ell}^q$ (Lemma 5.5). We *w.l.o.g.* assume that $q < 2p$: any bigger q can be approached in several steps. For $\vec{x} = \langle x, y \rangle \in \mathbb{R}^2$, let $\|\vec{x}\|_h = (x^h + y^h)^{1/h}$.

Lemma 5.12. *For any* $\frac{1}{2} < \alpha < 1$ *there is a continuous map* $K_\alpha : [0,1] \to [0,1]^2$ *and positive real numbers* $m < M$ *such that for all* $x < y$ *in* $[0,1]$ *we have* $m(y-x)^\alpha \le \|K_\alpha(y) - K_\alpha(x)\|_2 \le M(y-x)^\alpha$.

Proof (Lemma)**.** The construction of such a map K can be easier described in terms of fractal geometry rather than by an analytic expression. Let $r = 4^{-\alpha}$, so that $\frac{1}{4} < r < \frac{1}{2}$ and $\alpha = -\log_4 r$. Starting with the segment $[(0,0), (1,0)]$ of the horisontal axis of the cartesian plane, we replace it by four smaller segments of length r each (thin lines on Fig. 2, left). Each of them we replace by four segments of length r^2 (thin lines on Fig. 2, right). And so on, infinitely many steps. The resulting curve K is parametrized by giving the vertices of the polygons values equal to multiples of 4^{-n}, n being the number of the polygon. For instance, the vertices of the left polygon on Fig. 2 are given values $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

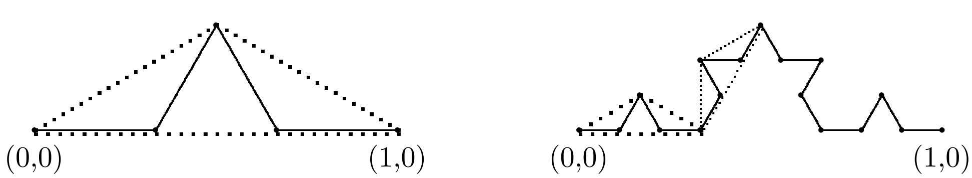

Figure 1: $r = \frac{1}{3}$, left: step 1, right: step 2

Note that the curve $K : [0,1] \to [0,1]^2$, approximated by the polygons, is bounded by certain triangles built on the sides of the polygons. For instance, the whole curve lies inside the triangle bounded by dotted lines in Fig. 2, left. (The dotted line that follows the basic side $[(0,0)\,,(1,0)]$ of the triangle is drawn slightly below its true position.) Further, the parts $0 \le t \le \frac{1}{4}$ and $\frac{1}{4} \le t \le \frac{1}{2}$ of the curve lie inside the triangles bounded by (slightly different) dotted lines in Fig. 2, right. And so on. Let us call those triangles *bounding triangles*.

To prove the inequality of the lemma, consider any pair of reals $x < y \in [0,1]$. Let n be the least number such that x, y belong to non-adjacent intervals, resp., $[\frac{i-1}{4^n}, \frac{i}{4^n}]$ and $[\frac{j-1}{4^n}, \frac{j}{4^n}]$, with $j > i+1$. Then $4^{-n} \le |y-x| \le 8 \cdot 4^{-n}$.

The points $K(x)$ and $K(y)$ then belong to one and the same side or adjacent sides of the $n-1$-th polygon. Let C be a common vertice of these sides. It is quite clear geometrically that the euclidean distances from $K(x)$ and $K(y)$ to C do not exceed r^{n-1} (the length of the side), thus $\|K(x)-K(y)\|_2 \le 2\,r^{n-1}$.

Estimation from below needs more work. The points $K(x)$, $K(y)$ belong to the bounding triangles built on the segments, resp., $[K(\frac{i-1}{4^n})\,, K(\frac{i}{4^n})]$ and $[K(\frac{j-1}{4^n})\,, K(\frac{j}{4^n})]$, and obviously $i+1 < j \le i+8$, so that there exist at most six bounding triangles between these two. Note that adjacent bounding triangles meet each other at only two possible angles (that depend on r but not on n), and taking it as geometrically evident that non-adjacent bounding triangles are disjoint, we conclude that there is a constant $c > 0$ (that depends on r but not on n) such that the distance between two non-adjacent bounding triangles of rank n, having at most 6 bounding triangles of rank n between them, does not exceed $c \cdot r^n$. In particular, $\|K(x)-K(y)\|_2 \ge c \cdot r^n$. Combining this with the inequalities above, we conclude that $m(y-x)^\alpha \le \|K(y)-K(x)\|_2 \le M(y-x)^\alpha$, where $m = \frac{c}{8^\alpha}$ and $M = \frac{2}{r}$ (and $\alpha = -\log_4 r$). □ (*Lemma*)

Coming back to the theorem, let $\alpha = p/q$, and let K_α be as in the lemma. Let $x = \langle x_0, x_1, x_2, \dots \rangle \in [0,1]^{\mathbb{N}}$. Then $K_\alpha(x_i) = \langle x'_i, x''_i \rangle \in [0,1]^2$. We put $\vartheta(x) = \langle x'_0, x''_0, x'_1, x''_1, x'_2, x''_2, \dots \rangle$. Prove that ϑ reduces $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}}$ to $\boldsymbol{\ell}^q$.

Let $x = \{x_i\}_{i\in\mathbb{N}}$ and $y = \{y_i\}_{i\in\mathbb{N}}$ belong to $[0,1]^{\mathbb{N}}$; we have to prove that $x \;\boldsymbol{\ell}^p\; y$ iff $\vartheta(x) \;\boldsymbol{\ell}^q\; \vartheta(y)$. To simplify the picture note the following:

$$2^{-1/2}\|w\|_2 \le \mathtt{max}\{w', w''\} \le \|w\|_q \le \|w\|_1 \le 2\|w\|_2$$

for any $w = \langle w', w'' \rangle \in \mathbb{R}^2$. The task takes the following form:

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i \|K_\alpha(x_i) - K_\alpha(y_i)\|_2{}^q < \infty\,.$$

Furthermore, by the choice of K_α, this converts to

$$\sum_i (x_i - y_i)^p < \infty \iff \sum_i (x_i - y_i)^{\alpha q} < \infty\,,$$

which holds because $\alpha q = p$. □ (*Theorem 5.11*)

5e $\boldsymbol{\ell}^\infty$: maximal $\mathbf{K}_\sigma$

Recall that $\mathbf{K}_\sigma$ denotes the class of all σ-compact sets in Polish spaces. Easy computations show that this class contains, among others, the equivalence relations E_1, E_∞, $\boldsymbol{\ell}^p$, $1 \le p \le \infty$, considered as sets of pairs in corresponding Polish spaces. Note that if E a $\mathbf{K}_\sigma$ equivalence on a Polish space $\mathbb{X}$ then $\mathbb{X}$ is $\mathbf{K}_\sigma$ as well since projections of compact sets are compact. Thus $\mathbf{K}_\sigma$ ERs on Polish spaces is one and the same as $\mathbf{\Sigma}^0_2$ ERs on $\mathbf{K}_\sigma$ Polish spaces.

Theorem 5.13. *Any $\mathbf{K}_\sigma$ equivalence relation on a Polish space, in particular, E_1, E_∞, $\boldsymbol{\ell}^p$, is Borel reducible to $\boldsymbol{\ell}^\infty$.* [3]

Proof (from Rosendal [52]**).** Let $\mathbb{A}$ be the set of all $\subseteq$-increasing sequences $a = \{a_n\}_{n\in\mathbb{N}}$ of subsets $a_n \subseteq \mathbb{N}$ — a closed subset of the Polish space $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Define an ER H on $\mathbb{A}$ by

$$\{a_n\}\,\mathsf{H}\,\{b_n\} \quad \text{iff} \quad \exists\, N\, \forall\, m\, (a_m \subseteq b_{N+m} \wedge b_m \subseteq a_{N+m}).$$

Claim 1: $\mathsf{H} \le_{\mathrm{B}} \boldsymbol{\ell}^\infty$. This is easy. Given a sequence $a = \{a_n\}_{n\in\mathbb{N}} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, define $\vartheta(a) \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ by $\vartheta(a)(n,k)$ to be the least $j \le k$ such that $n \in a_j$, or $\vartheta(a)(n,k) = k$ whenever $n \notin a_k$. Then $\{a_n\}\,\mathsf{H}\,\{b_n\}$ iff there is N such that $|\vartheta(a)(n,k) - \vartheta(b)(n,k)| \le N$ for all n,k.

Claim 2: any $\mathbf{K}_\sigma$ equivalence E on a Polish space $\mathbb{X}$ is Borel reducible to H. As a $\mathbf{K}_\sigma$ set, E has the form $\mathsf{E} = \bigcup_n E_n$, where each E_n is a compact subset of $\mathbb{X} \times \mathbb{X}$ (not necessarily an ER) and $E_n \subseteq E_{n+1}$. We can *w.l.o.g.* assume that each E_n is reflexive and symmetric on its domain $D_n = \mathtt{dom}\, E_n = \mathtt{ran}\, E_n$ (a compact set), in particular, $x \in D_n \Longrightarrow \langle x,x\rangle \in E_n$. Define $P_0 = E_0$ and

$$P_{n+1} = P_n \cup E_{n+1} \cup P_n^{(2)}, \quad \text{where } P_n^{(2)} = \{\langle x,y\rangle : \exists\, z\, (\langle x,z\rangle \in P_n \wedge \langle z,y\rangle \in P_n)\},$$

by induction. Thus all P_n are still compact subsets of $\mathbb{X}\times\mathbb{X}$, moreover, of E since E is an equivalence relation, and $E_n \subseteq P_n \subseteq P_{n+1}$, therefore $\mathsf{E} = \bigcup_n P_n$.

Let $\{U_k : k \in \mathbb{N}\}$ be a basis for the topology of $\mathbb{X}$. Put, for any $x \in \mathbb{X}$, $\vartheta_n(x) = \{k : U_k \cap R_n(x) \ne \varnothing\}$, where $R_n(x) = \{y : \langle x,y\rangle \in R_n\}$. Then obviously $\vartheta_n(x) \subseteq \vartheta_{n+1}(x)$, and hence $\vartheta(x) = \{\vartheta_n(x)\}_{n\in\mathbb{N}} \in \mathbb{A}$. Then ϑ reduces E to H.

Indeed if $x\,\mathsf{E}\,y$ then $\langle y,x\rangle \in P_n$ for some n, and for all m and $z \in \mathbb{X}$ we have $\langle x,z\rangle \in R_m \Longrightarrow \langle y,z\rangle \in R_{1+\mathtt{max}\{m,n\}}$. In other words, $R_m(x) \subseteq R_{1+\mathtt{max}\{m,n\}}(y)$ and hence $\vartheta_m(x) \subseteq \vartheta_{1+\mathtt{max}\{m,n\}}(y)$ hold for all m. Similarly, for some n' we have $\vartheta_m(y) \subseteq \vartheta_{1+\mathtt{max}\{m,n'\}}(y)$, $\forall\, m$. Thus $\vartheta(x)\,\mathsf{H}\,\vartheta(y)$.

Conversely, suppose that $\vartheta(x)\,\mathsf{H}\,\vartheta(y)$, thus, for some N, we have $R_m(x) \subseteq R_{N+m}(y)$ and $R_m(y) \subseteq R_{N+m}(x)$ for all m and y. Taking m big enough for P_m to contain $\langle x,x\rangle$, we obtain $x \in R_{N+m}(y)$, so that immediately $x\,\mathsf{E}\,y$. □

[3] The result for $\boldsymbol{\ell}^p$ is due to Su Gao [15]. He defines $d_p(x,s) = (\sum_{k=0}^{\mathtt{lh}\, s-1} |x(k) - s(k)|^p)^{\frac{1}{p}}$ for any $x \in \mathbb{R}^{\mathbb{N}}$ and $s \in \mathbb{Q}^{<\omega}$ (a finite sequence of rationals). Easily the $\boldsymbol{\ell}^p$-distance $(\sum_{k=0}^{\infty} |x(k) - y(k)|^p)^{\frac{1}{p}}$ between any pair of $x,y \in \mathbb{R}^{\mathbb{N}}$ is finite iff there is a constant C such that $|d_p(x,s) - d_p(y,s)| < C$ for all $s \in \mathbb{Q}^{<\omega}$. This yields a reduction required.

Chapter 6

Smooth, hyperfinite, countable

This Chapter is related to the domain $\le_B \mathsf{E}_\infty$ in the diagram on page 26. The following types of equivalence relations are relevant to this domain:

Definition 6.1. A Borel equivalence relation E on a (Borel) set $\mathbb{X}$ is:

– *countable*, if every E-class $[x]_{\mathsf{E}} = \{y \in X : x \mathrel{\mathsf{E}} y\}$, $x \in X$, is at most countable;

– *essentially countable*, if $\mathsf{E} \le_B \mathsf{F}$, where F is a countable Borel ER;

– *finite*, if every E-class $[x]_{\mathsf{E}} = \{y \in \mathbb{X} : x \mathrel{\mathsf{E}} y\}$, $x \in X$, is finite;

– *hyperfinite*, if $\mathsf{E} = \bigcup_n \mathsf{F}_n$ for an increasing sequence of Borel finite ERs F_n;

– *smooth*, if $\mathsf{E} \le_B \mathsf{EQ}_{2^{\mathbb{N}}}$;

– *hypersmooth*, if $\mathsf{E} = \bigcup_n \mathsf{F}_n$ for an increasing sequence of smooth ERs E_n. □

After a few rather simple results on smooth equivalence relations, we proceed to countable equivalences. We prove in Section 6b that every countable Borel ER is Borel reducible to E_∞, and hence the whole domain $\le_B \mathsf{E}_\infty$ is equal to the class of essentially countable Borel ERs.

Then we turn to hyperfinite equivalence relations, a very interesting subclass of countable Borel equivalences. A typical hyperfinite equivalence is E_0 — in fact the $\le_B$-largest, or *universal* hyperfinite ER. Hyperfinite ERs admit several different characterizations — some of them are presented by Theorem 6.5.

The equivalence relation E_∞ turns out to be (countable but) non-hyperfinite by Theorem 6.3. It follows that $\mathsf{E}_0 <_B \mathsf{E}_\infty$ strictly.

We finish with two separate theorems. One of them, Theorem 6.9, asserts that, given a countable equivalence relation satisfying $\mathsf{F} + \mathsf{F} \le_B \mathsf{F}$, the property “being Borel reducible to F” is σ-additive. Theorem 6.11 shows that Fin is the $\le_B$-least ideal.

6a Smooth and below

By definition an equivalence relation E is smooth iff there is a Borel map $\vartheta : X \to 2^{\mathbb{N}}$ such that the equivalence $x \mathrel{\mathsf{E}} y \iff \vartheta(x) = \vartheta(y)$ holds for all x, $y \in X = \mathtt{dom}\,\mathsf{E}$. In other words, it is required that the equivalence classes can be counted by reals (here: elements of $2^{\mathbb{N}}$) in Borel way. An important subspecies of smooth equivalence relations consists of those having a Borel *transversal*: a set with exactly one element in every equivalence class.

Lemma 6.2. (i) *Any Borel ER that has a Borel transversal is smooth*;

(ii) *any Borel finite (with finite classes) ER admits a Borel transversal*;

(iii) *any Borel countable smooth ER admits a Borel transversal;*

(iv) *any Borel ER* E *on a Polish space* $\mathbb{X}$, *such that every* E*-class is closed and the saturation* $[\mathscr{O}]_{\mathsf{E}}$ *of every open set* $\mathscr{O} \subseteq \mathbb{X}$ *is Borel, admits a Borel transversal, hence, is smooth.* [1]

(v) E_0 *is not smooth.*

(vi) *there exists a smooth ER* E *that does not have a Borel transversal.*

Proof. (i) Let T be a Borel transversal for E. The map $\vartheta(x) =$ "the only element of T E-equivalent to x" reduces E to EQ_T.

(ii) Consider the set of the $<$-least elements of E-classes, where $<$ is a fixed Borel linear order on the domain of E.

(iii) Use Countable-to-1 Uniformization (Theorem 1.6).

(iv) Since any uncountable Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$, we can assume that E is an equivalence relation on $\mathbb{N}^{\mathbb{N}}$. Then, for any $x \in \mathbb{N}^{\mathbb{N}}$, the equivalence class $[x]_{\mathsf{E}}$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$, naturally identified with a tree, say, $T_x \subseteq \mathbb{N}^{<\omega}$. Let $\vartheta(x)$ denote the leftmost branch of T_x. Then $x \mathrel{\mathsf{E}} \vartheta(x)$ and $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) = \vartheta(y)$, so that it remains to show that $Z = \{\vartheta(x) : x \in \mathbb{N}^{\mathbb{N}}\}$ is Borel. Note that

$$z \in Z \iff \forall\, m\, \forall\, s, t \in \mathbb{N}^m\, \big(s <_{\mathtt{lex}} t \wedge z \in \mathscr{O}_t \Longrightarrow [z]_{\mathsf{E}} \cap \mathscr{O}_t = \varnothing\big),$$

where $<_{\mathtt{lex}}$ is the lexicographical order on $\mathbb{N}^m$ and $\mathscr{O}_s = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}$. However $[x]_{\mathsf{E}} \cap \mathscr{O}_t = \varnothing$ iff $x \notin [\mathscr{O}_t]_{\mathsf{E}}$ and $[\mathscr{O}_t]_{\mathsf{E}}$ is Borel for any t.

(v) Otherwise E_0 has a Borel transversal T by (iii), which is a contradiction, see an argument after Example 3.7.

(vi) Take a closed set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $\mathtt{dom}\, P = \mathbb{N}^{\mathbb{N}}$ that is not uniformizable by a Borel set, and define $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ iff both $\langle x, y\rangle$ and $\langle x', y'\rangle$ belong to P and $x = x'$. □

[1] Srivastava [59] proved the result for ERs with $\mathbf{G}_\delta$ classes, which is the best possible as E_0 is a Borel ER, whose classes are $\mathbf{F}_\sigma$ and saturations of open sets are even open, but without any Borel transversal. See also [34, 18.20 iv)].

6b Countable equivalence relations

This class of equivalence relations is a subject of ongoing intence study. We present here the following important theorem ([11, Thm 1], [4, 1.8]) and a few more results below, leaving [23, 13, 38] as sources of further information regarding countable equivalence relations.

Theorem 6.3. *Any Borel countable ER* E *on a Polish space* $\mathbb{X}$:

(i) *is induced by a Polish action of a countable group* $\mathbb{G}$ *on* $\mathbb{X}$;

(ii) *satisfies* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty = \mathsf{E}(F_2, 2)$, *where* F_2 *is the free group with two generators and* $\mathsf{E}(F_2,2)$ *is the ER induced by the shift action of* F_2 *on* 2^{F_2}.

Proof. (i) We *w.l.o.g.* assume that $\mathbb{X} = 2^{\mathbb{N}}$. According to Countable-to-1 Enumeration (Theorem 1.4, in a relativized version, if necessary, see Remark 1.1), there is a sequence of Borel maps $f_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f_n(a) : n \in \mathbb{N}\}$ for each $a \in 2^{\mathbb{N}}$. Put $\Gamma'_n = \{\langle a, f_n(a)\rangle : a \in \mathbb{N}\}$ (the graph of f_n) and $\Gamma_n = \Gamma'_n \smallsetminus \bigcup_{k<n} \Gamma'_k$. The sets $P_{nk} = \Gamma_n \cap {\Gamma_k}^{-1}$ form a partition of (the graph of) E onto countably many Borel injective sets. Further define $\Delta = \{\langle a, a\rangle : a \in 2^{\mathbb{N}}\}$ and let $\{D_m\}_{m\in\mathbb{N}}$ be an enumeration of all non-empty sets of the form $P_{nk} \smallsetminus \Delta$. Intersecting the sets D_m with the rectangles of the form

$$R_s = \{\langle a, b\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : s^\wedge 0 \subset a \wedge s^\wedge 1 \subset b\} \quad \text{and} \quad {R_s}^{-1},$$

we reduce the general case to the case when $\mathtt{dom}\, D_m \cap \mathtt{ran}\, D_m = \varnothing$, $\forall\, m$.

Now, for any m define $h_m(a) = b$ whenever either $\langle a, b\rangle \in D_m$ or $\langle a, b\rangle \in {D_m}^{-1}$, or $a = b \notin \mathtt{dom}\, D_m \cup \mathtt{ran}\, D_m$. Clearly h_m is a Borel bijection $2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. Thus $\{h_m\}_{m\in\mathbb{N}}$ is a family of Borel automorphisms of $2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{h_m(a) : m \in \mathbb{N}\}$. It does not take much effort to expand this system to a Borel action of F_ω, the free group with countably many generators $a_1, a_2, a_3, \dots$, on $2^{\mathbb{N}}$, whose induced equivalence relation is E.

(ii) First of all, by (i), $\mathsf{E} \le_{\mathrm{B}} \mathsf{R}$, where R is induced by a Borel action $\cdot$ of F_ω on $2^{\mathbb{N}}$. The map $\vartheta(a) = \{g^{-1} \cdot a\}_{g\in F_\omega}$, $a \in 2^{\mathbb{N}}$ is a Borel reduction of R to $\mathsf{E}(F_\omega, 2^{\mathbb{N}})$. If now F_ω is a subgroup of a countable group H then $\mathsf{E}(F_\omega, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(H, 2^{\mathbb{N}})$ by means of the map sending any $\{a_g\}_{g\in F_\omega}$ to $\{b_h\}_{h\in H}$, where $b_g = a_g$ for $g \in F_\omega$ and b_h equal to any fixed $b' \in 2^{\mathbb{N}}$ for $h \in H \smallsetminus F_\omega$. As F_ω admits an injective homomorphism into F_2 [2] we conclude that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}(F_2, 2^{\mathbb{N}})$.

It remains to define a Borel reduction of $\mathsf{E}(F_2, 2^{\mathbb{N}})$ to $\mathsf{E}(F_2, 2)$. The inequality $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2, 2^{\mathbb{Z}\smallsetminus\{0\}})$ is clear. Further $\mathsf{E}(F_2, 2^{\mathbb{Z}\smallsetminus\{0\}}) \le_{\mathrm{B}} \mathsf{E}(F_2 \times \mathbb{Z}, 3)$, by means of the map sending any $\{a_g\}_{g\in F_2}$ ($a_g \in 2^{\mathbb{Z}\smallsetminus\{0\}}$) to $\{b_{gj}\}_{g\in F_2,\, j\in\mathbb{Z}}$, where $b_{gj} = a_g(j)$ for $j \ne 0$ and $b_{g0} = 2$. Further, for any group G it holds $\mathsf{E}(G, 3) \le_{\mathrm{B}}$

[2] Indeed, let F be the subgroup of F_2 generated by all elements of the form $\alpha_n = a^n b^n$ and $\alpha_n^{-1} = b^{-n} a^{-n}$. The map sending any a_n to α_n and accordingly a_n^{-1} to α_n^{-1} is an isomorphism of F_ω onto F.

$\mathsf{E}(G \times \mathbb{Z}_2, 2)$ by means of the map sending every element $\{a_g\}_{g \in G}$ $(a_g = 0, 1, 2)$ to $\{b_{gi}\}_{g \in G,\, i \in \mathbb{Z}_2}$, where

$$b_{gi} = \begin{cases} 0, & \text{if} \quad a_g = 0 \quad \textbf{or} \quad a_g = 1 \text{ and } i = 0, \\ 1, & \text{if} \quad a_g = 2 \quad \textbf{or} \quad a_g = 1 \text{ and } i = 1. \end{cases}$$

Thus $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2)$. However, $F_2 \times \mathbb{Z} \times \mathbb{Z}_2$ admits a homomorphism into the group F_ω, and then into F_2 by the above, so that $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2, 2)$, as required. □

We add here a technical lemma, attributed to Kechris in [17], that will be used in Chapter 13. Recall that equivalences Borel reducible to Borel countable ones are called *essentially countable*. The lemma shows that maps much weaker than reductions lead to the same class.

Lemma 6.4. *Suppose that A, X are Borel sets, E is a Borel ER on A, and $\rho : A \to X$ is a Borel map satisfying the following: first, the ρ-image of any E-class is at most countable, secong, ρ-images of different E-classes are pairwise disjoint. Then E is an essentially countable equivalence relation.*

Proof. The relation: $x \mathrel{\mathsf{R}} y$ iff $x, y \in Y$ belong to the ρ-image of one and the same E-class in A, is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on the set $Y = \mathtt{ran}\,\vartheta$. Moreover,

$$\mathsf{R} \subseteq P = \{\langle x, y \rangle : \neg\, \exists\, a, b \in A\ (a \not\mathrel{\mathsf{E}} b \wedge x = \rho(a) \wedge y = \rho(b))\}\,,$$

where P is $\boldsymbol{\Pi}^1_1$. Thus there is a Borel set U with $\mathsf{R} \subseteq U \subseteq P$. In particular, $U \cap (Y \times Y) = \mathsf{R}$. As all R-equivalence classes are at most countable, we can assume that all cross-sections of U are at most countable, too.

To prove the lemma it suffices to find a Borel equivalence relation F with $\mathsf{R} \subseteq \mathsf{F} \subseteq U$. Say that a set $Z \subseteq X$ is *stable* if $U \cap (Z \times Z)$ is an equivalence relation. For example, Y is stable. We observe that the set $D_0 = \{y : Y \cup \{y\} \text{ is stable}\}$ is $\boldsymbol{\Pi}^1_1$ and satisfies $Y \subseteq D_0$, hence, there is a Borel set Z_1 with $Y \subseteq Z_1 \subseteq D_0$. Similarly,

$$D_1 = \{y' \in Z_1 : Y \cup \{y, y'\} \text{ is stable for any } y \in Z_1\}$$

is $\boldsymbol{\Pi}^1_1$ and satisfies $Y \subseteq D_1$ by the definition of Z_1, so that there is a Borel set Z_2 with $Y \subseteq Z_2 \subseteq D_1$. Generally, we define

$$D_n = \{y' \in Z_n : Y \cup \{y_1, \dots, y_n, y'\} \text{ is stable for all } y_1, \dots, y_n \in Z_n\}$$

find that $Y \subseteq D_n$, and choose a Borel set Z_n with $Y \subseteq Z_n \subseteq D_n$. Then, by the construction, $Y \subseteq Z = \bigcap_n Z_n$, and, for any finite $Z' \subseteq Z$, the set $Y \cup Z'$ is stable, so that Z itself is stable, and we can take $\mathsf{F} = U \cap (Z \times Z)$. □

6c Hyperfinite equivalence relations

The class of Borel hyperfinite equivalence relations has been a topic of intense study since 1970s. Papers [4, 23, 38] give a comprehensive account of the results obtained regarding hyperfinite relations, with further references.

Theorem 6.5 (Theorems 5.1 and, partially, 7.1 in [4] and 12.1(ii) in [23]). *The following are equivalent for a Borel equivalence relation* E *on a Polish space* $\mathbb{X}$:

(i) $\mathsf{E} \le_{\mathrm{B}} \mathsf{E_0}$ *and* E *is countable*;

(ii) E *is hyperfinite*;

(iii) E *is hypersmooth and countable*;

(iv) *there is a Borel set* $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ *such that* $\mathsf{E_1} \restriction X$ *is a countable ER and* E *is isomorphic, via a Borel bijection of* $\mathbb{X}$ *onto* X, *to* $\mathsf{E_1} \restriction X$; [3]

(v) E *is induced by a Borel action of* $\mathbb{Z}$, *the additive group of the integers.*

(vi) *there exists a pair of Borel ERs* F, R *of type* 2 *such that* $\mathsf{E} = \mathsf{F} \vee \mathsf{R}$. [4]

Note that all Borel finite equivalence relations are smooth by Lemma 6.2. Accordingly, all hyperfinite ERs are hypersmooth. On the other hand, all finite and hyperfinite ERs are countable, of course. It follows from the theorem that, conversely, every hypersmooth countable equivalence relation is hyperfinite.

The theorem also shows that $\mathsf{E_0}$ is a universal hyperfinite equivalence. (To see that $\mathsf{E_0}$ is hyperfinite, define $x \mathrel{\mathsf{F}_n} y$ iff $x \mathbin{\Delta} y \subseteq [0, n)$ for x, $y \subseteq \mathbb{N}$.)

Some other characterizations of hyperfinite equivalence relations are known. For instance, for a Borel ER E to be hyperfinite it is necessary and sufficient that there is a Borel partial order $\prec$ on the domain of E that orders each E-class $[x]_{\mathsf{E}}$ linearly and with the order type being a suborder of $\mathbb{Z}$, the integers positive and negative. Thus the $\prec$-order type of $[x]_{\mathsf{E}}$ has to be either finite, or ω, or ω^* (the inverse ω), or $\omega^* + \omega$, the order type of $\mathbb{Z}$ itself. On this see the references given above.

Proof. It does not seem possible to prove the theorem by a simple cyclic argument. The structure of the proof will be the following:

$$\text{(i)} \implies \text{(iii)} \implies \text{(iv)} \implies \text{(v)} \implies \text{(i)} \;;$$
$$\text{(v)} \implies \text{(vi)} \implies \text{(v)} \;;$$
$$\text{(v)} \implies \text{(ii)} \implies \text{(iii)} \;.$$

[3] This transitional condition refers to $\mathsf{E_1}$, here an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x \mathrel{\mathsf{E_1}} y$ iff $x(n) = y(n)$ for all but finite n.

[4] An equivalence relation F is *of type* n if any F-class contains at most n elements. $\mathsf{F} \vee \mathsf{R}$ denotes the least ER which includes $\mathsf{F} \cup \mathsf{R}$.

The implications (ii) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iii) are quite elementary.

(iii) $\Longrightarrow$ (iv). Let $\mathsf{E} = \bigcup_n \mathsf{F}_n$ be a countable and hypersmooth ER on a space $\mathbb{X}$, all F_n being smooth (and countable), and $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, $\forall n$. We may assume that $\mathbb{X} = 2^{\mathbb{N}}$ and $\mathsf{F}_0 = \mathsf{EQ}_{2^{\mathbb{N}}}$. Let $T_n \subseteq \mathbb{X}$ be a Borel transversal for F_n (recall Lemma 6.2(iii)). Now let $\vartheta_n(x)$ be the only element of T_n with $v\,\mathsf{F}_n\,\vartheta_n(x)$. Then $x \mapsto \{\vartheta_n(x)\}_{n\in\mathbb{N}}$ is a $1-1$ Borel map $\mathbb{X} \to (2^{\mathbb{N}})^{\mathbb{N}}$ and $x\,\mathsf{E}\,y \Longleftrightarrow \vartheta(x)\,\mathsf{E}_1\,\vartheta(y)$. Take X to be the image of $\mathbb{X}$.

(iv) $\Longrightarrow$ (v). Let X be as indicated. For any $\mathbb{N}$-sequence x and $n \in \mathbb{N}$, let $x\restriction_{>n} = x \restriction (n,\infty)$. It follows from (the relativized versions of) Countable-to-1 Projection and Countable-to-1 Enumeration (theorems 1.3 and 1.4) that for any n the set $X\restriction_{>n} = \{x\restriction_{>n} : x \in X\}$ is Borel and there is a countable family of Borel functions $g_i^n : X\restriction_{>n} \to X$, $i \in \mathbb{N}$, such that the set $X_\xi = \{x \in X : x\restriction_{>n} = \xi\}$ is equal to $\{g_i^n(\xi) : i \in \mathbb{N}\}$ for any $\xi \in X\restriction_{>n}$. Then it holds $\{g_i^n(\xi)(n) : i \in \mathbb{N}\} = \{x(n) : x \in X_\xi\}$.

For any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ let $\varphi(x) = \{\varphi_n(x)\}_{n\in\mathbb{N}}$, where $\varphi_n(x)$ is the least number i such that $x(n) = f_i^n(x)(n)$; thus, $\varphi(x) \in \mathbb{N}^{\mathbb{N}}$. Let $\mu(x)$ be the sequence

$$\varphi_0(x), \varphi_0'(x), \varphi_1(x)+1, \varphi_1'(x)+1, \dots, \varphi_n(x)+n, \varphi_n'(x)+n, \dots,$$

where $\varphi_n'(x) = \mathtt{max}_{k\le n}\,\varphi_k(x)$. Easily if $x \neq y \in X$ satisfy $x\,\mathsf{E}_1\,y$, *i.e.*, $x\restriction_{>n} = y\restriction_{>n}$ for some n, then $\varphi(x)\restriction_{>n} = \varphi(y)\restriction_{>n}$ but $\varphi(x) \neq \varphi(y)$, $\mu(x) \neq \mu(y)$, and $\mu(x)\restriction_{>m} = \mu(y)\restriction_{>m}$ for some $m \ge n$.

Let $<_{\mathtt{alex}}$ be the anti-lexicographical partial order on $\mathbb{N}^{\mathbb{N}}$, *i.e.*, $a <_{\mathtt{alex}} b$ iff there is n such that $a\restriction_{>n} = b\restriction_{>n}$ and $a(n) < b(n)$. For $x,\ y \in X$ define $x <_0 y$ iff $\mu(x) <_{\mathtt{alex}} \mu(y)$. It follows from the above that $<_0$ linearly orders every E_1-class $[x]_{\mathsf{E}_1} \cap X$ of $x \in X$. Moreover, it follows from the definition of $\mu(x)$ that any $<_{\mathtt{alex}}$-interval between some $\mu(x) <_{\mathtt{alex}} \mu(y)$ contains only finitely many elements of the form $\mu(z)$. (For φ this would not be true.) We conclude that any class $[x]_{\mathsf{E}_1} \cap X$, $x \in X$, is linearly ordered by $<_0$ similarly to a subset of $\mathbb{Z}$, the integers. That $<_0$ can be converted to a required Borel action of $\mathbb{Z}$ on X is rather easy (however those E_1-classes in X ordered similarly to $\mathbb{N}$, the inverse of $\mathbb{N}$, or finite, should be treated separately).

(v) $\Longrightarrow$ (ii). Assume *w.l.o.g.* that $\mathbb{X} = 2^{\mathbb{N}}$. An increasing sequence of ERs F_n whose union is E is defined separately on each E-class C; they "integrate" into Borel ERs F_n defined on the whole of $2^{\mathbb{N}}$ because the action allows to replace quantifiers over a E-class C by quantifiers over $\mathbb{Z}$.

Let C be any E-class of $x \in X$. Note that if an element $x_C \in C$ can be chosen in some Borel-definable way then we can define $x\,\mathsf{F}_n\,y$ iff there exist integers $j,\ k \in \mathbb{Z}$ with $|j| \le n$, $|k| \le n$, and $x = j\cdot x_C$, $y = k\cdot x_C$. This applies, for instance, when C is finite, thus, we can assume that C is infinite. Let $<_{\mathtt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}}$, and $<_{\mathtt{act}}$ be the partial order induced by the action, *i.e.*, $x <_{\mathtt{act}} y$ iff $y = j\cdot x$, $j > 0$. By the same reason we can assume that neither of $a = \mathtt{inf}_{<_{\mathtt{lex}}} C$ and $b = \mathtt{sup}_{<_{\mathtt{lex}}} C$ belongs to C. Let C_n be the set of all $x \in C$ with $x \restriction n \neq a \restriction n$ and $x \restriction n \neq b \restriction n$. Define $x \mathsf{F}_n y$ iff x, y belong to one and the same $<_{\texttt{lex}}$-interval in C lying entirely within C_n, or just $x = y$. In our assumptions, any F_n has finite classes, and for any two $x, y \in C$ there is n with $x \mathsf{F}_n y$.

(v) $\Longrightarrow$ (i). This is more complicated. A preliminary step is to show that $\mathsf{E} \le_{\rm B} \mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}})$, where $\mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}})$ is the orbit equivalence induced by the shift action of $\mathbb{Z}$ on $(2^{\mathbb{N}})^{\mathbb{Z}}$: $k \cdot \{x_j\}_{j\in\mathbb{Z}} = \{x_{j-k}\}_{j\in\mathbb{Z}}$ for $k \in \mathbb{Z}$. Assuming *w.l.o.g.* that E is a ER on $2^{\mathbb{N}}$, we obtain a Borel reduction of E to $\mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}})$ by $\vartheta(x) = \{j \cdot x\}_{j\in\mathbb{Z}}$, where $\cdot$ is a Borel action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ which induces E. Then Theorem 7.1 in [4] proves that $\mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}}) \le_{\rm B} \mathsf{E}_0$.

(vi) $\Longrightarrow$ (v). Suppose that $\mathsf{E} = \mathsf{F} \vee \mathsf{R}$, where F, R are type-2 equivalence relations on $2^{\mathbb{N}}$. Let a F-pair be any pair $\{a, b\}$ in $2^{\mathbb{N}}$ such that $a \mathsf{F} b$. Let a F-singleton be any $x \in 2^{\mathbb{N}}$ F-equivalent only to itself. Then any $x \in 2^{\mathbb{N}}$ is either a F-singleton or a member of a unique F-pair.

Fix an arbitrary $x \in 2^{\mathbb{N}}$. We now define an oriented chain $\to$ on the equivalence class $[x]_{\mathsf{E}}$. For any F-pair $\langle a, b\rangle$ in $\mathbb{X}$ define $a \to b$ whenever $a <_{\texttt{lex}} b$, where $<_{\texttt{lex}}$ is the lexicographical order on $2^{\mathbb{N}}$. If $\{a <_{\texttt{lex}} b\}$ and $\{a' <_{\texttt{lex}} b'\}$ are different F-pairs then define $b \to a'$ whenever either $b \mathsf{R} a'$ or $b \mathsf{R} b'$. (These two conditions are obviously incompatible.) If c is a F-singleton then define $b \to c$ whenever $b \mathsf{R} c$, and $c \to a$ whenever $c \mathsf{R} a$. If finally $c \neq d$ are F-singletons then define $c \to d$ whenever $c \mathsf{R} d$ and $c <_{\texttt{lex}} d$.

If $[x]_{\mathsf{E}}$ has no endpoints in the sense of $\to$ then either

$$[x]_{\mathsf{E}} = \{\dots \to a_{-2} \to a_{-1} \to a_0 \to a_1 \to a_2 \to \dots\}$$

is a bi-infinite chain or $[x]_{\mathsf{E}} = \{a_1 \to a_2 \to a_3 \to \dots \to a_n \to a_1\}$ is a finite cyclic chain. In the first subcase we straightforwardly define an action of $\mathbb{Z}$ on $[x]_{\mathsf{E}}$ by $1 \cdot a_n = a_{n+1}, \forall\, n \in \mathbb{Z}$. In the second subcase put $1 \cdot a_k = a_{k+1}$ for $k < n$, and $1 \cdot a_n = a_1$. If $[x]_{\mathsf{E}} = \{a_1 \to a_2 \to a_3 \to \dots \to a_n\}$ is a chain with two endpoints then the action is defined the same way. If finally $[x]_{\mathsf{E}}$ is a chain with just one endpoint, say $[x]_{\mathsf{E}} = \{a_0 \to a_1 \to a_2 \to \dots\}$, then put $1 \cdot a_{2n} = a_{2n+2}$, $1 \cdot a_{2n+3} = a_{2n+1}$, and $1 \cdot a_1 = a_0$.

(v) $\Longrightarrow$ (vi). The authors of [23] present a short proof which refers to several difficult theorems on hyperfinite ERs. Here we give an elementary proof.

Let E be induced by a Borel action of $\mathbb{Z}$. We are going to define F and R on any E-class $C = [x]_{\mathsf{E}}$. If we can choose an element $x_C \in C$ in some uniform Borel-definable way then a rather easy construction is possible, which we leave to the reader. This applies, for instance, when C is finite, hence, let us assume that C is infinite. Then the linear order $<_{\texttt{act}}$ on C induced by the action of $\mathbb{Z}$ is obviously similar to $\mathbb{Z}$. Let $<_{\texttt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}} = \texttt{dom}\,\mathsf{E}$.

Our goal is to define F on C so that every F-class contains exactly two (distinct) elements. The ensuing definition of R is then rather simple. (First, order pairs $\{x, y\}$ of elements of C in accordance with the $<_{\texttt{act}}$-lexicographical ordering of pairs $\langle \mathtt{max}_{<_{\mathtt{act}}}\{x,y\}, \mathtt{min}_{<_{\mathtt{act}}}\{x,y\}\rangle$, this is still similar to $\mathbb{Z}$. Now, if $\{x,y\}$ and $\{x',y'\}$ are two F-classes, the latter being the next to the former in the sense just defined, and $x <_{\mathtt{act}} y$, $x' <_{\mathtt{act}} y'$, then define $y \mathrel{\mathsf{R}} x'$.)

Suppose that $W \subseteq C$. An element $z \in W$ iz *lmin* (locally minimal) in W if it is $<_{\mathtt{lex}}$-smaller than both of its $<_{\mathtt{act}}$-neighbours in W. Put $W_{\text{lmin}} = \{z \in W : z \text{ is lmin in } W\}$. If C_{lmin} is not unbounded in C in both directions then an appropriate choice of $x_C \in C$ is possible. (Take the $<_{\mathtt{act}}$-least or $<_{\mathtt{act}}$-largest point in C_{lmin}, or if $C_{\text{lmin}} = \varnothing$, so that, for instance, $<_{\mathtt{act}}$ and $<_{\mathtt{lex}}$ coincide on C, we can choose something like a $<_{\mathtt{lex}}$-middest element of C.) Thus, we can assume that C_{lmin} is unbounded in C in both directions.

Let a *lmin-interval* be any $<_{\mathtt{act}}$-semi-interval $[x,x')$ between two consecutive elements $x <_{\mathtt{act}} x'$ of C_{lmin}. Let $[x,x') = \{x_0, x_1, \dots, x_{m-1}\}$ be the enumeration in the $<_{\mathtt{act}}$-increasing order ($x_0 = x$). Define $x_{2k} \mathrel{\mathsf{F}} x_{2k+1}$ whenever $2k + 1 < m$. If m is odd then x_{m-1} remains unmatched. Let C^1 be the set of all unmatched elements. Now, the nontrivial case is when C^1 is unbounded in C in both directions. We define C^1_{lmin}, as above, and repeat the same construction, extending F to a part of C^1, with, perhaps, a remainder $C^2 \subsetneq C^1$ where F remains indefined. *Et cetera*.

Thus, we define a decreasing sequence $C = C^0 \supsetneq C^1 \supsetneq C^2 \supsetneq \dots$ of subsets of C, and the equivalence relation F on each difference $C^n \smallsetminus C^{n+1}$ whose classes contain exactly two points each, and the nontrivial case is when every C^n is $<_{\mathtt{act}}$-unbounded in C in both directions. (Otherwise there is an appropriate choice of $x_C \in C$.) If $C^\infty = \bigcap_n C^n = \varnothing$ then F is defined on C and we are done. If $C^\infty = \{x\}$ is a singleton then $x_C = x$ chooses an element in C. Finally, C^∞ cannot contain two different elements as otherwise one of C^n would contain two $<_{\mathtt{act}}$-neighbours $x <_{\mathtt{act}} y$ which survive in C^{n+1}, which is easily impossible. □

6d Non-hyperfinite countable equivalence relations

It follows from Theorem 6.5(i),(ii) that hyperfinite equivalence relations form an initial segment, in the sense of $\le_{\text{B}}$, within the collection of all Borel countable equivalence relations. Let us show that this is a proper initial segment, that is, not all Borel countable equivalence relations are hyperfinite.

Theorem 6.6. *The equivalence relation* E_∞ *is not hyperfinite, in particular* $\mathsf{E}_0 <_{\text{B}} \mathsf{E}_\infty$.

Proof. We present the original proof of this result given in [56]. There is another, more complicated proof, based on the fact that a certain property called *amenability* holds for all hyperfinite equivalence relations and associated groups like $\langle \mathbb{Z}\,;+\rangle$, but fails for E_∞ and the group F_2 — see [33, 23] and references there for details.

Given a pair of bijections $f,g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$, we define an action $\mathfrak{a}_{fg}$ of the free group F_2 with two generators a,b on $2^{\mathbb{N}}$ as follows: if $w = a_1 a_2 \dots a_n \in F_2$

then $\mathfrak{a}_{fg}(w,x) = w\cdot x = h_{a_1}(h_{a_2}(\dots(h_{a_n}(x))\dots))$, where $h_a = f$, $h_{a^{-1}} = f^{-1}$, $h_b = g$, $h_{b^{-1}} = g^{-1}$. Separately $\Lambda\cdot x = x$, where Λ, the empty word, is the neutral element of F_2. The maps f,g are *independent*, iff the action is free, that is, for any x, $w\cdot x = x$ implies $w = \Lambda$.

To prove the theorem we define a free action of F_2 on $2^{\mathbb{N}}$ by *Lipschitz homeomorphisms*, *i.e.* those homeomorphisms $f : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$ satisfying $x \restriction n = y \restriction n \iff f(x)\restriction n = f(y)\restriction n$ for all n and $y \in 2^{\mathbb{N}}$. Such an action can be extended to any set $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ so that $w\cdot(x\restriction n) = (w\cdot x)\restriction n$ for all $x \in 2^{\mathbb{N}}$.

Lemma 6.7. *There exists an independent pair of Lipschitz homeomorphisms* $f,g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$.

Proof. Define $f\restriction 2^n$ and $g\restriction 2^n$ by induction on n. We'll take care that

$$\mathtt{lh}\, f(s) = \mathtt{lh}\, g(s) = \mathtt{lh}\, s, \quad f(s) \subset f(s^\wedge i), \quad \text{and} \quad g(s) \subset g(s^\wedge i) \tag{1}$$

for all $s \in 2^{<\omega}$ and $i = 0,1$. Fix a linear ordering of length ω, of the set of all pairs $\langle w,s\rangle \in F_2 \times 2^{<\omega}$ such that $w \neq \Lambda$.

Put $f(\Lambda) = g(\Lambda) = \Lambda$ $(n=0)$ and $f(\langle i\rangle) = g(\langle i\rangle) = \langle 1-i\rangle$, $i = 0,1$.

To carry out the step $n \to n+1$, suppose that the values $f(s), g(s)$, and subsequently $w\cdot s$ for all $w \in F_2$, have been defined for all $w \in F_2$ and $s \in 2^{<\omega}$ with $\mathtt{lh}\, s \le n$. Let $\langle w_n, s_n\rangle$ be the least pair (in the sense of the ordering mentioned above) such that $k = \mathtt{lh}\, s_n \le n$, there is $t \in 2^n$ with $s_n \subseteq t$ and $w_n\cdot t = t$, and $u\cdot s_n \neq v\cdot s_n$ for all initial subwords[5] $u \neq v$ of w_n — except for the case when $u = \Lambda$ and $v = w_n$ or vice versa. (Pairs $\langle w,s\rangle$ of this kind do exist: as 2^n is finite, for any $s \in 2^n$ there is $w \in F_2 \smallsetminus \{\Lambda\}$ such that $w\cdot s = s$.)

We put $T_n = \{t \in 2^n : s_n \subseteq t \wedge w_n\cdot t = t\}$. The sets

$$C_t = \{u\cdot t : u \text{ is an initial subword of } w_n\}\,, \quad t \in T_n\,,$$

are pairwise disjoint. Indeed if $u\cdot t_1 = v\cdot t_2 = t'$, where u,v are initial subwords of w_n, then $u \neq v$ as otherwise $t_1 = u^{-1}\cdot t' = v^{-1}\cdot t') = t_2$. But then $u\cdot s_n = v\cdot s_n$ (as t_1,t_2 extend s_n), which contradicts the choice of s_n.

Consider any $t \in T_n$. The word w_n has the form $a_0\, a_1\, \dots\, a_{m-1}$ for some $m \ge 1$, where all a_ℓ belong to $\{a,b,a^{-1},b^{-1}\}$. Then $C_t = \{t_0,t_1,\dots,t_m\}$, where $t_0 = t$ and $t_{\ell+1} = a_\ell\cdot t_\ell$, $\forall\,\ell$. Easily $t_m = w_n\cdot t = t = t_0$, but $t_\ell \neq t_{\ell'}$ whenever $\ell < \ell' < m$. We define $a_0\cdot(t_0{}^\wedge i) = t_1{}^\wedge(1-i)$ for $i = 0,1$, but $a_\ell\cdot(t_\ell{}^\wedge i) = t_{\ell+1}{}^\wedge i$ whenever $1 \le \ell < m$. Then easily $w_n\cdot(t^\wedge i) = t^\wedge(1-i) \neq t$.

Note that this definition of **some** of the values of $a\cdot r, b\cdot r, a^{-1}\cdot r, b^{-1}\cdot r$, $r \in 2^{n+1}$, is self-consistent.[6] Thus it remains consistent on the union of all "cycles"

[5] Λ and w itself are considered as initial subwords of any word $w \in F_2$.

[6] The inconsistency would have appeared in the case $a_{m-1}^{-1} = a_0$. Then $a_0\cdot(t_0{}^\wedge i) = t_1{}^\wedge(1-i)$ while $a_{m-1}^{-1}\cdot(t_m{}^\wedge i) = t_{m-1}{}^\wedge i$, and $t_0 = t_m$. However $a_{m-1}^{-1} \neq a_0$, since otherwise $a_0^{-1}s_n = (a_0\,\dots\,a_{m-2})\cdot s_n$, contrary to the choice of s_n.

$C_t,\ t \in T_n$. It follows that the action of f and g can be defined on 2^{n+1} so that (1) holds, while the values of $a_\ell \cdot (t_\ell{}^\wedge i)$ coincide with the abovedefined ones within each cycle $C_t,\ t \in T_n$. Then $w_n \cdot (t^\wedge i) \neq t^\wedge i$ for all $t \in T_n,\ i = 0, 1$. It follows that there can be no pair $\langle w_{n'}, s_{n'} \rangle,\ n' > n$, equal to $\langle w_n, s_n \rangle$.

This definition results in a pair of Lipschitz homeomorphisms f, g of $2^{\mathbb{N}}$. To check the independence, suppose towards the contrary that $x \in 2^{\mathbb{N}},\ w \in F_2$, $w \neq \Lambda$, and $w \cdot x = x$, and there is no shorter word w of this sort. Then there exists $k \in \mathbb{N}$ such that $s = x \restriction k$ satisfies $u \cdot s \neq v \cdot s$ for all initial subwords $u \neq v$ of w except for the case $u = \Lambda$ and $v = w$ (or vice versa). The pair $\langle w, s \rangle$ is equal to $\langle w_n, s_n \rangle$ for some $n \geq k$. Then the set T_n contains the element $t = x \restriction n$. Put $i = x(n)$. Then by definition $w \cdot (t^\wedge i) = (w \cdot t)^\wedge (1 - i) = t^\wedge (1 - i) \neq t^\wedge i$, contary to the assumption $w \cdot x = x$. □ (*Lemma*)

Fix a pair of independent Lipschitz homeomorphisms $f, g : 2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. Define the action $\mathfrak{a}(w, x) = w \cdot x$ as above. This Polish (even "Lipschitz") action of F_2 on $2^{\mathbb{N}}$ induces a Borel countable equivalence relation $x \mathsf{E} y$ iff $\exists\, w \in F_2\, (y = w \cdot x)$. Let us show that E is not hyperfinite.

Suppose towards the contrary that $\mathsf{E} = \bigcup_n \mathsf{F}_n$ where $\{\mathsf{F}_n\}_{n \in \mathbb{N}}$ is a $\subseteq$-increasing sequence of finite Borel equivalence relations. For any x let n_x be the least n such that $\{f(x), g(x), f^{-1}(x), g^{-1}(x)\}$ is a subset of $[x]_{\mathsf{F}_n}$. Then there exist a number n and a closed $X \subseteq 2^{\mathbb{N}}$ such that $n_x \leq n$ for all $x \in X$, and $\mu(X) \geq 3/4$, where μ is the uniform probability measure on $2^{\mathbb{N}}$.

Define the subtree $T = \{x \restriction m : x \in X \wedge m \in \mathbb{N}\}$ of $2^{<\omega}$. We claim that the set U of all pairs $\langle w, s \rangle \in F_2 \times 2^{<\omega}$ such that $\mathtt{lh}\, w = \mathtt{lh}\, s$ and $u \cdot s \in T$ for any initial subword u of w (including Λ and w) is infinite.

To prove this fact fix $\ell \in \mathbb{N}$ and find $\langle w, s \rangle \in U$ such that $\mathtt{lh}\, s = \mathtt{lh}\, w \geq \ell$. By the independence of f, g, we have $w \cdot x \neq x$ for all $w \in W = \{a, b, a^{-1}, b^{-1}\}$ and $x \in 2^{\mathbb{N}}$, in addition $w \cdot x \neq w' \cdot x$ for any $w \neq w'$ in W. Then by König that there is a number $m \geq \ell$ such that $w \cdot s \neq s$ and $w \cdot s \neq w' \cdot s$ for all $w \neq w'$ in W and all $s \in 2^m$. Note that the graph

$$\Gamma = \{\{s, t\} : s, t \in 2^m \wedge \exists\, w \in W\ (w \cdot s = t)\}$$

on 2^m has exactly $2 \cdot 2^m$ edges: indeed, by the choice of m for every $s \in 2^m$ there exist exactly 4 different nodes $t \in 2^m$ such that $\{s, t\} \in \Gamma$.

Consider the subgraph $G = \{\{s, t\} \in \Gamma : s, t \in T\}$. The intersection $T \cap 2^m$ contains at least $\frac{3}{4} \cdot 2^m$ elements (as X is a set of measure $\geq 3/4$), accordingly the difference $2^m \smallsetminus T$ contains at most $\frac{1}{4} \cdot 2^m$ elements. Thus comparably to Γ the subgraph G loses at most $4 \cdot \frac{1}{4} \cdot 2^m = 2^m$ edges. In other words, G, a graph with $\leq 2^m$ nodes, has at least $2 \cdot 2^m - 2^m = 2^m$ edges.

Now we apply the following combinatorial fact.

Lemma 6.8. *Any graph G on a finite set Y, containing not more nodes than edges, has a cycle with at least three nodes.*

Proof (Sketch)**.** Otherwise Y contains an endpoint, that is, an element $y \in Y$ such that $\{y, y'\} \in G$ holds for at most one $y' \in Y \smallsetminus \{y\}$. This allows to use induction on the number of nodes. □

Thus G contains a cycle $s_0, s_1, \dots, s_{k_1}, s_k = s_0$. Here $k \ge 3$, all s_k belong to $T \cap 2^m$, s_i, $i < k$, are pairwise different, and for any $i < k$ there exists $a_i \in W = \{a, b, a^{-1}, b^{-1}\}$ such that $a_i \cdot s_i = s_{i+1}$. The word $u = a_0 a_1 \dots a_{k-1}$ is irreducible as otherwise $s_{i-1} = s_{i+1}$ for some $0 < i < k$. Moreover the word uu (the concatenation of two copies of u) is irreducible, too, as otherwise $s_1 = s_{k-1}$. Therefore u^m (the concatenation of m copies of u) is irreducible as well, and so is its initial subword $w = u^m \restriction m$. It follows that $\langle w, s_0 \rangle \in U$, as required.

As U is infinite, by König it contains an infinite branch, *i.e.* there is an (irreducible) word $w \in \{a, b, a^{-1}, b^{-1}\}^{\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$ such that $\langle w \restriction m, x \restriction m \rangle \in U$ for all m. Then clearly $(w \restriction m) \cdot x \in X$ for all m, and hence $x \mathsf{F}_n ((w \restriction m) \cdot x)$ by induction on m. Finally $(w \restriction m) \cdot x \ne (w \restriction m') \cdot x$ holds whenever $m \ne m'$ by the independence of f, g. Thus the equivalence class $[x]_{\mathsf{F}_n}$ is infinite, contradiction.

Thus E is a countable non-hyperfinite equivalence relation. Recall that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty$ by Theorem 6.3. Thus E_∞ itself is non-hyperfinite as well by the equivalence (i) $\Longleftrightarrow$ (ii) of Theorem 6.5. □ (*Theorem*)

6e Assembling countable equivalence relations

The following theorem shows that in certain cases the notion of being Borel reducible to a given countable Borel equivalence relation is σ-additive. The sum $\mathsf{F} + \mathsf{F}$ means the union of two Borel isomorphic copies of F defined on a pair of disjoint (and F-disconnected) Borel sets (in one and the same Polish space).

Theorem 6.9. *Let F be a countable Borel ER satisfying $\mathsf{F} + \mathsf{F} \le_{\mathrm{B}} \mathsf{F}$, and E be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all X_k also Borel. Suppose that $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{F}$ for each k. Then $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$.*

Proof. It obviously suffices to prove that if E is a Borel equivalence relation defined on the union $X \cup Y$ of disjoint Borel sets X, Y, F is a countable Borel equivalence relation defined on the union $P \cup Q$ of disjoint Borel sets P, Q, F-disconnected in the sense that $p \not\mathsf{F} q$ for all $p \in P$, $q \in Q$, and f, g are Borel reductions of resp. $\mathsf{E} \restriction X$, $\mathsf{E} \restriction Y$ to resp. $\mathsf{F} \restriction P$, $\mathsf{E} \restriction Q$ then there is a Borel reduction h of E to F. As X, Y are **not** assumed to be E-disconnected, the key problem is to define $h(y)$ in the case when $y \in Y$ satisfies $g(y) \in \mathtt{ran}\, U$, where

$$U = \{\langle p, q \rangle \in P \times Q : \exists x \in X \, \exists y \in Y \, (x \mathsf{E} y \wedge f(x) = p \wedge f(y) = q)\}$$

is a $\mathbf{\Sigma}^1_1$ set. As f, g are reductions to F, U is a subset of the $\mathbf{\Pi}^1_1$ set

$$W = \{\langle p, q \rangle \in P \times Q : \forall \langle p', q' \rangle \in U \, (p \mathsf{F} p' \Longleftrightarrow q \mathsf{F} q')\} .$$

Therefore by Separation there is an intermediate Borel set V, $U \subseteq V \subseteq W$.

The set U is 1−1 modulo F in the sense that the equivalence $p \,\mathsf{F}\, p' \iff q \,\mathsf{F}\, q'$ holds for any two pairs $\langle p,q\rangle$ and $\langle p',q'\rangle$ in U. The set V does not necessarily have this property. To obtain a Borel subset of V and still superset of U, $1-1$ modulo F, note that U is a subset of the $\mathbf{\Pi}^1_1$ set

$$R = \{\langle p',q'\rangle \in V : \forall\, \langle p,q\rangle \in V\ (p \,\mathsf{F}\, p' \iff q \,\mathsf{F}\, q')\}\,.$$

It follows that there exists a Borel set S with $U \subseteq S \subseteq R$. Clearly S is $1-1$ modulo F together with R. Since F is a countable equivalence relation, it follows by Countable-to-1 Projection and Countable-to-1 Enumeration (Theorems 1.3 and 1.4) that the set $Z = \mathtt{ran}\, S$ is Borel and there is a Borel map $\vartheta : Z \to P$ such that $\langle \vartheta(q), q\rangle \in S$ for every $q \in Z$.

In particular, we have $\mathtt{ran}\, U \subseteq Z$ and $p \,\mathsf{F}\, \vartheta(q)$ for all pairs $\langle p,q\rangle \in U$. In addition, it can be *w.l.o.g.* assumed that Z is F-invariant, *i.e.* $q \in Z \wedge q' \,\mathsf{F}\, q \Longrightarrow q' \in Z$. (Indeed consider the set $Z' = [Z]_{\mathsf{F}} = \{q' : \exists\, q \in Z\ (q \,\mathsf{F}\, q')\}$. Note that F is the orbit equivalence of a Polish action of a countable group by Theorem 6.3. It follows that there exists a countable system $\{\beta_n\}_{n\in\mathbb{N}}$ of Borel isomorphisms of the set $P \cup Q = \mathtt{dom}\, \mathsf{F}$ such that $Z' = \bigcup_n \{\beta_n(q) : q \in Z\}$. It follows that Z' is Borel by Countable-to-1 Projection, and by Countable-to-1 Enumeration there is a Borel map $\zeta : Z' \to Z$ such that $\zeta(q') \,\mathsf{F}\, q'$ for all $q' \in Z'$. Replace Z, ϑ by Z' and the map $\vartheta'(q') = \zeta(\vartheta(q'))$.)

This allows us to define a Borel reduction of E to F as follows. Naturally, put $h(x) = f(x)$ for $x \in X$. If $y \in Y$ and $g(y) \notin Z$ then put $h(y) = g(y)$, while in the case $g(y) \in Z$ we define $h(y) = \vartheta(g(y))$. □

The condition $\mathsf{F} + \mathsf{F} \le_{\mathrm{B}} \mathsf{F}$ holds for many naturally arising equivalence relations F. (In fact it is not clear how to cook up a Borel equivalence not satisfying this reduction.) In particular it holds for $\mathsf{F} = \mathsf{E}_0$ and the equalities $\mathsf{F} = \mathsf{EQ}_X$.

Corollary 6.10. *Suppose that E be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all X_k also Borel. If $\mathsf{E} \restriction X_k$ is smooth (resp. hyperfinite) for all k then E itself is smooth (resp. hyperfinite. If $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$ for all k then $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$.* □

6f Fin is the $\le_{\mathrm{B}}$-least ideal!

The proof of the following useful result is based on a short argument involved in many other results, including several proofs in Chapter 5.

Theorem 6.11. (i) [24, 47, 60] *If $\mathscr{I}$ is a* (nontrivial) *ideal on* $\mathbb{N}$, *with the Baire property in the topology of $\mathscr{P}(\mathbb{N})$, then* $\mathsf{Fin} \le^{++}_{\mathrm{RB}}$ *and* $\le_{\mathrm{RB}} \mathscr{I}$;

(ii) *however $\mathsf{EQ}_{2^{\mathbb{N}}} <_{\mathrm{B}} \mathsf{E}_0$ strictly, thus $\mathsf{EQ}_{2^{\mathbb{N}}}$ is not $\sim_{\mathrm{B}}$-equivalent to an equivalence relation of the form $\mathsf{E}_{\mathscr{I}}$* ;

(iii) *if* $\mathscr{I} \le^{+}_{\mathrm{RB}} \mathscr{J}$ *are Borel ideals, and there is an infinite set* $Z \subseteq \mathtt{dom}\,\mathscr{I}$ *such that* $\mathscr{I} \restriction Z = \mathscr{P}_{\mathtt{fin}}(Z)$, *then* $\mathscr{I} \le_{\mathrm{RB}} \mathscr{J}$.

Proof. (i) First of all $\mathscr{I}$ must be meager in $\mathscr{P}(\mathbb{N})$. (Otherwise $\mathscr{I}$ would be comeager somewhere, easily leading to contradiction.) Thus, all $X \subseteq \mathbb{N}$ "generic" [7] do not belong to $\mathscr{I}$. Now it suffices to define non-empty finite sets $w_i \subseteq \mathbb{N}$ with $\mathtt{max}\, w_i < \mathtt{min}\, w_{i+1}$ such that any union of infinitely many of them is "generic". Clearly the following observation yields the result: if D is an open dense subset of $\mathscr{P}(\mathbb{N})$ and $n \in \mathbb{N}$ then there is $m > n$ and a set $u \subseteq [n, m]$ with $m,\, n \in u$ such that any $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [n, m] = u$ belongs to D.

Thus we have $\mathsf{Fin} \le^{++}_{\mathrm{RB}} \mathscr{I}$. To derive $\mathsf{Fin} \le_{\mathrm{RB}} \mathscr{I}$ cover each w_k by a finite set u_k such that $\bigcup_{k\in\mathbb{N}} u_k = \mathbb{N}$ and still $u_k \cap u_l = \varnothing$ for $k \ne l$.

(ii) That $\mathsf{EQ}_{2^{\mathbb{N}}} \le_{\mathrm{B}} \mathsf{E}_0$ is witnessed by any perfect set $X \subseteq 2^{\mathbb{N}}$ which is a *partial* transversal for E_0 (*i.e.*, any $x \ne y$ in X are E_0-inequivalent). On the other hand, $\mathsf{EQ}_{2^{\mathbb{N}}}$ is smooth but E_0 is non-smooth by Lemma 6.2(v).

(iii) Assume *w.l.o.g.* that $\mathscr{I}, \mathscr{J}$ are ideals over $\mathbb{N}$. Let pairwise disjoint finite sets $w_k \subseteq \mathbb{N}$ witness $\mathscr{I} \le^{+}_{\mathrm{RB}} \mathscr{J}$. Put $Z' = \mathbb{N} \smallsetminus Z$, $X = \bigcup_{k\in Z} w_k$, and $Y = \bigcup_{k\in Z'} w_k$. The reduction via $\{w_k\}$ reduces $\mathscr{P}_{\mathtt{fin}}(Z)$ to $\mathscr{J} \restriction X$ and $\mathscr{I} \restriction Z'$ to $\mathscr{J} \restriction Y$. Keeping the latter, replace the former by a $\le_{\mathrm{RB}}$-like reduction of $\mathscr{P}_{\mathtt{fin}}(z)$ to $\mathscr{J} \restriction Y'$, where $Y' = \mathbb{N} \smallsetminus Y$, which exists by Theorem 6.11. □

[7] That is, Cohen generic in the sense a certain countable family of dense open subsets of $\mathscr{P}(\mathbb{N})$.

Chapter 7

The 1st and 2nd dichotomy theorems

The following two results are known as the first, or Silver, and 2nd, or "Glimm–Effros", dichotomy theorems.

Theorem 7.1 (Silver [55])**.** *Any* $\mathbf{\Pi}^1_1$ *(therefore any Borel) equivalence relation* E *on* $\mathbb{N}^{\mathbb{N}}$ **either** *has at most countably many equivalence classes* **or** *admits a perfect set of pairwise* E*-inequivalent reals.*

In other words, **either** $\mathsf{E} \le_{\mathrm{B}} \mathsf{EQ}_{\mathbb{N}}$ **or** $\mathsf{EQ}_{2^{\mathbb{N}}} \sqsubseteq_{\mathrm{C}} \mathsf{E}$*.*

Theorem 7.2 (Harrington, Kechris, Louveau [16])**.** *If* E *is a Borel equivalence relation then* **either** E *is smooth* **or** $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}$*.*

Recall that $\sqsubseteq_{\mathrm{C}}$ in the **or** part means the reducibility via a continuous injective map. Obviously $\sqsubseteq_{\mathrm{C}}$ implies $\le_{\mathrm{B}}$, and hence it follows from the first theorem that the union of the lower $\le_{\mathrm{B}}$-cone of $\mathsf{EQ}_{\mathbb{N}}$ and the upper $\le_{\mathrm{B}}$-cone of $\mathsf{EQ}_{2^{\mathbb{N}}}$ fully covers the whole class of Borel equivalence relations. As smoothness means simply $\mathsf{E} \le_{\mathrm{B}} \mathsf{EQ}_{2^{\mathbb{N}}}$, it follows from the second theorem that the union of the lower $\le_{\mathrm{B}}$-cone of $\mathsf{EQ}_{2^{\mathbb{N}}}$ and the upper $\le_{\mathrm{B}}$-cone of E_0 fully covers the whole class of Borel equivalence relations.

The proofs of these theorems follow below in this Chapter. They make heavy use of methods of effective descriptive set theory, in particular, the Gandy – Harrington topology. We begin with a brief introduction into this technical tool.

This Chapter ends with an introduction into an interesting forcing notion that consists of all uncountable Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is not smooth.

7a The Gandy – Harrington topology

The following notion is similar to the Choquet property but somewhat more convenient to provide the nonemptiness of countable intersections of pointsets.

Definition 7.3. A family $\mathscr{F}$ of sets in a topological space is *Polish–like* if there exists a countable collection $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of dense subsets $\mathscr{D}_n \subseteq \mathscr{F}$ such that we have $\bigcap_n F_n \neq \varnothing$ whenever $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ is a decreasing sequence of sets $F_n \in \mathscr{F}$ which intersects every $\mathscr{D}_n$.

Here, a set $\mathscr{D} \subseteq \mathscr{F}$ is *dense* if $\forall\, F \in \mathscr{F}\; \exists\, D \in \mathscr{D}\; (D \subseteq F)$. □

For instance if $\mathscr{X}$ is a Polish space then the collection of all its non-empty closed sets is Polish–like, for take $\mathscr{D}_n$ to be all closed sets of diameter $\le n^{-1}$. We'll make use of the following technical fact:

Theorem 7.4 (see *e.g.* Kanovei [28], Hjorth [17])**.** *The collection $\mathscr{F}$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$ is Polish–like.* □

Proof. For any $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ define $\mathtt{pr}\, P = \{x : \exists\, y\, P(x,y)\}$ (the projection). If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $s, t \in \mathbb{N}^{<\omega}$ then let $P_{st} = \{\langle x, y\rangle \in P : s \subset x \wedge t \subset y\}$. Let $\mathscr{D}(P,s,t)$ be the collection of all Σ^1_1 sets $\varnothing \neq X \subseteq \mathbb{N}^{\mathbb{N}}$ such that either $X \cap \mathtt{pr}\, P_{st} = \varnothing$ or $X \subseteq \mathtt{pr}\, P_{s^\wedge i,\, t^\wedge j}$ for some i, j. (Note that in the "or" case i is unique but j may be not unique.) Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be an arbitrary enumeration of all sets of the form $\mathscr{D}(P,s,t)$, where $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is Π^0_1. Note that in this case all sets of the form $\mathtt{pr}\, P_{st}$ are Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$, therefore, $\mathscr{D}(P,s,t)$ is easily a dense subset of $\mathscr{F}$, so that all $\mathscr{D}_n \subseteq \mathscr{F}$ are dense.

Now consider a decreasing sequence $X_0 \supseteq X_1 \supseteq \dots$ of non-empty Σ^1_1 sets $X_k \subseteq \mathbb{N}^{\mathbb{N}}$, which intersects every $\mathscr{D}_n$; prove that $\bigcap_n X_n \neq \varnothing$. Call a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ *positive* if there is n such that $X_n \subseteq X$. For any n, fix a Π^0_1 set $P^n \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $X_n = \mathtt{pr}\, P^n$. For any $s, t \in \mathbb{N}^{<\omega}$, if $\mathtt{pr}\, P^n_{st}$ is positive then, by the choice of the sequence of X_n, there is a unique i and some j such that $\mathtt{pr}\, P^n_{s^\wedge i,\, t^\wedge j}$ is also positive. It follows that there is a unique $x = x_n \in \mathbb{N}^{\mathbb{N}}$ and some $y = y_n \in \mathbb{N}^{\mathbb{N}}$ (perhaps not unique) such that $\mathtt{pr}\, P^n_{x\restriction k,\, y\restriction k}$ is positive for any k. As P^n is closed, we have $P^n(x,y)$, hence, $x_n = x \in X_n$.

It remains to show that $x_m = x_n$ for $m \neq n$. To see this note that if both P_{st} and $Q_{s't'}$ are positive then either $s \subseteq s'$ or $s' \subseteq s$. □

The collection of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$ is a base of the *Gandy – Harrington topology*, which has many remarkable applications in descriptive set theory. This topology is not Polish, even not metrizable at all, yet it shares the following important property of Polish topologies:

Corollary 7.5. *The Gandy – Harrington topology is Baire, that is every comeager set is dense.*

Proof. This can be proved using Choquet property of the topology, see [16], however, the Polish–likeness (Theorem 7.4) also immediately yields the result. □

7b The first dichotomy theorem.

Beginning the proof[1] of Theorem 7.1, let us fix a $\mathbf{\Pi}^1_1$ equivalence relation E on $\mathbb{N}^{\mathbb{N}}$. Then E belongs to $\Pi^1_1(p)$ for some parameter $p \in \mathbb{N}^{\mathbb{N}}$. As usual, we can suppose that E is in fact a lightface Π^1_1 relation; the case of an arbitrary p does not differ in any essential detail.

Case 1: every $x \in \mathbb{N}^{\mathbb{N}}$ belongs to a Δ^1_1 pairwise E-equivalent set X. (A set X is pairwise E-equivalent iff all elements of X are E-equivalent to each other, in other words, the saturation $[X]_{\mathsf{E}}$ is an equivalence class.) Then E has at most countably many equivalence classes.

Case 2: otherwise. Then the set H (*the domain of nontriviality*) of all $x \in \mathbb{N}^{\mathbb{N}}$ which do **not** belong to a Δ^1_1 pairwise E-equivalent set is non-empty.

Claim 7.6. *H is Σ^1_1. Any Σ^1_1 set $\varnothing \neq X \subseteq H$ is not pairwise E-equivalent.*

Proof. We make use of an enumeration of Δ^1_1 sets provided by Theorem 1.8. Suppose that $x \in \mathbb{N}^{\mathbb{N}}$. Then obviously $x \in H$ iff for any $e \in \mathbb{N}$: **if** e codes a Δ^1_1 set, say, $W_e \subseteq \mathbb{N}^{\mathbb{N}}$ and $x \in W_e$ **then** W_e is not E-equivalent. The **if** part of this characterization is Π^1_1 while the **then** part is Σ^1_1.

If $X \neq \varnothing$ is a pairwise E-equivalent Σ^1_1 set then $B = \bigcap_{x \in X} [x]_{\mathsf{E}}$ is a Π^1_1 E-equivalence class and $X \subseteq B$. By Separation (Theorem 1.2), there is a Δ^1_1 set C with $X \subseteq C \subseteq B$. Then, if $X \subseteq H$ then $C \subseteq H$ is a Δ^1_1 pairwise E-equivalent set, a contradiction to the definition of H. □ (*Claim*)

Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ (see Remark 5.9). We suppose that $\mathfrak{M}$ is an elementary submodel of the universe w.r.t. all analytic formulas[2]. We consider the set $\mathbb{P} = \{X \subseteq \mathbb{N}^{\mathbb{N}} : X$ is non-empty and $\Sigma^1_1\}$ as a forcing to extend $\mathfrak{M}$ (smaller sets are stronger conditions) — the *Gandy – Harrington forcing*. Obviously $\mathbb{P} \notin$ and $\not\subseteq \mathfrak{M}$, of course, but clearly $\mathbb{P}$ can be adequately coded in $\mathfrak{M}$, say, via a universal Σ^1_1 set.

Corollary 7.7 (from Theorem 7.4)**.** *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic, over $\mathfrak{M}$, set, then $\bigcap G$ contains a single real, denoted x_G.* □

Reals of the form x_G, G as in the corollary, are called $\mathbb{P}$-*generic* (over $\mathfrak{M}$). Let $\dot{x}$ be the name for x_G in the machinery of forcing $\mathbb{P}$. Then any condition $A \in \mathbb{P}$ forces that $\dot{x} \in A$.

The forcing product $\mathbb{P}^2$ consist of all rectangles $X \times Y$ with $X, Y \in \mathbb{P}$. It follows from the above by the product forcing lemmas that any set $G \subseteq \mathbb{P}^2$ $\mathbb{P}^2$-generic over $\mathfrak{M}$ produces a pair of reals (a $\mathbb{P}^2$-*generic pair*), say, $x^G_{\mathtt{left}}$ and

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[1] We present a forcing-style proof of Miller [49], with some simplifications. See [45] for another proof, based on the Gandy – Harrington topology. In fact both proofs involve very similar combinatorial arguments.

[2] Being an elementary submodel is useful to guarantee that relations like the inclusion orders of Σ^1_1 sets are absolute for $\mathfrak{M}$.

$x^G_{\mathtt{right}}$, so that $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}}\rangle \in W$ for any $W \in G$. Let $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$ be their names. The following is the key fact:

Lemma 7.8. *$H \times H$ $\mathbb{P}^2$-forces $\dot{x}_{\mathtt{left}} \not\mathsf{E}\, \dot{x}_{\mathtt{right}}$.*

Proof. Otherwise there is a condition $X \times Y \in \mathbb{P}^2$ with $X \cup Y \subseteq H$ that $\mathbb{P}^2$-forces $\dot{x}_{\mathtt{left}} \mathrel{\mathsf{E}} \dot{x}_{\mathtt{right}}$, and hence any $\mathbb{P}^2$-generic pair $\langle x, y\rangle \in X \times Y$ satisfies $x \mathrel{\mathsf{E}} y$. By the product forcing lemmas for any pair of $\mathbb{P}$-generic $x', x'' \in X$ there is $y \in Y$ such that both $\langle x, y\rangle$ and $\langle x', y\rangle$ are $\mathbb{P}^2$-generic pairs, therefore

$(*)$ $x' \mathrel{\mathsf{E}} x''$ holds for any points $x', x'' \in X$ separately $\mathbb{P}$-generic over $\mathfrak{M}$.

Note that the set $\mathbb{P}_2$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is just a copy of $\mathbb{P}$ (not of $\mathbb{P}^2$!) as a forcing. In particular, if a set $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic over $\mathfrak{M}$ then there is a unique pair of reals ($\mathbb{P}_2$-*generic pair*) $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}}\rangle$ which belongs to every W in G, and in this case, *both* $x^G_{\mathtt{left}}$ *and* $x^G_{\mathtt{right}}$ *are* $\mathbb{P}$-*generic*, because if $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic then the sets G' and G'' of all projections of sets $W \in G$ to resp. 1st and 2nd co-ordinate, are easily $\mathbb{P}$-generic. Now let $G \subseteq \mathbb{P}_2$ be a $\mathbb{P}_2$-generic set, over $\mathfrak{M}$, containing the Σ^1_1 set $P = X^2 \smallsetminus \mathsf{E}$. (Note that $P \neq \varnothing$ by Lemma 7.6.) Then $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}}\rangle \in P$, hence $x^G_{\mathtt{left}} \not\mathsf{E}\, x^G_{\mathtt{right}}$. However, as we observed, both $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$ are $\mathbb{P}$-generic elements of X (because $P \subseteq X \times X$), thus $x^G_{\mathtt{left}} \mathrel{\mathsf{E}} x^G_{\mathtt{right}}$ by $(*)$, contradiction. □ (*Lemma 7.8*)

Now to accomplish the proof of the theorem let us fix enumerations $\{\mathscr{D}(n)\}_{n\in\mathbb{N}}$ and $\{\mathscr{D}^2(n)\}_{n\in\mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}$ and $\mathbb{P}^2$ which are coded in $\mathfrak{M}$. Then there is a system $\{X_u\}_{u\in 2^{<\omega}}$ of sets X_u, satisfying

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$;

(ii) $X_u \in \mathscr{D}(n)$ whenever $u \in 2^n$;

(iii) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(iv) if $u \neq v \in 2^n$ then $X_u \times X_v \in \mathscr{D}^2(n)$.

It follows from (ii) that, for any $a \in 2^{\mathbb{N}}$, the set $\{X_{a\restriction m} : m \in \mathbb{N}\}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, hence, $\bigcap_m X_{a\restriction m}$ is a singleton by Corollary 7.7. Let x_a be its only element. The map $a \mapsto x_a$ is continuous because the diameters of sets X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$ by (i). In addition, by (iv) and Lemma 7.8, $x_a \not\mathsf{E}\, x_b$ holds for any pair $a \neq b$, in particular, $x_a \neq x_b$, hence, we have a perfect E-inequivalent set $Y = \{x_a : a \in 2^{\mathbb{N}}\}$.

□ (*Theorem 7.1*)

7c The second dichotomy theorem

Beginning the proof of Theorem 7.2 (it will be completed in Section 7f), we suppose, as usual, that E is a lightface Δ^1_1 equivalence relation on $\mathbb{N}^\mathbb{N}$. Similarly to Theorem 7.1, the proof employs the Gandy – Harrington topology, but is considerably more complicated.

Consider an auxiliary equivalence relation $x \mathrel{\widehat{\mathsf{E}}} y$ iff $x, y \in \mathbb{N}^\mathbb{N}$ belong to the same E-invariant Δ^1_1 sets. (A set X is E-*invariant* iff $X = [X]_\mathsf{E}$.) Easily $\mathsf{E} \subseteq \widehat{\mathsf{E}}$. In fact it follows from the next lemma that $\widehat{\mathsf{E}}$ is equal to the closure of E in the Gandy – Harrington topology.

Lemma 7.9. *If F is a Σ^1_1 ER on $\mathbb{N}^\mathbb{N}$, and $X, Y \subseteq \mathbb{N}^\mathbb{N}$ are disjoint F-invariant Σ^1_1 sets, then there is an F-invariant Δ^1_1 set X' separating X from Y.*

Proof. By $\mathsf{Separation}$, for any Σ^1_1 set A with $A \cap Y = \varnothing$ there is a Δ^1_1 set A' with $A \subseteq A'$ and $A' \cap Y = \varnothing$ — note that then $[A']_\mathsf{F} \cap Y = \varnothing$ because Y is F-invariant. It follows that that there is a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \dots$, where A'_i are Δ^1_1 sets, accordingly, $A_{i+1} = [A'_i]_\mathsf{F}$ are Σ^1_1 sets, and $A_i \cap Y = \varnothing$. Then $X' = \bigcup_n A_n = \bigcup_n A'_n$ and is an F-invariant Borel set which separates X from Y. To ensure that X' is Δ^1_1 we have to maintain the choice of sets A_n in effective manner.

Let $U \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}$ be a "good" universal Σ^1_1 set. (We make use of Theorem 1.10.) Then there is a recursive $h : \mathbb{N} \to \mathbb{N}$ such that $[U_n]_\mathsf{F} = U_{h(n)}$ for each n. Moreover, applying Lemma 1.11 (to the complement of U as a "good" universal Π^1_1 set, and with a code for Y fixed), we obtain a pair of recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for any n, if $U_n \cap Y = \varnothing$ then $U_{f(n)}, U_{g(n)}$ are complementary Σ^1_1 sets (hence, either of them is Δ^1_1) containing, resp., U_n and Y. A suitable iteration of h and f, g allows us to define a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \dots$ as above effectively enough for the union of those sets to be Δ^1_1. □ (*Lemma*)

Lemma 7.10. *$\widehat{\mathsf{E}}$ is a Σ^1_1 relation.*

Proof. Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}$ be as in Theorem 1.8. The formula $\texttt{inv}(e)$ saying that $e \in C$ and the set $W_e = W'_e$ is E-invariant, that is,

$$e \in C \ \wedge\ \forall\, a, b\, \big(a \in W_e \wedge b \notin W'_e \Longrightarrow a \mathrel{\not\mathsf{E}} b\big)$$

— is obviously a Π^1_1 formula. On the other hand, $x \mathrel{\widehat{\mathsf{E}}} y$ iff

$$\forall\, e\, \big(\texttt{inv}(e) \implies (x \in W_e \Longrightarrow y \in W'_e) \wedge (y \in W_e \Longrightarrow x \in W'_e)\big). \qquad \square\ (\textit{Lemma})$$

Let us return to the proof of Theorem 7.2. We have two cases.

Case 1: $\mathsf{E} = \widehat{\mathsf{E}}$, that is E is Gandy – Harrington closed. The next lemma shows that in this assumption we obtain the **either** case in Theorem 7.2.

Lemma 7.11. *If* $\mathsf{E} = \widehat{\mathsf{E}}$ *then there is a* $\varDelta^1_1$ *reduction of* E *to* $\mathsf{EQ}_{2^{\mathbb{N}}}$.

Proof. Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in Theorem 1.8. By Kreisel Selection (Theorem 1.7) there is a $\varDelta^1_1$ function $\varphi : X^2 \to C$ such that $W_{\varphi(x,y)} = W'_{\varphi(x,y)}$ is a E-invariant $\varDelta^1_1$ set containing x but not y whenever $x, y \in X$ are E-inequivalent. Then $R = \mathtt{ran}\,\varphi$ is a $\varSigma^1_1$ subset of C, hence, by Separation, there is a $\varDelta^1_1$ set N with $R \subseteq N \subseteq C$. The map $\vartheta(x) = \{n \in N : x \in D_n\}$ is a $\varDelta^1_1$ reduction of E to $\mathsf{EQ}_{2^{\mathbb{N}}}$. □ (*Lemma and Case 1*)

Case 2: $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$. Then the $\varSigma^1_1$ set $H = \{x : [x]_{\mathsf{E}} \subsetneqq [x]_{\widehat{\mathsf{E}}}\}$ (the union of all $\widehat{\mathsf{E}}$-classes containing more than one E-class) is non-empty. We are going to prove that this leads to the **or** case in Theorem 7.2. This will take some space. We begin with a couple of technical lemmas. The first of them says that the property $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$ holds hereditarily within the key domain H.

Lemma 7.12. *If* $X \subseteq H$ *is a* $\varSigma^1_1$ *set then* $\mathsf{E} \subsetneqq \widehat{\mathsf{E}}$ *on* X .

Proof. Suppose that $\mathsf{E} \restriction X = \widehat{\mathsf{E}} \restriction X$. Then $\mathsf{E} = \widehat{\mathsf{E}}$ on $Y = [X]_{\mathsf{E}}$ as well. (If $y, y' \in Y$ then there are $x, x' \in X$ such that $x \mathrel{\mathsf{E}} y$ and $x' \mathrel{\mathsf{E}} y'$, so that if $y \mathrel{\widehat{\mathsf{E}}} y'$ then $x \mathrel{\widehat{\mathsf{E}}} x'$ by transitivity, hence, $x \mathrel{\mathsf{E}} x'$, and $y \mathrel{\mathsf{E}} y'$ again by transitivity.) It follows that $\mathsf{E} = \widehat{\mathsf{E}}$ on an even bigger set, $Z = [X]_{\widehat{\mathsf{E}}}$. (Otherwise the $\varSigma^1_1$ set $Y' = Z \smallsetminus Y = \{z : \exists x \in X\,(x \mathrel{\widehat{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y)\}$ is non-empty and E-invariant, together with Y, hence by Lemma 7.9 there is a E-invariant $\varDelta^1_1$ set B with $Y \subseteq B$ and $Y' \cap B = \varnothing$, which implies that no point in Y is $\widehat{\mathsf{E}}$-equivalent to a point in Y', contradiction.) Then by definition $Z \cap H = \varnothing$. □ (*Lemma*)

Lemma 7.13. *If* $A, B \subseteq H$ *are non-empty* $\varSigma^1_1$ *sets with* $A \mathrel{\mathsf{E}} B$ *then there exist non-empty* **disjoint** $\varSigma^1_1$ *sets* $A' \subseteq A$ *and* $B' \subseteq B$ *still satisfying* $A' \mathrel{\mathsf{E}} B'$.

Recall that $A \mathrel{\mathsf{E}} B$ means that $[A]_{\mathsf{E}} = [B]_{\mathsf{E}}$.

Proof. We assert that there are points $a \in A$ and $b \in B$ with $a \neq b$ and $a \mathrel{\mathsf{E}} b$.

(Otherwise E is the equality on $X = A \cup B$. Prove that then $\mathsf{E} = \widehat{\mathsf{E}}$ on X, a contradiction to Lemma 7.12. Take any $x \neq y$ in X. Let U be a clopen set containing x but not y. Then $A = [U \cap X]_{\mathsf{E}}$ and $C = [X \smallsetminus U]_{\mathsf{E}}$ are two disjoint E-invariant $\varSigma^1_1$ sets containing resp. x, y. Then $x \mathrel{\widehat{\mathsf{E}}} y$ fails by Lemma 7.9.)

Thus let a, b be as indicated. Let U be a clopen set containing a but not b. Put $A' = A \cap U \cap [U^{\complement}]_{\mathsf{E}}$ and $B' = B \cap U^{\complement} \cap [U]_{\mathsf{E}}$. □ (*Lemma*)

7d Restricted product forcing

In continuation of the proof of Theorem 7.2 (Case 2), we come back to the forcing notions $\mathbb{P}$ and $\mathbb{P}_2$ introduced in Section 7b. Let us fix a countable model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ chosen as in Section 7b.

Let $\mathbb{P}^2\restriction\mathsf{E}$ be the collection of all sets of the form $X \times Y$, where $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ are non-empty Σ^1_1 sets and $X \mathsf{E} Y$ (which means here that $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$). Easily $\mathbb{P}_2 \subseteq \mathbb{P}^2\restriction\mathsf{E} \subseteq \mathbb{P}^2$. The forcing $\mathbb{P}^2\restriction\mathsf{E}$ is not really a product, yet if $X \times Z \in \mathbb{P}^2\restriction\mathsf{E}$ and $\varnothing \neq X' \subseteq X$ is Σ^1_1 then $Z' = Z \cap [X']_{\mathsf{E}}$ is Σ^1_1 and $X' \times Z' \in \mathbb{P}^2\restriction\mathsf{E}$. It follows that any set $G \subseteq \mathbb{P}^2\restriction\mathsf{E}$, $\mathbb{P}^2\restriction\mathsf{E}$-generic over $\mathfrak{M}$, still produces a pair of $\mathbb{P}$-generic sets $G_{\mathtt{left}} = \{\operatorname{\mathtt{dom}} P : P \in G\}$ and $G_{\mathtt{right}} = \{\operatorname{\mathtt{ran}} P : P \in G\}$, therefore produces a pair of $\mathbb{P}$-generic reals $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, whose names will be $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$ as above.

Lemma 7.14. *In the sense of the forcing $\mathbb{P}^2\restriction\mathsf{E}$, any condition $P = X \times Z$ in $\mathbb{P}^2\restriction\mathsf{E}$ forces $\langle \dot{x}_{\mathtt{left}}, \dot{x}_{\mathtt{right}} \rangle \in P$ and forces $\dot{x}_{\mathtt{left}} \mathrel{\widehat{\mathsf{E}}} \dot{x}_{\mathtt{right}}$, but $H \times H$ forces $\dot{x}_{\mathtt{left}} \mathrel{\not\mathsf{E}} \dot{x}_{\mathtt{right}}$.*

Proof. To see that $\dot{x}_{\mathtt{left}} \mathrel{\widehat{\mathsf{E}}} \dot{x}_{\mathtt{right}}$ is forced suppose otherwise. Then, by the definition of $\widehat{\mathsf{E}}$, there is a condition $P = X \times Z \in \mathbb{P}^2\restriction\mathsf{E}$ and an E-invariant Δ^1_1 set B such that P forces $\dot{x}_{\mathtt{left}} \in B$ but $\dot{x}_{\mathtt{right}} \notin B$. Then easily $X \subseteq B$ but $Z \cap B = \varnothing$, a contradiction with $[X]_{\mathsf{E}} = [Z]_{\mathsf{E}}$.

To see that $H \times H$ forces $\dot{x}_{\mathtt{left}} \mathrel{\not\mathsf{E}} \dot{x}_{\mathtt{right}}$ suppose towards the contrary that some $P = X \times Z \in \mathbb{P}^2\restriction\mathsf{E}$ with $X \cup Z \subseteq H$ forces $\dot{x}_{\mathtt{left}} \mathsf{E} \dot{x}_{\mathtt{right}}$, thus,

(1) $x \mathsf{E} z$ holds for every $\mathbb{P}^2\restriction\mathsf{E}$-generic pair $\langle x, z \rangle \in P$.

Claim 7.15. *If $x, y \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$, and $x \mathrel{\widehat{\mathsf{E}}} y$, then $x \mathsf{E} y$.*

Proof. We assert that

(2) $x \in A \Longleftrightarrow y \in A$ holds for each E-invariant Σ^1_1 set A.

Indeed, if, say, $x \in A$ but $y \notin A$ then by the genericity of y there is a Σ^1_1 set C with $y \in C$ and $A \cap C = \varnothing$. As A is E-invariant, Lemma 7.9 yields an E-invariant Δ^1_1 set B such that $C \subseteq B$ but $A \cap B = \varnothing$. Then $x \notin B$ but $y \in B$, a contradiction to $x \mathrel{\widehat{\mathsf{E}}} y$.

Let $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ be an enumeration of all dense subsets of $\mathbb{P}^2\restriction\mathsf{E}$ which are coded in $\mathfrak{M}$. We define two sequences $P_0 \supseteq P_1 \supseteq \dots$ and $Q_0 \supseteq Q_1 \supseteq \dots$ of conditions $P_n = X_n \times Z_n$ and $Q_n = Y_n \times Z_n$ in $\mathbb{P}^2\restriction\mathsf{E}$, so that $P_0 = Q_0 = P$, $x \in X_n$ and $y \in Y_n$ for any n, and finally $P_n, Q_n \in \mathscr{D}_{n-1}$ for $n \geq 1$. If this is done then we have a real z (the only element of $\bigcap_n Z_n$) such that both $\langle x, z \rangle$ and $\langle y, z \rangle$ are $\mathbb{P}^2\restriction\mathsf{E}$-generic, hence, $x \mathsf{E} z$ and $y \mathsf{E} z$ by (1), hence, $x \mathsf{E} y$.

Suppose that P_n and Q_n have been defined. As x is generic, there is (we leave details for the reader) a condition $P' = A \times C \in \mathscr{D}_n$ and $\subseteq P_n$ such that $x \in A$. Let $B = Y_n \cap [A]_{\mathsf{E}}$: then $y \in B$ by (2), and easily $[B]_{\mathsf{E}} = [C]_{\mathsf{E}} = [A]_{\mathsf{E}}$ (as $[X_n]_{\mathsf{E}} = [Z_n]_{\mathsf{E}} = [Y_n]_{\mathsf{E}}$), thus, $B \times C \in \mathbb{P}^2\restriction\mathsf{E}$, so there is a condition $Q' = V \times W \in \mathscr{D}_n$ and $\subseteq B \times C \subseteq Q_n$ such that $y \in V$. Put $Y_{n+1} = V$, $Z_{n+1} = W$, and $X_{n+1} = A \cap [W]_{\mathsf{E}}$. □ (*Claim*)

It follows that $\mathsf{E} = \widehat{\mathsf{E}}$ on X. (Otherwise $S = \{\langle x,y\rangle \in X^2 : x \mathrel{\widehat{\mathsf{E}}} y \wedge x \mathrel{\not{\mathsf{E}}} y\}$ is a non-empty Σ^1_1 set, and any $\mathbb{P}_2$-generic pair $\langle x,y\rangle \in S$ implies a contradiction to Claim 7.15. Recall that $\mathbb{P}_2 =$ all non-empty Σ^1_1 subsets of $(\mathbb{N}^{\mathbb{N}})^2$.) But this implies $X \cap H = \varnothing$ by Lemma 7.12, contradiction. □ (*Lemma 7.14*)

7e Splitting system

The **or** case of Theorem 7.2, that is $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}$, means that E_0 has a continuous "copy" of the form $\mathsf{E} \restriction X$, X being a closed set in $\mathbb{N}^{\mathbb{N}}$. To obtain such a set, we define a splitting system of sets in $\mathbb{P}$ satisfying certain requirements.

Let us fix enumerations $\{\mathscr{D}(n)\}_{n\in\mathbb{N}}$, $\{\mathscr{D}_2(n)\}_{n\in\mathbb{N}}$, $\{\mathscr{D}^2(n)\}_{n\in\mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}$, $\mathbb{P}_2$, $\mathbb{P}^2 \restriction \mathsf{E}$, which belong to the model $\mathfrak{M}$ fixed above. We assume that $\mathscr{D}(n+1) \subseteq \mathscr{D}(n)$, $\mathscr{D}_2(n+1) \subseteq \mathscr{D}_2(n)$, and $\mathscr{D}^2(n+1) \subseteq \mathscr{D}^2(n)$. If $u,v \in 2^m$ (binary sequences of length m) have the form $u = 0^k{}^\wedge 0^\wedge w$ and $v = 0^k{}^\wedge 1^\wedge w$ for some $k < m$ and $w \in 2^{m-k-1}$ then we call $\langle u,v\rangle$ a *crucial pair*. It can be proved by induction on m that 2^m is a connected tree (*i.e.*, a connected graph without cycles) of crucial pairs, with sequences beginning with 1 as the endpoints of the graph.

We define a system of sets X_u ($u \in 2^{<\omega}$) and R_{uv}, $\langle u,v\rangle$ being a crucial pair, so that the following requirements are satisfied:

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$;

(ii) $X_u \in \mathscr{D}(n)$ for any $u \in 2^n$;

(iii) $X_{u^\wedge i} \subseteq X_u$ for all u and i;

(iv) $\mathsf{R}_{uv} \in \mathbb{P}_2$, moreover, $\mathsf{R}_{uv} \in \mathscr{D}_2(n)$ for any crucial pair $\langle u,v\rangle$ in 2^n;

(v) $\mathsf{R}_{uv} \subseteq \mathsf{E}$ and $X_u \mathrel{\mathsf{R}_{uv}} X_v$ for any crucial pair $\langle u,v\rangle$ in 2^n;

(vi) $\mathsf{R}_{u^\wedge i, v^\wedge i} \subseteq \mathsf{R}_{uv}$;

(vii) if $u,v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $X_u \times X_v \in \mathscr{D}^2(n)$ and also $X_u \cap X_v = \varnothing$.

Note that (v) implies that $X_u \mathrel{\mathsf{E}} X_v$ for any crucial pair $\langle u,v\rangle$, hence, also for any pair in 2^n because any $u,v \in 2^n$ are connected by a unique chain of crucial pairs. It follows that $X_u \times X_v \in \mathbb{P}^2 \restriction \mathsf{E}$ for any pair of $u,v \in 2^n$, for any n.

Assume that such a system has been defined. Then for any $a \in 2^{\mathbb{N}}$ the sequence $\{X_{a\restriction n}\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ by (ii), therefore $\bigcap_n X_{a\restriction n} = \{x_a\}$, where x_a is $\mathbb{P}$-generic, and the map $a \mapsto x_a$ is continuous since diameters of X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$ by (i), and is $1-1$ by the last condition of (vii).

Let $a,b \in 2^{\mathbb{N}}$. If $a \mathrel{\not{\mathsf{E}}_0} b$ then, by (vii), $\langle x_a, x_b\rangle$ is a $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair, hence, $x_a \mathrel{\not{\mathsf{E}}} x_b$ by Lemma 7.14. Now suppose that $a \mathrel{\mathsf{E}_0} b$, prove that then $x_a \mathrel{\mathsf{E}} x_b$. We can suppose that $a = w^\wedge 0^\wedge c$ and $b = w^\wedge 0^\wedge c$, where $w \in 2^{<\omega}$ and $c \in 2^{\mathbb{N}}$ (indeed if $a \mathsf{E}_0 b$ then a, b can be connected by a finite chain of such special pairs). Then $\langle x_a, x_b \rangle$ is $\mathbb{P}_2$-generic, actually, the only member of the intersection $\bigcap_n \mathsf{R}_{w^\wedge 0^\wedge (c\restriction n), w^\wedge 1^\wedge (c\restriction n)}$ by (iv) and (v), in particular, $x_a \mathsf{E}\, x_b$ because we have $R_{uv} \subseteq \mathsf{E}$ for all u, v.

Thus we have a continuous $1-1$ reduction of E_0 to E.

□ (*Case 2 in Theorem 7.2 modulo the construction*)

7f Construction of a splitting system

Thus it remains to define a splitting system of sets satisfying (i) – (vii).

Let X_Λ be any set in $\mathscr{D}(0)$ such that $X_\Lambda \subseteq H$.

Now suppose that X_s and R_{st} have been defined for all $s \in 2^n$ and all crucial pairs in 2^n, and extend the construction on 2^{n+1}. Temporarily, define $X_{s^\wedge i} = X_s$ and $\mathsf{R}_{s^\wedge i, t^\wedge i} = \mathsf{R}_{st}$: this leaves $\mathsf{R}_{0^n{}^\wedge 0, 0^n{}^\wedge 1}$ still undefined, so we put $\mathsf{R}_{0^n{}^\wedge 0, 0^n{}^\wedge 1} = \mathsf{E} \cap (X_{0^n} \times X_{0^n})$. Note that the system of sets X_u and relations R_{uv} defined this way at level $n+1$ satisfies all requirements of (i) – (vii) except for the requirements of membership in the dense sets in (ii), (iv), (vii) — say in this case that the system is "coherent". It remains to produce a still "coherent" system of smaller sets and relations which also satisfies the membership in the dense sets. This will be achieved in several steps.

Step 1: achieve that $X_u \in \mathscr{D}(n+1)$ for any $u \in 2^{n+1}$. Take any particular $u_0 \in 2^{n+1}$. There is, by the density, $X'_{u_0} \in \mathscr{D}(n+1)$ and $\subseteq X_{u_0}$. Suppose that $\langle u_0, v \rangle$ is a crucial pair. Put $\mathsf{R}'_{u_0,v} = \{\langle x, y \rangle \in \mathsf{R}_{u_0,v} : x \in X'_{u_0}\}$ and $X'_v = \mathtt{ran}\, \mathsf{R}'_{u_0,v}$. This shows how the change spreads along the whole set 2^{n+1} viewed as the tree of crucial pairs. Finally we obtain a coherent system with the additional requirement that $X'_{u_0} \in \mathscr{D}(n+1)$. Do this consecutively for all $u_0 \in 2^{n+1}$. The total result – we re-denote it as still X_u and R_{uv} – is a "coherent" system with $X_u \in \mathscr{D}(n+1)$ for all u. Note that still $X_{0^n{}^\wedge 0} = X_{0^n{}^\wedge 1}$ and

$$\mathsf{R}_{0^n{}^\wedge 0, 0^n{}^\wedge 1} = \mathsf{E} \cap (X_{0^n{}^\wedge 0} \times X_{0^n{}^\wedge 1}) . \tag{$*$}$$

Step 2: achieve that $X_{s^\wedge 0} \times X_{t^\wedge 1} \in \mathscr{D}^2(n+1)$ for all $s, t \in 2^{n+1}$. Consider a pair of $u_0 = s_0{}^\wedge 0$ and $v_0 = t_0{}^\wedge 1$ in 2^{n+1}. By the density there is a set $X'_{u_0} \times X'_{v_0} \in \mathscr{D}^2(n+1)$ and $\subseteq X_{u_0} \times X_{v_0}$. By definition we have $X'_{u_0} \mathsf{E}\, X'_{v_0}$, but, due to Lemma 7.13 we can maintain that $X'_{u_0} \cap X'_{v_0} = \varnothing$. The two "shockwaves", from the changes at nodes u_0 and v_0, as in Step 1, meet only at the pair $0^m{}^\wedge 0$, $0^m{}^\wedge 1$, where the new sets satisfy $X'_{0^m{}^\wedge 0} \mathsf{E}\, X'_{0^m{}^\wedge 1}$ just because E-equivalence is everywhere preserved though the changes. Now, in view of $(*)$, we can define $\mathsf{R}'_{0^n{}^\wedge 0, 0^n{}^\wedge 1} = \mathsf{E} \cap (X'_{0^n{}^\wedge 0} \times X'_{0^n{}^\wedge 1})$, preserving $(*)$ as well. When all pairs are considered, we will be left with a coherent system of sets and relations, re-denoted as X_u and R_{uv}, which satisfies the $\mathscr{D}(n+1)$-requirements in (ii) and (vii).

Step 3: achieve that $\mathsf{R}_{uv} \in \mathscr{D}_2(n+1)$ for any crucial pair at level $n+1$, and also that $X'_{0^n{}^\wedge 0} \cap X'_{0^n{}^\wedge 1} = \varnothing$. Consider any crucial pair $\langle u_0, v_0\rangle$. If this is not $\langle 0^n{}^\wedge 0, 0^n{}^\wedge 1p\rangle$ then let $\mathsf{R}'_{u_0v_0} \subseteq \mathsf{R}_{u_0v_0}$ be any set in $\mathscr{D}_2(n+1)$. If this is $u_0 = 0^n{}^\wedge 0$ and $v_0 = 0^n{}^\wedge 1$ then first we choose (Lemma 7.13) disjoint non-empty Σ^1_1 sets $U \subseteq X_{0^n{}^\wedge 0}$ and $V \subseteq X_{0^n{}^\wedge 1}$ still with $U \mathrel{\mathsf{E}} V$, and only then a set $\mathsf{R}'_{u_0v_0} \subseteq \mathsf{E} \cap (U \times V)$ which belongs to $\in \mathscr{D}_2(n+1)$. In both cases, put $X'_{u_0} = \mathtt{dom}\,\mathsf{R}'_{u_0v_0}$ and $X'_{v_0} = \mathtt{ran}\,\mathsf{R}'_{u_0v_0}$. It remains to spread the changes, along the chain of crucial pairs, to the left of u_0 and to the right of v_0, exactly as in Case 1. Executing such a reduction for all crucial pairs $\langle u_0, v_0\rangle$ at level $n+1$ one by one, we end up with a system of sets fully satisfying (i) – (vii).

□ (*Theorem 7.2*)

7g A forcing notion associated with E_0

We here consider a forcing notion $\mathbb{P}_{\mathsf{E}_0}$ that consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is non-smooth. A related ideal $\mathscr{I}_{\mathsf{E}_0}$ (this time an ideal on $2^{\mathbb{N}}$) consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is smooth. Alternatively for a Borel $X \subseteq 2^{\mathbb{N}}$ to be in $\mathscr{I}_{\mathsf{E}_0}$ it is necessary and sufficient that $\mathsf{E}_0 \restriction X$ has a Borel transversal — this is by Lemma 6.2.

Forcings like $\mathbb{P}_{\mathsf{E}_0}$, that is those defined in the form of a collection of all Borel sets X such that a given Borel equivalence relation E satisfies $\mathsf{E} \le_{\mathrm{B}} \mathsf{E} \restriction X$, are still work in progress, their applications not yet established.

Lemma 7.16. (i) *$\mathscr{I}_{\mathsf{E}_0}$ is a σ-additive ideal. Let $X \subseteq 2^{\mathbb{N}}$ be a Borel set.*

(ii) *X belongs to $\mathbb{P}_{\mathsf{E}_0}$ iff $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ (by a continuous injection).*

(iii) *X belongs to $\mathscr{I}_{\mathsf{E}_0}$ iff $\mathsf{E}_0 \restriction X$ admits a Borel transversal.*

Proof. (i) immediately follows from Corollary 6.10. In (ii), if $X \in \mathbb{P}_{\mathsf{E}_0}$ then $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ by Theorem 7.2, while if $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ then $\mathsf{E}_0 \restriction X$ is not smooth since E_0 itself is not smooth by Lemma 6.2(v). In (iii), if $\mathsf{E}_0 \restriction X$ admits a Borel transversal then it is smooth by Lemma 6.2(i) and hence X belongs to $\mathscr{I}_{\mathsf{E}_0}$. To prove the converse apply Lemma 6.2(iii). □

Note that any $X \in \mathbb{P}_{\mathsf{E}_0}$ contains a closed subset $Y \subseteq X$ also in $\mathbb{P}_{\mathsf{E}_0}$ by Theorem 7.2. (Apply the theorem for $\mathsf{E} = \mathsf{E}_0 \restriction X$. As $\mathsf{E}_0 \restriction X$ is not smooth, we have $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$, by a continuous reduction ϑ. Take as Y the full image of ϑ. Y is compact, hence closed.) Such sets Y can be chosen in a special family.

Definition 7.17. Suppose that two binary sequences $u^0_n \ne u^1_n \in 2^{<\omega}$ of equal length $\mathtt{lh}\, u^0_n = \mathtt{lh}\, u^1_n \ge 1$ are chosen for each n, together with one more sequence $u_0 \in 2^{<\omega}$. Define $\vartheta(a) = u_0{}^\wedge u_0^{a(0)}{}^\wedge u_1^{a(1)}{}^\wedge \dots$ for any $a \in 2^{\mathbb{N}}$. Easily ϑ is a continuous injection $2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $Y = \mathtt{ran}\,\vartheta$ is a closed set in $2^{\mathbb{N}}$, ϑ witnesses $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction Y$, and hence $Y \in \mathbb{P}_{\mathsf{E}_0}$.

Let $\mathbb{P}'_{\mathsf{E}_0}$ denote the collection of all sets Y definable in such a form. □

The next theorem gives a necessary and sufficient condition for a Borel set $X \subseteq 2^{\mathbb{N}}$ in the class $\varDelta^1_1$ to belong to $\mathbb{P}_{\mathsf{E}_0}$, in terms related to the Gandy – Harrington forcing. Relativization to $\varDelta^1_1(p)$ for an arbitrary parameter $p \in 2^{\mathbb{N}}$ is obvious. The theorem also proves the density of the subset $\mathbb{P}'_{\mathsf{E}_0}$ of much more transparent "conditions" in $\mathbb{P}_{\mathsf{E}_0}$.

Theorem 7.18. *Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\varDelta^1_1$ set. Then $X \in \mathbb{P}_{\mathsf{E}_0}$ iff X is not covered by the union of all pairwise E_0-inequivalent $\varDelta^1_1$ sets. In addition,*

(i) (Zapletal [64]) *$\mathbb{P}'_{\mathsf{E}_0}$ is a dense subset of $\mathbb{P}_{\mathsf{E}_0}$: for any $X \in \mathbb{P}_{\mathsf{E}_0}$ there exists $Y \in \mathbb{P}'_{\mathsf{E}_0}$ such that $Y \subseteq X$;*

(ii) (Zapletal [64]) *$\mathbb{P}_{\mathsf{E}_0}$ forces that the "old" continuum $\mathfrak{c}$ remains uncountable.*

Proof. *The "if" claim.* This is easy. It is quite clear that $\mathsf{E}_0 \restriction Y$ is smooth whenever Y is a Borel pairwise E_0-inequivalent $\varDelta^1_1$ set. However countable unions preserve smoothness by Corollary 6.10.

The "only if" claim. Suppose that X is not covered by the union U of all pairwise E_0-inequivalent $\varDelta^1_1$ sets. As in the proofs of the 1st and 2nd dichotomy theorems above, U is a $\varPi^1_1$ set, and hence $A = X \smallsetminus U$ is a non-empty $\varSigma^1_1$ set.

The key property of A is that it does not intersect any pairwise E_0-inequivalent $\varSigma^1_1$ set. (To prove this one has to establish that any pairwise E_0-inequivalent $\varSigma^1_1$ set can be covered by a pairwise E_0-inequivalent $\varDelta^1_1$ set.) It follows that

($*$) any non-empty $\varSigma^1_1$ set $Y \subseteq A$ is not pairwise E_0-inequivalent, *i.e.* it contains a pair of points $x \neq y$ with $x \mathrel{\mathsf{E}_0} y$.

For any sequences $r, w \in 2^{<\omega}$ with $\mathtt{lh}\, r \le \mathtt{lh}\, w$, define $r \cdot w \in 2^{<\omega}$ (the r-shift of w) so that $\mathtt{lh}\,(r \cdot w) = \mathtt{lh}\, w$ and $(r \cdot w)(k) = 1 - w(k)$ whenever $k < \mathtt{lh}\, r$ and $r(k) = 1$, and $(r \cdot w)(k) = w(k)$ otherwise. Clearly $r \cdot (r \cdot w) = w$. Similarly define $r \cdot a \in 2^{\mathbb{N}}$ for $a \in 2^{\mathbb{N}}$, and $r \cdot X = \{r \cdot a : a \in X\}$ for any set $X \subseteq 2^{\mathbb{N}}$.

We are going to define sequences $u \in 2^{<\omega}$ and $u^0_n \neq u^1_n \in 2^{<\omega}$ ($n \in \mathbb{N}$) such that $\mathtt{lh}\, u^0_n = \mathtt{lh}\, u^1_n$, as in Definition 7.17, and also a system of $\varSigma^1_1$ sets $X_s \in \mathbb{P}_{\mathsf{E}_0}$ ($s \in 2^{<\omega}$) satisfying the following:

(1) $X_\Lambda \subseteq A$, $X_{s^\wedge i} \subseteq X_s$, and $\mathtt{diam}\, X_s \le 2^{-\mathtt{lh}\, s}$;

(2) a condition in terms of the Gandy – Harrington forcing, similar to (ii) in Section 7b or (ii) in Section 7e, such that, as a consequence, $\bigcap_n X_{a \restriction n} \neq \varnothing$ for any $a \in 2^{\mathbb{N}}$;

(3) $X_s \subseteq \mathscr{O}_{w_s}$, where $w_s = u_0{}^\wedge u_0^{s(0)}{}^\wedge u_1^{s(1)}{}^\wedge \dots {}^\wedge u_{k-1}^{s(k-1)} \in 2^{<\omega}$, $k = \mathtt{lh}\, s$, and $\mathscr{O}_w = \{a \in 2^{\mathbb{N}} : w \subset a\}$ for $w \in 2^{<\omega}$;

(4) if $s, t \in 2^n$ for some n then $X_t = w_t \cdot w_s \cdot X_s$.

Then define the map ϑ as in Definition 7.17. The set $Y = \mathtt{ran}\,\vartheta = \bigcap_n \bigcup_{s\in 2^n} X_s \subseteq X$ belongs to $\mathbb{P}'_{\mathsf{E_0}}$, hence to $\mathbb{P}_{\mathsf{E_0}}$, proving that $X \in \mathbb{P}_{\mathsf{E_0}}$ as well.

This argument also proves claim (i) of the theorem. Indeed suppose that $X \subseteq 2^{\mathbb{N}}$ is a Δ^1_1 set. (As usual the relativization to any $\Delta^1_1(p)$ is routine.) It follows from the "if" claim of the theorem that $X \not\subseteq U$, and hence we are in the domain of the "only if" claim, thus there is a subset $Y \subseteq X$, $Y \in \mathbb{P}'_{\mathsf{E_0}}$.

It remains to carry out the construction of sets X_s.

Step 0. We put $X_\Lambda = A$ and let $u_0 \in 2^{<\omega}$ be the largest sequence such that $X_\Lambda \subseteq \mathscr{O}_{u_0}$. Let $\ell_0 = \mathtt{lh}\,u_0$.

Step 1. Here we define u_0^i and $X_{\langle i\rangle}$ for $i = 0, 1$. It follows from $(*)$ above that there exist points $x' \neq y' \in X_\Lambda$ such that $x' \mathsf{E_0}\, y'$. This means that there exist two different sequences $u_0^0 \neq u_0^1$ of equal length $\mathtt{lh}\,u_0^0 = \mathtt{lh}\,u_0^1$ such that $u_0{}^\wedge u_0^0 \subset x'$, $u_0{}^\wedge u_0^1 \subset y'$, and $x'(k) = y'(k)$ for all $k \geq \ell_1 = \ell_0 + \mathtt{lh}\,u_0^i$. Put $w_{\langle 0\rangle} = u_0{}^\wedge u_0^0$ and $w_{\langle 1\rangle} = u_0{}^\wedge u_1^0$. Then the sets

$$\begin{aligned} X_{\langle 0\rangle} &= \{x \in X_\Lambda : w_{\langle 0\rangle} \subset x \wedge \exists\, y \in X_\Lambda\, (w_{\langle 1\rangle} \subset y \wedge x \mathsf{E_0}\, y)\}, \quad \text{and} \\ X_{\langle 1\rangle} &= \{y \in X_\Lambda : w_{\langle 1\rangle} \subset y \wedge \exists\, x \in X_\Lambda\, (w_{\langle 0\rangle} \subset x \wedge x \mathsf{E_0}\, y)\} \end{aligned}$$

are still nonempty Σ^1_1 sets (containing resp. x, y), and they satisfy (3) and (4).

Finally replace $X_{\langle 0\rangle}$ by a suitable smaller Σ^1_1 set $X'_{\langle 0\rangle}$ in order to fulfill (2), and put $X'_{\langle 1\rangle} = w_{\langle 0\rangle} \cdot w_{\langle 1\rangle} \cdot X'_{\langle 0\rangle}$. Now choose suitable smaller Σ^1_1 set $X''_{\langle 1\rangle} \subseteq X'_{\langle 1\rangle}$ in order to fulfill (2), and put $X''_{\langle 0\rangle} = w_{\langle 1\rangle} \cdot w_{\langle 0\rangle} \cdot X''_{\langle 1\rangle}$. Re-denote the sets $X''_{\langle 0\rangle}$, $X''_{\langle 1\rangle}$ again by $X_{\langle 0\rangle}$, $X_{\langle 1\rangle}$.

Step 2. Here we define u_1^i for $i = 0, 1$ and X_s for $s \in 2^{<\omega}$ with $\mathtt{lh}\,s = 2$. Once again there exist points $x' \neq y' \in X_{\langle 0\rangle}$ such that $x' \mathsf{E_0}\, y'$. This means that there exist two different sequences $u_1^0 \neq u_1^1$ of equal length $\mathtt{lh}\,u_1^0 = \mathtt{lh}\,u_1^1$ such that $u_0{}^\wedge u_0^0{}^\wedge u_1^0 \subset x'$, $u_0{}^\wedge u_0^1{}^\wedge u_1^1 \subset y'$, and $x'(k) = y'(k)$ for all $k \geq \ell_2 = \ell_1 + \mathtt{lh}\,u_1^i$. Put $w_{\langle i,j\rangle} = u_0{}^\wedge u_0^i{}^\wedge u_1^j$ for $i, j \in \{0, 1\}$. Then the sets

$$\begin{aligned} X_{\langle 0,0\rangle} &= \{x \in X_\Lambda : w_{\langle 0,0\rangle} \subset x \wedge \exists\, y \in X_\Lambda\, (w_{\langle 0,1\rangle} \subset y \wedge x \mathsf{E_0}\, y)\}, \quad \text{and} \\ X_{\langle 0,1\rangle} &= \{y \in X_\Lambda : w_{\langle 0,1\rangle} \subset y \wedge \exists\, x \in X_\Lambda\, (w_{\langle 0,0\rangle} \subset x \wedge x \mathsf{E_0}\, y)\} \end{aligned}$$

are still nonempty Σ^1_1 sets satisfying (3) and (4). There is no need in an additional split of $X_{\langle 1\rangle}$ in order to define the sets $X_{\langle 1,0\rangle}$, $X_{\langle 1,1\rangle}$: just put $X_{\langle 1,0\rangle} = w_{\langle 0\rangle} \cdot w_{\langle 1\rangle} \cdot X_{\langle 0,0\rangle}$ and $X_{\langle 1,1\rangle} = w_{\langle 0\rangle} \cdot w_{\langle 1\rangle} \cdot X_{\langle 0,1\rangle}$.

It remains to shrink the sets $X_{\langle i,j\rangle}$ in several (that is, four) rounds in order to fulfill (2), applying the actions of $w_{\langle i,j\rangle}$ as required by (4) to define intermediate sets.

Steps ≥ 3. Suppose that all sets X_s, $s \in 2^n$, have been suitably defined. Let $\subseteq\in 2^n$ be the sequence of n zeros. We define sets $X_{\sigma^\wedge 0}$ and $X_{\sigma^\wedge 1}$ by splitting X_σ as above, and then split every other X_s applying $w_\sigma \cdot w_s$.

The construction results in a system of sets and sequences satisfying requirements (1), (3), (4), as required.

(ii) It suffices to prove the same result for the subforcing $\mathbb{P}'_{\mathsf{E}_0}$. Given a sequence of dense sets $D_n \subseteq \mathbb{P}'_{\mathsf{E}_0}$, we carry out a splitting construction similar to the one given above, with the following amendments. First, each set X_s belongs to $D_{\mathtt{lh}\, s}$, hence to $\mathbb{P}'_{\mathsf{E}_0}$, therefore is a closed set in $2^{\mathbb{N}}$. Second, condition (2) is abolished, of course. That any set $X \in \mathbb{P}'_{\mathsf{E}_0}$ satisfies $(*)$ (that is, it contains a pair of points $x \neq y$ with $x \mathrel{\mathsf{E}_0} y$) is obvious. □

We observe that $\mathbb{P}_{\mathsf{E}_0}$ as a forcing is somewhat closer to Silver rather than Sacks forcing. The property of minimality of the generic real, common to both Sacks and Silver forcings, holds for $\mathbb{P}_{\mathsf{E}_0}$ as well, the proof resembles known arguments, but in addition the following is applied: if $X \in \mathbb{P}_{\mathsf{E}_0}$ and $f : X \to 2^{\mathbb{N}}$ is a Borel E_0-invariant map (that is, $x \mathrel{\mathsf{E}_0} y \Longrightarrow f(x) = f(y)$) then f is constant on a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq X$. [3]

[3] Suppose, for the sake of brevity, that $X = 2^{\mathbb{N}}$. For any n, the set $Y_n^0 = \{a : f(a)(n) = 0\}$ is Borel and E_0-invariant. It follows that Y_n^0 is either meager or comeager. Put $b(n) = 0$ iff Y_n^0 is comeager. Then $D = \{a : f(a) = b\}$ is comeager. A splitting construction as in the proof of Theorem 7.18 yields a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq D$.

Chapter 8

Ideal $\mathscr{I}_1$ and P-ideals

By definition the ideal $\mathtt{Fin} \times 0 = \mathscr{I}_1$ consists of all sets $x \subseteq \mathscr{P}(\mathbb{N} \times \mathbb{N})$ such that all, except for finitely many, cross-sections $(x)_n = \{k : \langle n, k \rangle \in x\}$ are empty. This Chapter contains proofs of some key results related to this ideal. First of all we show following Kechris that there exist essentially only three types of ideals Borel reducible to $\mathscr{I}_1$, two of them being $\mathtt{Fin}$ and $\mathscr{I}_1$ itself. Then a proof of Solecki's theorem, that characterizes P-ideals in terms of LSC submeasures and polishability and shows that $\mathscr{I}_1$ is the least Borel non-polishable ideal, will be given.

8a Ideals below $\mathscr{I}_1$

Recall that $\mathscr{I} \cong \mathscr{J}$ means the isomorphism of ideals $\mathscr{I}, \mathscr{J}$ via a bijection between the underlying sets. The ideal $\mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$ (the disjoint sum in the sense of Section 2d) in the next theorem is isomorphic to the ideal $\mathtt{Fin}_{\mathrm{ODD}} = \{x \subseteq \mathbb{N} : x \cap 2\mathbb{N} \in \mathtt{Fin}\}$, where $2\mathbb{N} =$ all odd numbers.[1]

Theorem 8.1 (Kechris [35]). *If $\mathscr{I}$ is a Borel (nontrivial) ideal on $\mathbb{N}$ and $\mathscr{I} \le_{\mathrm{B}} \mathscr{I}_1$ then $\mathscr{I}$ is isomorphic to one of the following three ideals: $\mathscr{I}_1$, $\mathtt{Fin}$, $\mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$.*

Thus there exist only three different ideals Borel reducible to $\mathscr{I}_1$, they are $\mathtt{Fin}$, the disjoint sum $\mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$, and $\mathscr{I}_1$ itself.

Proof. We begin with another version of the method used in the proof of Theorem 6.11. Suppose that $\{\mathscr{B}_k\}_{k \in \mathbb{N}}$ is a fixed system of Borel subsets of $\mathscr{P}(\mathbb{N})$. (It will be specified later.) Then there exists an increasing sequence of integers $0 = n_0 < n_1 < n_2 < \dots$ and sets $s_k \subseteq [n_k, n_{k+1})$ such that

(1) any $x \subseteq \mathbb{N}$ with $\forall^\infty k\, (x \cap [n_k, n_{k+1}) = s_k)$ is "generic"[2];

[1] Ideals isomorphic to any of $\mathscr{I}, \mathscr{I} \oplus \mathscr{P}(\mathbb{N})$ were called *trivial variations* of $\mathscr{I}$ in [35].

[2] We mean, Cohen generic over a certain countable family of dense open subsets of $\mathscr{P}(\mathbb{N})$ that depends on the choice of the family of sets $\mathscr{B}_k$.

(2) if $k' \ge k$ and $u \subseteq [0, n_{k'})$ then $u \cup s_{k'}$ decides $\mathscr{B}_k$ in the sense that either any "generic" $x \in \mathscr{P}(\mathbb{N})$ with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ belongs to $\mathscr{B}_k$ or any "generic" x with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ does not belong to $\mathscr{B}_k$.

Now put $\mathscr{D}_0 = \{x \cup S_1 : x \subseteq Z_0\}$ and $\mathscr{D}_1 = \{x \cup S_0 : x \subseteq Z_1\}$, where

$$S_0 = \bigcup_k s_{2k} \subseteq Z_0 = \bigcup_k [n_{2k}, n_{2k+1}), \quad S_1 = \bigcup_k s_{2k+1} \subseteq Z_1 = \bigcup_k [n_{2k+1}, n_{2k+2}).$$

Clearly any $x \in \mathscr{D}_0 \cup \mathscr{D}_1$ is "generic" by (1), hence it follows from (2) that

(3) each $\mathscr{B}_k$ is clopen on both $\mathscr{D}_0$ and $\mathscr{D}_1$.

As $\mathscr{I} \le_{\mathrm{B}} \mathscr{I}_1$, it follows from Lemma 4.3 (and the trivial fact that $\mathscr{I}_1 \oplus \mathscr{I}_1 \cong \mathscr{I}_1$) that there exists a *continuous* reduction $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N} \times \mathbb{N})$ of $\mathscr{I}$ to $\mathscr{I}_1$. Thus $\mathsf{E}_{\mathscr{I}}$ is the union of an increasing sequence of (topologically) closed equivalence relations $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$ just because $\mathscr{I}_1$ admits such a form. We now require that $\{\mathscr{B}_k\}$ includes all sets $B^m_l = \{x \in \mathscr{P}(\mathbb{N}) : \forall\, s \subseteq [0, l)\; x\, \mathsf{R}_m\, (x \,\Delta\, s)\}$. Then by (3) and the compactness of $\mathscr{D}_i$ for any l there is $m(l) \ge l$ satisfying

(4) $\forall\, x \in \mathscr{D}_0 \cup \mathscr{D}_1\, \forall\, s \subseteq [0, l)\, \big(x\, \mathsf{R}_{m(l)}\, (x \,\Delta\, s)\big)$.

To prove the theorem it suffices to obtain a sequence $x_0 \subseteq x_1 \subseteq x_2 \subseteq \dots$ of sets $x_k \in \mathscr{I}$ with $\mathscr{I} = \bigcup_n \mathscr{P}(x_n)$: that in this case $\mathscr{I}$ is as required is an easy exercise. As any topologically closed ideal is easily $\mathscr{P}(x)$ for some $x \subseteq \mathbb{N}$, it suffices to show that $\mathscr{I}$ is a union of a countable sequence of closed subideals. It suffices to demonstrate this fact separately for $\mathscr{I} \restriction Z_0$ and $\mathscr{I} \restriction Z_1$. Prove that $\mathscr{I} \restriction Z_0$ is a countable union of closed subideals, ending the proof of the theorem.

If $m \in \mathbb{N}$ and $s \subseteq u \subseteq Z_0$ are finite then let

$$I^m_{us} = \{A \subseteq Z_0 : \forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow \big(x \cup (A \smallsetminus u)\big)\, \mathsf{R}_m\, x)\}\,.$$

Lemma 8.2. *Sets I^m_{us} are closed topologically and under $\cup$, and $I^m_{us} \subseteq \mathscr{I}$.*

Proof. I^m_{us} are topologically closed because so are R_m.

Suppose that $A, B \in I^m_{us}$. To prove that $A \cup B \in I^m_{us}$, let $x \in \mathscr{D}_0$ satisfy $x \cap u = s$. Then $x' = x \cup (A \smallsetminus u) \in \mathscr{D}_0$ satisfies $x' \cap u = s$, too, hence, as $B \in I^m_{us}$, we have $\big(x' \cup (B \smallsetminus u)\big)\, \mathsf{R}_m\, x'$, thus, $\big(x \cup ((A \cup B) \smallsetminus u)\big)\, \mathsf{R}_m\, x'$. However $x'\, \mathsf{R}_m\, x$ just because $A \in I^m_{us}$. It remains to recall that R_m is a ER.

To prove that any $A \in I^m_{us}$ belongs to $\mathscr{I}$ take $x = s \cup S_1$. Then we have $x \cup (A \smallsetminus u)\, \mathsf{R}_m\, x$, thus, $A \in \mathscr{I}$ as s is finite and $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$. □ (*Lemma*)

Lemma 8.3. $\mathscr{I} \restriction Z_0 = \bigcup_{m,u,s} I^m_{us}$.

Proof. Let $A \in \mathscr{I}$, $A \subseteq Z_0$. The sets $Q_m = \{x \in \mathscr{D}_0 : (x \cup A)\, \mathsf{R}_m\, x\}$ are closed and satisfy $\mathscr{D}_0 = \bigcup_m Q_m$. It follows that one of them has a non-empty interior in $\mathscr{D}_0$, thus, there exist finite sets $s \subseteq u \subseteq Z_0$ and some m_0 with

$$\forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow (x \cup A)\, \mathsf{R}_{m_0}\, x)\,.$$

This is not exactly what we need, however, by (4), there exists a number $m = \mathtt{max}\{m_0, m(\mathtt{sup}\, u)\}$ big enough for

$$\forall\, x \in \mathscr{D}_0 : (x \cup A) \mathsf{R}_m (x \cup (A \smallsetminus u)) .$$

It follows that $A \in I^m_{su}$, as required. □ (*Lemma*)

Let J^m_{su} be the hereditary hull of I^m_{su} (all subsets of sets in I^m_{su}). It follows from Lemma 8.2 that any J^m_{su} is a topologically closed subideal of $\mathscr{I} \restriction Z_0$, however, $\mathscr{I} \restriction Z_0$ is the union of those ideals by Lemma 8.3, as required. □

Corollary 8.4. *The ERs E_2 and E_3 are Borel irreducible to E_1. It follows that they are Borel irreducible to E_0, and hence $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_2$ and $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$.*

Proof. It is quite clear that neither $\mathscr{I}_2$ nor $\mathscr{I}_3$ belong to the types of ideals mentioned in Theorem 8.1. □

That $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1$ strictly, and even that E_1 is not essentially countable (formally $\mathsf{E}_1 \not\leq_{\mathrm{B}} \mathsf{E}_\infty$), will be established by Lemma 9.3 below.

8b $\mathscr{I}_1$ and P-ideals

The next theorem claims that the ideal $\mathscr{I}_1$ is the $\leq_{\mathrm{RB}}$-least among all Borel ideals which are not P-ideals. That it is the $\leq_{\mathrm{B}}$-least in this family will be shown in the next Chapter.

Recall that *analytic* means $\mathbf{\Sigma}^1_1$ while the notions of polishable ideals and P-ideals were introduced in Chapter 2.

Theorem 8.5. *The following conditions are equivalent for any ideal on $\mathbb{N}$:*

(i) *$\mathscr{I}$ has the form $\mathtt{Exh}_\varphi$, where φ is a* LSC *submeasure on $\mathbb{N}$;*

(ii) *$\mathscr{I}$ is a polishable ideal;*

(iii) *$\mathscr{I}$ is an analytic P-ideal;*

(iv) *$\mathscr{I}$ is an analytic ideal such that all countable unions of $\mathscr{I}$-small sets are $\mathscr{I}$-small,* where a set $X \subseteq \mathscr{P}(\mathbb{N})$ is *$\mathscr{I}$-small* if there is $A \in \mathscr{I}$ such that $X \restriction A = \{x \cap A : x \in X\} \subseteq \mathscr{P}(A)$ is meager in $\mathscr{P}(A)$;

(v) *$\mathscr{I}$ is an analytic ideal satisfying $\mathscr{I}_1 \not\leq_{\mathrm{RB}} \mathscr{I}$;*

(vi) *$\mathscr{I}$ is an analytic ideal satisfying $\mathscr{I}_1 \not\leq_{\mathrm{B}} \mathscr{I}$.*

By the way it follows that all analytic P-ideals actually belong to $\mathbf{\Pi}^0_3$, simply because any ideal of type (i) is easily $\mathbf{\Pi}^0_3$.

Corollary 8.6. *If $\mathscr{I}_1$ is a Borel ideal then $\mathscr{I}_1 \leq_{\mathrm{RB}} \mathscr{I}$ iff $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.* □

Corollary 8.7. *Suppose that $\mathscr{J}$ is an analytic P-ideal. Then any ideal $\mathscr{I}$ satisfying $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ is an analytic P-ideal, too.*

Proof. Use equivalence (vi) $\Longleftrightarrow$ (iii) of the theorem. □

Proof (Theorem). We begin with the proof of the equivalence of the first five conditions, the result of Solecki [57, 58]. First of all, comparably simple (but tricky in some points) equivalences (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (iv) and implication (i) $\Longrightarrow$ (iii) will be established. The hard part will be the implication (iv) $\Longrightarrow$ (i) that follows in Section 8c. The last condition (vi) (Kechris and Louveau [37]) will be addes to the equivalence by Lemma 9.10 based on several complicated theorems in the next Chapter.

(i) $\Longrightarrow$ (ii) If $\varphi(\{n\}) > 0$ for all n then the required metric on $\mathscr{I} = \mathtt{Exh}_\varphi$ can be defined by $d_\varphi(x, y) = \varphi(x \,\Delta\, y)$. Then any set $U \subseteq \mathscr{I}$ open in the sense of the ordinary topology (the one inherited from $\mathscr{P}(\mathbb{N})$) is d_φ-open, while any d_φ-open set is Borel in the ordinary sense. In the general case we assemble the required metric of d_φ on the domain $\{n : \varphi(\{n\}) > 0\}$ and the ordinary Polish metric on $\mathscr{P}(\mathbb{N})$ on the complementary domain.

(ii) $\Longrightarrow$ (i) Let τ be a Polish group topology on $\mathscr{I}$, generated by a Δ-invariant compatible metric d. It can be shown (Solecki [58, p. 60]) that $\varphi(x) = \mathtt{sup}_{y \in \mathscr{I},\, y \subseteq x}\, d(\varnothing, x)$ is a LSC submeasure with $\mathscr{I} = \mathtt{Exh}_\varphi$. The key observation is that for any $x \in \mathscr{I}$ the sequence $\{x \cap [0, n)\}_{n \in \mathbb{N}}$ d-converges to x by the last statement of Lemma 2.5, which implies both that φ is LSC (because the supremum above can be restricted to finite sets y) and that $\mathscr{I} = \mathtt{Exh}_\varphi$ (where the inclusion $\supseteq$ needs another "identity map" argument).

(i) $\Longrightarrow$ (iii) That any $\mathscr{I} = \mathtt{Exh}_\varphi$, φ being LSC, is a P-ideal, is an easy exercise: if $x_1, x_2, x_3, \dots \in \mathscr{I}$ then define an increasing sequence of numbers $n_i \in x_i$ with $\varphi(x_i \cap [n_i, \infty)) \le 2^{-n}$ and put $x = \bigcup_i (x_i \cap [n_i, \infty))$.

(iii) $\Longrightarrow$ (v) This is because $\mathscr{I}_1$ easily does not satisfy (iii).

(v) $\Longrightarrow$ (iv) Suppose that sets $X_n \subseteq \mathscr{P}(\mathbb{N})$ are $\mathscr{I}$-small, so that $X_n \restriction A_n$ is meager in $\mathscr{P}(A_n)$ for some $A_n \in \mathscr{I}$, but $X = \bigcup_n X_n$ is not $\mathscr{I}$-small, and prove $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$. Arguing as in the proof of Theorem 6.11, we use the meagerness to find, for any n, a sequence of pairwise disjoint non-empty finite sets $w^n_k \subseteq A_n$, $k \in \mathbb{N}$, and subsets $u^n_k \subseteq w^n_k$, such that

(a) if $x \subseteq \mathbb{N}$ and $\exists^\infty k\, (x \cap w^n_k = u^n_k)$ then $x \notin X_n$.

Dropping some sets w^n_k away and reenumerating the rest, we can strengthen the disjointness to the following: $w^n_k \cap w^m_l = \varnothing$ unless both $n = m$ and $k = l$.

Now put $w^n_{ij} = w^n_{2^i(2j+1)-1}$. The sets $\overline{w}_{ij} = \bigcup_{n \le i} w^n_{ij}$ are still pairwise disjoint, and satisfy the following two properties:

(b) $\bigcup_j \overline{w}_{ij} \subseteq A_0 \cup \dots \cup A_i$, hence, $\in \mathscr{I}$, for any i;

(c) if a set $Z \subseteq \mathbb{N} \times \mathbb{N}$ does not belong to $\mathscr{I}_1$, *i.e.*, $\exists^\infty i \, \exists j \, (\langle i,j\rangle \in Z)$, then $\forall n \, \exists^\infty k \, (w_k^n \subseteq \overline{w}_Z)$, where $\overline{w}_Z = \bigcup_{\langle i,j\rangle \in Z} \overline{w}_{ij})$.

We assert that the map $\langle i,j\rangle \mapsto \overline{w}_{ij}$ witnesses $\mathscr{I}_1 \le^+_{\rm RB} \mathscr{I}$. (Then a simple argument, as in the proof of Theorem 6.11, gives $\mathscr{I}_1 \le_{\rm RB} \mathscr{I}$.)

Indeed if $Z \subseteq \mathbb{N} \times \mathbb{N}$ belongs to $\mathscr{I}_1$ then $\overline{w}_Z \in \mathscr{I}$ by (b). Suppose that $Z \notin \mathscr{I}_1$. It suffices to show that $X_n \restriction \overline{w}_Z$ is meager in $\mathscr{P}(\overline{w}_Z)$ for any n. Note that by (c) the set $K = \{k : w_k^n \subseteq \overline{w}_Z\}$ is infinite and in fact $\overline{w}_Z \cap A_n = \bigcup_{k\in K} w_k^n$. Therefore, any $x \subseteq \overline{w}_Z$ satisfying $x \cap w_k^n = u_k^n$ for infinitely many $k \in K$, does not belong to X_n by (a). Now the meagerness of $X_n \restriction \overline{w}_Z$ is clear.

(iv) $\Longrightarrow$ (iii) This also is quite easy: if a sequence of sets $Z_n \in \mathscr{I}$ witnesses that $\mathscr{I}$ is not a P-ideal, then the union of $\mathscr{I}$-small sets $\mathscr{P}(Z_n)$ is not $\mathscr{I}$-small.

8c The hard part

We finally prove (iv) $\Longrightarrow$ (i), the hard part of Theorem 8.5. A couple of definitions precede the key lemma.

- Let $C(\mathscr{I})$ be the collection of all hereditary (*i.e.*, $y \subseteq x \in K \Longrightarrow y \in K$) compact $\mathscr{I}$-large sets $K \subseteq \mathscr{P}(\mathbb{N})$. (By definition a set $K \subseteq \mathscr{P}(\mathbb{N})$ is $\mathscr{I}$-*large* iff it is not $\mathscr{I}$-small in the sense of (iv) of Theorem 8.5.)

Note that if $K \subseteq \mathscr{P}(\mathbb{N})$ is hereditary and compact then for $K \in C(\mathscr{I})$ it is necessary and sufficient that for any $A \in \mathscr{I}$ there is n such that $A \cap [n,\infty) \in K$.

- Given sets $X, Y \subseteq \mathscr{P}(\mathbb{N})$, let $X + Y = \{x \cup y : x \in X \wedge y \in Y\}$.

Lemma 8.8. *Assume that $\mathscr{I}$ is of type* (iv) *of Theorem 8.5. Then there is a countable sequence of sets $K_m \in C(\mathscr{I})$ such that for any set $K \in C(\mathscr{I})$ there exist numbers m,n with $K_m + K_n \subseteq K$.*

Proof. As $\mathscr{I}$ is a $\mathbf{\Sigma}^1_1$ subset of $\mathscr{P}(\mathbb{N})$, there exists a continuous map $f : \mathbb{N}^\mathbb{N} \xrightarrow{\text{onto}} \mathscr{I}$. For any $s \in \mathbb{N}^{<\omega}$, we define

$$N_s = \{a \in \mathbb{N}^\mathbb{N} : s \subset a\} \quad \text{and} \quad B_s = f\text{"}N_s \quad (\text{the } f\text{-image of } N_s)\,.$$

Consider the set $T = \{s : B_s \text{ is } \mathscr{I}\text{-large}\}$. As $\mathscr{I}$ itself is clearly $\mathscr{I}$-large, $\Lambda \in T$. On the other hand, the assumption (iv) easily implies that T has no endpoints and no isolated branches, hence, $P = \{a \in \mathbb{N}^\mathbb{N} : \forall n \, (a \restriction n \in T)\}$ is a perfect set. Moreover, $F_s = f\text{"}(P \cap N_s)$ is $\mathscr{I}$-large for any $s \in T$ because $B_s \smallsetminus F_s$ is a countable union of $\mathscr{I}$-small sets.

Now consider any set $K \in C(\mathscr{I})$. By definition, if $x,y \in \mathscr{I}$ then $z = x \cup y \in \mathscr{I}$, thus, $K \restriction z$ is not meager in $\mathscr{P}(z)$, hence, by the compactness, $K \restriction z$ includes a basic nbhd of $\mathscr{P}(z)$, hence, by the hereditarity, there is a number n such that

$Z \cap [n,\infty) \in K$. We conclude that $P^2 = \bigcup_n Q_n$, where each $Q_n = \{\langle a,b\rangle \in P^2 : (f(a) \cup f(b)) \cap [n,\infty) \in K\}$ is closed in P because so is K and f is continuous. Thus, there are $s,t \in T$ such that $P^2 \cap (N_s \times N_t) \subseteq Q_n$, in other words, $(F_s + F_t) \restriction [n,\infty) \subseteq K$, hence, $(\widehat{F_s} + \widehat{F_t}) \restriction [n,\infty) \subseteq K$, where $\widehat{\dots}$ denotes the topological closure of the hereditary hull. Thus we can take, as $\{K_m\}$, all sets of the form $K_{sn} = \widehat{F_s} \restriction n$. □

As $C(\mathscr{I})$ is obviously a filter, we can transform (still in the assumption that $\mathscr{I}$ is of type (iv)) the sequence of sets given by the lemma into a $\subseteq$-decreasing sequence of sets $K_n \in C(\mathscr{I})$ such that

(1) for any $K \in C(\mathscr{I})$ there is n with $K_n \subseteq K$,

and $K_{n+1} + K_{n+1} \subseteq K_n$ for any n. Taking any other term of the sequence, we can sharpen the latter requirement to

(2) for any n : $K_{n+1} + K_{n+1} + K_{n+1} \subseteq K_n$.

This is the starting point for the construction of a LSC submeasure φ with $\mathscr{I} = \mathtt{Exh}_\varphi$. Assuming that, in addition, $K_0 = \mathscr{P}(\mathbb{N})$, let, for any $x \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$,

$$\begin{aligned} \varphi_1(x) &= \mathtt{inf}\,\{ 2^{-n} : x \in K_n \} , \text{ and} \\ \varphi_2(x) &= \mathtt{inf}\,\{ \textstyle\sum_{i=1}^m \varphi_1(x_i) : m \geq 1 \wedge x_i \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N}) \wedge x \subseteq \bigcup_{i=1}^m x_i \} . \end{aligned}$$

Then set $\varphi(x) = \mathtt{sup}_n\, \varphi_2(x \cap [0,n))$ for any $x \subseteq \mathbb{N}$. A routine verification shows that φ submeasure and that $\mathscr{I} = \mathtt{Exh}_\varphi$. (See Solecki [58]. To check that any $x \in \mathtt{Exh}_\varphi$ belongs to $\mathscr{I}$ we use the following observation: $x \in \mathscr{I}$ iff for any $K \in C(\mathscr{I})$ there is n such that $x \cap [n,\infty) \in K$.)

□ (*Theorem 8.5 without* (vi))

Chapter 9

Equivalence relation E_1

The ideal $\mathscr{I}_1$ naturally defines the ER $\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$ on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ so that $x \mathrel{\mathsf{E}_1} y$ iff $x \,\Delta\, y \in \mathscr{I}_1$. We can as well consider E_1 as an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$, or even on $\mathbb{X}^{\mathbb{N}}$ for any uncountable Polish space $\mathbb{X}$, defined as $x \mathrel{\mathsf{E}_1} y$ iff $x(k) = y(k)$ for all but finite k.

The following notation will be quite useful in our study of subsets of spaces of the form $X^{\mathbb{N}}$. If x is a function defined on $\mathbb{N}$ then, for any n, let

$$x \restriction_{<n} = x \restriction [0,n)\,, \quad x \restriction_{\le n} = x \restriction [0,n]\,, \quad x \restriction_{>n} = x \restriction (n,\infty)\,, \quad x \restriction_{\ge n} = x \restriction [n,\infty)\,.$$

For any set X of $\mathbb{N}$-sequences, let $X \restriction_{<n} = \{x \restriction_{<n} : x \in X\}$, and similarly for $\le$, $>$, $\ge$. If $\xi \in X \restriction_{>n}$ then let $\mathbf{S}_X(\xi) = \{x(n) : x \in X \wedge x \restriction_{>n} = \xi\}$.

9a E_1 : hypersmoothness and non-countability

Recall that a hypersmooth equivalence relation is a countable increasing union of Borel smooth ERs. This Section contains a several results which describe the relationships between hypersmooth and countable equivalence relations. The following lemma shows that E_1 is universal in this class.

Lemma 9.1. *For a Borel ER E to be hypersmooth it is necessary and sufficient that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$.*

Proof. Let $\mathbb{X}$ be the domain of E. Assume that E is hypersmooth, *i.e.*, $\mathsf{E} = \bigcup_n \mathsf{E}_n$, where $x \mathrel{\mathsf{E}_n} y$ iff $\vartheta_n(x) = \vartheta_n(y)$, each $\vartheta_n : \mathbb{X} \to 2^{\mathbb{N}}$ is Borel, and $\mathsf{E}_n \subseteq \mathsf{E}_{n+1}$, $\forall\, n$. Then $\vartheta(x) = \{\vartheta_n(x)\}_{n\in\mathbb{N}}$ witnesses $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$. Conversely, if $\vartheta : \mathbb{X} \to (2^{\mathbb{N}})^{\mathbb{N}}$ is a Borel reduction of E to E_1 then the sequence of ERs $x \mathrel{\mathsf{E}_n} y$ iff $\vartheta(x) \restriction_{\ge n} = \vartheta(y) \restriction_{\ge n}$ witnesses that E is hypersmooth. □

Corollary 9.2. $\mathsf{E}_\infty \not\le_{\mathrm{B}} \mathsf{E}_1$.

Proof. Otherwise E_∞ is a hypersmooth equivalence relation by Lemma 9.1. But E_∞ is countable as well. It follows that $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_0$ by Theorem 6.5. This contradicts Theorem 6.6. □

The following result is given in [37] with a reference to earlier papers.

Lemma 9.3. (i) E_1 *is not essentially countable, that is, there is no Borel countable* (with at most countable classes) *ER* E *such that* $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}$.

(ii) $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1$, *in other words,* $\mathtt{Fin} <_{\mathrm{B}} \mathscr{I}_1$.

Proof. (i) (A version of the argument in [37], 1.4 and 1.5.) Let $\mathbb{X}$ be the domain of E, and $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{X}$ a Borel map satisfying $x \mathrel{\mathsf{E}_1} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$. We begin with a few definitions. Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ (a big enough fragment of $\mathbf{ZFC}$, see Remark 5.9), which contains codes for D, $\vartheta \restriction D$, $\mathbb{X}$.

We are going to define, for any k, a pair of points $a_k \ne b_k \in 2^{\mathbb{N}}$, a number $\ell(k)$ and a tuple $\tau_k \in (2^{\mathbb{N}})^{\ell(k)}$ such that

(1) both $x = \langle a_0\rangle^\wedge\tau_0{}^\wedge\langle a_1\rangle^\wedge\tau_1{}^\wedge\dots$ and $y = \langle b_0\rangle^\wedge\tau_0{}^\wedge\langle b_1\rangle^\wedge\tau_1{}^\wedge\dots$ are elements of $(2^{\mathbb{N}})^{\mathbb{N}}$ Cohen generic over $\mathfrak{M}$;

(2) for any k, $\zeta_k = \langle a_0, b_0\rangle^\wedge\tau_0{}^\wedge\langle a_1, b_1\rangle^\wedge\tau_1{}^\wedge\dots{}^\wedge\langle a_k, b_k\rangle^\wedge\tau_k$ is Cohen generic over $\mathfrak{M}$, hence so are the subsequences $\xi_k = \langle a_0\rangle^\wedge\tau_0{}^\wedge\dots{}^\wedge\langle a_k\rangle^\wedge\tau_k$ and $\eta_k = \langle b_0\rangle^\wedge\tau_0{}^\wedge\dots{}^\wedge\langle b_k\rangle^\wedge\tau_k$;

(3) for any k and any $z \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\zeta_k{}^\wedge z$ is generic over $\mathfrak{M}$ we have $\vartheta(\xi_k{}^\wedge z) = \vartheta(\eta_k{}^\wedge z)$.

If this is done then by (2) choose for any k a point $z_k \in (2^{\mathbb{N}})^{\mathbb{N}}$ Cohen generic over $\mathfrak{M}[\zeta_k]$. Then $\zeta_k{}^\wedge z_k$ is Cohen generic over $\mathfrak{M}$ by the product forcing theorem. It follows by (3) that $\vartheta(x_k) = \vartheta(y_k)$, where $x_k = \xi_k{}^\wedge z_k$ and $y_k = \eta_k{}^\wedge z_k$. Note that $x_k \to x$ and $y_k \to y$ in $(2^{\mathbb{N}})^{\mathbb{N}}$ with $k \to \infty$, and on the other hand, all of x_k, x, y_k, y belong to D because of the genericity. It follows that $\vartheta(x) = \vartheta(y)$ by the choice of D. However obviously $\neg\, x \mathrel{\mathsf{E}_1} y$, so that ϑ is not a reduction, as required.

To define a_0, b_0, τ_0 note that there exist a perfect set $X \subseteq 2^{\mathbb{N}}$ and a point $z \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\langle a, b\rangle^\wedge z$ is Cohen generic over $\mathfrak{M}$ for any two $a \ne b \in X$. (Indeed let $\langle w, z\rangle \in 2^{2^{<\omega}} \times (2^{\mathbb{N}})^{\mathbb{N}}$ be Cohen generic over $\mathfrak{M}$. Put $X = \{w_a : a \in 2^{\mathbb{N}}\}$, where $w_a \in 2^{\mathbb{N}}$ is defined by $w_a(k) = w(a \restriction k)$, $\forall\, k$.) In particular, $\langle a\rangle^\wedge z$ is Cohen generic over $\mathfrak{M}$ for any $a \in X$. However all points of the form $\langle a\rangle^\wedge z$ are pairwise E_1-equivalent. Thus ϑ sends all of them into one and the same F-class, which is a countable set by the choice of F. It follows that there is a pair of $a \ne b$ in X such that $\vartheta(\langle a\rangle^\wedge z) \ne \vartheta(\langle b\rangle^\wedge z)$. This equality is a property of the generic point $\langle a, b\rangle^\wedge z$, hence, it is forced in the sense that there is a number ℓ such that $\vartheta(\langle a\rangle^\wedge \hat z) = \vartheta(\langle b\rangle^\wedge \hat z)$ whenever $z \in (2^{\mathbb{N}})^{\mathbb{N}}$, $\langle a, b\rangle^\wedge \hat z$ is Cohen generic over $\mathfrak{M}$, and $\hat z \restriction \ell = z \restriction \ell$. Put $a_0 = a$, $b_0 = b$, $\tau_0 = z \restriction \ell$.

The induction step is carried out by a similar argument. For instance to define a_1, b_1, τ_1 we find points $a' \ne b' \in 2^{\mathbb{N}}$ and $z' \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\langle a', b'\rangle^\wedge z'$ is Cohen generic over $\mathfrak{M}[a_0, b_0, z]$ and $\vartheta(\langle a_0\rangle^\wedge\tau_0{}^\wedge\langle a'\rangle^\wedge z') = \vartheta(\langle a_0\rangle^\wedge\tau_0{}^\wedge\langle b'\rangle^\wedge z')$.

Yet we have $\vartheta(\langle a_0\rangle^\wedge\tau_0{}^\wedge\langle b'\rangle^\wedge z') = \vartheta(\langle b_0\rangle^\wedge\tau_0{}^\wedge\langle b'\rangle^\wedge z')$ by the choice of ℓ (take $\hat z = \tau_0{}^\wedge\langle b'\rangle^\wedge z'$). Thus $\vartheta(\langle a_0\rangle^\wedge\tau_0{}^\wedge\langle a'\rangle^\wedge z') = \vartheta(\langle b_0\rangle^\wedge\tau_0{}^\wedge\langle b'\rangle^\wedge z')$. It follows that there is a number ℓ' satisfying $\vartheta(\langle a_0\rangle^\wedge\tau_0{}^\wedge\langle a'\rangle^\wedge\hat z) = \vartheta(\langle b_0\rangle^\wedge\tau_0{}^\wedge\langle b'\rangle^\wedge\hat z)$ for any $\hat z \in (2^{\mathbb N})^{\mathbb N}$ such that $\langle a_0, b_0\rangle^\wedge\tau_0{}^\wedge\langle a', b'\rangle^\wedge\hat z$ is Cohen generic over $\mathfrak M$ and $\hat z \restriction \ell' = z' \restriction \ell'$. Put $a_1 = a'$, $b_1 = b'$, $\tau_1 = z' \restriction \ell'$.

(ii) That $\mathsf E_0 \le_{\mathrm B} \mathsf E_1$ is witnessed by the map $f(x) = \{\langle 0, n\rangle : n \in x\}$. □

While $\mathsf E_1$ is not countable, the conjunction of hypersmoothness and countability characterizes the essentially more primitive class of hyperfinite equivalence relations.

9b The 3rd dichotomy

The following major result is called the 3rd dichotomy theorem.

Theorem 9.4 (Kechris and Louveau [37]). *Suppose that* $\mathsf E$ *is a Borel ER on some Polish space, and* $\mathsf E \le_{\mathrm B} \mathsf E_1$. *Then* **either** $\mathsf E \le_{\mathrm B} \mathsf E_0$ **or** $\mathsf E_1 \le_{\mathrm B} \mathsf E$.

Proof. Starting the proof, we may assume that $\mathsf E$ is a Δ^1_1 ER on $2^{\mathbb N}$, and that there is a reduction $\rho : 2^{\mathbb N} \to (2^{\mathbb N})^{\mathbb N}$ of $\mathsf E$ to $\mathsf E_1$, of class Δ^1_1. In fact it can be assumed that ρ is a bijection. Indeed define another map $\varphi : 2^{\mathbb N} \to (2^{\mathbb N})^{\mathbb N}$ so that $\varphi(x)(0) = x$ and $\varphi(x)(n+1) = \rho(x)(n)$ for all $x \in 2^{\mathbb N}$ and all n. Then φ is a bijection and still a Δ^1_1 reduction of $\mathsf E$ to $\mathsf E_1$.

Then $R = \mathtt{ran}\,\rho$ is a Δ^1_1 subset of $(2^{\mathbb N})^{\mathbb N}$. The idea behind the proof is to show that the set R is either small enough for $\mathsf E_1 \restriction R$ to be Borel reducible to $\mathsf E_0$, or otherwise it is big enough to contain a closed subset X such that $\mathsf E_1 \restriction X$ is Borel isomorphic to $\mathsf E_1$.

Relations $\prec$ and $\preccurlyeq$ will denote the inverse order relations on $\mathbb N$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$, and $m \prec n$ iff $n < m$. If $x \in (2^{\mathbb N})^{\mathbb N}$ then $x \restriction_{\preccurlyeq n}$ denotes the restriction of x (a function defined on $\mathbb N$) on the domain $\preccurlyeq n$, *i.e.*, $[n, \infty)$. If $X \subseteq (2^{\mathbb N})^{\mathbb N}$ then let $X \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in X\}$. Define $x \restriction_{\prec n}$ and $X \restriction_{\prec n}$ similarly. In particular, $(2^{\mathbb N})^{\mathbb N} \restriction_{\preccurlyeq n} = (2^{\mathbb N})^{\preccurlyeq n} = (2^{\mathbb N})^{[n,\infty)}$.

For any sequence $x \in (2^{\mathbb N})^{\preccurlyeq n}$, let $\mathtt{dep}\,x$ (the *depth* of x) be the number (finite or ∞) of elements of the set $\nabla(x) = \{j \preccurlyeq n : x(j) \notin \Delta^1_1(x \restriction_{\prec j})\}$. The formula $\mathtt{dep}\,x \ge d$ (of two variables, d running over $\mathbb N \cup \{\infty\}$) is obviously Σ^1_1.

We have two cases:

Case 1: all $x \in R = \mathtt{ran}\,\rho$ satisfy $\mathtt{dep}\,x < \infty$.

Case 2: there exist $x \in R$ with $\mathtt{dep}\,x = \infty$.

Case 1 is the easier case. The following lemma proves that the Case 1 assumption implies the **either** case of Theorem 9.4.

Lemma 9.5. *Suppose that* $X \subseteq (2^{\mathbb N})^{\mathbb N}$ *is a* Δ^1_1 *set and any* $x \in X$ *satisfies* $\mathtt{dep}\,x < \infty$. *Then* $\mathsf E_1 \restriction X \le_{\mathrm B} \mathsf E_0$.

Proof. By the choice of X for any $x \in X$ there is a number n such that $\forall m \preccurlyeq n \, (x(m) \in \Delta^1_1(x \restriction_{\prec m}))$. As the relation between x and n here is clearly Π^1_1, the Kreisel selection theorem (Theorem 1.7) yields a Δ^1_1 map $\nu : X \to \mathbb{N}$ such that $x(m) \in \Delta^1_1(x \restriction_{\prec n})$ holds whenever $x \in X$ and $m \preccurlyeq \nu(x)$. Now define, for each $x \in X$, $\vartheta(x) \in (2^{\mathbb{N}})^{\mathbb{N}}$ as follows: $\vartheta(x) \restriction_{\preccurlyeq \nu(x)} = x \restriction_{\preccurlyeq \nu(x)}$, but $\vartheta(x)(j) = \varnothing$ for all $j < \nu(x)$. Note that $x \mathrel{\mathsf{E}_1} \vartheta(x)$ for any $x \in X$.

The other important thing is that $\mathtt{ran}\,\vartheta \subseteq Z = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \mathtt{dep}\, x = 0\}$, where Z is a Π^1_1 set, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\,\vartheta \subseteq Y \subseteq Z$. In particular ϑ reduces $\mathsf{E}_1 \restriction X$ to $\mathsf{E}_1 \restriction Y$. We observe that $\mathsf{E}_1 \restriction Y$ is a countable equivalence relation: any E_1-class in $(2^{\mathbb{N}})^{\mathbb{N}}$ intersects Y by an at most countable set (as so is the property of Z, a bigger set). Thus, $\mathsf{E}_1 \restriction Y$ is hyperfinite by Theorem 6.5. □

9c Case 2

We are going to prove that then the Δ^1_1 set $R = \mathtt{ran}\,\rho$ contains a Δ^1_1 subset $X \subseteq R$ with $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$. This implies the **or** case of Theorem 9.4. Indeed as ρ is a Borel bijection, there exists the inverse map ρ^{-1}, and it obviously witnesses $\mathsf{E}_1 \restriction R \le_{\mathrm{B}} \mathsf{E}$. On the other hand, $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X \le_{\mathrm{B}} \mathsf{E}_1 \restriction R$.

The required subset X of R will be defined with the help of a splitting construction developed in [29] for the study of "ill"founded Sacks iterations.

We shall define a map $\varphi : \mathbb{N} \to \mathbb{N}$, which assumes infinitely many values and assumes each its value infinitely many times (but $\mathtt{ran}\,\varphi$ may be a proper subset of $\mathbb{N}$), and, for each $u \in 2^{<\omega}$, a non-empty Σ^1_1 subset $X_u \subseteq R$, which satisfy a quite long list of properties. First of all, if φ is already defined at least on $[0, n)$ and $u \ne v \in 2^{<\omega}$ then let $\nu_\varphi[u, v] = \mathtt{min}_{\preccurlyeq}\{\varphi(k) : k < n \wedge u(k) \ne v(k)\}$. (Note that the minimum is taken in the sense of $\preccurlyeq$, hence, it is $\mathtt{max}$ in the sense of $\le$, the usual order). Separately, put $\varphi[u, u] = -1$ for any u.

Now we give the list of requirements.

(i) if $\varphi(n) \notin \{\varphi(k) : k < n\}$ then $\varphi(n) \prec \varphi(k)$ for any $k < n$;

(ii) every X_u is a non-empty Σ^1_1 subset of R;

(iii) if $u \in 2^n$, $x \in X_u$, and $k < n$, then $\varphi(k) \in \nabla(x)$;

(iv) if $u, v \in 2^n$ then $X_u \restriction_{\prec \nu_\varphi[u,v]} = X_v \restriction_{\prec \nu_\varphi[u,v]}$;

(v) if $u, v \in 2^n$ then $X_u \restriction_{\preccurlyeq \nu_\varphi[u,v]} \cap X_v \restriction_{\preccurlyeq \nu_\varphi[u,v]} = \varnothing$;

(vi) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(vii) $\mathtt{max}_{u \in 2^n}\, \mathtt{diam}\, X_u \to 0$ as $n \to \infty$ (a reasonable Polish metric on $(2^{\mathbb{N}})^{\mathbb{N}}$ is assumed to be fixed);

(viii) for any n, a certain condition, in terms of the Gandy – Harrington forcing, similar to (ii) in Section 7b or (ii) in Section 7e, related to all sets X_u, $u \in 2^n$, so that, as a consequence, $\bigcap_n X_{a \restriction n} \ne \varnothing$ for any $a \in 2^{\mathbb{N}}$.

Let us demonstrate how such a system of sets and a function φ accomplish Case 2. According to (vii) and (viii), for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a\restriction n}$ contains a single point, let it be $F(a)$, and F is continuous and $1-1$.

Put $J = \mathtt{ran}\,\varphi = \{j_m : m \in \mathbb{N}\}$, in the $<$-increasing order; $J \subseteq \mathbb{N}$ is infinite. Let $n \in \mathbb{N}$. Then $\varphi(n) = j_m$ for some (unique) m : we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(j_m)$ is an infinite subset of $\mathbb{N}$ for any m. This allows us to define a parallel system of sets $Y_u \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = (2^{\mathbb{N}})^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$. Put $j = \varphi(n) = j_{\psi(n)}$. Let K be the number of all indices $k < n$ still satisfying $\varphi(k) = j$, perhaps $K = 0$. Put $Y_{u^\wedge i} = \{x \in Y_u : x(j)(K) = i\}$ for $i = 0, 1$.

Each of Y_u is clearly a basic clopen set in $(2^{\mathbb{N}})^{\mathbb{N}}$, and one easily verifies that conditions (i) – (vii), except for (iii), are satisfied for the sets Y_u (instead of X_u) and the map ψ (instead of φ), in particular, for any $a \in 2^{\mathbb{N}}$, $\bigcap_n Y_{a\restriction n} = \{G(a)\}$ is a singleton, and the map G is continuous and $1-1$. (We can, of course, define G explicitly: $G(a)(m)(l) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = m$ and there is exactly l numbers $k < n$ with $\psi(k) = m$.) Note finally that $\{G(a) : a \in 2^{\mathbb{N}}\} = (2^{\mathbb{N}})^{\mathbb{N}}$ since by definition $Y_{u^\wedge 1} \cup Y_{u^\wedge 0} = Y_u$ for all u.

We conclude that the map $\vartheta(x) = F(G^{-1}(x))$ is a continuous bijection (hence, in this case, a homeomorphism by compactness) $(2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow{\text{onto}} X$. We further assert that ϑ satisfying the following: for each $y, y' \in (2^{\mathbb{N}})^{\mathbb{N}}$ and m,

$$y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m} \quad \text{iff} \quad \vartheta(y) \restriction_{\preccurlyeq j_m} = \vartheta(y') \restriction_{\preccurlyeq j_m} . \qquad (*)$$

Indeed, let $y = G(a)$ and $x = F(a) = \vartheta(y)$, and similarly $y' = G(a')$ and $x' = F(a') = \vartheta(y')$, where $a, a' \in 2^{\mathbb{N}}$. Suppose that $y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m}$. According to (v) for ψ and the sets Y_u, we then have $m \prec \nu_\psi[a \restriction n, a' \restriction n]$ for any n. It follows, by the definition of ψ, that $j_m \prec \nu_\varphi[a \restriction n, a' \restriction n]$ for any n, hence, $X_{a\restriction n} \restriction_{\preccurlyeq j_m} = X_{a\restriction n} \restriction_{\preccurlyeq j_m}$ for any n by (iv). Assuming now that Polish metrics on all spaces $(2^{\mathbb{N}})^{\preccurlyeq j}$ are chosen so that $\mathtt{diam}\, Z \ge \mathtt{diam}\,(Z \restriction_{\preccurlyeq j})$ for all $Z \subseteq 2^{\mathbb{N}}$ and j, we easily obtain that $x \restriction_{\preccurlyeq j_m} = x' \restriction_{\preccurlyeq j_m}$, *i.e.*, the right-hand side of $(*)$. The inverse implication in $(*)$ is proved similarly.

Thus we have $(*)$, but this means that ϑ is a continuous reduction of E_1 to $\mathsf{E}_1 \restriction X$, thus, $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$, as required.

□ (*Theorem 9.4 modulo the construction* (i) – (viii))

9d The construction

Recall that $R \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is a fixed non-empty Σ^1_1 set such that $\mathtt{dep}\, x = \infty$ for each $x \in R$. Set $X_\Lambda = R$.

Now suppose that the sets $X_u \subseteq R$ with $u \in 2^n$ have been defined and satisfy the applicable part of (i) – (viii).

Step 1. Our 1st task is to choose $\varphi(n)$. Let $\{j_1 < \cdots < j_m\} = \{\varphi(k) : k < n\}$. For any $1 \le p \le m$, let N_p be the number of all $k < n$ with $\varphi(k) = j_p$.

Case 1a. If some numbers N_p are $< m$ then choose $\varphi(n)$ among j_p with the least N_p, and among them the least one.

Case 1b: $N_p \geq m$ (then actually $N_p = m$) for all $p \leq m$. It follows from our assumptions, in particular (iv), that $X_u \restriction_{\prec j_m} = X_v \restriction_{\prec j_m}$ for all $u, v \in 2^n$. Let $Y = X_u \restriction_{\prec j_m}$ for any such u. Take any $y \in Y$. Then $\nabla(y)$ is infinite, hence, there is some $j \in \nabla(y)$ with $j \prec j_m$. Put $\varphi(n) = j$.

We have something else to do in this case. Let $X'_u = \{x \in X_u : j \in \nabla(y)\}$ for any $u \in 2^m$. Then we easily have $X'_u = \{x \in X_u : x \restriction_{\prec j_m} \in Y'\}$, where $Y' = \{y \in Y : j \in \nabla(y)\}$ is a non-empty Σ^1_1 set, so that the sets $X'_u \subseteq X_u$ are non-empty Σ^1_1. Moreover, as j_m is the $\preccurlyeq$-least in $\{\varphi(k) : k < n\}$, we can easily show that the system of sets X'_u still satisfies (iv). This allows us to assume, without any loss of generality, that, in Case 1b, $X'_u = X_u$ for all u, or, in other words, that any $x \in X_u$ for any $u \in 2^n$ satisfies $j = \varphi(n) \in \nabla(x)$. (This is true in Case 1a, of course, because then $\varphi(n) = \varphi(k)$ for some $k < n$.)

Note that this manner to choose $\varphi(n)$ implies (i) and also implies that φ takes infinitely many values and takes each its value infinitely many times.

The continuation of the construction requires the following

Lemma 9.6. *If* $u_0 \in 2^n$ *and* $X' \subseteq X_{u_0}$ *is a non-empty* Σ^1_1 *set then there is a system of* Σ^1_1 *sets* $\varnothing \neq X'_u \subseteq X_u$ *with* $X'_{u_0} = X'$, *which still satisfies* (iv).

Proof. For any $u \in 2^n$, let $X'_u = \{x \in X_u : x \restriction_{\prec n(u)} \in X' \restriction_{\prec n(u)}\}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$, because $\nu_\varphi[u_0, u_0] = -1$. The sets X'_u are as required, via a routine verification. □ (*Lemma*)

Step 2. First of all put $j = \varphi(n)$ and $Y_u = X_u \restriction_{\prec j}$. (All Y_u are equal to Y in Case 1b, but the argument pretends to make no difference between 1a and 1b). Take any $u_1 \in 2^n$. By the construction any element $x \in X_{u_1}$ satisfies $j \in \nabla(x)$, so that $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$. As X_{u_1} is a Σ^1_1 set, it follows that $\{x'(j) : x' \in X_{u_1} \wedge x' \restriction_{\prec j} = x \restriction_{\prec j}\}$ is not a singleton, in fact is uncountable. It follows that there is a number l_{u_1} having the property that the Σ^1_1 set

$$Y'_{u_1} = \{y \in Y_{u_1} : \exists\, x, x' \in X_{u_1}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))\}$$

is non-empty. We now put $X' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and define Σ^1_1 sets $\varnothing \neq X'_u \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1} \restriction_{\prec j} = Y'_{u_1}$, still (iv) is satisfied, and in addition

$$\forall\, y \in X'_{u_1} \restriction_{\prec j}\, \exists\, x, x' \in X'_{u_1}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j)) \quad (1)$$

Now take some other $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j \prec \nu$ then $X_{u_1} \restriction_{\prec j} = X_{u_2} \restriction_{\prec j}$, so that we already have, for $l_{u_2} = l_{u_1}$, that

$$\forall\, y \in X'_{u_2} \restriction_{\prec j}\, \exists\, x, x' \in X'_{u_2}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\,, \quad (2)$$

and can pass to some $u_3 \in 2^n$. Suppose that $\nu \preccurlyeq j$. Now things are somewhat nastier. As above there is a number l_{u_2} such that

$$Y'_{u_2} = \{y \in Y_{u_2} : \exists\, x, x' \in X_{u_2}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\}$$

is a non-empty Σ^1_1 set, thus, we can define $X'' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and maintain the construction of Lemma 9.6, getting non-empty Σ^1_1 sets $X''_u \subseteq X'_u$ still satisfying (iv) and $X''_{u_2} = X''$, therefore, we still have (2) for the set X''_{u_2}.

Yet it is most important in this case that (1) is preserved, *i.e.*, it still holds for the set X''_{u_1} instead of X'_{u_1}! Why is this ? Indeed, according to the construction in the proof of Lemma 9.6, we have $X''_{u_1} = \{x \in X'_{u_1} : x \restriction_{\prec \nu} \in X'' \restriction_{\prec \nu}\}$. Thus, although, in principle, X''_{u_1} is smaller than X'_{u_1}, for any $y \in X''_{u_1} \restriction_{\prec j}$ we have

$$\{x \in X''_{u_1} : x \restriction_{\prec j} = y\} = \{x \in X'_{u_1} : x \restriction_{\prec j} = y\},$$

simply because now we assume that $\nu \preccurlyeq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty Σ^1_1 sets, let us call them "new" X_u, but they are subsets of the "original" X_u, still satisfying (iv), still satisfying that $\varphi(n) \in \nabla(x)$ for each $x \in \bigcap_{u \in 2^n} X_u$, and, in addition, for any $u \in 2^n$ there is a number l_u such that $j \prec \nu_\varphi[u,v] \Longrightarrow l_u = l_v$ and

$$\forall\, y \in X_u \restriction_{\prec j}\, \exists\, x, x' \in X_u\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_u \in x(j) \wedge l_u \notin x'(j)). \qquad (*)$$

Step 3. We define the $(n+1)$-th level of sets by $X_{u^\wedge 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u^\wedge 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from $(*)$ that all these Σ^1_1 sets are non-empty.

Lemma 9.7. *The system of sets* X_s, $s \in 2^{n+1}$ *just defined satisfies* (iv), (v).

Proof. Let $s = u^\wedge i$ and $t = v^\wedge i'$ belong to 2^{n+1}, so that $u, v \in 2^n$ and $i, i' \in \{0,1\}$. Let $\nu = \nu_\varphi[u,v]$ and $\nu' = \nu_\varphi[s,t]$.

Case 3a: $\nu \preccurlyeq j = \varphi(n)$. Then easily $\nu = \nu'$, so that (v) immediately follows from (v) at level n for X_u and X_v. As for (iv), we have $X_s \restriction_{\prec \nu} = X_u \restriction_{\prec \nu}$ (because by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$), and similarly $X_t \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$, therefore, $X_t \restriction_{\prec \nu'} = X_s \restriction_{\prec \nu'}$ since $X_u \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$ by (iv) at level n.

Case 3b: $j \prec \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have (v). Further, $X_u \restriction_{\prec \nu} = X_v \restriction_{\prec \nu}$ by (iv) at level n, hence, $X_u \restriction_{\preccurlyeq j} = X_v \restriction_{\preccurlyeq j}$, hence, $l_u = l_v$ (see above). Now, assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that

$$X_s \restriction_{\prec \nu'} = \{y \in X_u \restriction_{\prec \nu} : l \in y(j)\} = \{y \in X_v \restriction_{\prec \nu} : l \in y(j)\} = X_t \restriction_{\prec \nu'}\ .$$

Case 3c: $j \prec \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$. Now $\nu' = j$. Yet by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$ and $X_t \restriction_{\prec j} = X_v \restriction_{\prec j}$, so it remains to apply (iv) for level n. As for (v), note that by definition $l \notin x(j)$ for any $x \in X_s = X_{u^\wedge 0}$ while $l \in x(j)$ for any $x \in X_t = X_{v^\wedge 1}$, where $l = l_u = l_v$. □ (*Lemma*)

Step 4. In addition to (iv) and (v), we already have (i), (ii), (iii), (vi) at level $n+1$. To achieve the remaining properties (vii) and (viii), it suffices to consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep, a non-empty Σ^1_1 subset of X_s which is consistent with the requirements of (vii) and (viii) (for instance, for (vii), just take it so the diameter is $\le 2^{-n}$), and then reducing all other sets X_t by Lemma 9.6 at level $n+1$.

□ (*Construction and Theorem 9.4*)

9e Above E_1

Recall that an embedding is a $1-1$ reduction, and an invariant embedding is an embedding ϑ such that its range is an invariant set, see Section 4a.

Theorem 9.8 (Kechris and Louveau [37]). *Suppose that* $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{F}$, *where* F *is an analytic ER on a Polish space* $\mathbb{Y}$. *Then both* $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{F}$ *and* $\mathsf{E}_1 \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$.

Proof. To prove the first statement, let $\preccurlyeq$ be the inverted order on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$. Let $\mathfrak{P}$ be the collection of all sets $P \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ such that there is a continuous $1-1$ map $\eta : (2^{\mathbb{N}})^{\mathbb{N}} \xrightarrow{\text{onto}} P$ satisfying

$$x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \eta(x) \restriction_{\preccurlyeq n} = \eta(y) \restriction_{\preccurlyeq n}$$

for all n and $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, where $x \restriction_{\preccurlyeq n} = \{x(i)\}_{i \preccurlyeq n}$ for any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. Clearly any such a map is a continuous embedding of E_1 into itself.

This set $\mathfrak{P}$ can be used as a forcing notion to extend the universe by a sequence of reals x_i so that each x_n is Sacks–generic over $\{x_i\}_{i \prec n}$. This is an example of iterated Sacks extensions with an ill-founded "skeleton" of iteration, which we defined in [29]. (See [39] on more recent developments on ill-iterated forcing.) Here, the "skeleton" is $\mathbb{N}$ with the inverted order $\preccurlyeq$.

The method of [29] contains a study of continuous and Borel functions on sets in $\mathfrak{P}$. In particular it is shown there that Borel maps admit the following *cofinal classification* on sets in $\mathfrak{P}$: if $\mathbb{Y}$ is Polish, $P' \in \mathfrak{P}$, and $\vartheta : P' \to \mathbb{Y}$ is Borel then there is a set $P \in \mathfrak{P}$, $P \subseteq P'$, on which ϑ is continuous, and either a constant or, for some n, $1-1$ on $P \restriction_{\preccurlyeq n}$ in the sense that,

$$\text{for all } x, y \in P : \quad x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \vartheta(x) = \vartheta(y)\,. \qquad (*)$$

We apply this to a Borel map $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{Y}$ which reduces E_1 to F. We begin with $P' = (2^{\mathbb{N}})^{\mathbb{N}}$ and find a set $P \in \mathfrak{P}$ as indicated. Since ϑ cannot be a constant on P (indeed, any $P \in \mathfrak{P}$ contains many pairwise E_1-inequivalent elements), we have $(*)$ for some n. In other words, there is a $1-1$ continuous map $f : P \restriction_{\preccurlyeq n} \to \mathbb{Y}$ (where $P \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in P\}$) such that $\vartheta(x) = f(x \restriction_{\preccurlyeq n})$ for all $x \in P$. Now, suppose that $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. Define $\zeta(x) = z \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $z(i) = \mathbb{N} \times \{0\}$ for $i < n$ and $z(n+i) = x(i)$ for all i. Finally set

$\vartheta'(x) = f(\eta(\zeta(x))\restriction_{\preccurlyeq n})$ for all $x \in (2^{\mathbb{N}})^{\mathbb{N}}$: this map turns out to be a continuous embedding of E_1 in F.

Now we prove the second claim. We can assume that $\mathbb{Y} = 2^{\mathbb{N}}$ and that $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ is already a continuous embedding of E_1 into F. Let $Y = \mathtt{ran}\,\vartheta$ and $Z = [Y]_{\mathsf{F}}$. Normally Y, Z are analytic, but in this case they are even Borel. Indeed Z is the projection of $P = \{\langle z, x\rangle : z \mathrel{\mathsf{F}} \vartheta(x)\}$, a Borel subset of $2^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$ whose all cross-sections are E_1-equivalence classes, *i.e.*, σ-compact sets. It is known that in this case Z is Borel and, moreover, there is a Borel map $f : Z \to (2^{\mathbb{N}})^{\mathbb{N}}$ such that $f(z) \mathrel{\mathsf{E}_1} x$ whenever $z \mathrel{\mathsf{F}} \vartheta(x)$.

We can convert f to a $1-1$ map $g : Z \to (2^{\mathbb{N}})^{\mathbb{N}}$ with the same properties: $g(z)(n) = f(z)(n)$ for $n \ge 1$, but $g(z)(0) = z$. Then $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to Z \subseteq 2^{\mathbb{N}}$ and $g : Z \to (2^{\mathbb{N}})^{\mathbb{N}}$ are Borel $1-1$ maps (ϑ is even continuous, but this does not matter now), and, for any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, ϑ maps $[x]_{\mathsf{E}_1}$ into $[\vartheta(x)]_{\mathsf{F}} \subseteq Z$, and g maps $[\vartheta(x)]_{\mathsf{F}}$ back into $[x]_{\mathsf{E}_1}$. It remains to apply the construction from the Cantor – Bendixson theorem, to get a Borel embedding, say, f of E_1 into F with $\mathtt{ran}\, f = Z$, that is an invariant embedding. □

The following theorem shows that orbit equivalence relations of Polish group actions cannot reduce E_1.

Theorem 9.9 (Kechris and Louveau [37]). *Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a Borel $\mathbb{G}$-space. Then E_1 is **not** Borel reducible to $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$.*

Proof. Towards the contrary, let $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{X}$ be a Borel reduction of E_1 to E. We can assume, by Theorem 9.8, that ϑ is in fact an invariant embedding, *i.e.*, $1-1$ and $Y = \mathtt{ran}\,\vartheta$ is an E-invariant set. Define, for $g \in \mathbb{G}$ and $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, $g \cdot x = \vartheta^{-1}(g \cdot \vartheta(x))$. Then this is a Borel action of $\mathbb{G}$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ such that the induced relation $\mathsf{E}^{(2^{\mathbb{N}})^{\mathbb{N}}}_{\mathbb{G}}$ coincides with E_1.

Let us fix $x \in (2^{\mathbb{N}})^{\mathbb{N}}$.

Consider any $y = \{y_n\}_n \in [x]_{\mathsf{E}_1}$. Then $[x]_{\mathsf{E}_1} = \bigcup_n C_n(y)$, where each set $C_n(y) = \{y' \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall\, m \ge n\ (y(n) = y'(n))\}$ is Borel (even compact). It follows that $\mathbb{G} = \bigcup_n G_n(y)$, where each $G_n(y) = \{g \in \mathbb{G} : g(x) \in C_n(y)\}$ is Borel. Thus, as $\mathbb{G}$ is Polish, there is a number n such that $G_n(y)$ is not meager in $\mathbb{G}$ (then this will hold for all $n' \ge n$, of course). Let $n(y)$ be the least such an n.

We assert that for any n the set $Y_n(x) = \{y \restriction [n, \infty) : y \in [x]_{\mathsf{E}_1} \wedge n(x) = n\}$ is at most countable. Indeed suppose that $Y_n(x)$ is not countable. Note that if y_1 and y_2 in $[x]_{\mathsf{E}_1}$ have different restrictions $y_i \restriction [n, \infty)$ then the sets $C_n(y_1)$ and $C_n(y_2)$ are disjoint, therefore, the sets $G_n(y_1)$ and $G_n(y_2)$ are disjoint, so we would have uncountably many pairwise disjoint non-meager sets in $\mathbb{G}$, contradiction. Thus all sets $Y_n(x)$ are countable.

It is most important that $Y_n(x)$ depends on $[x]_{\mathsf{E}_1}$ rather than x itself. More exactly, if $x' \in [x]_{\mathsf{E}_1}$ then $Y_n(x) = Y_n(x')$: this is because any set $G_n(y)$ in the sense of x' is just a shift, within $\mathbb{G}$, of the set $G_n(y)$ in the sense of x. Therefore, putting $Y(x) = \bigcup_n \{\bar{u} : u \in Y_n(x)\}$, where, for $u \in (2^{\mathbb{N}})^{[n,\infty)}$, $\bar{u} \in (2^{\mathbb{N}})^{\mathbb{N}}$ is defined by $\bar{u} \restriction [n,\infty) = u$ and $\bar{u}(k)(j) = 0$ for $k < n$ and all j, we obtain a set $Y = \bigcup_{x \in (2^{\mathbb{N}})^{\mathbb{N}}} Y(x)$ with the property that $Y \cap [x]_{\mathsf{E}_1}$ is non-empty and at most countable for any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$.

The other important fact is that the relation $y \in Y(x)$ is Borel: this is because it is assembled from Borel relations via the Vaught quantifier "there exists nonmeager-many", known to preserve the borelness. It follows that

$$Y = \{y : \exists\, x\, (y \in Y_x)\} = \{y : \forall\, x\, (x \in [y]_{\mathsf{E}_1} \Longrightarrow y \in Y(x)\}$$

is a Borel subset of $(2^{\mathbb{N}})^{\mathbb{N}}$. By the uniformization theorem for Borel sets with countable sections, there is a Borel map f defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ so that $f(x) \in Y(x)$ for any x. This implies $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction Y$. On the other hand, $\mathsf{E}_1 \restriction Y$ is a countable equivalence relation by the above, which is a contradiction to Lemma 9.3. □

The theorem just proved allows us to accomplish the proof of Theorem 8.5 by adding its last condition (vi) to the equivalence of its first five conditions established in Chapter 8. Since $\le_{\mathrm{RB}}$ implies $\le_{\mathrm{B}}$, the following lemma implies the result required.

Lemma 9.10. *If $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a polishable ideal then $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.*

Proof. Recall that if $\mathscr{I}$ is polishable then $\mathsf{E}_{\mathscr{I}}$ is induced by a Polish action of the Δ-group of $\mathscr{I}$ on $\mathscr{P}(\mathbb{N})$. It remains to apply Theorem 9.9. □

We are able now to also give another proof of a result already obtained by different method. (See Corollary 9.2.)

Corollary 9.11. $\mathsf{E}_\infty \not\le_{\mathrm{B}} \mathsf{E}_1$.

Proof. If $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_1$ then by Theorem 9.4 either $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_0$ or $\mathsf{E}_\infty \sim_{\mathrm{B}} \mathsf{E}_1$. The "either" case contradicts Theorem 6.6. The "or" case contradicts Theorem 9.9 since E_∞ is induced by a Polish action of F_2. □

Chapter 10

Actions of the infinite symmetric group

This Section is connected with the next one (on turbulence). We concentrate on a main result in this area, due to Hjorth, that turbulent ERs are not reducible to those induced by actions of S_∞. In particular, we shall prove the following:

I. Lopez-Escobar: any invariant Borel set of countable models is the truth domain of a formula of $\mathscr{L}_{\omega_1\omega}$.

II. Any orbit ER of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures (up to isomorphism).

III. Any ER, classifiable by countable structures, is Borel reducible to isomorphism of countable ordered graphs.

IV. Any <u>Borel</u> ER, classifiable by countable structures, is Borel reducible to one of ERs T_ξ.

V. Any ER, classifiable by countable structures and induced by a Polish action (of a Polish group), is Borel reducible to one of ERs T_ξ on a comeager set.

VI. Any "turbulent" ER E is generically T_ξ-ergodic for any $\xi < \omega_1$, in particular, E is not Borel reducible to T_ξ.

VII. Any "turbulent" ER is not classifiable by countable structures: a corollary of VI and V.

VIII. A generalization of VII: any "turbulent" ER is not Borel reducible to a ER that can be obtained from the equality $\mathsf{EQ}_\mathbb{N}$ using operations defined in Section 3b.

Scott's analysis, involved in proofs of IV and V, appears only in a rather mild and self-contained version.

10a Infinite symmetric group S_∞

Let S_∞ be the group of all permutations (*i.e.*, 1–1 maps $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$) of $\mathbb{N}$, with the superposition as the group operation. Clearly S_∞ is a $\mathbf{G}_\delta$ subset of $\mathbb{N}^\mathbb{N}$, hence, a Polish group. A compatible complete metric on S_∞ can be defined by $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, where d is the ordinary complete metric of $\mathbb{N}^\mathbb{N}$, *i.e.*, $d(x,y) = 2^{-m-1}$, where m is the least such that $x(m) \neq y(m)$. Yet S_∞ admits no compatible left-invariant complete metric [3, 1.5].

For instance isomorphism relations of various kinds of countable structures are orbit ERs induced by S_∞. Indeed, suppose that $\mathscr{L} = \{R_i\}_{i\in I}$ is a countable relational language, *i.e.*, $0 < \mathtt{card}\, I \le \aleph_0$ and each R_i is an m_i-ary relational symbol. We put [1] $\mathtt{Mod}_\mathscr{L} = \prod_{i\in I} \mathscr{P}(\mathbb{N}^{m_i})$, the space of (coded) *$\mathscr{L}$-structures* on $\mathbb{N}$. The *logic action* $j_\mathscr{L}$ of S_∞ on $\mathtt{Mod}_\mathscr{L}$ is defined as follows: if $x = \{x_i\}_{i\in I} \in \mathtt{Mod}_\mathscr{L}$ and $g \in S_\infty$ then $y = j_\mathscr{L}(g,x) = g\cdot x = \{y_i\}_{i\in I} \in \mathtt{Mod}_\mathscr{L}$, where we have

$$\langle k_1,\dots,k_{m_i}\rangle \in x_i \iff \langle g(k_1),\dots,g(k_{m_i})\rangle \in y_i$$

for all $i \in I$ and $\langle k_1,\dots,k_{m_i}\rangle \in \mathbb{N}^{m_i}$. Then $\langle \mathtt{Mod}_\mathscr{L}\,; j_\mathscr{L}\rangle$ is a Polish S_∞-space and $j_\mathscr{L}$-orbits in $\mathtt{Mod}_\mathscr{L}$ are exactly the isomorphism classes of $\mathscr{L}$-structures, which is a reason to denote the associated equivalence relation $\mathsf{E}_{j_\mathscr{L}}^{\mathtt{Mod}_\mathscr{L}}$ as $\cong_\mathscr{L}$.

If G is a subgroup of S_∞ then $j_\mathscr{L}$ restricted to G is still an action of G on $\mathtt{Mod}_\mathscr{L}$, whose orbit ER will be denoted by $\cong_\mathscr{L}^G$, *i.e.*, $x \cong_\mathscr{L}^G y$ iff $\exists\, g \in G\,(g\cdot x = y)$.

10b Borel invariant sets

A set $M \subseteq \mathtt{Mod}_\mathscr{L}$ is *invariant* if $[M]_{\cong_\mathscr{L}} = M$. There is a convenient characterization of *Borel* invariant sets, in terms of $\mathscr{L}_{\omega_1\omega}$, an infinitary extension of $\mathscr{L} = \{R_i\}_{i\in I}$ by countable conjunctions and disjunctions. To be more exact,

1) any $R_i(v_0,\dots,v_{m_i-1})$ is an atomic formula of $\mathscr{L}_{\omega_1\omega}$ (all v_i being variables over $\mathbb{N}$ and m_i is the arity of R_i), and propositional connectives and quantifiers $\exists, \forall$ can be applied as usual;

2) if φ_i, $i \in \mathbb{N}$, are formulas of $\mathscr{L}_{\omega_1\omega}$ whose free variables are among a finite list $v_0,\dots,v_n$ then $\bigvee_i \varphi_i$ and $\bigwedge_i \varphi_i$ are formulas of $\mathscr{L}_{\omega_1\omega}$.

If $x \in \mathtt{Mod}_\mathscr{L}$, $\varphi(v_1,\dots,v_n)$ is a formula of $\mathscr{L}_{\omega_1\omega}$, and $i_1,\dots,i_n \in \mathbb{N}$, then $x \models \varphi(i_1,\dots,i_n)$ means that $\varphi(i_1,\dots,i_n)$ is satisfied on x, in the usual sense that involves transfinite induction on the "depth" of φ, see [34, 16.C].

Theorem 10.1 (Lopez-Escobar, see [34, 16.8]). *A set $M \subseteq \mathtt{Mod}_\mathscr{L}$ is invariant and Borel iff $M = \{x \in \mathtt{Mod}_\mathscr{L} : x \models \varphi\}$ for a closed formula φ of $\mathscr{L}_{\omega_1\omega}$.*

[1] $X_\mathscr{L}$ is often used to denote $\mathtt{Mod}_\mathscr{L}$.

Proof. To prove the nontrivial direction let $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ be invariant and Borel. Put $B_s = \{g \in S_\infty : s \subset g\}$ for any injective $s \in \mathbb{N}^{<\omega}$ (*i.e.*, $s_i \neq s_j$ for $i \neq j$), this is a clopen subset of S_∞ (in the Polish topology of S_∞ inherited from $\mathbb{N}^{\mathbb{N}}$). If $A \subseteq S_\infty$ then let $s \Vdash A(\mathring{g})$ mean that the set $B_s \cap A$ is co-meager in B_s, *i.e.*, $g \in A$ holds for a. a. $g \in S_\infty$ with $s \subset g$. The proof consists of two parts:

(i) $M = \{x \in \mathtt{Mod}_{\mathscr{L}} : \Lambda \Vdash \mathring{g}\cdot x \in M\}$ (where $g\cdot x = j_{\mathscr{L}}(g,x)$, see above);

(ii) For any Borel $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ and any n there is a formula $\varphi^n_M(v_0,\dots,v_{n-1})$ of $\mathscr{L}_{\omega_1\omega}$ such that we have, for every $x \in \mathtt{Mod}_{\mathscr{L}}$ and every injective $s \in \mathbb{N}^n$: $x \models \varphi^n_M(s_0,\dots,s_{n-1})$ iff $s \Vdash \mathring{g}^{-1}\cdot x \in M$.

(i) is clear: since M is invariant, we have $g\cdot x \in M$ for all $x \in M$ and $g \in S_\infty$, on the other hand, if $g\cdot x \in M$ for at least one $g \in S_\infty$ then $x \in M$.

To prove (ii) we argue by induction on the Borel complexity of M. Suppose, for the sake of simplicity, that $\mathscr{L}$ contains a single binary predicate, say, $R(\cdot,\cdot)$; then $\mathtt{Mod}_{\mathscr{L}} = \mathscr{P}(\mathbb{N}^2)$. If $M = \{x \subseteq \mathbb{N}^2 : \langle k,l\rangle \notin x\}$ for some $k,l \in \mathbb{N}$ then take

$$\forall u_0 \dots \forall u_m \left(\textstyle\bigwedge_{i<j\le m}(u_i \neq u_j) \wedge \bigwedge_{i<n}(u_i = v_i) \implies \neg R(u_k,u_l)\right),$$

where $m = \mathtt{max}\{l,k,n\}$, as $\varphi^n_M(v_0,\dots,v_{n-1})$. Further, take

$$\textstyle\bigwedge_{k\ge n} \forall u_0 \dots \forall u_{k-1} \bigvee_{m\ge k} \exists w_0 \dots \exists w_{m-1} \left(\bigwedge_{i<j<k}(u_i \neq u_j) \wedge \bigwedge_{i<n}(u_i = v_i) \right.$$
$$\left. \implies \textstyle\bigwedge_{i<j<m}(w_i \neq w_j) \wedge \bigwedge_{i<k}(w_i = v_i) \wedge \varphi^m_M(w_0,\dots,w_{m-1})\right)$$

as $\varphi^n_{\neg M}(v_0,\dots,v_{n-1})$. Finally, if $M = \bigcap_j M_j$ then we take $\bigwedge_j \varphi^n_{M_j}(v_0,\dots,v_{n-1})$ as $\varphi^n_M(v_0,\dots,v_{n-1})$. □ (*Theorem 10.1*)

10c ERs classifiable by countable structures

The classifiability by countable structures means that we can associate, in a Borel way, a countable $\mathscr{L}$-structure, say, $\vartheta(x)$ with any point $x \in \mathbb{X} = \mathtt{dom}\,\mathsf{E}$ so that $x \mathrel{\mathsf{E}} y$ iff $\vartheta(x)$ and $\vartheta(y)$ are isomorphic.

Definition 10.2 (Hjorth [19, 2.38]**).** An ER E is *classifiable by countable structures* if there is a countable relational language $\mathscr{L}$ such that $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{L}}$. □

Remark 10.3. Any E classifiable by countable structures is $\mathbf{\Sigma}^1_1$, of course, and many of them are Borel. The equivalence relations T_2, E_3, all countable Borel ERs (see the diagram on page 26) are classifiable by countable structures, but E_1, E_2, Tsirelson ERs are not. □

Theorem 10.4 (Becker and Kechris [3]**).** *Any orbit ER of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures.*

Thus all orbit ERs of Polish actions of S_∞ and its closed subgroups are Borel reducible to a very special kind of actions of S_∞.

Proof. First show that any orbit ER of a Polish action of S_∞ itself is classifiable by countable structures. Hjorth's simplified argument [19, 6.19] is as follows. Let $\mathbb{X}$ be a Polish S_∞-space with basis $\{U_l\}_{l\in\mathbb{N}}$, and let $\mathscr{L}$ be the language with relations R_{lk} where each R_{lk} has arity k. If $x\in\mathbb{X}$ then define $\vartheta(x)\in\mathtt{Mod}_{\mathscr{L}}$ by stipulation that $\vartheta(x)\models R_{lk}(s_0,\dots,s_{k-1})$ iff 1) $s_i\ne s_j$ whenever $i<j<k$, and 2) $\forall\, g\in B_s\,(g^{-1}\cdot x\in U_l)$, where $B_s=\{g\in S_\infty: s\subset g\}$ and $s=\langle s_0,\dots,s_{k-1}\rangle\in\mathbb{N}^k$. Then ϑ reduces $\mathsf{E}^{\mathbb{X}}_{S_\infty}$ to $\cong_{\mathscr{L}}$.

To accomplish the proof of the theorem, it remains to apply the following result (an immediate corollary of Theorem 2.3.5b in [3]):

Proposition 10.5. *If $\mathbb{G}$ is a closed subgroup of a Polish group $\mathbb{H}$ and $\mathbb{X}$ is a Polish $\mathbb{G}$-space then there is a Polish $\mathbb{H}$-space $\mathbb{Y}$ such that $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\le_{\mathrm{B}}\mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$.*

Proof. Hjorth [19, 7.18] outlines a proof as follows. Let $Y=\mathbb{X}\times\mathbb{H}$; define $\langle x,h\rangle\approx\langle x',h'\rangle$ if $x'=g\cdot x$ and $h'=gh$ for some $g\in\mathbb{G}$, and consider the quotient space $\mathbb{Y}=Y/{\approx}$ with the topology induced by the Polish topology of Y via the surjection $\langle x,h\rangle\mapsto[\langle x,h\rangle]_\approx$, on which $\mathbb{H}$ acts by $h'\cdot[\langle x,h\rangle]_\approx=[\langle x,hh'^{-1}\rangle]_\approx$. Obviously $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\le_{\mathrm{B}}\mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$ via the map $x\mapsto[\langle x,1\rangle]_\approx$, hence, it remains to prove that $\mathbb{Y}$ is a Polish $\mathbb{H}$-space, which is not really elementary — we refer the reader to [19, 7.18] or [3, 2.3.5b]. □ (*Proposition*)

To bypass 10.5 in the proof of Theorem 10.4, we can use a characterization of all closed subgroups of S_∞. Let $\mathscr{L}$ be a language as above, and $x\in\mathtt{Mod}_{\mathscr{L}}$. Define $\mathtt{Aut}_x=\{g\in S_\infty: g\cdot x=x\}$: the group of all automorphisms of x.

Proposition 10.6 (see [3, 1.5]). *$G\subseteq S_\infty$ is a closed subgroup of S_∞ iff there is an $\mathscr{L}$-structure $x\in\mathtt{Mod}_{\mathscr{L}}$ of a countable language $\mathscr{L}$, such that $G=\mathtt{Aut}_x$.*

Proof. For the nontrivial direction, let G be a closed subgroup of S_∞. For any $n\ge1$, let I_n be the set of all G-orbits in $\mathbb{N}^n$, *i.e.*, equivalence classes of the ER $s\sim t$ iff $\exists\, g\in G\,(t=g\circ s)$, thus, I_n is an at most countable subset of $\mathscr{P}(\mathbb{N}^n)$. Let $I=\bigcup_n I_n$, and, for any $i\in I_n$, let R_i be an n-ary relational symbol, and $\mathscr{L}=\{R_i\}_{i\in I}$. Let $x\in\mathtt{Mod}_{\mathscr{L}}$ be defined as follows: if $i\in I_n$ then $x\models R_i(k_0,\dots,k_{n-1})$ iff $\langle k_0,\dots,k_{n-1}\rangle\in i$. Then $G=\mathtt{Aut}_x$, actually, if G is not necessarily closed subgroup then $\mathtt{Aut}_x=\overline{G}$. □ (*Proposition*)

Now come back to Theorem 10.4. The same argument as in the beginning of the proof shows that any orbit ER of a Polish action of G, a closed subgroup of S_∞, is $\le_{\mathrm{B}}\ \cong^G_{\mathscr{L}}$ for an appropriate countable language $\mathscr{L}$. Yet, by 10.6, $G=\mathtt{Aut}_{y_0}$ where $y_0\in\mathtt{Mod}_{\mathscr{L}'}$ and $\mathscr{L}'$ is a countable language disjoint from $\mathscr{L}$. The map $x\longmapsto\langle x,y_0\rangle$ witnesses that $\cong^G_{\mathscr{L}}\le_{\mathrm{B}}\ \cong_{\mathscr{L}\cup\mathscr{L}'}$.

□ (*Theorem 10.4*)

10d Reduction to countable graphs

It could be expected that the more complicated a language $\mathscr{L}$ is accordingly the more complicated isomorphism equivalence relation $\cong_{\mathscr{L}}$ it produces. However this is not the case. Let $\mathscr{G}$ be the language of (oriented binary) graphs, *i.e.*, $\mathscr{G}$ contains a single binary predicate, say $R(\cdot,\cdot)$.

Theorem 10.7. *If $\mathscr{L}$ is a countable relational language then $\cong_{\mathscr{L}} \le_{\mathrm{B}} \cong_{\mathscr{G}}$. Therefore, an ER E is classifiable by countable structures iff $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$. In other words, a single binary relation can code structures of any countable language.*

Becker and Kechris [3, 6.1.4] outline a proof based on coding in terms of lattices, unlike the following argument, yet it may in fact involve the same idea.

Proof. Let $\mathrm{HF}(\mathbb{N})$ be the set of all hereditarily finite sets over the set $\mathbb{N}$ considered as the set of atoms, and ε be the associated "membership" (any $n \in \mathbb{N}$ has no ε-elements, $\{0,1\}$ is different from 2, *etc.*). Let $\simeq_{\mathrm{HF}(\mathbb{N})}$ be the $\mathrm{HF}(\mathbb{N})$ version of $\cong_{\mathscr{G}}$, *i.e.*, if $P,Q \subseteq \mathrm{HF}(\mathbb{N})^2$ then $P \simeq_{\mathrm{HF}(\mathbb{N})} Q$ means that there is a bijection b of $\mathrm{HF}(\mathbb{N})$ such that $Q = b\cdot P = \{\langle b(s), b(t)\rangle : \langle s,t\rangle \in P\}$. Obviously $(\cong_{\mathscr{G}}) \sim_{\mathrm{B}} (\simeq_{\mathrm{HF}(\mathbb{N})})$, thus, we have to prove that $\cong_{\mathscr{L}} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $\mathscr{L}$.

An action of S_∞ on $\mathrm{HF}(\mathbb{N})$ is defined as follows. If $g \in S_\infty$ then $g \circ n = g(n)$ for any $n \in \mathbb{N}$, and, by ε-induction, $g \circ \{a_1,\dots,a_n\} = \{g\circ a_1,\dots,g\circ a_n\}$ for all $a_1,\dots,a_n \in \mathrm{HF}(\mathbb{N})$. Clearly the map $a \mapsto g \circ a$ $(a \in \mathrm{HF}(\mathbb{N}))$ is an ε-isomorphism of $\mathrm{HF}(\mathbb{N})$, for any fixed $g \in S_\infty$.

Lemma 10.8. *Suppose that $X, Y \subseteq \mathrm{HF}(\mathbb{N})$ are ε-transitive subsets of $\mathrm{HF}(\mathbb{N})$, the sets $\mathbb{N} \smallsetminus X$ and $\mathbb{N} \smallsetminus Y$ are infinite, and $\varepsilon \restriction X \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$. Then there is $f \in S_\infty$ such that $Y = f \circ X = \{f \circ s : s \in X\}$.*

Proof. It follows from the assumption $\varepsilon \restriction X \cong_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$ that there is an ε-isomorphism $\pi : X \xrightarrow{\text{onto}} Y$. Easily $\pi \restriction (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto $Y_0 = Y \cap \mathbb{N}$, hence, there is $f \in S_\infty$ such that $f \restriction X_0 = \pi \restriction X_0$, and then we have $f \circ s = \pi(s)$ for any $s \in X$. □ (*Lemma*)

Coming back to the proof of Theorem 10.7, we first show that $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $m \ge 3$, where $\mathscr{G}(m)$ is the language with a single m-ary predicate. Note that $\langle i_1,\dots,i_m\rangle \in \mathrm{HF}(\mathbb{N})$ whenever $i_1,\dots,i_m \in \mathbb{N}$.

Put $\Theta(x) = \{\vartheta(s) : s \in x\}$ for every element $x \in \mathtt{Mod}_{\mathscr{G}(m)} = \mathscr{P}(\mathbb{N}^m)$, where $\vartheta(s) = \mathrm{TC}_\varepsilon(\{\langle 2i_1,\dots,2i_m\rangle\})$ for each $s = \langle i_1,\dots,i_m\rangle \in \mathbb{N}^m$, and finally, for $X \subseteq \mathrm{HF}(\mathbb{N})$, $\mathrm{TC}_\varepsilon(X)$ is the least ε-transitive set $T \subseteq \mathrm{HF}(\mathbb{N})$ with $X \subseteq T$. It easily follows from Lemma 10.7 that $x \cong_{\mathscr{G}(m)} y$ iff $\varepsilon \restriction \Theta(x) \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction \Theta(y)$. This ends the proof of $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathscr{L}'} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$, where $\mathscr{L}'$ is the language with infinitely many binary predicates. In this case $\mathtt{Mod}_{\mathscr{L}'} = \mathscr{P}(\mathbb{N}^2)^{\mathbb{N}}$, so that we can assume that every $x \in \mathtt{Mod}_{\mathscr{L}'}$ has the form $x = \{x_n\}_{n \ge 1}$, with $x_n \subseteq (\mathbb{N} \setminus \{0\})^2$ for all n. Let $\Theta(x) = \{s_n(k,l) : n \ge 1 \wedge \langle k,l \rangle \in x_n\}$ for any such x, where

$$s_n(k,l) = \mathrm{TC}_\varepsilon(\{\{\dots\{\langle k,l\rangle\}\dots\},0\})\,,\ \text{with } n+2 \text{ pairs of brackets } \{\,,\}\,.$$

Then Θ is a continuous reduction of $\cong_{\mathscr{L}'}$ to $\simeq_{\mathrm{HF}(\mathbb{N})}$. □ (*Theorem*)

10e Borel countably classified ERs: reduction to T_ξ

Equivalence relations T_ξ of Section 3b offer a perfect calibration tool for those Borel ERs which admit classification by countable structures. First of all,

Proposition 10.9. *Every equivalence relation T_ξ admits classification by countable structures.*

Proof. T_0, the equality on $\mathbb{N}$, is the orbit ER of the action of S_∞ by $g \cdot x = x$ for all g, x. The operation (o2) of Section 3b (countable disjoint union) easily preserves the property of being Borel reducible to an orbit ER of continuous action of S_∞.

Now consider operation (o5) of countable power. Suppose that a ER E on a Polish space $\mathbb{X}$ is Borel reducible to F, the orbit relation of a continuous action of S_∞ on some Polish $\mathbb{Y}$. Let D be the set of all points $x = \{x_k\}_{k\in\mathbb{N}} \in \mathbb{X}^{\mathbb{N}}$ such that either $x_k \not\mathsf{E}\, x_l$ whenever $k \ne l$, or there is m such that $x_k \mathsf{E}\, x_l$ iff m divides $|k-l|$. Then $\mathsf{E}^+ \le_{\mathrm{B}} (\mathsf{E}^+ \restriction D)$ (via a Borel map $\vartheta : \mathbb{X}^{\mathbb{N}} \to D$ such that $x \,\mathsf{E}^+\, \vartheta(x)$ for all x). On the other hand, obviously $(\mathsf{E}^+ \restriction D) \le_{\mathrm{B}} \mathsf{F}'$, where, for $y, y' \in \mathbb{Y}^{\mathbb{N}}$, $y \,\mathsf{F}'\, y'$ means that there is $f \in S_\infty$ such that $y_k \,\mathsf{F}\, y'_{f(k)}$ for all k. Finally, F' is the orbit ER of a continuous action of $S_\infty \times {S_\infty}^{\mathbb{N}}$, which can be realized as a closed subgroup of S_∞, so it remains to apply Theorem 10.5. □

The relations T_α are known in different versions, which reflect the same idea of coding sets of α-th cumulative level over $\mathbb{N}$, as, *e.g.*, in [22, § 1], where results similar to Proposition 10.9 are obtained in much more precise form.

Theorem 10.10. *If E is a Borel ER classifiable by countable structures then $\mathsf{E} \le_{\mathrm{B}} \mathsf{T}_\xi$ for some $\xi < \omega_1$.*

Proof. The proof (a version of the proof in [12]) is based on Scott's analysis. Define, by induction on $\alpha < \omega_1$, a family of Borel ERs $\equiv^\alpha$ on $\mathbb{N}^{<\omega} \times \mathscr{P}(\mathbb{N}^2)$:

$*$ $A \equiv^\alpha_{st} B$ means $\langle s, A\rangle \equiv^\alpha \langle t, B\rangle$;

thus, all $\equiv^\alpha_{st}$ ($s, t \in \mathbb{N}^{<\omega}$) are binary relations on $\mathscr{P}(\mathbb{N}^2)$, and among them all relations $\equiv^\alpha_{ss}$ are ERs. We define them by transfinite induction on α.

- $A \equiv^0_{st} B$ iff $A(s_i, s_j) \Longleftrightarrow B(t_i, t_j)$ for all $i, j < \mathtt{lh}\, s = \mathtt{lh}\, t$;

- $A \equiv_{st}^{\alpha+1} B$ iff $\forall\, k\, \exists\, l\, (A \equiv_{s^\wedge k\,,\,t^\wedge l}^{\alpha} B)$ and $\forall\, l\, \exists\, k\, (A \equiv_{s^\wedge k\,,\,t^\wedge l}^{\alpha} B)$;
- if $\lambda < \omega_1$ is limit then: $A \equiv_{st}^{\lambda} B$ iff $A \equiv_{st}^{\alpha} B$ for all $\alpha < \lambda$.

Easily $\equiv^\beta \subseteq \equiv^\alpha$ whenever $\alpha < \beta$.

Recall that, for $A, B \subseteq \mathbb{N}^2$, $A \cong_{\mathscr{G}} B$ means that there is $f \in S_\infty$ with $A(k,l) \Longleftrightarrow B(f(k), f(l))$ for all k, l. Then we have $\cong_{\mathscr{G}} \subseteq \bigcap_{\alpha<\omega_1} \equiv_{\Lambda\Lambda}^{\alpha}$ by induction on α (in fact $=$ rather than $\subseteq$, see below), where Λ is the empty sequence. Call a set $P \subseteq \mathscr{P}(\mathbb{N}^2) \times \mathscr{P}(\mathbb{N}^2)$ *unbounded* if $P \cap \equiv_{\Lambda\Lambda}^{\alpha} \neq \varnothing$ for all $\alpha < \omega_1$.

Lemma 10.11. *Any unbounded* $\mathbf{\Sigma}_1^1$ *set* P *contains a pair* $\langle A, B\rangle \in P$ *such that* $A \cong_{\mathscr{G}} B$.

It follows that $A \cong_{\mathscr{G}} B$ iff $A \equiv_{\Lambda\Lambda}^{\alpha} B$ for all $\alpha < \omega_1$ (take $P = \{\langle A, B\rangle\}$).

Proof. Since P is $\mathbf{\Sigma}_1^1$, there is a continuous map $F : \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{onto}} P$. For $u \in \mathbb{N}^{<\omega}$, let $P_u = \{F(a) : u \subset a \in \mathbb{N}^{\mathbb{N}}\}$. There is a number n_0 such that $P_{\langle n_0\rangle}$ is still unbounded. Let $k_0 = 0$. By a simple cofinality argument, there is l_0 such that $P_{\langle n_0\rangle}$ is still unbounded *over* $\langle k_0\rangle, \langle l_0\rangle$ in the sense that there is no ordinal $\alpha < \omega_1$ such that $P_{\langle i_0\rangle} \cap \equiv_{\langle k_0\rangle\langle l_0\rangle}^{\alpha} = \varnothing$. Following this idea, we can define infinite sequences of numbers n_m, k_m, l_m such that both $\{k_m\}_{m\in\mathbb{N}}$ and $\{l_m\}_{m\in\mathbb{N}}$ are permutations of $\mathbb{N}$ and, for any m, the set $P_{\langle n_0,\dots,n_m\rangle}$ is still unbounded over $\langle k_0, \dots, k_m\rangle, \langle l_0, \dots, l_m\rangle$ in the same sense. Note that $a = \{n_m\}_{m\in\mathbb{N}} \in \mathbb{N}$ and $F(a) = \langle A, B\rangle \in P$. (Both A, B are subsets of $\mathbb{N}^2$.)

Prove that the map $f(k_m) = l_m$ witnesses $A \cong_{\mathscr{G}} B$, i.e., $A(k_j, k_i)$ iff $B(l_j, l_i)$ for all j, i. Take $m > \mathtt{max}\{j, i\}$ big enough for the following: if $\langle A', B'\rangle \in P_{\langle i_0,\dots,i_m\rangle}$ then $A(k_j, k_i)$ iff $A'(k_j, k_i)$, and similarly $B(l_j, l_i)$ iff $B'(l_j, l_i)$. By the construction, there is a pair $\langle A', B'\rangle \in P_{\langle i_0,\dots,i_m\rangle}$ with $A' \equiv_{\langle k_0,\dots,k_m\rangle\langle l_0,\dots,l_m\rangle}^{0} B'$, in particular, $A'(k_j, k_i)$ iff $B'(l_j, l_i)$, as required. □ (*Lemma*)

Corollary 10.12 (See, e.g., Friedman [12]). *If* E *is a Borel ER and* $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$ *then* $\mathsf{E} \le_{\mathrm{B}} \equiv_{\Lambda\Lambda}^{\alpha}$ *for some* $\alpha < \omega_1$.

Proof. Let ϑ be a Borel reduction of E to $\cong_{\mathscr{G}}$. Then $\{\langle\vartheta(x), \vartheta(y)\rangle : x \not\mathsf{E}\, y\}$ is a $\mathbf{\Sigma}_1^1$ subset of $\mathscr{P}(\mathbb{N}^2) \times \mathscr{P}(\mathbb{N}^2)$ which does not intersect $\cong_{\mathscr{G}}$, hence, it is bounded by Lemma 10.11. Take an ordinal $\alpha < \omega_1$ which witnesses the boundedness. □

Now, if E is a Borel ER classifiable by countable structures then $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$ by Theorem 10.7, hence, it remains to establish the following:

Proposition 10.13. *Any ER* $\equiv^\alpha$ *is Borel reducible to some* T_ξ.

Proof. We have $\equiv^0 \le_{\mathrm{B}} \mathsf{T}_0$ since $\equiv^0$ has countably many equivalence classes, all of which are clopen sets. To carry out the step $\alpha \mapsto \alpha + 1$ note that the map $\langle s, A\rangle \mapsto \{\langle s^\wedge k, A\rangle\}_{k\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^\alpha)^\infty$. To carry out the limit step, let $\lambda = \{\alpha_n : n \in \mathbb{N}\}$ be a limit ordinal, and $\mathsf{R} = \bigvee_{n\in\mathbb{N}} \equiv^{\alpha_n}$, *i.e.*, R is a ER on $\mathbb{N} \times \mathbb{N}^{<\omega} \times \mathscr{P}(\mathbb{N}^2)$ defined so that $\langle m, s, A\rangle \,\mathsf{R}\, \langle n, t, B\rangle$ iff $m = n$ and $A \equiv^{\alpha_m}_{st} B$. However the map $\langle s, A\rangle \mapsto \{\langle m, s, A\rangle\}_{m\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\lambda}$ to R^{∞}. □ (*Proposition*)

□ (*Theorem 10.10*)

Chapter 11

Turbulent group actions

This Section accomplishes the proof of irreducibility of the equivalence relations E_2 and $\mathbf{c}_0$ to T_2 (see Subsection 4c). In fact it will be established that all relations in one family of equivalence relations are Borel irreducible to Borel relations in another family. The second family contains all Borel orbit equivalence relations which admit classification by countable structures (see Section 10), and in fact many more equivalences, see below. The first family consists of orbit equivalences induced by *turbulent* actions.

11a Local orbits and turbulence

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$. If $G \subseteq \mathbb{G}$ and $X \subseteq \mathbb{X}$ then let

$$\mathsf{R}_G^X = \{\langle x, y\rangle \in X^2 : \exists\, g \in G\,(x = g \cdot y)\}$$

and let $\sim_G^X$ denote the ER-hull of R_G^X, *i.e.*, the $\subseteq$-least equivalence relation on X such that $x \mathrel{\mathsf{R}_G^X} y \Longrightarrow x \sim_G^X y$. In particular $\sim_{\mathbb{G}}^{\mathbb{X}} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$, but generally we have $\sim_G^X \subsetneqq \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} \restriction X$. Finally, define $\mathscr{O}(x, X, G) = [x]_{\sim_G^X} = \{y \in X : x \sim_G^X y\}$ for $x \in X$ – the *local orbit* of x. In particular, $[x]_{\mathbb{G}} = [x]_{\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}} = \mathscr{O}(x, \mathbb{X}, \mathbb{G})$ is the full $\mathbb{G}$-orbit of a point $x \in \mathbb{X}$.

Definition 11.1 (This particular version taken from Kechris [36, § 8]**).** Suppose that $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting on $\mathbb{X}$ continuously.

(t1) A point $x \in \mathbb{X}$ is *turbulent* if for any non-empty open set $X \subseteq \mathbb{X}$ containing x and any nbhd $G \subseteq \mathbb{G}$ (not necessarily a subgroup) of $1_{\mathbb{G}}$, the local orbit $\mathscr{O}(x, X, G)$ is somewhere dense (that is, not a nowhere dense set) in $\mathbb{X}$.

(t2) An orbit $[x]_{\mathbb{G}}$ is *turbulent* if x is such (then all $y \in [x]_{\mathbb{G}}$ are turbulent since this notion is invariant w. r. t. homeomorphisms).

(t3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *generically* [1], or *gen. turbulent* and $\mathbb{X}$ is a *gen. turbulent* Polish $\mathbb{G}$-space, if the union of all dense (topologically), turbulent, and meager orbits $[x]_{\mathbb{G}}$ is comeager. □

Thus turbulence means that orbits, and even local orbits of the action considered behave rather chaotically in some exact sense. According to the following theorem, this property is incompatible with the classifiability by countable structures.

Theorem 11.2 (Hjorth [19])**.** *Suppose that* $\mathbb{G}$ *is a Polish group,* $\mathbb{X}$ *is a gen. turbulent Polish* $\mathbb{G}$*-space. Then* $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ *is* **not** *Baire measurable reducible* [2] *to a Polish action of* S_∞*, hence,* **not** *classifiable by countable structures.*

The proof given below is based on general ideas in [19, § 3.2], [36, § 12], [12]. Yet it is designed so that only quite common tools of descriptive set theory are involved. It will also be shown that "turbulent" equivalence relations are not reducible actually to a much bigger family of relations than orbit equivalences of Polish actions of S_∞.

11b Shift actions of summable ideals are turbulent

Quite a lot of examples of turbulent actions is known (see *e.g.* [19]). The following example will be used in the proof of some Borel irreducibility results in the end of this Section. Recall that any summable ideal $\mathscr{S}_{\{r_n\}} = \{x \subseteq \mathbb{N} : \sum_{n\in x} r_n < +\infty\}$ (where $r_n \ge 0$ for all n) generates the equivalence relation $\mathsf{S}_{\{r_n\}} = \mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ on $\mathscr{P}(\mathbb{N})$, defined so that $x \,\mathsf{S}_{\{r_n\}}\, y$ iff $x \,\Delta\, y \in \mathscr{S}_{\{r_n\}}$.

Theorem 11.3. *If* $r_n > 0$, $\{r_n\} \to 0$, *and* $\sum_n r_n = +\infty$ *then the* Δ*-action of* $\mathscr{S}_{\{r_n\}}$ *on* $\mathscr{P}(\mathbb{N})$ *is Polish and gen. turbulent.*

The condition $\{r_n\} \to 0$ here implies that $\mathscr{S}_{\{r_n\}}$ contains some infinite sets. The condition $\sum_n r_n = +\infty$ means that $\mathscr{S}_{\{r_n\}}$ does not contain co-infinite sets.

Proof. Show that $\langle \mathscr{S}_{\{r_n\}} ; \Delta\rangle$ is a Polish group with the distance $d_{\{r_n\}}(a,b) = \varphi_{\{r_n\}}(a \,\Delta\, b)$, where

$$\varphi_{\{r_n\}}(x) = \sum_{n\in x} r_n \text{ for } x \in \mathscr{P}(\mathbb{N}), \text{ hence } \mathscr{S}_{\{r_n\}} = \{x : \varphi_{\{r_n\}}(x) < +\infty\}.$$

To prove that the operation is continuous, let $x, y \in \mathscr{P}(\mathbb{N})$. Fix a real $\delta > 0$, and let $\varepsilon = \frac{\delta}{2}$. If x', y' belong to the ε-nbhds of x, y in $\mathscr{S}_{\{r_n\}}$ with the distance $d_{\{r_n\}}$, then $(x' \,\Delta\, y') \,\Delta\, (x \,\Delta\, y) \subseteq (x \,\Delta\, x') \cup (y \,\Delta\, y')$, therefore $d_{\{r_n\}}(x' \,\Delta\, y', x \,\Delta\, y) \le d_{\{r_n\}}(x,x') + d_{\{r_n\}}(y,y') = \delta$.

[1] In this research direction, "generically", or, in our abbreviation, "gen." (property) intends to mean that (property) holds on a comeager domain.

[2] Reducible via a Baire measurable function. This is weaker than the Borel reducibility.

Now prove that the Δ-action of $\mathscr{S}_{\{r_n\}}$ on $\mathscr{P}(\mathbb{N})$ is continuous in the sense of the $d_{\{r_n\}}$-topology of $\mathscr{S}_{\{r_n\}}$ and the ordinary Polish product topology on $\mathscr{P}(\mathbb{N})$. Suppose that $g \in \mathscr{S}_{\{r_n\}}$, $x \in \mathscr{P}(\mathbb{N})$, and fix a Polish nbhd $V = \{y \in \mathscr{P}(\mathbb{N}) : y \cap n = (g \cdot x) \cap n\}$ of $g \cdot x$ in $\mathscr{P}(\mathbb{N})$, where $n \in \mathbb{N}$. Consider the corresponding nbhd $U = \{x' \in \mathscr{P}(\mathbb{N}) : x' \cap n = x \cap n\}$ of x. Let $\varepsilon = \mathtt{min}\{r_k : k < n\}$. Then any element $g' \in \mathscr{S}_{\{r_n\}}$ of the ε-nbhd of g in the $d_{\{r_n\}}$-topology satisfies $g \,\Delta\, g' \subseteq [n, \infty)$, therefore $g' \,\Delta\, x' \in V$ for any $x' \in U$.

Finally prove the turbulence of the action.

Let $x \in \mathscr{P}(\mathbb{N})$. That $[x]_{\mathscr{S}_{\{r_n\}}} = \mathscr{S}_{\{r_n\}} \,\Delta\, x$ is dense and meager is an easy exercise. Thus it suffices to check that x is turbulent. Consider an open nbhd $X = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0, k) = u\}$ of x, where $k \in \mathbb{N}$ and $u = x \cap [0, k)$, and a $d_{\{r_n\}}$-nbhd $G = \{g \in \mathscr{S}_{\{r_n\}} : \varphi(g) < \varepsilon\}$ of $\varnothing$ (the neutral element), where $\varepsilon > 0$. Prove that the local orbit $\mathscr{O}(x, X, G)$ is somewhere dense in X.

Let $l \geq k$ be large enough for $r_n < \varepsilon$ to hold for all $n \geq l$. Prove that the orbit $\mathscr{O}(x, X, G)$ is dense in $Y = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0, l) = v\}$, where $v = x \cap [0, l)$. Consider an open set $Z = \{z \in Y : z \cap [l, j) = w\}$, where $j \geq l$, $w \subseteq [l, j)$. Let z be the only point of Z satisfying $z \cap [j, +\infty) = x \cap [j, +\infty)$. Thus $x \,\Delta\, z = \{l_1, \dots, l_m\} \subseteq [l, j)$. Note that every element of the form $g_i = \{l_i\}$ belongs to G by the choice of l since $l_i \geq l$. Moreover, $x_i = g_i \,\Delta\, g_{i-1} \,\Delta \dots \Delta\, g_1 \,\Delta\, x = \{l_1, \dots, l_i\} \,\Delta\, x$ belongs to X for each $i = 1, \dots, m$. On the other hand $x_m = z$. It follows that $z \in \mathscr{O}(x, X, G)$, as required. □

A suitable modification of this argument can be used to prove the turbulence of the Δ-action of some other ideals including the density ideal $\mathscr{Z}_0$, but as far as some irreducibility results are concerned, the turbulence of summable ideals will suffice!

11c Ergodicity

The non-reducibility in Theorem 11.2 will be established in a special stronger form. Let E, F be ERs on Polish spaces resp. $\mathbb{X}, \mathbb{Y}$. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is

- $(\mathsf{E} \to \mathsf{F})$-invariant if $x \,\mathsf{E}\, y \Longrightarrow \vartheta(x) \,\mathsf{F}\, \vartheta(y)$ for all $x, y \in \mathbb{X}$;[3]
- *gen.* $(\mathsf{E} \to \mathsf{F})$-invariant if the implication $x \,\mathsf{E}\, y \Longrightarrow \vartheta(x) \,\mathsf{F}\, \vartheta(y)$ holds for all x, y in a comeager subset of $\mathbb{X}$;
- *gen. reduction of* E *to* F if the equivalence $x \,\mathsf{E}\, y \Longleftrightarrow \vartheta(x) \,\mathsf{F}\, \vartheta(y)$ holds for all x, y in a comeager subset of $\mathbb{X}$;
- *gen.* F-constant if $\vartheta(x) \,\mathsf{F}\, \vartheta(y)$ for all x, y in a comeager subset of $\mathbb{X}$.

[3] Recall that 'gen. ' means 'generic' or 'generically'.

Finally, following Hjorth and Kechris, say that E is *gen.* F*-ergodic* if every Borel gen. $(\mathsf{E} \to \mathsf{F})$-invariant map is gen. F-constant.

The ergodicity preserves $\le_{\mathrm{B}}$ in the sense of the next lemma.

Lemma 11.4. *If* $\mathsf{E}, \mathsf{F}, \mathsf{F}'$ *are Borel equivalence relations,* E *is gen.* F*-ergodic, and* $\mathsf{F}' \le_{\mathrm{B}} \mathsf{F}$*, then* E *is gen.* F'*-ergodic as well.*

Proof. Let ϑ be a Borel reduction of F' to F. Given a Borel gen. $(\mathsf{E} \to \mathsf{F}')$-invariant map f, the map $f'(x) = \vartheta(f(x))$ is obviously gen. $(\mathsf{E} \to \mathsf{F})$-invariant, hence it is a gen. F-constant — then easily a gen. F'-constant, too. □

The following lemma shows that ergodicity implies irreducibility.

Lemma 11.5. *If an equivalence relation* E *is gen.* F*-ergodic and does not have co-meager equivalence classes then* E *does not admit a Borel gen. reduction to* F*. In addition* E *does not admit a Baire measurable reduction to* F*.*

Proof. Suppose towards the contrary that a Borel map $\vartheta : \mathbb{X} \to \mathbb{Y}$ (where $\mathbb{X}$, $\mathbb{Y}$ are the domains of resp. E, F) is a gen. reduction of E to F, that is, ϑ is a true reduction on a co-meager set $C \subseteq \mathbb{X}$. Then ϑ is a gen. F-constant by the ergodicity, that is, there exists a co-meager set $C' \subseteq \mathbb{X}$ such that $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$ for all $x, x' \in C'$. Thye set $D = C \cap C'$ is co-meager as well, hence there exist $x, x' \in D$ such that $x \mathrel{\not\mathsf{E}} x'$. Then $\vartheta(x) \mathrel{\not\mathsf{F}} \vartheta(x')$ holds since ϑ is a reduction on C. On the other hand, we know that $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$, contradiction.

The additional result follows because it is known that any Baire measurable map is continuous on a co-meager set. □

The proof of Theorem 11.2 consists of the next two lemmas. [4]

Lemma 11.6. *If* $\mathbb{G}$ *is a Polish group,* $\mathbb{X}$ *is a gen. turbulent Polish* $\mathbb{G}$*-space, and* $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ *is Baire measurable reducible to a Polish action of* S_∞ *then* $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ *admits a Borel gen. reduction to an equivalence relation of the form* T_ξ*.*

Saying it differently, any equivalence relation, Baire measurable reducible to a Polish action of S_∞, is Borel reducible to one of T_ξ on a co-meager set. Note that any equivalence relation, Borel reducible (in proper sense) to one of T_ξ, is Borel itself. Yet this cannot be applied to $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ in the lemma, since only a generic (on a co-meager set) reduction is claimed.

Lemma 11.7. *Every equivalence relation induced by a gen. turbulent Polish action of a Polish group is gen.* T_ξ*-ergodic for all* ξ*.*

[4] There are slightly different ways to the same goal. Hjorth [19, 3.18] proves outright and with different technique, that any gen. turbulent equivalence relation is gen. ergodic w.r.t. any Polish action of S_∞. Kechris [36, § 12] proves that 1) any gen. T_2-ergodic equivalence is gen. ergodic w.r.t. any Polish action of S_∞, and 2) any turbulent one is gen. T_2-ergodic.

Proof of Theorem 11.2 from lemmas 11.6 and 11.7. If $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Baire measurable reducible to a Polish action of S_∞ then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ also is Borel gen. reducible to one of T_ξ by Lemma 11.6. On the other hand, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is gen. T_ξ-ergodic by Lemma 11.7. Thus $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ has a co-meager equivalence class by Lemma 11.5. But this contradicts the assumption of gen. turbulence.

□ (*Theorem 11.2 from lemmas 11.6 and 11.7*)

The proof of the lemmas follows below in this Section.

11d "Generic" reduction to T_ξ

Here, we prove Lemma 11.6. Suppose that $\mathbb{G}$ is a Polish group, $\mathbb{X}$ a gen. turbulent Polish $\mathbb{G}$-space. In particular, the set W_0 of all points $x \in \mathbb{X}$ that belong to dense turbulent orbits $[x]_G$ is comeager in $\mathbb{X}$. It follows that there exists a dense $\mathbf{G}_\delta$ set $W \subseteq W_0$.

Assume further that the orbit equivalence relation $\mathsf{E} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Baire measurable reducible to a Polish action of S_∞. As the latter is Borel reducible to the isomorphism $\cong_{\mathscr{G}}$ of binary relations on $\mathbb{N}$ according to Theorems 10.4 and 10.7, E itself admits a Baire measurable reduction $\rho : \mathbb{X} \to \mathscr{P}(\mathbb{N}^2)$ to $\cong_{\mathscr{G}}$. The remainder of the argument borrows notation from the proof of Theorem 10.10.

There is a dense $\mathbf{G}_\delta$ set $D_0 \subseteq \mathbb{X}$ such that the restricted map $\vartheta = \rho \restriction D_0$ is continuous on D_0. By definition, we have

$$x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \cong_{\mathscr{G}} \vartheta(y) \quad \text{and} \quad x \mathrel{\not\mathsf{E}} y \Longrightarrow \vartheta(x) \not\cong_{\mathscr{G}} \vartheta(y)$$

for all $x, y \in D_0$. We are mostly interested in the second implication, and the aim is to find a dense $\mathbf{G}_\delta$ set $D \subseteq D_0$ such that, for some $\alpha < \omega_1$:

$(*)$ $x \mathrel{\not\mathsf{E}} y \Longrightarrow \vartheta(x) \not\equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$ holds for all $x, y \in D$.

Recall that $A \cong_{\mathscr{G}} B$ iff $\forall\, \alpha < \omega_1\ A \equiv_{\Lambda\Lambda}^{\alpha} B$. (See a remark after Lemma 10.11.) It follows that $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$ holds for all $x, y \in D_0$. Thus $(*)$ implies that $\mathsf{E} \restriction D$ is Borel reducible to $\equiv_{\Lambda\Lambda}^{\alpha}$. Now to end the proof of Lemma 11.6 apply Proposition 10.13.

To find an ordinal α satisfying $(*)$ we make use of **Cohen forcing**. Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$, *i.e.*, **ZFC** minus the Power Set axiom but plus the axiom: "every set belongs to $\mathrm{HC} = \{x : x \text{ is hereditarily countable}\}$".

We shall assume that $\mathbb{X}$ is coded in $\mathfrak{M}$ in the sense that there is a set $D_{\mathbb{X}} \in \mathfrak{M}$ which is a dense (countable outside of $\mathfrak{M}$) subset of $\mathbb{X}$, and $d_{\mathbb{X}} \restriction D_{\mathbb{X}}$ (the distance function of $\mathbb{X}$ restricted to $D_{\mathbb{X}}$) also belongs to $\mathfrak{M}$. Further, $\mathbb{G}$, the action, D_0, the map ϑ, and the $\mathbf{G}_\delta$ set W defined above — are also assumed to be coded in $\mathfrak{M}$ in a similar sense.

In these assumption, the notion of a point of $\mathbb{X}$ or of $\mathbb{G}$ *Cohen generic over* $\mathfrak{M}$ makes sense, and, as usual, the set D of all Cohen generic, over $\mathfrak{M}$, points of $\mathbb{X}$ is a dense $\mathbf{G}_\delta$ subset of $\mathbb{X}$ and $D \subseteq D_0$. We are going to prove that D fulfills $(*)$.

Suppose that $x, y \in D$, and $\langle x, y \rangle$ is a Cohen generic pair over $\mathfrak{M}$. If $x \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} y$ is false then $\vartheta(x) \not\cong_{\mathscr{G}} \vartheta(y)$. Moreover, this fact holds in the extended model $\mathfrak{M}[x, y]$ by the Mostowski absoluteness. This allows us to find, arguing in $\mathfrak{M}[x, y]$ (which is still a model of $\mathbf{ZFC}^-$), an ordinal $\alpha \in \mathtt{Ord}^{\mathfrak{M}} = \mathtt{Ord}^{\mathfrak{M}[x,y]}$ such that $\vartheta(x) \not\equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$. Moreover, since the Cohen forcing satisfies CCC, there is an ordinal $\alpha \in \mathfrak{M}$ such that $\vartheta(x) \not\equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$ holds for **every** Cohen generic, over $\mathfrak{M}$, pair $\langle x, y \rangle \in D^2$ such that $x \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} y$ is false. It remains to show that this also holds provided $x, y \in D$ (are generic separately, but) do not necessarily form a pair Cohen generic over $\mathfrak{M}$. Now we prove

Lemma 11.8. *If $\mathfrak{N}$ is a countable transitive model of $\mathbf{ZFC}^-$ with $\mathfrak{M} \subseteq \mathfrak{N}$, a point $x \in \mathbb{X} \cap \mathfrak{N}$ is Cohen generic over $\mathfrak{M}$, and an element $g \in \mathbb{G}$ is Cohen generic over $\mathfrak{N}$, then $x' = g \cdot x$ is Cohen generic over $\mathfrak{N}$.*

Proof. It follows from the genericity that x belongs to the set W introduced in the beginning of Subsection 11d. Thus the $\mathbb{G}$-orbit $\{g' \cdot x : g' \in G\}$ is turbulent, in particular dense in $\mathbb{X}$.

Now consider any dense open set $X \subseteq \mathbb{X}$ coded in $\mathfrak{N}$. The set $H = \{g' \in \mathbb{G} : g' \cdot x \in X\}$ is also open and coded in $\mathfrak{N}$, Moreover H is dense in $\mathbb{G}$. (Indeed otherwise there is an open non-empty set $G \subseteq \mathbb{G}$ such that the partial orbit $G \cdot x = \{g \cdot x : g \in G\}$ is nowhere dense. This leads to a contradiction with the turbulence of x.) We conclude that $g \in H$, and further $g \cdot x \in X$, as required. □

To make use of the lemma, let $\mathfrak{N}$ be a countable transitive model of $\mathbf{ZFC}^-$ containing x, y, and all sets in $\mathfrak{M}$. Note that $\mathfrak{N}$ may contain more ordinals than $\mathfrak{M}$ does since the pair $\langle x, y \rangle$ is not assumed to be generic over $\mathfrak{M}$.

Fix an element $g \in \mathbb{G}$ Cohen generic over $\mathfrak{N}$. Then $x' = g \cdot x$ is Cohen generic over $\mathfrak{N}$ by the lemma, hence over $\mathfrak{M}[y]$. Yet y is generic over $\mathfrak{M}$, thus the pair $\langle x', y \rangle$ is Cohen generic over $\mathfrak{M}$. This implies $\vartheta(x') \not\equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$ by the choice of α. On the other hand we have $x' \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} x$ and hence $\vartheta(x) \equiv_{\Lambda\Lambda}^{\alpha} \vartheta(x')$. Thus we finally obtain $\vartheta(x') \not\equiv_{\Lambda\Lambda}^{\alpha} \vartheta(y)$, as required.

□ (*Lemma 11.6*)

11e Ergodicity of turbulent actions w.r.t. T_ξ

Here, we prove Lemma 11.7.

We begin with two rather simple technical results of topological nature, involved in the proof of the lemma.

Lemma 11.10. *Suppose that $\mathbb{G}$ is a Polish group, and $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Let $\varnothing \neq X \subseteq \mathbb{X}$ be an open set, $G \subseteq \mathbb{G}$ be a nbhd of $1_{\mathbb{G}}$, and*

$\mathscr{O}(x,X,G)$ be dense in X for X-comeager many $x \in X$. Let $U, U' \subseteq X$ be non-empty open and $D \subseteq X$ be comeager in X. Then there exist points $x \in D \cap U$ and $x' \in D \cap U'$ with $x \sim_G^X x'$.

Proof. Under our assumptions there exist points $x_0 \in U$ and $x_0' \in U'$ with $x_0 \sim_G^X x_0'$, that is, there exist elements $g_1, \dots, g_n \in G \cup G^{-1}$ such that $x_0' = g_n g_{n-1} \dots g_1 \cdot x_0$ and in addition $g_k \dots g_1 \cdot x_0 \in X$ for all $k \le n$. Since the action is continuous, there is a nbhd $U_0 \subseteq U$ of x_0 such that $g_k \dots g_1 \cdot x \in X$ for all k and $g_n g_{n-1} \dots g_1 \cdot x \in U'$ for all $x \in U_0$. Since D is comeager, easily there is $x \in U_0 \cap D$ such that $x' = g_n g_{n-1} \dots g_1 \cdot x \in U' \cap D$. □ (*Lemma*)

Lemma 11.11. *Suppose that $\mathbb{G}$ is a Polish group, and $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Then for any open non-empty $U \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ with $1_{\mathbb{G}} \in G$ there is an open non-empty $U' \subseteq U$ such that the local orbit $\mathscr{O}(x,U',G)$ is dense in U' for U'-comeager many $x \in U'$.*

Proof. Let $\operatorname{Int}\overline{X}$ be the interior of the closure of X. If $x \in U$ and $\mathscr{O}(x,U,G)$ is somewhere dense (in U) then the set $U_x = U \cap \operatorname{Int}\overline{\mathscr{O}(x,U,G)} \subseteq U$ is open and $\sim_G^U$-invariant (an observation made, *e.g.*, in [36, proof of 8.4]), moreover, $\mathscr{O}(x,U,G) \subseteq U_x$, hence, $\mathscr{O}(x,U,G) = \mathscr{O}(x,U_x,G)$. It follows from the invariance that the sets U_x are pairwise disjoint, and it follows from the turbulence that the union of them is dense in U. Take any non-empty U_x as U'. □ (*Lemma*)

The proof of Lemma 11.7 involves a somewhat stronger property than gen. ergodicity in Section 11c. Suppose that F is an ER on a Polish space $\mathbb{X}$.

- An action of $\mathbb{G}$ on $\mathbb{X}$ and the induced equivalence relation $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ are *locally generically* (*loc. gen.,* for brevity) F-ergodic if the equivalence relation $\sim_G^X$ is generically F-ergodic whenever $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a non-empty open set containing $1_{\mathbb{G}}$, and the local orbit $\mathscr{O}(x,X,G)$ is dense in X for comeager (in X) many $x \in X$.

This obviously implies gen. F-ergodicity of $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ provided the action is gen. turbulent. Therefore, Lemma 11.7 is a corollary of the following theorem:

Theorem 11.12. *Let $\mathbb{X}$ be a gen. turbulent Polish $\mathbb{G}$-space. Suppose that an equivalence relation F belongs to $\mathscr{F}_0$, the least collection of equivalence relations containing $\mathsf{EQ}_{\mathbb{N}}$* (the equality on $\mathbb{N}$) *and closed under operations (o1) – (o5) in Section 3b. Then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is loc. gen. F-ergodic, in particular, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is not Borel reducible to F.*

Due to the operation of Fubini product, the family $\mathscr{F}_0$ contains a lot of equivalence relations very different from T_ξ, among them some Borel equivalences which do not admit classification by countable structures, *e.g.*, all equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$, where $\mathscr{I}$ is one of Fréchet ideals, indecomposable ideals, or Weiss ideals of Section 2e. (In fact it is not so easy to show that ideals of the two last families produce relations in $\mathscr{F}_0$.) In particular, it follows that *no gen. turbulent equivalence relation is Borel reducible to a Fréchet, or indecomposable, or Weiss ideal.*

Our proof of Theorem 11.12 goes on by induction on the number of applications of the basic operations, in several following subsections.

Right now, we begin with the initial step: prove that, under the assumptions of the theorem, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is loc. gen. $\mathsf{EQ}_{\mathbb{N}}$-ergodic. Suppose that $X \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ are non-empty open sets, $1_{\mathbb{G}} \in G$, and $\mathscr{O}(x, X, G)$ is dense in X for X-comeager many $x \in X$, and prove that $\sim_G^X$ is generically $\mathsf{EQ}_{\mathbb{N}}$-ergodic.

Consider a Borel gen. $(\sim_G^X \to \mathsf{EQ}_{\mathbb{N}})$-invariant map $\vartheta : X \to \mathbb{N}$. Suppose towards the contrary that ϑ is not gen. $\mathsf{EQ}_{\mathbb{N}}$-constant. Then there exist two open non-empty sets $U_1, U_2 \subseteq X$, two numbers $\ell_1 \neq \ell_2$, and a comeager set $D \subseteq X$ such that $\vartheta(x) = \ell_1$ for all $x \in D \cap U_1$, $\vartheta(x) = \ell_2$ for all $x \in D \cap U_2$, and $\vartheta \restriction D$ is "strictly" $(\sim_G^X \to \mathsf{EQ}_{\mathbb{N}})$-invariant. Lemma 11.10 yields a pair of points $x_1 \in U_1 \cap D$ and $x_2 \in U_2 \cap D$ with $x_1 \sim_G^X x_2$, contradiction.

11f Inductive step of countable power

To carry out this step in the proof of Theorem 11.12, suppose that

- $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, F is a Borel ER on a Polish space $\mathbb{Y}$, and the action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F-ergodic,

and prove that the action is loc. gen. F^∞-ergodic. Fix a nonempty open set $X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that $\mathscr{O}(x, X_0, G_0)$ is dense in X_0 for all x in a comeager $\mathbf{G}_\delta$-set $D_0 \subseteq X_0$. Consider a Borel function $\vartheta : X_0 \to \mathbb{Y}^{\mathbb{N}}$, continuous and $(\sim_{G_0}^{X_0} \to \mathsf{F}^\infty)$-invariant on D_0, so that

$$x \sim_{G_0}^{X_0} x' \implies \forall\, k\, \exists\, l\, \big(\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(x')\big) \quad : \qquad \text{for all } x, x' \in D_0\,,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, $\vartheta_k : X_0 \to \mathbb{Y}$, and prove that ϑ is gen. F^∞-constant.

Let us fix a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ (see above), which contains all relevant objects or their codes, in particular, codes of the topologies of $\mathbb{X}, \mathbb{G}, \mathbb{Y}$, of the set D_0, and of the Borel map ϑ. Then every point $x \in X_0$ Cohen generic over $\mathfrak{M}$ belongs to D_0, hence ϑ is $(\sim_{G_0}^{X_0} \to \mathsf{F}^\infty)$-invariant on Cohen generic points of X_0, and local orbits $\mathscr{O}(x, X_0, G_0)$ of Cohen generic points $x \in X_0$ are dense in X_0.

Coming back to the step of countable power, fix $k \in \mathbb{N}$. Consider any open non-empty set $U_0 \subseteq X_0$.

Lemma 11.13. *There exist a number l and open non-empty sets $U \subseteq U_0$ and $H \subseteq G_0$ such that both $g \cdot x \in X_0$ and $\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(g \cdot x)$ hold for any pair $\langle g, x \rangle$ in $H \times U$ Cohen generic over $\mathfrak{M}$.*

Proof. Consider any point $x_0 \in U_0$ Cohen generic over $\mathfrak{M}$. Note that $1_{\mathbb{G}} \cdot x_0 = x_0 \in X_0$, hence there exist a nbhd $U_1 \subseteq U_0$ of x_0 and a nbhd $G_1 \subseteq G_0$ of $1_{\mathbb{G}}$ such that $G_1 \cdot U_1 \subseteq U_0$, *i.e.* $g \cdot x \in X_0$ for all $g \in G_1$ and $x \in U_1$.

Consider any pair $\langle g, x\rangle \in G_1 \times U_1$ Cohen generic over $\mathfrak{M}$. Then $g \cdot x \in U_0$. In addition, x is Cohen generic over $\mathfrak{M}$ while g is Cohen generic over $\mathfrak{M}[x]$ by the forcing product theorem. It follows that $g \cdot x$ is Cohen generic over $\mathfrak{M}[x]$, and hence over $\mathfrak{M}$, by Lemma 11.8.

Furthermore, we have $x \sim_{G_0}^{X_0} g \cdot x$. By the invariance of ϑ on generic points this implies $\vartheta(x) \mathsf{F}^+ \vartheta(g \cdot x)$. It follows that there is an index l such that $\vartheta_k(x) \mathsf{F} \vartheta_l(g \cdot x)$. Thus there exist Cohen conditions, *i.e.* non-empty open sets $U \subseteq U_1$ and $H \subseteq G_1$ such that $x \in U$, $g \in H$, and any pair $\langle g', x'\rangle \in H \times U$ Cohen generic over $\mathfrak{M}$ satisfies $g' \cdot x' \in X_0$ and $\vartheta_k(x') \,\mathsf{F}\, \vartheta_l(g' \cdot x')$. □ (*Lemma*)

Fix l, U, H as provided by the lemma. Since H is open, there is $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_1$ of $1_{\mathbb{G}}$ such that $g_0\, G \subseteq H$.

Lemma 11.14 (The key point of the turbulence**).** *If $x, x' \in U$ are Cohen generic over $\mathfrak{M}$ and $x \sim_G^U x'$ then we have $\vartheta_k(x) \,\mathsf{F}\, \vartheta_k(x')$.*

Proof. We argue by induction on $n(x, x') =$ the least number n such that there exist $g_1, \dots, g_n \in G$ (recall: $G = G^{-1}$) satisfying

$$(*) \quad x' = g_n\, g_{n-1} \dots g_1 \cdot x, \quad \text{and} \quad g_k \dots g_1 \cdot x \in U' \quad \text{for all} \quad k \le n\,.$$

Suppose that $n(x, x') = 1$, thus, $x = g \cdot x'$ for some $g \in G$. Let $\mathfrak{N}$ be any countable transitive model of $\mathbf{ZFC}^-$ containing x, x', g, and all sets in $\mathfrak{M}$. Consider any element $h \in H$ Cohen generic over $\mathfrak{N}$ and close enough to h_0 for $h' = hg^{-1}$ to belong to H. (Note that $h_0 g^{-1} \in H$ by the choice of G.) Then h is generic over $\mathfrak{M}[x]$, too, and hence $\langle h, x\rangle \in H \times U$ is Cohen generic over $\mathfrak{M}$ by the product forcing theorem. It follows, by the choice of H, that $h \cdot x \in X_0$ and $\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(h \cdot x)$.

Moreover, $h' = hg^{-1}$ also is $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{N}$ (because $g \in \mathfrak{N}$), so that $\vartheta_k(x') \,\mathsf{F}\, \vartheta_l(h' \cdot x')$ by the same argument. Finally $g' \cdot x' = gh^{-1} \cdot (h \cdot x) = g \cdot x$, and hence $\vartheta_k(x') \,\mathsf{F}\, \vartheta_k(x)$, as required.

As for the inductive step, prove that $(*)$ holds for some $n \ge 2$ assuming that it holds for $n-1$. Consider an element $g'_1 \in G$ close enough to g_1 for $g'_2 = g_2\, g_1\, {g'_1}^{-1}$ to belong to G and for $x^* = g'_1 \cdot x$ to belong to U, and Cohen generic over a fixed transitive countable model $\mathfrak{N}$ of $\mathbf{ZFC}^-$ containing x, x', g_1, g_2. Then as above g'_2 is Cohen generic over $\mathfrak{N}$ while x^* is Cohen generic over $\mathfrak{M}$, and obviously $n(x^*, x') \le n-1$ because $g'_2 \cdot x^* = g_2 \cdot g_1 \cdot x$. It remains to use the induction hypothesis. □ (*Lemma*)

To summarize, we have shown that for any k and any open $\varnothing \ne U_0 \subseteq X_0$ there exist: an open non-empty set $U \subseteq U_0$, and an open $G \subseteq G_0$ with $1_{\mathbb{G}} \in G$, such that the map ϑ_k is $(\sim_G^U \to \mathsf{F})$-invariant on U. We can also assume that the orbit $\mathscr{O}(x,U,G)$ is dense in U for U-comeager many $x \in U$ by Lemma 11.11. Then, by the loc. gen. F-ergodicity of the action considered, ϑ_k is gen. F-constant on U, that is, there exist a comeager $\mathbf{G}_\delta$ set $D \subseteq U$ and a point $y \in \mathbb{Y}$ such that $\vartheta_k(x)\,\mathsf{F}\,y$ for all $x \in D$.

We conclude that there exist: an X_0-comeager set $D \subseteq X_0$, and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{Y}$ such that, for any k and for any $x \in D$ there is j with $\vartheta_k(x)\,\mathsf{F}\,y_j$. Put $\eta(x) = \bigcup_{k\in\mathbb{N}}\{j : \vartheta_k(x)\,\mathsf{F}\,y_j\}$. Then, for any pair $x, x' \in D$, we have $\vartheta(x)\,\mathsf{F}^\infty\,\vartheta(x')$ iff $\eta(x) = \eta(x')$, so that, by the invariance of ϑ,

$$x \sim^{U_0}_{G_0} x' \implies \eta(x) = \eta(x') \quad : \quad \text{for all } x, x' \in D. \tag{†}$$

It remains to show that η is a constant on a comeager subset of D.

Suppose, on the contrary, that there exist two non-empty open sets U_1, $U_2 \subseteq U_0$, a number $j \in \mathbb{N}$, and a comeager set $D' \subseteq D$ such that $j \in \eta(x_1)$ and $j \notin \eta(x_2)$ for all $x_1 \in D' \cap U_1$ and $x_2 \in D' \cap U_2$. Now Lemma 11.10 yields a contradiction to (†), as in the end of Subsection 11e.

□ (*Inductive step of countable power in Theorem 11.12*)

11g Inductive step of the Fubini product

To carry out this step in the proof of Theorem 11.12, suppose that

- $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, for any k, F_k be a Borel ER on a Polish space $\mathbb{Y}_k$, the action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F_k-ergodic for any k, and $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$ is, accordingly, a Borel ER on $\mathbb{Y} = \prod_k \mathbb{Y}_k$,

and prove that the action is loc. gen. F-ergodic.

Fix a nonempty open set $X_0 \subseteq \mathbb{X}$, a nbhd G_0 of $1_\mathbb{G}$ in $\mathbb{G}$, and a comeager $\mathbf{G}_\delta$ set $D_0 \subseteq X_0$ such that all local orbits $\mathscr{O}(x, X_0, G_0)$ with $x \in D_0$ are dense in X_0. Consider a Borel function $\vartheta : U_0 \to \mathbb{Y}$, $(\sim^{U_0}_{G_0} \to \mathsf{F})$-invariant on D_0, *i.e.*,

$$x \sim^{U_0}_{G_0} y \implies \exists\, k_0\; \forall\, k \ge k_0\; (\vartheta_k(x)\,\mathsf{F}_k\,\vartheta_k(y)) \quad : \quad \text{for all } x, y \in D_0\,,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, and prove that ϑ is gen. F-constant.

Choose a countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ as in 11f.

Consider an open non-empty set $U_0 \subseteq X_0$. Similarly to Lemma 11.13, there exist non-empty open sets $U \subseteq U_0$ and $H \subseteq G_0$, and a number k_0, such that both $g \cdot x \in X_0$ and $\vartheta_k(x)\,\mathsf{F}_k\,\vartheta_k(g \cdot x)$ hold for all indices $k \ge k_0$ and for all pairs $\langle g, x\rangle \in H \times U$ Cohen generic over $\mathfrak{M}$.

As H is open, there exist an element $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_\mathbb{G}$ such that $h_0\,G \subseteq H$.

Lemma 11.15. *If $k \ge k_0$, points $x, y \in U$ are Cohen generic over $\mathfrak{M}$, and $x \sim^U_G y$, then $\vartheta_k(x)\,\mathsf{F}_k\,\vartheta_k(y)$.* (Similarly to Lemma 11.14.) □

Thus, for any open non-empty set $U_0 \subseteq X_0$ there exist: a number k_0, an open non-empty $U \subseteq U_0$, and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$, such that $\vartheta_k(x)$ is gen. $(\sim_G^U \to \mathsf{F}_k)$-invariant on U for every $k \ge k_0$. We can assume that U-comeager many orbits $\mathscr{O}(x, U, G)$ are dense in U, by Lemma 11.11. Now, by F_k-ergodicity, any ϑ_k with $k \ge k_0$ is gen. F_k-constant on such a set U, hence, ϑ itself is gen. F-constant on U because $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$. It remains to show that these constants are F-equivalent to each other.

Suppose, on the contrary, that there exist two non-empty open sets U_1, $U_2 \subseteq U_0$ and a pair of $y_1 \not\mathsf{F}\, y_2$ in $\mathbb{Y}$ such that $\vartheta(x_1) \mathrel{\mathsf{F}} y_1$ and $\vartheta(x_2) \mathrel{\mathsf{F}} y_2$ for comeager many $x_1 \in U_1$ and $x_2 \in U_2$. Contradiction follows as in the end of Subsection 11f.

□ (*Inductive step of Fubini product in Theorem 11.12*)

11h Other inductive steps

Here, we accomplish the proof of Theorem 11.12, by carrying out induction steps, related to operations (o1), (o2), (o3) of Subsection 3b.

Countable union. Suppose that $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3, \dots$ are Borel equivalence relations on a Polish space $\mathbb{Y}$, and $\mathsf{F} = \bigcup_k \mathsf{F}_k$ is still an equivalence relation, and the Polish and gen. turbulent action of $\mathbb{G}$ on $\mathbb{X}$ is loc. gen. F_k-ergodic for any k.. Prove that the action remains loc. gen. F-ergodic.

Fix a nonempty open set $X_0 \subseteq \mathbb{X}$, a co-meager $\mathbf{G}_\delta$ set $D_0 \subseteq X_0$, and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$ such that all local orbits $\mathscr{O}(x, X_0, G_0)$ with $x \in D_0$ are dense in U_0. Consider a Borel function $\vartheta : U_0 \to \mathbb{Y}$, $(\sim_{G_0}^{U_0} \to \mathsf{F})$-invariant on D_0. It follows from the invariance that for any open non-empty $U \subseteq U_0$ there exist: a number k and open non-empty sets $U \subseteq U_0$ and $H \subseteq G_0$ such that both $g \cdot x \in X_0$ and $\vartheta(x) \mathrel{\mathsf{F}_k} \vartheta(g \cdot x)$ hold for any pair $\langle x, g\rangle \in U \times H$ Cohen generic over a fixed countable transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ chosen as above. Further, there exist $h_0 \in H \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $h_0 G \subseteq H$.

Similarly to Lemmas 11.14 and 11.15, $\vartheta(x) \mathrel{\mathsf{F}_k} \vartheta(x')$ holds for any pair of elements $x, x' \in U$ Cohen generic over $\mathfrak{M}$ and satisfying $x \sim_G^U x'$. It follows, by the ergodicity, that ϑ is F_k-constant, hence, F-constant, on a comeager subset of U. It remains to show that these F-constants are F-equivalent to each other, which is demonstrated exactly as in the end of Section 11f.

Disjoint union. Let F_k be Borel ERs on Polish spaces $\mathbb{Y}_k$, $k = 0, 1, 2, \dots$. By definition, $\bigvee_k \mathsf{F}_k = \bigcup_k \mathsf{F}'_k$, where each F'_k is a Borel equivalence relation defined on the space $\mathbb{Y} = \bigcup_k \{k\} \times \mathbb{Y}_k$ as follows: $\langle l, y\rangle \mathrel{\mathsf{F}'_k} \langle l', y'\rangle$ iff either $l = l'$ and $y = y'$ or $l = l' = k$ and $y \mathrel{\mathsf{F}_k} y'$.

Countable product. Let F_k be equivalence relations on a Polish spaces $\mathbb{Y}_k$. Then $\mathsf{F} = \prod_k \mathsf{F}_k$ is an equivalence relation on the space $\mathbb{Y} = \prod_k \mathbb{Y}_k$. For any map $\vartheta : \mathbb{X} \to \mathbb{Y}$, to be gen. $(\mathsf{E} \to \mathsf{F})$-invariant (where E is any equivalence on $\mathbb{X}$) it is necessary and sufficient that every co-ordinate map $\vartheta_k(x) = \vartheta(x)(k)$ is gen. $(\mathsf{E} \to \mathsf{F}_k)$-invariant. This allows to easily accomplish this induction step.

□ (*Theorem 11.12, Lemma 11.7, Theorem 11.2*)

11i Applications to the shift action of ideals

We are going to apply the results of this Section in order to prove that equivalence relations generated by many Borel ideals (in particular almost all polishable ideals) are not Borel reducible to Borel actions of the permutation group S_∞, and hence not classifiable by countable structures. The difficult problem of verification of the turbulence can fortunately be circumpassed by reference to theorems 11.12 and 11.3 (the turbulence of summable ideals).

Say that a Borel ideal $\mathscr{Z} \subseteq \mathscr{P}(\mathbb{N})$ is *special* if there is a sequence of reals $r_n > 0$ with $\{r_n\} \to 0$, such that $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. *Nontrivial* in the next theorem means: containing no cofinite sets. In the context of summable ideals the nontriviality means simply that $\sum_n r_n = +\infty$.

Theorem 11.16. *Suppose that $\mathscr{Z}$ is a nontrivial Borel special ideal, and F belongs to the family $\mathscr{F}_0$ of Theorem 11.12. Then $\mathsf{E}_{\mathscr{Z}}$ is generically F-ergodic, hence, is not Borel reducible to F.*

Proof. The "hence" statement follows because by the nontriviality all $\mathsf{E}_{\mathscr{Z}}$-equivalence classes are meager subsets of $\mathscr{P}(\mathbb{N})$.

As $\mathscr{Z}$ is special, let $\{r_k\} \to 0$ be a sequence of positive reals such that $\sum_n r_n = +\infty$ and $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. Note that $x \mathrel{\mathsf{S}_{\{r_n\}}} y$ implies $x \mathrel{\mathsf{E}_{\mathscr{Z}}} y$, and hence any gen. $(\mathsf{E}_{\mathscr{Z}} \to \mathsf{F})$-invariant map is gen. $(\mathsf{S}_{\{r_n\}} \to \mathsf{F})$-invariant as well (on the same co-meager set). Thus it suffices to prove that $\mathsf{S}_{\{r_n\}} = \mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ is gen. F-ergodic.

Recall that the shift action of $\mathscr{S}_{\{r_n\}}$ on $\mathscr{P}(\mathbb{N})$ is Polish and gen. turbulent by Theorem 11.3. Thus $\mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ is gen. F-ergodic by Theorem 11.12, as required. □

The next corollary returns us to the discussion in the end of Subsection 4c.

Corollary 11.17. *The equivalence relations $\mathbf{c}_0$ and E_2 are not Borel reducible to any ideal F in the family $\mathscr{F}_0$, in particular, are not Borel reducible to T_2.*

Proof. According to lemmas 5.6 and 5.7, it suffices to prove that the ideals $\mathscr{Z}_0$ (density 0) and $\mathscr{S}_{\{1/n\}}$ are special. (Their nontriviality is obvious.) The ideal $\mathscr{S}_{\{1/n\}}$ is special by definition. As for $\mathscr{Z}_0$, it suffices to prove that $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$. Consider a set $x \subseteq \mathbb{N}$, $x \notin \mathscr{Z}_0$. There is a real $\varepsilon > 0$ such that $\frac{\#(x \cap [0,n))}{n} > 2\varepsilon$ for infinitely many numbers n. One easily defines an increasing sequence $n_0 < n_1 < n_2 < \dots$ such that $n_{i+1} \geq 2n_i$ and $\frac{\#(x \cap [n_i, n_{i+1}))}{n_{i+1} - n_i} > \varepsilon$ for all i. Then $\sum_{n \in x} \frac{1}{n} \geq \varepsilon \sum_i \frac{n_{i+1} - n_i}{n_{i+1}} = +\infty$, hence $x \notin \mathscr{S}_{\{1/n\}}$. □

The next theorem shows that, with three exceptions, there exist no polishable ideals Borel reducible to equivalence relations in $\mathscr{F}_0$. (Note that $\mathscr{F}_0$ contains various equivalence relations of the form $\mathsf{E}_{\mathscr{I}}$, generated by non-polishable ideals $\mathscr{I}$, for instance, by Fréchet ideals.) Kechris [35] proved a similar theorem, with the assumption of reducibility to a relation in $\mathscr{F}_0$ the reducibility to a Borel action of S_∞ is considered. Recall that $\mathscr{I} \cong \mathscr{J}$ means isomorphism via bijection between the ground sets of the ideals.

Theorem 11.18. *If $\mathscr{I}$ is a nontrivial Borel polishable ideal on $\mathbb{N}$, F an equivalence relation in $\mathscr{F}_0$, and $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{F}$, then $\mathscr{I}$ is isomorphic to one of the following three ideals: $\mathscr{I}_3$, $\mathtt{Fin}$, $\mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$.*

Note that in each of the three cases $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_3$ holds.

Proof. It follows from Theorem 8.5 that $\mathscr{I} = \mathtt{Exh}_\varphi$ for a LSC submeasure φ on $\mathbb{N}$. We can assume that $\varphi(x) \le 1$ for all $x \in \mathscr{P}(\mathbb{N})$. (Otherwise put $\varphi'(x) = \mathtt{min}\{1, \varphi(x)\}$.) Consider the sets $U_n = \{k : \varphi(\{k\}) \le \frac{1}{n}\}$ and $U_\infty = \{k : \varphi(\{k\}) = 0\}$. Clearly $U_{n+1} \subseteq U_n$ and $\varphi(U_\infty) = 0$, therefore $\varphi(x) = \varphi(x \smallsetminus U_\infty)$ for all $x \in \mathscr{P}(\mathbb{N})$.

We claim that $\mathtt{lim}_{n\to\infty} \varphi(U_n) = 0$.

Suppose towards the contrary that there exists $\varepsilon > 0$ such that $\varphi(U_n) > \varepsilon$ for all n. By definition for every m there is $n \ge m$ satisfying $U_n \subseteq [m, \infty) \cup U_\infty$ — then $\varphi(U_n \smallsetminus m) > \varepsilon$ as well. Moreover there exists $n' \ge n$ satisfying $\varphi(U_n \cap [m, n')) > \varepsilon$. This leads to a sequence $n_1 < n_2 < n_3 < \dots$ of numbers and a sequence of finite sets $w_j \subseteq U_{n_j} \smallsetminus U_{n_{j+1}}$ such that $\varphi(w_j) > \varepsilon$. The sets w_j are pairwise disjoint, hence every "tail" $W \cap [n, \infty)$ of their union $W = \bigcup_j w_j$ includes at least one of w_j as a subset. It follows that $W \notin \mathscr{I} = \mathtt{Exh}_\varphi$. The ideal $\mathscr{J} = \mathscr{I} \cap \mathscr{P}(W)$ on W is then nontrivial. We also have $\{\varphi(\{k\})\}_{k\in W} \to 0$ and $\sum_k \varphi(\{k\}) = +\infty$ since for any n all but finite sets w_l satisfy $w_l \subseteq W$. Finally the equivalence $x \,\Delta\, y \in \mathscr{I} \iff x \,\Delta\, y \in \mathscr{J}$ holds for all $x, y \subseteq W$. It follows that $\mathsf{E}_{\mathscr{J}} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ by means of the identity map.

Since φ is a LSC submeasure, we have $\varphi(y) \le \sum_{k\in y} \varphi(\{k\})$ for all $y \subseteq \mathbb{N}$. It follows that every set $x \subseteq W$ satisfying $\sum_{k\in x} \varphi(\{k\}) < +\infty$ belongs to $\mathscr{I}$, hence to $\mathscr{J}$ as well. Thus $\mathscr{J}$ is isomorphic to a special ideal via a bijection of W onto $\mathbb{N}$. We conclude that $\mathsf{E}_{\mathscr{J}}$, and hence $\mathsf{E}_{\mathscr{I}}$, are Borel irreducible to relations in the family $\mathscr{F}_0$ by Theorem 11.16, contradiction.

Thus $\varphi(U_n) \to 0$. It follows that for any set $x \in \mathscr{P}(\mathbb{N})$ to belong to $\mathscr{I}$ it is necessary and sufficient that $x \cap (U_n \smallsetminus U_{n+1})$ is finite for every m. This observation allows us to accomplish the proof: if the difference $U_n \smallsetminus U_{n+1}$ is infinite for infinitely many indices n then $\mathscr{I} \cong \mathscr{I}_3$; if there exist only finitely many infinite differences $U_n \smallsetminus U_{n+1}$ and their union is co-finite in $\mathbb{N}$ then $\mathscr{I} \cong \mathtt{Fin}$; and finally $\mathscr{I} \cong \mathtt{Fin} \oplus \mathscr{P}(\mathbb{N})$ iff there exist only finitely many (but > 0) infinite differences $U_n \smallsetminus U_{n+1}$ but their union is co-infinite in $\mathbb{N}$. □

Corollary 11.19. *There is no Borel ideal* $\mathscr{I}$ *such that* $\mathsf{E}_{\mathscr{I}} \sim_{\mathrm{B}} \mathsf{T}_2$.

Proof. Suppose towards the contrary that $\mathscr{I}$ is such an ideal. Then $\mathscr{I}$ is polishable. (Indeed otherwise $\mathsf{E}_1 \le_{\mathrm{B}} \mathscr{I}$ by Theorem 8.5, and hence $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{T}_2$. But this contradicts Theorem 9.9 since T_2 is easily Borel reducible to a Polish action.) Thus $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_3$ by Theorem 11.18. On the other hand, recall that recall that the ideal $\mathscr{I}_3 = 0 \times \mathsf{Fin}$ is a P-ideal (Example 2.3), hence it is polishable by Theorem 8.5. Thus $\mathsf{T}_2 \not\le_{\mathrm{B}} \mathsf{E}_3$ by Theorem 15.3, which is applicable in this case because the Δ-group of $\mathscr{I}_3$ (basically, of any ideal) is abelian. Therefore $\mathsf{T}_2 \not\le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$, as required. □

The next application of Theorem 11.16 is related to the structure of ideals Borel reducible to E_3. The result is similar to Theorem 8.1. We begin with the following irreducibility lemma:

Lemma 11.20. $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$. *Equivalence relations* E_3 *and* E_1 *are* $\le_{\mathrm{B}}$*-incomparable. Equivalence relations* E_2 *and* E_1 *are* $\le_{\mathrm{B}}$*-incomparable as well.*

Proof. It is quite obvious that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_3$ and $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_1$. Thus $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_3$ strictly since we have $\mathsf{E}_3 \not\le_{\mathrm{B}} \mathsf{E}_1$ by Corollary 8.4. To prove $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_3$ recall that the ideal $\mathscr{I}_3$ is polishable (see above). Now $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_3$ follows from Theorem 8.5.

The proof of the second claim is similar. □

The following result of Kechris [35] should be compared with Theorem 8.1.

Corollary 11.21. *If* $\mathscr{I}$ *is a nontrivial Borel ideal on* $\mathbb{N}$ *and* $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_3$ *then* $\mathscr{I}$ *is isomorphic to one of the following three ideals*: $\mathscr{I}_3$, Fin, $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$.

Proof. We have $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ by Lemma 11.20. Therefore $\mathscr{I}$ is a polishable ideal by Theorem 8.5. It remains to apply Theorem 11.18. □

Chapter 12

Ideal $\mathscr{I}_3$ and the equivalence relation E_3

This Chapter is devoted to the ideal $\mathscr{I}_3$ and the corresponding equivalence relation E_3. Recall that $\mathscr{I}_3$ (also denoted by or $0 \times \mathtt{Fin}$) consists of all sets $x \subseteq \mathscr{P}(\mathbb{N} \times \mathbb{N})$ such that all cross-sections $(x)_n = \{k : \langle n, k\rangle \in x\}$ are finite. Accordingly the relation $\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$ is defined on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ by $x \mathrel{\mathsf{E}_3} y$ iff $x \,\Delta\, y \in \mathscr{I}_3$. But we'll rather consider E_3 as an equivalence on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined so that $x \mathrel{\mathsf{E}_3} y$ iff $x(n) \mathrel{\mathsf{E}_0} y(n)$ for all n : here x, y belong to $(2^{\mathbb{N}})^{\mathbb{N}}$. More detaily, $x \mathrel{\mathsf{E}_3} y$ holds iff

$$\forall\, n\, \exists\, k_0\, \forall\, k \geq k_0\ (x(n,k) = y(n,k)).$$

The main goal of this Section will be the proof of the following theorem of Hjorth and Kechris [20, 21] known as the 6th dichotomy theorem.

Theorem 12.1. *If* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_3$ *is a Borel equivalence relation then* **either** $\mathsf{E} \leq_{\mathrm{B}} \mathsf{E}_0$ **or** $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

Thus similarly to E_1, the ER E_3 is an immediate successor of E_0 in a rather strong sense. Let us mention an immediate corollary.

Corollary 12.2. $\mathsf{E}_\infty \not\leq_{\mathrm{B}} \mathsf{E}_3$.

Proof. If $\mathsf{E}_\infty \leq_{\mathrm{B}} \mathsf{E}_3$ then by Theorem 12.1 either $\mathsf{E}_\infty \leq_{\mathrm{B}} \mathsf{E}_0$ or $\mathsf{E}_\infty \sim_{\mathrm{B}} \mathsf{E}_3$. The **either** case contradicts Theorem 6.6. To derive a contradiction from the **or** case recall that $\mathsf{E}_\infty \leq_{\mathrm{B}} \boldsymbol{\ell}^\infty$ by Theorem 5.13 but on the other hand $\mathsf{E}_3 \not\leq_{\mathrm{B}} \boldsymbol{\ell}^\infty$ by Lemma 5.1. □

The proof of Theorem 12.1 employs the Gandy – Harrington topology in a manner rather similar to the proof of Theorem 9.4 (3rd dichotomy). The scheme of the proof given here is designed on the base of the proofs of Theorems 7.2 and 7.3 in [21]. The first of them contains a different dichotomy while the second theorem contains a result thal allows to derive Theorem 12.1 from the first theorem. To present Theorem 7.2 in [21] recall that E_∞ is a $\le_{\mathrm{B}}$-largest countable Borel equivalence relation, realized in the form of a certain equivalence on the Polish space 2^{F_2}, where F_2 is the free group with two generators. (Here it is essential only that F_2 is a countable set.) Let $(\mathsf{E}_\infty)^{\aleph_0}$ denote the equivalence relation on $(2^{F_2})^{\mathbb{N}}$, defined so that $x \, (\mathsf{E}_\infty)^{\aleph_0} \, y$ iff $x(n) \mathrel{\mathsf{E}_\infty} y(n)$ for all n. Thus $(\mathsf{E}_\infty)^{\aleph_0}$ is related to E_∞ just as E_3 to E_0. Theorem 7.2 in [21] asserts that any Borel equivalence relation E such that $\mathsf{E} \le_{\mathrm{B}} (\mathsf{E}_\infty)^{\aleph_0}$ satisfies either $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_\infty$ or $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}$.

12a Continual assembling of equivalence relations

The next theorem will be used in the proof of Theorem 12.1. The result is somewhat similar to Theorem 6.9 in that it evaluates the type of an equivalence relation E on the base of the types of certain fragments of E. But in this case the number of fragments can be continual.

Theorem 12.3. *Suppose that $\mathbb{X}, \mathbb{Y}$ are Polish spaces, $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a Borel set, E is a Borel equivalence relation on P, and $\mathbb{G}$ is a countable group acting on $\mathbb{X}$ in a Borel way, and $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ implies $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$.*

Finally, assume that $\mathsf{E} \restriction P(x)$ is smooth for each $x \in \mathbb{X}$, where $P(x) = \{\langle x', y\rangle \in P : x' = x\}$. Then E is Borel-reducible to a Borel action of $\mathbb{G}$.

Proof. We can assume that $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$ and both P and E are $\varDelta^1_1$. We can also assume that the action of $\mathbb{G}$ (a countable group) is $\varDelta^1_1$. Then clearly $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x' \Longrightarrow \varDelta^1_1(x) = \varDelta^1_1(x')$. Define $P^*(x) = \bigcup_{a\in\mathbb{G}} P(a\cdot x)$ for $x \in \mathbb{X}$.

Claim 1. *Suppose that pairs $\langle x, y\rangle$ and $\langle x', y'\rangle$ belong to P and $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$. Then $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ iff the equivalence $\langle x, y\rangle \in U \Longleftrightarrow \langle x', y'\rangle \in U$ holds for any $\mathsf{E} \restriction P^*(x)$--invariant $\varDelta^1_1(x)$ set $U \subseteq P^*(x)$.*

Proof. Note that $\mathsf{E} \restriction P^*(x)$ is still smooth by Corollary 6.10 because $\mathbb{G}$ is countable. In addition $\mathsf{E} \restriction P^*(x)$ is $\varDelta^1_1(x)$. This observation yields the result. Indeed otherwise the equivalence relation, defined om $P^*(x)$ by intersections with $\mathsf{E} \restriction P^*(x)$-invariant $\varDelta^1_1(x)$ sets, is coarser than $\mathsf{E} \restriction P^*(x)$. It follows (see the proof of the 2nd dichotomy theorem, Theorem 7.2) that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E} \restriction P^*(x)$, a contradiction with the smoothness. $\dashv$ (*Claim*)

In the continuation of the proof of Theorem 12.3 we make use of a standard enumeration of $\varDelta^1_1$ sets. It follows from Theorem 1.9 that there exist $\varPi^1_1$ sets $C \subseteq \mathbb{X} \times \mathbb{N}$ and $W \subseteq \mathbb{X} \times \mathbb{N} \times \mathbb{X} \times \mathbb{Y}$ and a $\varSigma^1_1$ set $W' \subseteq \mathbb{X} \times \mathbb{N} \times \mathbb{X} \times \mathbb{Y}$ such that the sets

$$W_{xe} = \{\langle x', y'\rangle : \langle x, e, x', y'\rangle \in W\} \quad \text{and} \quad W'_{xe} = \{\langle x', y'\rangle : \langle x, e, x', y'\rangle \in W'\}$$

coincide whenever $\langle x,e\rangle \in C$, and for any $x \in \mathbb{X}$ a set $R \subseteq \mathbb{X}\times\mathbb{Y}$ is $\Delta^1_1(x)$ iff there is $e \in C_x = \{e : \langle x,e\rangle \in C\}$ such that $\langle x,e\rangle \in C$ and $X = W_{xe} = W'_{xe}$.

Let $\mathtt{inv}(x,e)$ be the formula

$$x \in \mathbb{X} \wedge e \in C_x \wedge W_{xe} \subseteq P^*(x) \wedge W_{xe} \text{ is } \mathsf{E}\restriction P^*(x)\text{-invariant}\,.$$

Corollary 2. *Let* $\langle x,y\rangle, \langle x',y'\rangle$ *be as in Claim 1. Then* $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ *iff the equivalence* $\langle x,y\rangle \in W_{xe} \Longleftrightarrow \langle x',y'\rangle \in W_{xe}$ *holds for any* e *with* $\mathtt{inv}(x,e)$. ⊢

Let us change "iff" here to $\Longleftarrow$. Such a reduced claim can be formally represented in the form $(P\times P)\cap \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \subseteq U \cap \mathsf{E}$, where $U = \bigcup_{e\in\mathbb{N}} U_e$ and

$$U_e = \{\langle\langle x,y\rangle,\langle x',y'\rangle\rangle : \langle x,e\rangle \in J \wedge \neg\,(\langle x,y\rangle \in W_{xe} \Longleftrightarrow \langle x',y'\rangle \in W_{xe})\}.$$

As $J \subseteq C$, we can re-write the negation of $\Longleftrightarrow$ in the last formula as follows:

$$\big(\langle x,y\rangle \in W_{xe} \wedge \langle x',y'\rangle \notin W'_{xe}\big) \wedge \big(\langle x,y\rangle \notin W'_{xe} \wedge \langle x',y'\rangle \in W_{xe}\big).$$

Thus the inclusion $(P\times P)\cap \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \subseteq U\cap\mathsf{E}$ as a property of a Π^1_1 set J is Π^1_1 in the codes. It follows by Theorem 1.12 (Reflection) that there is a Δ^1_1 set $J' \subseteq J$ such that $(P\times P)\cap \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \subseteq U'\cap\mathsf{E}$ holds, where U' is defined in terms of J' similarly to the definition of U in terms of J.

Corollary 3. *Let* $\langle x,y\rangle, \langle x',y'\rangle$ *be as in Claim 1. Then* $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ *iff the equivalence* $\langle x,y\rangle \in W_{xe} \Longleftrightarrow \langle x',y'\rangle \in W_{xe}$ *holds for any* e *with* $\langle x,e\rangle \in J'$. ⊢

To continue the proof of the theorem, define, for any $\langle x,y\rangle \in P$,

$$D_{xy} = \{\langle a,e\rangle : a \in \mathbb{G} \wedge \langle a{\cdot}x,e\rangle \in J' \wedge \langle x,y\rangle \in W_{a\cdot x,e}\}\,.$$

Clearly $\langle x,y\rangle \mapsto D_{x,y}$ is a Δ^1_1 map $P \to \mathscr{P}(\mathbb{G}\times\mathbb{N})$.

If $D \subseteq \mathbb{G}\times\mathbb{N}$ and $b \in \mathbb{G}$ then put $b\circ D = \{\langle ab^{-1},e\rangle : \langle a,e\rangle \in D\}$.

Claim 4. *Suppose that* $\langle x,y\rangle$ *and* $\langle x',y'\rangle$ *belong to* P, $b\in\mathbb{G}$, *and* $x' = b{\cdot}x$. *Then* $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ *iff* $b\circ D_{xy} = D_{x'y'}$.

Proof. Assume that $b\circ D_{xy} = D_{x'y'}$. According to Corollary 3, to establish $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ it suffices to prove that $\langle x,y\rangle \in W_{xe} \Longleftrightarrow \langle x',y'\rangle \in W_{xe}$ holds whenever $\langle x,e\rangle \in J'$. We have

$$\langle x,y\rangle \in W_{xe} \Leftrightarrow \langle \Lambda,e\rangle \in D_{xy} \Leftrightarrow \langle b^{-1},e\rangle \in D_{x'y'} \Leftrightarrow \langle x',y'\rangle \in W_{b^{-1}\cdot x',e} = W_{xe}\,,$$

as required. Conversely, let $\langle x,y\rangle\mathsf{E}\langle x',y'\rangle$. If $\langle a,e\rangle \in D_{xy}$ then $\langle a{\cdot}x,e\rangle \in J'$ and $\langle x,y\rangle \in W_{a\cdot x,e}$, hence, $\langle x',y'\rangle \in W_{a\cdot x,e}$, too, because the set $W_{a\cdot x,e}$ is invariant and $\langle x,y\rangle\mathsf{E}\langle x',y'\rangle$. Yet $a{\cdot}x = ab^{-1}{\cdot}x'$, therefore, by definition, $\langle ab^{-1},e\rangle \in D_{x'y'}$. The same argument can be carried out in the opposite direction, so that $\langle a,e\rangle \in D_{xy}$ iff $\langle ab^{-1},e\rangle \in D_{x'y'}$, that means $b\circ D_{xy} = D_{x'y'}$. ⊢ (*Claim*)

To end the proof of the theorem, consider $\mathbb{S} = \mathbb{X} \times \mathscr{P}(\mathbb{G} \times \mathbb{N})$, a Polish space. Define a Borel action $b\cdot\langle x, D\rangle = \langle b\cdot x, b \circ D\rangle$ of $\mathbb{G}$ on $\mathbb{S}$. We assert that $\vartheta(x,y) = \langle x, D_{xy}\rangle$ is a Borel reduction of $\mathsf{E} \restriction P$ to the action $\mathsf{E}^{\mathbb{S}}_{\mathbb{G}}$. Indeed, let $\langle x,y\rangle$ and $\langle x',y'\rangle$ belong to P. Suppose that $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$. Then $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$, so that $x' = b\cdot x$ for some $b \in \mathbb{G}$. Moreover, $b \circ D_{xy} = D_{x'y'}$ by Claim 4, hence, $\vartheta(x',y') = b\cdot\vartheta(x,y)$. Let, conversely, $\vartheta(x',y') = b\cdot\vartheta(x,y)$, so that $x' = b\cdot x$ and $D_{x'y'} = b \circ D_{xy}$. Then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ by Claim 4, as required. □

12b The two cases

Here we begin the proof of Theorem 12.1.

We may assume that E is a $\varDelta^1_1$ equivalence relation on the Cantor space $2^{\mathbb{N}}$, and there is a $\varDelta^1_1$ reduction $\vartheta : 2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ of E to E_3. In this case, it can be *w.l.o.g.* assumed that in fact ϑ is a $\varDelta^1_1$ bijection. Indeed, define $\varphi : \mathbb{X} \to (2^{\mathbb{N}})^{\mathbb{N}}$ so that for any $x \in 2^{\mathbb{N}}$: $\varphi(x)(n)(0) = x(n)$ for all n and $\varphi(x)(n)(k+1) = \vartheta(x)(n)(k)$ for all n and k. The map φ is a bijection and still a Borel reduction of E to E_3.

Define $R = \mathtt{ran}\,\vartheta$, a $\varDelta^1_1$ subset of $(2^{\mathbb{N}})^{\mathbb{N}}$. (That R is $\varDelta^1_1$ follows from the assumption that ϑ is a Borel bijection.)

For $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $x \equiv_n y$ iff $x \mathrel{\mathsf{E}_3} y$ and $x \restriction_{\le n} = y \restriction_{\le n}$ (the latter requirement means $x(k) = y(k)$ for all $k \le n$). Put

$$\mathscr{A}^n_{kp} = \{A \subseteq (2^{\mathbb{N}})^{\mathbb{N}} : A \text{ is } \varSigma^1_1 \wedge \forall\, x,y \in A\ (x \equiv_n y \Longrightarrow x(k)\cdot y(k) \subseteq [0,p))\}$$

for all $n,k,p \in \mathbb{N}$, [1] where $a\cdot b \in 2^{\mathbb{N}}$ is defined for any pair of $a,b \in 2^{\mathbb{N}}$ so that $(a\cdot b)(k) = 0$ whenever $a(k) = b(k)$ and $(a\cdot b)(k) = 1$ otherwise — for all $k \in \mathbb{N}$. Thus for a $\varSigma^1_1$ set $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ to belong to $\bigcup_p \mathscr{A}^n_{kp}$ it is necessary and sufficient that for any $x \in A$ the set $\{y(k) : y \in A \wedge y \equiv_n x\}$ is finite. To strengthen here finiteness to being a singleton with the help of something like Theorem 1.4 (Countable-to-1 Enumeration) is hardly possible.

Lemma 12.4. *If $A \in \mathscr{A}^n_{kp}$ then there is a $\varDelta^1_1$ set $B \in \mathscr{A}^n_{kp}$ with $A \subseteq B$.*

Proof. The definition of $A \in \mathscr{A}^n_{kp}$ as a property of a $\varSigma^1_1$ set A is obviously $\varPi^1_1$ in the codes. Therefore Theorem 1.12 (Reflection) implies the result required. There is a more pedestrian proof based on the Separation rather than Reflection theorem. First consider the $\varPi^1_1$ set

$$P = \{y \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall\, x \in A\ (x \equiv_n y \Longrightarrow x(k)\cdot y(k) \subseteq [0,p))\}\,.$$

[1] Hjorth and Kechris [21] define $\mathscr{A}^n_{kp}$ with $\forall\, x,y \in R \cap A$ instead of $\forall\, x,y \in A$. Let us use $\mathscr{A}'_{nkp}$ to denote their version, thus, $\mathscr{A}^n_{kp} \subseteq \mathscr{A}'_{nkp}$. However if Case 1 holds in the sense of $\mathscr{A}'_{nkp}$ then it also holds in the sense of $\mathscr{A}^n_{kp}$ because $A \in \mathscr{A}'_{nkp}$ iff $A \cap R \in \mathscr{A}^n_{kp}$.

Then $A \subseteq P$ since $A \in \mathscr{A}^n_{kp}$. Take a Δ^1_1 set D such that $A \subseteq D \subseteq P$. Now consider the Π^1_1 set

$$P' = \{x \in D : \forall\, y \in D\, (x \equiv_n y \Longrightarrow x(k) \cdot y(k) \subseteq [0,p))\}\,.$$

Then $A \subseteq P'$ since $A \subseteq D \subseteq P$. Any Δ^1_1 set B such that $A \subseteq B \subseteq P$ is as required. □

Corollary 12.5. *The sets* $A^n_{kp} = \bigcup \mathscr{A}^n_{kp}$ [2] *belong to* Π^1_1 *uniformly on* n, k, p. *Therefore the set* $\widehat{A} = \bigcup_n \bigcap_{k>n} \bigcup_p A^n_{kp}$ *also belongs to* Π^1_1.

Proof. The result follows from Lemma 12.4 by standard computations based on the coding of Δ^1_1 sets (see Chapter 1) and Theorem 1.8. □

This leads us to the following partition onto cases.

Case 1: $R \subseteq \widehat{A}$.

Case 2: otherwise.

12c Case 1

We are going to prove that in this case $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$. The proof of the next theorem shows that the Case 1 condition makes all E_3-classes inside the domain $R = \mathtt{ran}\,\vartheta$ looking in a sense similar to E_0-classes. This will allow to employ Theorem 12.3 to obtain the result required.

Thus we are going to prove:

Theorem 12.6. *In all asssumptions above,* $\mathsf{E}_3 \restriction R \le_{\mathrm{B}} \mathsf{E}_0$.

Proof. By Kreisel Selection (Theorem 1.7) there exists a Δ^1_1 map $\nu : R \to \mathbb{N}$ such that for any $x \in R$ we have

$$\forall\, k > \nu(x)\, \exists\, p\, \exists\, B \in \mathscr{A}^{\nu(x)}_{k,p}\, (x \in B \in \Delta^1_1)\,.$$

Let $R_n = \{x \in R : \nu(x) \le n\}$, these are increasing Δ^1_1 subsets of R, and $R = \bigcup_n R_n$. According to Corollary 6.10, it suffices to prove that $\mathsf{E}_3 \restriction R_n \le_{\mathrm{B}} \mathsf{E}_0$ for any n. Thus let us fix n. Then by definition

$$\forall\, x \in R_n\, \forall\, k > n\, \exists\, p\, \exists\, B \in \mathscr{A}^n_{kp}\, (x \in B \in \Delta^1_1)\,. \qquad (*)$$

Recall that $\mathbf{C}$ is the least class of sets containing all open sets and closed under the A-operation and the complement. A map f is called $\mathbf{C}$-*measurable* iff all f-preimages of open sets belong to $\mathbf{C}$.

[2] That is, $A^n_{kp} = \bigcup\{A : A \in \mathscr{A}^n_{kp}\}$, the union of all sets in $\mathscr{A}^n_{kp}$.

Lemma 12.7. *For any* n *there is a* **C**-*measurable map* $f : R_n \to (2^{\mathbb{N}})^{\mathbb{N}}$ *such that* $f(x) = f(y) \equiv_n x$ *whenever* $x, y \in R_n$ *satisfy* $x \equiv_n y$.

Proof. Let $C \subseteq \mathbb{N}$ be the Π^1_1 set of all codes of Δ^1_1 subsets of $(2^{\mathbb{N}})^{\mathbb{N}}$, and let $W_e \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be the Δ^1_1 set coded by $e \in C$. We have, by $(*)$,

$$\forall\, x \in R_n\, \forall\, k > n\, \exists\, p\, \exists\, e \in C\ (x \in W_e \in \mathscr{A}^n_{kp}).$$

Now a straightforward application of Kreisel Selection (Theorem 1.7) yields a pair of Δ^1_1 maps $\pi, \varepsilon : R_n \times \mathbb{N} \to \mathbb{N}$ such that $\varepsilon(x,k) \in C$ and $x \in W_{\varepsilon(x,k)} \in \mathscr{A}^n_{k,\pi(x,k)}$ hold whenever $x \in R_n$ and $k > n$. Let $\tilde\pi(x,k)$ and $\tilde\varepsilon(x,k)$ be the least, in the sense of any fixed recursive ω-long wellordering of $\mathbb{N} \times \mathbb{N}$, of all possible pairs $\pi(x',k)$ and $\varepsilon(x',k)$ with $x' \in R_n \cap [x]_{\equiv_n}$. Then $\tilde\pi$ and $\tilde\varepsilon$ are $\equiv_n$-invariant in the 1st argument. In addition, we have $W_{\tilde\varepsilon(x,k)} \in \mathscr{A}^n_{k,\tilde\pi(x,k)}$ and the set $Z_{xk} = R_n \cap [x]_{\equiv_n} \cap W_{\tilde\varepsilon(x,k)}$ is nonempty, whenever $x \in R_n$ and $k > n$.

Let $x \in R_n$. For any $k > n$, the set $Y_{xk} = \{y(k) : y \in Z_{xk}\} \subseteq 2^{\mathbb{N}}$ is finite (and nonempty) by the definition of $\mathscr{A}^n_{kp}$. Let $f_k(x)$ be the least member of Y_{xk} in the sense of the lexicographical order of $2^{\mathbb{N}}$. Define $f(x) \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $f(x)(k) = x(k)$ for $k \le n$ and $f(x)(k) = f_k(x)$ for $k > n$.

That $f(x) = f(y)$ whenever $x \equiv_n y$ follows from the invariance of ε and π. To see that $f(x) \equiv_n x$ note that by definition $f_k(x)\ \mathsf{E}_0\ x_k$ for $k > n$: indeed, $f_k(x) = y_k$ for some $y \in [x]_{\equiv_n}$, but $x \equiv_n y$ implies $x_k\ \mathsf{E}_0\ y_k$ for all k. Finally, the **C**-measurability needs a routine check. □ (*Lemma*)

For any $u \in (2^{\mathbb{N}})^n$ define $R_n(u) = \{x \in R_n : x \restriction_{\le n} = u\}$.

Lemma 12.8. *If* $u \in (2^{\mathbb{N}})^n$ *then* $\mathsf{E}_3 \restriction R_n(u)$ *is smooth.*

Proof. As E_3 and $\equiv_n$ coincide on $R_n(u)$, the relation $\mathsf{E}_3 \restriction R_n(u)$ is smooth by means of a **C**-measurable, hence, a Baire-measurable map. Suppose, towards the contrary, that it is not really smooth, *i.e.*, smooth by means of a Borel map. Then, by the 2-nd dichotomy theorem, we have $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_3 \restriction R_n(u)$, hence, E_0 turns out to be smooth by means of a Baire-measurable map, which is easily impossible. □ (*Lemma*)

To complete the proof of the theorem, let $\mathbb{G}$ denote the group $\mathscr{P}_{\mathtt{fin}}(\mathbb{N})^n$, that is, the product of n copies of $\langle \mathscr{P}_{\mathtt{fin}}(\mathbb{N}); \cdot \rangle$. Let $\mathbb{G}$ act on $\mathbb{X} = (2^{\mathbb{N}})^n$ componentwise and by $\cdot$ on each of the n co-ordinates. (Recall that $(a \cdot b)(k) = 0$ iff $a(k) = b(k)$ whenever $a, b \in 2^{\mathbb{N}}$ and $k \in \mathbb{N}$.) Then, for any $u, v \in \mathbb{X}$, $u\ \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\ v$ is equivalent to $u(k)\ \mathsf{E}_0\ v(k)$ for all $k < n$. Let us apply Theorem 12.3 with $\mathbb{G}$ and $\mathbb{X}$ as indicated, and $P = R_n$ and $\mathsf{E} = \mathsf{E}_3 \restriction R_n$, Lemma 12.8 witnesses the principal requirement. Thus $\mathsf{E}_3 \restriction R_n$ is Borel reducible to an equivalence relation induced by a Borel action of $\mathbb{G}$. Yet $\mathbb{G}$ is the increasing union of a countable sequence of its finite subgroups, hence any ER induced by a Borel action of $\mathbb{G}$ is hyperfinite, therefore Borel reducible to E_0.

□ (*Theorem 12.6 and Case 1 in Theorem 12.1*)

12d Case 2

Then the Σ^1_1 set $H = R \smallsetminus \widehat{A}$ is non-empty. A rather typical example is

$$R = \{x \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall\, n, k, l\, (x(\prec n, k\succ) = x(\prec n, l\succ))\},$$

where $n, k \mapsto \prec n, k\succ$ is a recursive pairing function on $\mathbb{N}$, see below. Thus members of R are those infinite sequences of elements of $2^{\mathbb{N}}$ in which every term is duplicated in infinitely many copies. It can be verified that the intersection $\widehat{A} \cap R$ consists of all sequences $x \in R$ that contain a finite number of terms $x(0), \dots, x(n)$ such that any other term is Δ^1_1 in $x(0), \dots, x(n)$. Obviously the difference $R \smallsetminus \widehat{A}$ is non-empty.

We are going to prove

Theorem 12.9. *In all asssumptions above, including the Case 2 assumption, there exists a Borel subset X of H such that $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}_3 \restriction X$.*

This result leads to the **or** case of Theorem 12.1. Indeed the Borel map ϑ that reduces E to E_3, actually to $\mathsf{E}_3 \restriction R$, is a bijection (see the beginning of Section 12b), therefore there is an inverse map $\varphi = \vartheta^{-1} : R \to \mathbb{X} = \mathtt{dom}\,\mathsf{E}$, also Borel, of course. The map φ then witnesses $\mathsf{E}_3 \restriction R \le_{\mathrm{B}} \mathsf{E}$. On the other hand, $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{E}_3 \restriction X \le_{\mathrm{B}} \mathsf{E}_3 \restriction R$.

Proof. By definition, $H = \bigcap_n \bigcup_{k>n} H^n_k$, where $H^n_k = H \smallsetminus \bigcup_p A^n_{kp}$. Note that

$$\begin{aligned} H^n_k &= \{x \in H : \forall\, p\, \forall\, A \in \mathscr{A}^n_{kp}\, (x \notin A)\} \\ &= \{x \in H : \forall\, p\, \forall\, A \in \Delta^1_1\, (x \in A \Longrightarrow A \notin \mathscr{A}^n_{kp})\} \end{aligned} \tag{1}$$

by Lemma 12.4, and hence H^n_k is Σ^1_1 by rather elementary computation. Note also that for any Σ^1_1 set A and any n, k, p the following holds:

$$A \notin \mathscr{A}^n_{kp} \iff \exists\, y, z \in A\, \exists\, j \ge p\, (y \equiv_n z \wedge y(k)(j) \ne z(k)(j))\,. \tag{2}$$

To prove the theorem, we are going to define a rather complicated splitting system of non-empty Σ^1_1 subsets of $(2^{\mathbb{N}})^{\mathbb{N}}$. Let us take some space for technical notation involved in the construction of the splitting system.

Put $\prec r, q\succ = 2^r(2q+1) - 1$ for all $r, q \in \mathbb{N}$. Thus $\langle r, q\rangle \mapsto \prec r, q\succ$ is a recursive bijection $\mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$, increasing in each argument. Put

$$L(n) = \mathtt{max}\{r : \exists\, q\, (\prec r, q\succ \le n)\} = \{r : 2^r - 1 \le n\}$$

for any n — for instance $L(0) = 0$ and $L(1) = L(2) = 1$. For any $r \le L(n)$ define $(n)_r = \{q : \prec r, q\succ \le n\}$ — this is a natural number ≥ 1 (assuming $r \le L(n)$). For instance $(0)_0 = 1$ (since $\prec 0, 0\succ = 0$), $(1)_0 = 2$, and $(1)_1 = 1$. Obviously $n = \sum_{r=0}^{L(n)-1} (n)_r$.

Suppose that $n \in \mathbb{N}$ and $s \in 2^n$ (a dyadic sequence of length n). For any $r < L(n)$ define $(s)_r \in 2^{(n)_r}$ so that $(s)_r(q) = s(\prec r, q \succ)$ for all $q < (n)_r$. Thus the original sequence $s \in 2^{<\omega}$ of length $\mathtt{lh}\, s = n$ is split into a $L(n)$-sequence of dyadic sequences of lengths $\mathtt{lh}\,(s)_r = (n)_r$. Formally this secondary sequence $\{(s)_r\}_{r<L(n)}$ belongs to the product set $\prod_{r=0}^{L(n)-1} 2^{(n)_r}$.

We consider $2^{\mathbb{N}}$ as a group with the componentwise operation, that is, if $a, b \in 2^{\mathbb{N}}$ then $a \cdot b \in 2^{\mathbb{N}}$ and $(a \cdot b)(k) = a(k) +_2 b(k)$, $\forall\, k$, where $+_2$ is the addition modulo 2. The neutral element is the constant-0 sequence $\mathbf{0} = \mathbb{N} \times \{0\}$ (that is, $\mathbf{0}(k) = 0$, $\forall\, k$), clearly $\mathbf{0} \cdot a = a$ for all $a \in 2^{\mathbb{N}}$.

Accordingly consider $(2^{\mathbb{N}})^{\mathbb{N}}$ as the product of $\mathbb{N}$-many copies of $2^{\mathbb{N}}$, a group with the componentwise operation still denoted by $\cdot$, so that $(f \cdot g)(n)(k) = f(n)(k) +_2 g(n)(k)$ for all n, k. The neutral element is the constant-$\mathbf{0}$ sequence $\mathbf{0}^{\mathbb{N}} \in (2^{\mathbb{N}})^{\mathbb{N}}$. Define $\mathtt{supp}\, g = \{n \in \mathbb{N} : g(n) \neq \mathbf{0}\}$, the domain of non-triviality of $g \in (2^{\mathbb{N}})^{\mathbb{N}}$.

The group $(2^{\mathbb{N}})^{\mathbb{N}}$ contains the subgroups

$$\mathbf{F} = \{g \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall\, n\, \exists\, k_0\, \forall\, k \geq k_0\, (f(n)(k) = 0)\},$$

essentially the ideal $\mathscr{I}_3$, acting on $(2^{\mathbb{N}})^{\mathbb{N}}$ by the group operation $\cdot$, and

$$\begin{aligned}
\mathbf{F}_{>n} &= \{g \in \mathbf{F} : \mathtt{supp}\, g \subseteq (n, \infty)\} &&= \{g \in \mathbf{F} : \forall\, k \leq n\, (g(k) = \mathbf{0})\}, \\
\mathbf{F}_{\geq n} &= \{g \in \mathbf{F} : \mathtt{supp}\, g \subseteq [n, \infty)\} &&= \{g \in \mathbf{F} : \forall\, k < n\, (g(k) = \mathbf{0})\}, \\
\mathbf{F}_{\leq n} &= \{g \in \mathbf{F} : \mathtt{supp}\, g \subseteq [0, n]\} &&= \{g \in \mathbf{F} : \forall\, k > n\, (g(k) = \mathbf{0})\}.
\end{aligned}$$

for any n. Obviously $x \mathrel{\mathsf{E}_3} y$ iff $y \in \mathbf{F} \cdot x$, and $x \equiv_n y$ iff $y \in \mathbf{F}_{>n} \cdot x$,

Finally if $X \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ then put $g \cdot X = \{g \cdot x : x \in X\}$.

The splitting system used here will contain non-empty Σ^1_1 sets $X_s \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $s \in 2^{<\omega}$, the increasing sequence of numbers $k_0 < k_1 < k_2 < \cdots \in \mathbb{N}$, a collection of natural numbers p_{mj}, $m, j \in \mathbb{N}$, and elements $g_{st} \in \mathbf{F}$, where $s, t \in 2^{<\omega}$, $\mathtt{lh}\, s = \mathtt{lh}\, t$, satisfying the following requirements (i) – (ix):

(i) $X_\Lambda \subseteq H$, $X_{s^\wedge i} \subseteq X_s$, $\mathtt{diam}\, X_s \leq 2^{-\mathtt{lh}\, s}$.

(ii) A certain condition similar to (viii) in Section 9c holds, connecting each $X_{s^\wedge i}$ with X_s so that, as a consequence, $\bigcap_n X_{a \restriction n} \neq \varnothing$ for any $a \in 2^{\mathbb{N}}$.

(iii) If $s \in 2^{n+1}$ then $X_s \subseteq \bigcap_{r \leq L(n)} H^r_{k_r}$.

(iv) If $s, t \in 2^{n+1}$ then $\mathtt{supp}\, g_{st} \subseteq [0, k_{L(n)}]$, that is, $g_{st} \in \mathbf{F}_{\leq k_{L(n)}}$.

(v) $k_0 < k_1 < k_2 < \ldots$, and $p_{m0} < p_{m1} < p_{m2} < \ldots$ for any m.

(vi) $g_{su} = g_{tu} \cdot g_{st}$ for all $s, t, u \in 2^{n+1}$. It easily follows that $g_{ss} = \mathbf{0}^{\mathbb{N}}$, $\forall\, s$.

(vii) For any $s, t \in 2^{n+1}$, we have $g_{st} \cdot X_s \equiv_{k_{L(n)}} X_t$.

We define $X \equiv_m Y$ (for any sets $X, Y \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$) iff $[X]_{\equiv_m} = [Y]_{\equiv_m}$, that is, for any $x \in X$ there exists $y \in Y$ satisfying $x \equiv_m y$ and *vice versa* for any $y \in Y$ there exists $x \in X$ satisfying $x \equiv_m y$. This is equivalent to $\mathbf{F}_{>m} \cdot X = \mathbf{F}_{>m} \cdot Y$.

(viii) For any $s, t \in 2^{n+1}$, if $\ell \le L(n)$, $n' \le n$, and $s', t' \in 2^{n'}$ satisfy $s' \subseteq s$, $t' \subseteq t$, and the equality $(s)_r(q) = (t)_r(q)$ holds whenever $r \le \ell$ and $q \in \mathbb{N}$ satisfy $n' \le \prec r, q \succ \le n$, then $g_{st}(i) = g_{s't'}(i)$ for any $i \le \ell$.

(ix) For any $s, t \in 2^{n+1}$, if $s(n) = 0 \ne 1 = t(n)$, and $n = \prec m, j \succ$, then $x(k_m)(p_{mj}) = 0$ for all $x \in X_s$ but $y(k_m)(p_{mj}) = 1$ for all $y \in X_t$.

12e The embedding

Suppose that a system of sets X_s, elements g_{st}, and numbers k_m and p_{mj} satisfying (i) – (ix) has been defined. Let us show that this leads to the proof of Theorem 12.9.

As usual it follows from (i) and (ii) that for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a \restriction n}$ is a singleton. Let us denote by $\vartheta(a) = \{\vartheta_n(a)\}_{n \in \mathbb{N}}$ its only element. Thus $a \mapsto \vartheta(a)$ is a map $2^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ while each ϑ_n is a map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$. In addition both ϑ and all ϑ_n are continuous (in the Polish product topology).

On the other hand for any $a \in 2^{\mathbb{N}}$ there is a unique point $\tilde{a} = \{(a)_n\}_{n \in \mathbb{N}} \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $(a)_n(k) = a(\prec n, k \succ)$ for all n, k. The map $a \mapsto \tilde{a}$ is a homeomorphism of $2^{\mathbb{N}}$ onto $(2^{\mathbb{N}})^{\mathbb{N}}$, while each $a \mapsto (a)_n$ is a continuous map $2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Thus the following lemma suffices to prove Theorem 12.9:

Lemma 12.10. *For any* $a, b \in 2^{\mathbb{N}}$, *we have*: $\tilde{a} \mathsf{E}_3 \tilde{b}$ *iff* $\vartheta(a) \mathsf{E}_3 \vartheta(b)$.

Proof. Assume that $\tilde{a} \mathsf{E}_3 \tilde{b}$, take an arbitrary $\ell \in \mathbb{N}$ and prove that $\vartheta_\ell(a) \mathsf{E}_0 \vartheta_\ell(b)$. In our assumptions there exists a number n' such that $\ell \le L(n')$ and for any $r \le \ell$ and q, if $\prec r, q \succ \ge n'$ then $a(\prec r, q \succ) = b(\prec r, q \succ)$. Put $s' = a \restriction n'$ and $t' = b \restriction n'$. Then $g' = g_{s't'} \in \mathbf{F}$. Our goal is to prove that $\vartheta_\ell(b) = (g')_\ell \cdot \vartheta_\ell(a)$, that obviously implies $\vartheta_\ell(a) \mathsf{E}_0 \vartheta_\ell(b)$.

It suffices to show that $g_{s't'} \cdot X_s \equiv_\ell X_t$ holds for any $n > n'$, where $s = a \restriction n$ and $t = b \restriction n$. We observe that $g_{st} \cdot X_s \equiv_\ell X_t$ by (vii) because $\ell \le L(n') \le k_{L(n')} \le k_{L(n)}$. On the other hand, $g_{st}(i) = g_{s't'}(i)$ for any $i \le \ell$ by (viii) and the choice of n'. It follows that $g_{s't'} \cdot X_s = g_{st} \cdot X_s \equiv_\ell X_t$, as required.

To prove the converse suppose that $\tilde{a} \mathsf{E}_3 \tilde{b}$ fails, and hence there is at least one index m such that $(a)_m \mathsf{E}_0 (b)_m$ fails as well, meaning that $a(\prec m, j \succ) \ne b(\prec m, j \succ)$ holds for infinitely many numbers $j \in \mathbb{N}$. Then by (ix) we obtain $\vartheta_{k_m}(a)(p_{mj}) = 0 \ne 1 = \vartheta_{k_m}(b)(p_{mj})$ for all j, and hence $\vartheta_{k_m}(a) \mathsf{E}_0 \vartheta_{k_m}(b)$ fails since the numbers p_{mj}, $j \in \mathbb{N}$, form a strictly increasing sequence by (v).

□ (*Lemma 12.10*)

12f The construction of a splitting system: warmup

Now to prove Theorem 12.9 it remains to carry out the construction of a system of sets X_s and g_{st} and numbers k_m and p_{mj} satisfying conditions (i) – (ix) of Section 12d. The construction goes on by induction on n, so that at each step n we define the sets X_s, $s \in 2^n$ and elements g_{st}, $s,t \in 2^n$. Here we present only the transition from 0 to 1 as a warmup.

Put $X_\Lambda = H$ and by default $g_{\Lambda\Lambda} = \mathbf{0}^{\mathbb{N}}$ for the only sequence Λ of length 0.

At the next stage, we have to define Σ^1_1 sets $X_{\langle 0\rangle}, X_{\langle 1\rangle} \subseteq X_\Lambda$, an element $g_{\langle 0\rangle\langle 1\rangle} = g_{\langle 1\rangle\langle 0\rangle} \in \mathbf{F}$, and numbers k_0 and p_{00} such that a relevant fragment of (i) – (ix) is satisfied. Note that $L(0) = 0$.

Stage 1. We shrink X_Λ to make sure that conditions (i) and (ii) are satisfied; the resulting Σ^1_1 set is still denoted by X_Λ.

Stage 2. Consider any $x \in X_\Lambda$. Then $x \in \bigcap_{k>0} H^0_k$ (see the beginning of the proof of Theorem 12.9). Fix a number $k = k_0 > 0$ such that $x \in H^0_{k_0}$. The set $X'_\Lambda = X_\Lambda \cap H^0_{k_0}$ is still of class Σ^1_1, and for any p it does not belong to the family $\mathscr{A}^0_{k_0,p}$ by (1) in Section 12d. Thus by (2) there exist points y_0, $z_0 \in X'_\Lambda$ satisfying $y_0 \equiv_{L(0)} z_0$ and numbers $k_0 > 0 = L(0)$ and p_{00} such that $y_0(k_0)(p_{00}) = 0 \neq 1 = z_0(k_0)(p_{00})$. The Σ^1_1 sets

$$\begin{aligned} Y &= \{y \in X'_\Lambda : y \equiv_{L(0)} y_0 \wedge y(k_0)(p_{00}) = 0\}, \quad \text{and} \\ Z &= \{z \in X'_\Lambda : z \equiv_{L(0)} z_0 \wedge z(k_0)(p_{00}) = 1\} \end{aligned}$$

still contain resp. y_0, z_0, therefore so do the Σ^1_1 sets

$$Y' = \{y' \in Y : \exists z \in Z\ (y' \equiv_{L(0)} z)\} \quad \text{and} \quad Z' = \{z' \in Z : \exists y \in Y\ (y \equiv_{L(0)} z')\}.$$

Finally define $g_{\langle 0\rangle\langle 1\rangle} = g_{\langle 1\rangle\langle 0\rangle} \in \mathbf{F}$ so that $g_{\langle 0\rangle\langle 1\rangle}(k_0)(p_{00}) = 1$ and $g_{\langle 0\rangle\langle 1\rangle}(m)(j) = 0$ for any other pair of m, j. Then easily $g_{\langle 0\rangle\langle 1\rangle} \cdot y_0 \equiv_{k_0} z_0$, hence $g_{\langle 0\rangle\langle 1\rangle} \cdot Y' \equiv_{k_0} Z'$. Thus we get a pair of sets $X_{\langle 0\rangle} = Y'$ and $X_{\langle 1\rangle} = Z'$ compatible with (vii). This ends the construction for $n = 1$.

12g The construction of a splitting system: the step

Now suppose that $n = \prec m, j \succ \geq 1$, and the construction has been accomplished up to the level n, that is, there exist sets $X_s \subseteq H$ and elements $g_{st} \in \mathbf{F}$, where $s,t \in 2^{n'}$, $n' \leq n$, and numbers $k_0, \dots, k_{L(n-1)}$ and $p_{m'j'}$, where $\prec m', j' \succ < n$, such that conditions (i) – (ix) are satisfied in this domain. The goal is to define X_s and g_{st}, where $s,t \in 2^{n+1}$, and numbers k_n and p_{mj}, such that conditions (i) – (ix) are satisfied in the extended domain.

The numbers n, m, j are fixed in the course of the arguments in this Section.

Lemma 12.11. *Suppose that collections of* Σ^1_1 *sets* $P_s \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, $s \in 2^n$ *and elements* $g_{st} \in \mathbf{F}$, $s,t \in 2^n$, *satisfy both (vi) and (vii) for a fixed* k, *that is,* $g_{su} = g_{tu} \cdot g_{st}$ *and* $g_{st} \cdot P_s \equiv_k P_t$ *for all* $s,t,u \in 2^{n+1}$.

If $\sigma \in 2^n$ *and* $P \subseteq P_\sigma$ *is a non-empty* Σ^1_1 *set then the sets*

$$P'_s = \{x \in P_s : \exists\, y \in P\, (g_{\sigma s} \cdot y \equiv_k x)\}, \quad s \in 2^n,$$

are non-empty Σ^1_1 *sets still satisfying (vii), i.e.* $g_{st} \cdot P'_s \equiv_k P'_t$ *for all* $s, t \in 2^{n+1}$.

Proof. Fix $s, t \in 2^n$. To show that $g_{st} \cdot X'_s \equiv_k X'_t$ consider any $x \in X'_s$, so that $g_{st} \cdot x \in g_{st} \cdot X'_s$. By definition there exists $y \in X$ satisfying $g_{\sigma s} \cdot y \equiv_k x$. It follows from (iv) that $g_{st} \in \mathbf{F}_{\le k}$, therefore $g_{st} \cdot g_{\sigma s} \cdot y \equiv_k g_{st} \cdot x$, that is, $g_{\sigma t} \cdot y \equiv_k g_{st} \cdot x$ by (vi). However by definition $g_{\sigma t} \cdot y \in X'_t$, as required.

The converse is similar. □ (*Lemma*)

It follows from (iv) (in the domain 2^n) that there is a number $\mu \in \mathbb{N}$ such that $g_{st}(r)(q) = 1$ holds only in the case when both $r \le k_{L(n-1)}$ and $q \le \mu$. We proceed with several stages of successive reduction and splitting of the Σ^1_1 sets X_s, $s \in 2^n$. These further stages depend on whether the number $n = \prec m, j \succ$ considered opens a "new" axis k_m of splitting.

Case A: $j > 0$.

Then $n' = \prec m, j-1 \succ < n$, thus m is "old". Moreover, $L(n) = L(n-1)$. We have to define p_{mj} but needn't to define any new k_r.

Stage 1. Fix an arbitrary sequence $\sigma \in 2^n$; this can be *e.g.* the sequence 0^n of n zeros. Consider any $x \in X_\sigma$. Then $x \in H^m_{k_m}$ by (iii), and hence there exist points $y_0, z_0 \in X_\sigma$ and a number $p_{mj} > \mu$ such that $y_0 \equiv_{m-1} z_0$ and $y_0(k_m)(p_{mj}) = 0$ but $z_0(k_m)(p_{mj}) = 1$. Easily $p_{mj} > p_{m,j-1}$: indeed $p_{m,j-1} \le \mu$ by the choice of μ.

Stage 2. Define $g \in (2^{\mathbb{N}})^{\mathbb{N}}$ so that $g(r)(q) = 1$ iff both $m \le r \le k_{L(n)}$ and $y_0(r)(q) \ne z_0(r)(q)$. Then $g \in \mathbf{F}$ since $y\, \mathsf{E}_3\, z$. Moreover we have $\mathsf{supp}\, g \subseteq [m, k_{L(n)}]$, in other words, $g \in \mathbf{F}_{\ge m} \cap \mathbf{F}_{\le k_{L(n)}}$. In addition $g(k_m)(p_{mj}) = 1$.

We observe that by definition $g \cdot y_0 \equiv_{k_{L(n)}} z_0$. Thus the Σ^1_1 sets

$$\begin{aligned} Y &= \{y \in X_\sigma : y(k_m)(p_{mj}) = 0 \land \exists\, z \in Z\, (z(k_m)(p_{mj}) = 1 \land g \cdot y \equiv_{k_{L(n)}} z)\}\,, \\ Z &= \{z \in X_\sigma : z(k_m)(p_{mj}) = 1 \land \exists\, y \in Y\, (y(k_m)(p_{mj}) = 0 \land g \cdot y \equiv_{k_{L(n)}} z)\} \end{aligned}$$

are still non-empty (contain resp. y_0, z_0) and satisfy $g \cdot Y \equiv_{k_{L(n)}} Z$; in addition $y(k_m)(p_{mj}) = 0$ and $z(k_m)(p_{mj}) = 1$ for all $y \in Y$ and $z \in Z$.

As a matter of fact we can *w.l.o.g.* assume that $Y \cup Z = X_\sigma$: indeed otherwise put $P = Y \cup Z$ and apply Lemma 12.11.

Stage 3. Put $X_{\sigma^\wedge 0} = Y$ and $X_{\sigma^\wedge 1} = Z$, thus

$$g \cdot X_{\sigma^\wedge 0} \equiv_{k_{L(n)}} X_{\sigma^\wedge 1}, \tag{3}$$

and then

$$X_{s^\wedge \xi} = \{x \in X_s : \exists\, y \in X_{\sigma^\wedge \xi}\, (g_{\sigma s}(y) \equiv_{k_{L(n)}} x)\}$$

for all $s \in 2^n$ and $\xi = 0, 1$. It follows, by (vii) at the level n, that

$$X_{s^\wedge\xi} \equiv_{k_{L(n)}} g_{\sigma s} \cdot X_{\sigma^\wedge\xi} \quad \text{for all } s \in 2^n \text{ and } \xi = 0, 1\,. \tag{4}$$

Put $g_{s^\wedge\xi\,,\,t^\wedge\xi} = g_{st}$ but $g_{s^\wedge\xi\,,\,t^\wedge(1-\xi)} = g_{st} \cdot g$ for all $s, t \in 2^n$ and $\xi = 0, 1$, [3] or saying it differently

$$g_{s^\wedge\xi\,,\,t^\wedge\eta} = g_{st} \cdot g^{\xi-\eta} \quad \text{for all } s, t \in 2^n \text{ and } \xi, \eta = 0, 1 \tag{5}$$

where $g^1 = g^{-1} = g$ while $g^0 = \mathbf{0}^{\mathbb{N}}$ is the neutral element in $\langle (2^{\mathbb{N}})^{\mathbb{N}}\,; \cdot \rangle$.

Stage 4. Lemma 12.11 allows us to reduce the sets X_s, $s \in 2^{n+1}$, in several rounds to make sure that conditions (i) and (ii) are satisfied at level $n+1$; the resulting Σ^1_1 sets are still denoted by X_s.

This ends the transition from n to $n+1$. It remains to show that conditions (i) – (ix) are satisfied in the extended $(\le n+1)$-domain.

Verification. As (i) and (ii) are explicitly fulfilled, (iii) in Case 1 is vacuous, and (iv), (v) clearly hold by definition, we begin with (vi). We have to prove that

$$g_{s^\wedge\xi\,,\,u^\wedge\zeta} = g_{t^\wedge\eta\,,\,u^\wedge\zeta} \cdot g_{s^\wedge\xi\,,\,t^\wedge\eta}$$

for all $s, t, u \in 2^n$ and $\xi, \eta, \zeta = 0, 1$. By definition this equality is equivalent to $g_{su} \cdot g^{\xi-\zeta} = g_{tu} \cdot g^{\eta-\zeta} \cdot g_{st} \cdot g^{\xi-\eta}$. However obviously $g^{\xi-\zeta} = g^{\eta-\zeta} \cdot g^{\xi-\eta}$, and on the other hand in our assumptions $g_{su} = g_{tu} \cdot g_{st}$ by (vi) at level n.

Let us check (vii), that is, $g_{s^\wedge\xi\,,\,t^\wedge\eta} \cdot X_{s^\wedge\xi} \equiv_{k_{L(n)}} X_{t^\wedge\eta}$ for all $s, t \in 2^n$ and $\xi, \eta = 0, 1$. It follows from (4) that the left-hand side is $\equiv_{k_{L(n)}}$-equivalent to $g_{st} \cdot g^{\xi-\eta} \cdot g_{\sigma s} \cdot X_{\sigma^\wedge\xi}$ while the right-hand side is $\equiv_{k_{L(n)}}$-equivalent to $g_{\sigma t} \cdot X_{\sigma^\wedge\eta}$. On the other hand it follows from (3) that $g^{\xi-\eta} \cdot X_{\sigma^\wedge\xi} \equiv_{k_{L(n)}} X_{\sigma^\wedge\eta}$. This allows to easily get the result required.

Let us check (viii). Suppose that s, t, ℓ, n', s', t' are as indicated in (viii). Then $s = \bar{s}^\wedge\xi$ and $t = \bar{t}^\wedge\eta$, where $\bar{s}, \bar{t} \in 2^n$ while $\xi = s(n)$ and $\eta = t(n)$ are numbers in $\{0, 1\}$. Then $g_{\bar{s}\bar{t}}(i) = g_{s't'}(i)$ for any $i \le \ell$ by (viii) in the domain 2^n. Thus if $\xi = \eta$ then the result holds immediately because then $g_{st} = g_{\bar{s}\bar{t}}$ by (5). Assume that *e.g.* $\xi = 0$ and $\eta = 1$. Then $\ell < m$ in the assumptions of (viii), and hence the set $\mathtt{supp}\, g$ does not contain numbers $i \le \ell$, in other words, $g(i) = \mathbf{0}$ for any $i \le \ell$. It follows that $g_{st}(i) = g_{\bar{s}\bar{t}}(i)$ for any $i \le \ell$, as required.

We finally check (ix). Suppose that $s^\wedge\xi$ and $t^\wedge\eta$ belong to 2^{n+1} and $\xi \ne \eta$, say $\xi = 0 \ne 1 = \eta$. We have to prove that $x(k_m)(p_{mj}) = \xi$ for all $\xi = 0, 1$ and $x \in X_{s^\wedge\xi}$. First of all note that by definition $x(k_m)(p_{mj}) = \xi$ for all $x \in X_{\sigma^\wedge\xi}$. On the other hand $g_{\sigma s}(k_m)(p_{mj}) = 0$ since $p_{mj} > \mu$ by the construction. Thus $(g_{\sigma s} \cdot x)(k_m)(p_{mj}) = \xi$ for all $x \in X_{\sigma^\wedge\xi}$. It remains to use (4).

[3] In the definition of g_{st} we make use of the fact that $\langle (2^{\mathbb{N}})^{\mathbb{N}}\,; \cdot \rangle$ is an abelian group. In the non-abelian case we would have to define $g_{s^\wedge i\,,\,t^\wedge(1-i)} = g_{\sigma t} \cdot g \cdot g_{s\sigma}$ and accordingly render some other related definitions in somewhat more complicated way.

Case B: $j = 0$.

Then there is no number $n' = \prec m', j' \succ < n$ such that $m' = m$ — in other words, m is "new". Obviously $m = L(n-1) + 1 = L(n)$ in this case.

Stage 1. The first goal is to appropriately choose a number k_m. Let us fix an arbitrary $\sigma \in 2^n$. Consider any $x \in X_\sigma$. As $X_\sigma \subseteq X_\Lambda \subseteq H = \bigcap_n \bigcup_{k>n} H^n_k$, it follows from (1) in Section 12d that $x \in H^{k_{L(n-1)}+1}_{k_m}$ for some $k_m > k_{L(n-1)} + 1$. In particular $k_m > k_{m-1}$, $k_m > L(n)$, and $x \in H^{L(n)}_{k_m}$.

It can be *w.l.o.g.* assumed that $X_\sigma \subseteq H^{L(n)}_{k_m}$. (Indeed otherwise we can replace the set X_σ by $X'_\sigma = X_\sigma \cap H^{L(n)}_{k_m}$, still a non-empty $\varSigma^1_1$ set, and apply Lemma 12.11 to shrink all sets X_s, $s \in 2^n$, accordingly.)

Lemma 12.12. *In this assumption,* $X_s \subseteq H^{L(n)}_{k_m}$ *for all* $s \in 2^n$.

Proof. Consider an arbitrary point $x_0 \in X_s$ and prove that $x_0 \in H^{L(n)}_{k_m}$. Fix any number p and a $\varDelta^1_1$ set $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ containing x_0; we have to show that $A \notin \mathscr{A}^{L(n)}_{k_m,p}$.

Recall that $X_s \equiv_{k_{L(n-1)}} g_{\sigma s} \cdot X_\sigma$ by (vii) in the domain 2^n, therefore there is a point $y_0 \in X_\sigma$ satisfying $x_0 \equiv_{k_{L(n-1)}} g_{\sigma s} \cdot y_0$. Then there exists an element $g \in \mathbf{F}$ with $\mathsf{supp}\, g \subseteq [0, k_m]$ such that $x_0 \equiv_{k_m} g \cdot y_0$. And it is clear that g extends $g_{\sigma s}$ in the sense that $g(r) = g_{\sigma s}(r)$ for all $r \le L(n-1)$.

The pre-image $B = \{y \in (2^{\mathbb{N}})^{\mathbb{N}} : \exists x \in A\, (g \cdot y \equiv_{k_m} x)\}$ is a $\varSigma^1_1$ set containing y_0. But in our assumptions $y_0 \in X_\sigma \subseteq H^{L(n)}_{k_m}$, and hence there exist points y, $y' \in B$ such that $y \equiv_{L(n)} y'$ but $y'(k_m) \cdot y(k_m) \not\subseteq [0, p]$. In other words, there is a number $j > p$ with $y'(k_m)(j) \ne y(k_m)(j)$. By definition there exist poits x, $x' \in A$ such that $g \cdot y \equiv_{k_m} x$ and $g \cdot y' \equiv_{k_m} x'$. In particular $x(r) = g(r) \cdot y(r)$ and $x'(r) = g(r) \cdot y'(r)$ for all $r \le k_n$. We conclude that $x \equiv_{L(n)} x'$ but $x'(k_m)(j) \ne x(k_m)(j)$. It follows that $A \notin \mathscr{A}^{L(n)}_{k_m,p}$, as required. $\square$ (*Lemma*)

Stage 2. It follows from (2) in Section 12d that there exist points $y_0, z_0 \in X_\sigma$ and a number $p_{m0} \in \mathbb{N}$ such that $y_0 \equiv_{L(n)} z_0$ and $y_0(k_m)(p_{m0}) = 0 \ne 1 = z_0(k_m)(p_{m0})$. Following the construction in Case A, define $g \in \mathbf{F}_{\ge m} \cap \mathbf{F}_{\le k_{L(n)}}$ so that $g \cdot y_0 \equiv_{k_{L(n)}} z_0$, in particular, $g(k_m)(p_{m0}) = 1$. Then the $\varSigma^1_1$ sets

$$
\begin{aligned}
Y &= \{y \in X_\sigma : y(k_m)(p_{m0}) = 0 \wedge \exists z \in Z\ (z(k_m)(p_{m0}) = 1 \wedge g \cdot y \equiv_{k_{L(n)}} z)\}, \\
Z &= \{z \in X_\sigma : z(k_m)(p_{m0}) = 1 \wedge \exists y \in Y\ (y(k_m)(p_{m0}) = 0 \wedge g \cdot y \equiv_{k_{L(n)}} z)\}
\end{aligned}
$$

are still non-empty sets containing resp. y_0, z_0 and satisfying $g \cdot Y \equiv_{k_{L(n)}} Z$; in addition $y(k_m)(p_{m0}) = 0$ and $z(k_m)(p_{m0}) = 1$ for all $y \in Y$ and $z \in Z$. And still we can *w.l.o.g.* assume that $Y \cup Z = X_\sigma$.

Stage 3. We define the sets $X_{s^\wedge\xi} \subseteq X_s$ and elements $g_{s^\wedge\xi, t^\wedge\eta}$ ($s, t \in 2^n$ and $\xi = 0, 1$) exactly as on Stage 3 of Case A. Conditions (3), (4), (5) still hold and by the same reasons.

Stage 4. Shrink the sets X_s, $s \in 2^{n+1}$, with the help of Lemma 12.11, in several rounds, so that the resulting Σ^1_1 sets, still denoted by X_s, satisfy (i) and (ii) in the domain 2^{n+1}. This completes the transition from n to $n+1$.

Verification. A new feature here in comparison to Case A is the non-vacuous character of condition (iii). It suffices to show that $X_{s^\wedge\xi} \in H^{L(n)}_{k_m}$ for all $s \in 2^n$ and $\xi = 0, 1$, or, that is sufficient, $X_s \in H^{L(n)}_{k_m}$ for all $s \in 2^n$ — but this follows from Lemma 12.12. The verification of (iv) – (ix) is quite similar to the verification in Case 1, we leave it to the reader.

$\square$ (*Theorem 12.9 and Case 2 in Theorem 12.1*)

$\square$ (*Theorem 12.1*)

Chapter 13

Summable ideals and equivalence relations

Given a sequence of nonnegative reals r_n with $\sum_{n=0}^{\infty} r_n = +\infty$, the summable ideal $\mathscr{S}_{\{r_n\}}$ consists of all sets $x \subseteq \mathbb{N}$ such that $\mu_{\{r_n\}}(x) = \sum_{n \in x} r_n < +\infty$. The corresponding equivalence relation $\mathsf{S}_{\{r_n\}}$ is defined on $\mathscr{P}(\mathbb{N})$ so that $x \mathrel{\mathsf{S}_{\{r_n\}}} y$ iff $x \,\Delta\, y \in \mathscr{S}_{\{r_n\}}$. Equivalently $\mathsf{S}_{\{r_n\}}$ is defined on $2^{\mathbb{N}}$ the same way, with $a \,\Delta\, b = \{n : a(n) \neq b(n)\}$ for $a, b \in 2^{\mathbb{N}}$.

Farah [7, § 1.12] gives the following classification of summable ideals based on the distribution of reals r_n:

(S1) *Atomic* ideals: there is $\varepsilon > 0$ such that the set $A_\varepsilon = \{n : r_n \geq \varepsilon\}$ is infinite and satisfies $\mu_{\{r_n\}}(\complement A_\varepsilon) < +\infty$. In this case $\mathscr{S}_{\{r_n\}} = \{x : x \cap A_\varepsilon \in \mathtt{Fin}\}$; so this is what Kechris [35] called *trivial variations of* $\mathtt{Fin}$, see Footnote 1 in ChapteridI1.

(S2) *Dense* (summable) ideals: $r_n \to 0$.

(S3) There is a decreasing sequence of positive reals $\varepsilon_n \to 0$ sich that all sets $D_n = A_{\varepsilon_{n+1}} \smallsetminus A_{\varepsilon_n}$ are infinite.

(S4) Ideals of the form $\mathtt{Fin} \oplus \text{dense}$: there is a real $\varepsilon > 0$ such that the set A_ε is infinite, $\mu_{\{r_n\}}(\complement A_\varepsilon) = +\infty$, and $\mathtt{lim}_{n \to \infty,\, n \in \complement A_\varepsilon} r_n = 0$.

In the sense of $\leq_{\mathrm{B}}$, all ideals of types (S2), (S3), (S4) are equivalent to each other, and all ideals of type (S1) are equivalent to each other, so that we have just 2 summable ideals modulo $\sim_{\mathrm{B}}$, namely $\mathtt{Fin}$ and, say, $\mathscr{S}_{\{1/n\}}$. The structure under $\leq_{\mathrm{RB}}$ (which we don't consider here) is much more complicated.

This Chapter is mainly devoted to the following theorem of Hjorth [17], often called the 4th dichotomy theorem.

Theorem 13.1. *Let* E *be a Borel ER on a Polish space* $\mathbb{X}$, *and* $\mathsf{E} \leq_{\mathrm{B}} \mathsf{S}_{\{1/n\}}$. *Then either* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{S}_{\{1/n\}}$ *or* E *is essentially countable.*

13a Grainy sets

We begin the proof of Theorem 13.1 with a few definitions.

For $a, b \in 2^{\mathbb{N}}$ put $a \,\Delta\, b = \{n : a(n) \neq b(n)\}$ (identified with the function $c(n) = 1$ iff $a(n) \neq b(n)$) and $\boldsymbol{\delta}(a, b) = \sum_{n \in a \Delta b,\, n \geq 1} \frac{1}{n}$ — this can be a nonnegative real or $+\infty$. Generally, we define $\boldsymbol{\delta}_k^m(a, b) = \sum_{n \in a \Delta b\,,\, k \leq n \leq m} \frac{1}{n}$ for $1 \leq k \leq m$, and accordingly $\boldsymbol{\delta}_k^\infty(a, b) = \sum_{n \in a \Delta b\,,\, k \leq n < \infty} \frac{1}{n}$.

Define $\boldsymbol{\delta}(a) = \sum_{a(n)=1,\, n \geq 1} \frac{1}{n}$ and similarly $\boldsymbol{\delta}_k^m(a)$ and $\boldsymbol{\delta}_k^\infty(a)$.

Recall that the *summable ideal* is defined as

$$\mathscr{S}_{\{1/n\}} = \{a \in 2^{\mathbb{N}} : \boldsymbol{\delta}(a) < +\infty\}\,.$$

(The notation $\mathscr{I}_2$ and $\mathscr{I}_0$ is also used.) $\mathsf{S}_{\{1/n\}}$ will denote the associated Borel ER on $2^{\mathbb{N}}$, *i.e.*, $a \,\mathsf{S}_{\{1/n\}}\, b$ iff $\boldsymbol{\delta}(a, b) < +\infty$.

Suppose that $\vartheta : \mathbb{X} \to 2^{\mathbb{N}}$ is a Borel reduction of E to $\mathsf{S}_{\{1/n\}}$. We can assume that ϑ is in fact continuous. Indeed it is known that there is a stronger Polish topology on $\mathbb{X}$ which makes ϑ continuous but does not add new Borel subsets of $\mathbb{X}$. Moreover, as any Polish space $\mathbb{X}$ is a $1-1$ continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, we can assume that $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$.

Finally, we can assume that ϑ is Δ^1_1, not merely Borel.

If $a \in A \subseteq 2^{\mathbb{N}}$ and $q \in \mathbb{Q}^+$ then let $\mathtt{Gal}^q_A(a)$ be the set of all $b \in A$ such that there is a finite chain $a = a_0, a_1, \dots, a_n = b$ of reals $a_i \in A$ such that $\boldsymbol{\delta}(a_i, a_{i+1}) < q$ for all i, the *q-galaxy of a in A*.

Definition 13.2. A set $A \subseteq 2^{\mathbb{N}}$ is *q-"grainy"*, where $q \in \mathbb{Q}^+$, iff $\boldsymbol{\delta}(a, b) < 1$ for all $a \in A$ and $b \in \mathtt{Gal}^q_A(a)$. A set A is *"grainy"* if it is q-"grainy" for some $q \in \mathbb{Q}^+$. (In other words it is required that the galaxies are rather small.) □

Claim 13.3. *Any q-"grainy" Σ^1_1 set $A \subseteq 2^{\mathbb{N}}$ is covered by a q-"grainy" Δ^1_1 set.*

Proof. [1] The set $D_0 = \{b \in 2^{\mathbb{N}} : A \cup \{b\}$ is q-"grainy"$\}$ is Π^1_1 and $A \subseteq D_0$, hence, there is a Δ^1_1 set B_1 with $A \subseteq B_1 \subseteq D_0$. Note that $A \cup \{a\}$ is q-"grainy" for any $a \in B_1$. It follows that the Π^1_1 set

$$D_1 = \{b \in B_1 : A \cup \{a, b\} \text{ is } q\text{-"grainy" for any } a \in B_1\}$$

still contains A, hence, there is a Δ^1_1 set B_2 with $A \subseteq B_2 \subseteq D_1 \subseteq B_1$. Note that $A \cup \{a_1, a_2\}$ is q-"grainy" for any $a_1, a_2 \in B_2$. In general, as soon as we have got a Δ^1_1 set B_n with $A \subseteq B_n$ and such that $A \cup \{a_1, \dots, a_n\}$ is q-"grainy" for any $a_1, \dots, a_n \in B_n$, then the Π^1_1 set

$$D_n = \{b \in B_n : A \cup \{a_1, \dots, a_n, b\} \text{ is } q\text{-"grainy" for any } a_1, \dots, a_n \in B_n\}$$

[1] The result can be achieved as a routine application of a reflection principle, yet we would like to show how it works with a low level technique.

contains A, hence, there is a $\varDelta^1_1$ set B_{n+1} with $A \subseteq B_{n+1} \subseteq D_n \subseteq B_n$.

As usual in similar cases, the choice of the sets B_n can be made effective enough for the set $B = \bigcap_n B_n$ to be still $\varDelta^1_1$, not merely Borel. On the other hand, $A \subseteq B$ and B is q-"grainy". □ (*Claim*)

Coming back to the proof of the theorem, let C be the union of all "grainy" $\varDelta^1_1$ sets. An ordinary computation shows that C is $\varPi^1_1$. We have two cases.

Case 1: $\mathtt{ran}\,\vartheta \subseteq C$.

Case 2: otherwise.

13b Case 1

We are going to prove that, in this case, E is essentially countable. First note that, by Separation, there is a $\varDelta^1_1$ set $H^* \subseteq 2^{\mathbb{N}}$ with $\mathtt{ran}\,\vartheta \subseteq H^* \subseteq C$.

Fix a standard enumeration $\{W_e\}_{e\in E}$ of all $\varDelta^1_1$ subsets of $2^{\mathbb{N}}$, where, as usual, $E \subseteq \mathbb{N}$ is a $\varPi^1_1$ set. By Kreisel Selection, there exist $\varDelta^1_1$ functions $a \longmapsto e(a)$ and $a \longmapsto q(a)$, defined on H^*, such that for any $a \in H^*$ the $\varDelta^1_1$ set $W(a) = W_{e(a)}$ contains a and is $q(a)$-"grainy". The final point of our argument will be an application of Lemma 6.4, where ρ will be a derivate of the function $G(a) = \mathtt{Gal}^{q(a)}_{W(a)}(a)$. We prove

Claim 13.4. *If $a \in H^*$ then $\gamma_a = \{G(b) : b \in [a]_{\mathsf{S}_{\{1/n\}}} \cap H^*\}$ is at most countable.*

Proof. Otherwise there is a pair of $e \in E$ and $q \in \mathbb{Q}^+$ and an uncountable set $B \subseteq [a]_{\mathsf{S}_{\{1/n\}}} \cap H^*$ such that $q(b) = q$ and $e(b) = e$ for any $b \in B$ and $G(b') \neq G(b)$ for any two different $b, b' \in B$. Note that any $G(b)$, $b \in B$, is a q-galaxy in one and the same set $W(a) = W(b) = W_e$, therefore, if $b \neq b' \in B$ then $b' \notin G(b)$ and $\boldsymbol{\delta}(b, b') \geq q$. On the other hand, as $B \subseteq [a]_{\mathsf{S}_{\{1/n\}}}$, we have $\boldsymbol{\delta}(a, b) < +\infty$ for all $b \in B$, hence, there is m and a still uncountable set $B' \subseteq B$ such that $\boldsymbol{\delta}^\infty_m(a, b) < q/2$ for all $b \in B'$. Now take a pair of $b \neq b' \in B'$ with $b \restriction [0, m) = b' \restriction [0, m)$: then $\boldsymbol{\delta}(b, b') < q$, contradiction. □ (*Claim*)

It follows that $x \mapsto G(\vartheta(x))$ maps any E-class into a countable set of galaxies $G(a)$. To code the galaxies by single points, let $S(a) = \bigcup_m \{b \restriction m : b \in G(a)\}$. Thus $S(a) \subseteq 2^{<\omega}$ codes the Polish topological closure of the galaxy $G(a)$.

Claim 13.5. *If $a, b \in H^*$ and $\neg\ a\ \mathsf{S}_{\{1/n\}}\ b$ then b does not belong to the (topological) closure of $G(a)$, in particular, $b \restriction m \notin S(a)$ for some m.*

Proof. Take m big enough for $\boldsymbol{\delta}^{m-1}_0(a, b) \geq 2$. Then $s = b \restriction m$ does not belong to $S(a)$ because any $a' \in G(a)$ satisfies $\boldsymbol{\delta}(a, a') < 1$. □ (*Claim*)

Elementary computation shows that the sets

$$\mathbf{G} = \{\langle a, b\rangle : a \in H^* \wedge b \in G(a)\} \quad \text{and} \quad \mathbf{S} = \{\langle a, s\rangle : a \in H^* \wedge s \in S(a)\}\,.$$

belong to Σ^1_1, but this is not enough to claim that $a \mapsto S(a)$ is a Borel map. Yet we can change it appropriately to get a Borel map with similar properties. First of all define the following Σ^1_1 ER on H^*:

$$a \mathsf{F} b \qquad \text{iff} \qquad e(a) = e(b) \wedge q(a) = q(b) \wedge G(a) = G(b)\,.$$

(To see that F is Σ^1_1 note that here $G(a) = G(b)$ is equivalent to $b \in G(a)$, and that $\mathbf{G}$ is Σ^1_1.) It follows from Claim 13.5 and Kreisel Selection that there is a Δ^1_1 function $\mu : H^* \times H^* \to \mathbb{N}$ such that for any pair of $a, b \in H^*$ with $a \not\mathsf{S}_{\{1/n\}} b$ we have $b \restriction \mu(a, b) \notin S(a)$. Then the set

$$R(a) = \{b \restriction \mu(a', b) : a', b \in H^* \wedge a \mathsf{F} a' \wedge a' \not\mathsf{S}_{\{1/n\}} b)\} \subseteq 2^{<\omega}$$

does not intersect $S(a)$, for any $a \in H^*$, hence, the Σ^1_1 set

$$\mathbf{R} = \{\langle a, s\rangle : a \in H^* \wedge s \in R(a)\}$$

does not intersect $\mathbf{S}$. Note that by definition $\mathbf{R}$ is F-invariant w.r.t. the 1st argument, *i.e.*, if $a, a' \in H^*$ satisfy $a \mathsf{F} a'$ then $R(a) = R(a')$. It follows from Lemma 7.9 that there is a Δ^1_1 set $\mathbf{Q} \subseteq H^* \times 2^{<\omega}$ with $\mathbf{S} \subseteq \mathbf{Q}$ but $\mathbf{R} \cap \mathbf{Q} = \varnothing$, F-invariant in the same sense. Then the map $a \mapsto Q(a) = \{s : \mathbf{Q}(a, s)\}$ is Δ^1_1.

Claim 13.6. *Suppose that* $a, b \in H^*$. *Then*: $a \mathsf{F} b$ *implies* $Q(a) = Q(b)$ *and* $a \not\mathsf{S}_{\{1/n\}} b$ *implies* $Q(a) \neq Q(b)$.

Proof. The first statement holds just because Q is F-invariant. Now suppose that $a \not\mathsf{S}_{\{1/n\}} b$. Then by definition $s = b \restriction \mu(a, b) \in R(a)$, hence, $s \notin Q(a)$. On the other hand, $s \in S(b) \subseteq Q(b)$. □ (*Claim*)

Define $\tau(x) = Q(\vartheta(x))$ for $x \in \mathbb{N}^{\mathbb{N}}$, so that τ is a Δ^1_1 map $\mathbb{N}^{\mathbb{N}} \to \mathscr{P}(2^{<\omega})$.

Claim 13.7. *If* $x \in \mathbb{N}^{\mathbb{N}}$ *then* $T_a = \{\tau(y) : y \in [x]_{\mathsf{E}}\}$ *is at most countable.*

Proof. Suppose that $y, z \in [x]_{\mathsf{E}}$. Then $a = \vartheta(x)$, $b = \vartheta(y)$, and $c = \vartheta(z)$ belong to H^*, and $b, c \in [a]_{\mathsf{S}_{\{1/n\}}}$. It follows from Claim 13.6 that if $G(b) = G(c)$, $e(b) = e(c)$, and $q(b) = q(c)$, then $Q(b) = Q(c)$. It remains to note that G takes only countably many values on $H^* \cap [a]_{\mathsf{S}_{\{1/n\}}}$ by Claim 13.4. □ (*Claim*)

Finally note that, if $x \not\mathsf{E}\, y \in \mathbb{N}^{\mathbb{N}}$ then $\vartheta(x), \vartheta(y)$ belong to H^* and satisfy $\vartheta(x) \not\mathsf{S}_{\{1/n\}} \vartheta(y)$, hence, $\tau(x) \neq \tau(y)$ by Claim 13.6. Thus, the Borel map τ witnesses that the given ER E is essentially countable by Lemma 6.4.

13c Case 2

Thus we suppose that the Σ^1_1 set $B^* = \mathtt{ran}\,\vartheta \smallsetminus C$ is non-empty. Note that, by Claim 13.3, there is no non-empty Σ^1_1 "grainy" set $A \subseteq B^*$.

Let $\mathscr{B}_s = \{a \in 2^{\mathbb{N}} : s \subset a\}$ for $s \in 2^{<\omega}$ and $\mathscr{N}_u = \{x \in \mathbb{N}^{\mathbb{N}} : u \subset x\}$ for $u \in \mathbb{N}^{<\omega}$ (basic open nbhds in $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$).

If $A, B \subseteq 2^{\mathbb{N}}$ and $m, k \in \mathbb{N}$, then $A \mathsf{R}^m_{\geq k} B$ will mean that for any $a \in A$ there is $b \in B$ with $\boldsymbol{\delta}^\infty_k(a,b) < 2^{-m}$, and conversely, for any $b \in B$ there is $a \in A$ with $\boldsymbol{\delta}^\infty_k(a,b) < 2^{-m}$. This is not a ER, of course, yet the conjunction of $A \mathsf{R}^m_{\geq k} B$ and $B \mathsf{R}^m_{\geq k} C$ implies $A \mathsf{R}^{m-1}_{\geq k} C$.

0^m will denote the sequence of m zeros.

To prove that $\mathsf{S}_{\{1/n\}} \leq_{\mathrm{B}} \mathsf{E}$ in Case 2, we define an increasing sequence of natural numbers $0 = k_0 < k_1 < k_2 < \dots$, and also objects A_s, g_s, v_s for any $s \in 2^{<\omega}$, which satisfy the following list of requirements (i) – (ix).

(i) if $s \in 2^m$ then $g_s \in 2^{k_m}$, and $s \subset t \Longrightarrow g_s \subset g_t$;

(ii) $\varnothing \neq A_s \subseteq B^* \cap \mathscr{B}_{g_s}$, A_s is Σ^1_1, and $s \subset t \Longrightarrow A_t \subseteq A_s$.

(iii) if $s \in 2^n$ then $A_{0^n} \mathsf{R}^{n+2}_{\geq k_n} A_s$;

(iv) if $s \in 2^n$, $m < n$, $s(m) = 0$, then $\boldsymbol{\delta}^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) < 2^{-m-1}$;

(v) if $s \in 2^n$, $m < n$, $s(m) = 1$, then $|\boldsymbol{\delta}^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) - \frac{1}{m+1}| < 2^{-m-1}$;

(vi) if $s, t \in 2^n$, $m < n$, $s(m) = t(m)$, then $|\boldsymbol{\delta}^{k_{m+1}-1}_{k_m}(g_s, g_t)| < 2^{-m}$;

(vii) for any n, a certain condition, in terms of the Gandy – Harrington forcing, similar to (ii) in Section 7b or (ii) in Section 7e, related to all sets A_s, $s \in 2^n$, so that, as a consequence, $\bigcap_n A_{a \restriction n} \neq \varnothing$ for any $a \in 2^{\mathbb{N}}$;

(viii) if $s \in 2^n$ then $v_s \in \mathbb{N}^n$, and $s \subset t \Longrightarrow v_s \subset v_t$;

(ix) $A_s \subseteq \{a \in B^* : \vartheta^{-1}(a) \cap \mathscr{N}_{v_s} \neq \varnothing\}$.

We can now accomplish Case 2 as follows. For any $a \in 2^{\mathbb{N}}$ define $F(a) = \bigcup_n g_{a \restriction n} \in 2^{\mathbb{N}}$ (the only element satisfying $g_{a \restriction n} \subset F(a)$ for all n) and $\rho(a) = \bigcup_n v_{a \restriction n} \in \mathbb{N}^{\mathbb{N}}$. It follows, by (ix) and the continuity of ϑ, that $F(a) = \vartheta(\rho(a))$ for any $a \in 2^{\mathbb{N}}$. Thus the next claim proves that ρ is a Borel (in fact, here continuous) reduction $\mathsf{S}_{\{1/n\}}$ to E and ends Case 2.

Lemma 13.8. *The map F reduces $\mathsf{S}_{\{1/n\}}$ to $\mathsf{S}_{\{1/n\}}$, that is, the equivalence $a \mathsf{S}_{\{1/n\}} b \Longleftrightarrow F(a) \mathsf{S}_{\{1/n\}} F(b)$ holds for all $a, b \in 2^{\mathbb{N}}$.*

Proof. By definition $\boldsymbol{\delta}(F(a), F(b)) = \mathtt{lim}_{n\to\infty} \boldsymbol{\delta}^{k_n-1}_0(g_{a\restriction n}, g_{b\restriction n})$. However it follows from (iv), (v), (vi) that

$$|\boldsymbol{\delta}^{k_n-1}_0(g_{a\restriction n}, g_{b\restriction n}) - \boldsymbol{\delta}^{n-1}_0(a \restriction n, b \restriction n)| \leq \textstyle\sum_{m<n} 2^{-m} < 2\,.$$

We conclude that $|\boldsymbol{\delta}(F(a), F(b)) - \boldsymbol{\delta}(a,b)| \leq 2$, as required. □ (*Lemma*)

13d Construction

The construction of a system of sets satisfying (i) – (ix) goes on by induction. To begin with we set $k_0 = 0$, $g_\Lambda = \Lambda$ and $A_\Lambda = B^*$. Suppose that, for some n, we have the objects as required for all $n' \le n$, and extend the construction on the level $n+1$.

As A_{0^n} is not "grainy" (see above), there is a pair of elements $a^0, a^1 \in A_{0^n}$ such that $|\boldsymbol{\delta}(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2}$. Note that $a^0 \restriction k_n = a^1 \restriction k_n$ by (i) and (ii), therefore there is $k_{n+1} > k_n$ such that $|\boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2}$. According to (iii), for any $s \in 2^n$ there exist $b_s^0, b_s^1 \in A_s$ such that and $\boldsymbol{\delta}_{k_n}^{\infty}(a^i, b_s^i) < 2^{-n-2}$ for $i = 0, 1$; we can, of course, assume that $b_{0^n}^i = a^i$. Moreover, the number k_{n+1} can be chosen big enough for the following to hold:

$$\boldsymbol{\delta}_{k_{n+1}}^{\infty}(b_s^i, a^0) < 2^{-n-3} \quad \text{— for all} \quad s \in 2^n \quad \text{and} \quad i = 0, 1. \tag{1}$$

We let $g_{s^\wedge i} = b_s^i \restriction k_{n+1}$ for all $s^\wedge i \in 2^{n+1}$. This definition preserves (i). To check (iv) for $s' = s^\wedge 0 \in 2^{n+1}$ and $m = n$, note that

$$\boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(g_{s'}, g_{0^{n+1}}) = \boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(b_s^0, a^0) < 2^{-n-2}.$$

To check (v) for $s' = s^\wedge 1 \in 2^{n+1}$ and $m = n$, note that

$$|\boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(g_{s'}, g_{0^{n+1}}) - \tfrac{1}{n+1}| \le \boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(b_s^1, a^1) + |\boldsymbol{\delta}_{k_n}^{k_{n+1}-1}(a^0, a^1) - \tfrac{1}{n+1}| < 2^{-n-1}.$$

To fulfill (viii), choose, for any $s^\wedge i \in 2^{n+1}$, a sequence $v_{s^\wedge i} \in \mathbb{N}^{n+1}$ so that $v_s \subset v_{s^\wedge i}$ and there is $\mathscr{N}_{v_{s^\wedge i}} \cap \vartheta^{-1}(b_s^i) \ne \varnothing$.

Let us finally define the sets $A_{s'} \subseteq A_s$, for all $s' = s^\wedge i \in 2^{n+1}$ (so that $s \in 2^n$ and $i = 0, 1$). To fulfill (ii) and (ix), we begin with

$$A'_{s^\wedge i} = \{a \in A_s \cap \mathscr{B}_{g_{s^\wedge i}} : \vartheta^{-1}(a) \cap \mathscr{N}_{v_{s^\wedge i}} \ne \varnothing\}.$$

This is a Σ_1^1 subset of A_s, containing b_s^i. To fulfill (iii), we define $A_{0^{n+1}}$ to be the set of all $a \in A'_{0^{n+1}}$ such that

$$\forall\, s' = s^\wedge i \in 2^{n+1}\ \exists\, b \in A'_{s'}\ \big(\boldsymbol{\delta}_{k_{n+1}}^{\infty}(a, b) < 2^{-n-3}\big)\,;$$

this is still a Σ_1^1 set containing $b_{0^n}^0 = a^0$ by (1). It remains to define, for any $s^\wedge i \ne 0^{n+1}$, $A_{s^\wedge i}$ to be the set of all $b \in A'_{s^\wedge i}$ such that

$$\exists\, b \in A_{0^{n+1}}\ \big(\boldsymbol{\delta}_{k_{n+1}}^{\infty}(a, b) < 2^{-n-3}\big)\,.$$

This ends the definition for the level $n+1$.

□ (*Construction and Theorem 13.1*)

Chapter 14

$\mathbf{c}_0$-equalities

Recall that the equivalence relation c_0 is defined on $\mathbb{R}^{\mathbb{N}}$ as follows: $x \;\mathsf{c}_0\; y$ iff $x(n) - y(n) \to 0$ with $n \to \infty$. This definition admits a straightforward generalization.

Definition 14.1 (Farah [9]). Suppose that K is a non-empty index set, and $\langle X_k\,; d_k\rangle$ is a metric space for any index $k \in K$. An equivalence relation [1] $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle_{k\in K})$ on the cartesian product $X = \prod_k X_k$ is defined so that $x \;\mathsf{D}\; y$ iff $\mathtt{lim}\; d_n(x(n), y(n)) = 0$, where the limit is associated with the filter of all finite subsets of K. [2]

If $K = \mathbb{N}$ (the most typical case below) then we'll write $\mathsf{D}(X_k\,; d_k)$ instead of $\mathsf{D}(\langle X_k\,; d_k\rangle_{k\in\mathbb{N}\in\mathbb{N}})$ for the sake of brevity.

We'll be mostly interested in the case when

(∗) X_k are Borel sets in Polish spaces $\mathbb{X}_k$, and the distance functions d_k are Borel maps $X_k \times X_k \to \mathbb{R}^+$, not necessarily equal to the restrictions of Polish metrics of $\mathbb{X}_k$.

Then $\mathsf{D}(X_k\,; d_k)$ is obviously a Borel equivalence relation on $X = \prod_k X_k$.

The equivalence relation $\mathsf{D}(X_k\,; d_k)$ is *nontrivial* if $\mathtt{lim\,sup}_{k\to\infty}\,\mathtt{diam}(X_k) > 0$. (Otherwise $\mathsf{D}(X_k\,; d_k)$ obviously makes everything equivalent.)

A c_0-*equality* is any equivalence relation of the form $\mathsf{D}(\langle X_k\,; d_k\rangle_{k\in\mathbb{N}})$, where all sets X_k are finite. □

Every c_0-equality is easily a Borel equivalence relation, more exactly, of type $\mathbf{\Pi}^0_3$. The equivalence relation c_0 itself is essentially a c_0-equality (see below) — this explains the meaning of the term "c_0-equality".

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[1] The letter D in this context is due to Farah [9]. One has to suppress any association with the diagonal, *i.e.*, the true equality.

[2] Thus $\mathtt{lim}\; d_n(x(n), y(n)) = 0$ iff for any $\varepsilon > 0$ there exist only finitely many indices $k \in K$ such that $d_n(x(n), y(n)) > \varepsilon$.

The $\le_{\rm B}$-properties of these ERs are largely unknown, except for the case of σ-compact metric spaces $\langle X_k\,; d_k\rangle$, easily reducible to the case of X_k finite (= $\mathbf{c}_0$-equalities). This case is presented in this Chapter. We prove that Borel reducibility of a $\mathbf{c}_0$-equality to another one implies a stronger additive reducibility of an infinitely generated $\mathbf{c}_0$-subequality (Theorem 14.6), show that $\mathbf{c}_0$ is a $\le_{\rm B}$-maximal $\mathbf{c}_0$-equality (Theorem 14.7), prove Theorem 14.9 that shows the turbulence of $\mathbf{c}_0$-equalities except those $\sim_{\rm B}$-equivalent to E_0 and E_3, and finally show that the $\le_{\rm B}$-structure of $\mathbf{c}_0$-equalities includes a substructure similar to $\langle \mathscr{P}(\mathbb{N})\,; \subseteq^*\rangle$ (Theorem 14.12).

14a Some examples and simple results

The following examples show that many typical equivalence relations can be defined in the form of $\mathbf{c}_0$-equalities.

Example 14.2. (i) Let $X_k = \{0,1\}$ with $d_k(0,1) = 1$ for all k. Then clearly the relation $\mathsf{D}(X_k\,; d_k)$ on $2^{\mathbb{N}} = \prod_k \{0,1\}$ is just E_0.

(ii) Let $X_{kl} = \{0,1\}$ with $d_{kl}(0,1) = k^{-1}$ for all $k, l \in \mathbb{N}$. Then the relation $\mathsf{D}(\langle X_{kl}\,; d_{kl}\rangle_{k,l\in\mathbb{N}})$ on $2^{\mathbb{N}\times\mathbb{N}} = \prod_{k,l}\{0,1\}$ is exactly E_3.

(iii) Generally, if $0 = n_0 < n_1 < n_2 < \dots$ and φ_i is a submeasure on $[n_i, n_{i+1})$, then let $X_i = \mathscr{P}([n_i, n_{i+1}))$ and $d_i(u,v) = \varphi_i(u \,\Delta\, v)$ for $u, v \subseteq [n_i, n_{i+1})$. Then $\mathsf{D}(X_i\,; d_i)$ is isomorphic to $\mathsf{E}_{\mathscr{I}}$, where

$$\mathscr{I} = \mathtt{Exh}(\varphi) = \{x \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(x \cap [n,\infty)) = 0\}$$

and $\varphi(x) = \mathtt{sup}_i\, \varphi_i(x \cap [n_i, n_{i+1}))$.

(iv) Let, for all k, $X_k = \mathbb{R}$ with d_k being the usual distance on $\mathbb{R}$. Then the relation $\mathsf{D}(\langle X_k\,; d_k\rangle_{k\in\mathbb{N}})$ on $\mathbb{R}^{\mathbb{N}}$ is just $\mathbf{c}_0$. □

Lemma 14.3 (Farah [9] with a reference to Hjorth). *Every $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(X_k\,; d_k)$ is induced by a continuous action of a Polish group.*

The domain $\mathbb{X} = \prod_k X_k$ of D is considered with the product topology.

Proof (sketch). For any k let S_k be the (finite) group of all permutations of X_k, with the distance $\rho_k(s,t) = \mathtt{max}_{x\in X_k}\, d_k(s(x), t(x))$. Then

$$\mathbb{G} = \{g \in \textstyle\prod_k S_k : \lim_{k\to\infty} \rho_k(g_k, e_k) = 0\}\,, \quad \text{where } e_k \in S_k \text{ is the identity}\,,$$

is easily a subgroup of $\prod_k S_k$. Moreover, the distance $d(g,h) = \mathtt{sup}_k\, \rho_k(g_k, h_k)$ converts $\mathbb{G}$ into a Polish group, the natural action of which on $\mathbb{X}$, that is, $(g\cdot x)_k = g_k(x_k)$, $\forall\, k$, is continuous and induces D. □

Let us finally show that the case of σ-compact spaces X_k does not give anything beyond the case of $\mathbf{c}_0$-equalities.

Lemma 14.4. *Suppose that in the assumptions of 14.1(∗) $\langle X_k\,; d_k\rangle$ are σ-compact spaces. Then $\mathsf{D}(X_k; d_k)$ is $\sim_{\mathrm{B}}$-equivalent to a $\mathbf{c}_0$-equality.*

Proof. Suppose that all spaces X_k are compact. Then for any k there exists a finite $\frac{1}{k}$-net $X'_k \subseteq X_k$. Given $x \in X = \prod_k X_k$, we define $\vartheta(x) \in X' = \prod_k X'_k$ so that $\vartheta(x)(k)$ is the d_k-closest to $x(k)$ element of X'_k (or the least, in the sense of a fixed ordering of X'_k, of such closest elements, whenever there exist two or more of them) for each k. Then ϑ is obviously a Borel reduction of $\mathsf{D}(X_k; d_k)$ to the $\mathbf{c}_0$-equality $\mathsf{D}(X'_k; d_k)$.

The general σ-compact case can be reduced to the compact case by the same trick as in the beginning of the proof of Lemma 5.5. □

14b $\mathbf{c}_0$-equalities and additive reducibility

The structure of $\mathbf{c}_0$-equalities tend to be connected more with the additive reducibility $\le_{\mathrm{A}}$ than with the general Borel reducibility.[3] In particular, we have

Lemma 14.5. *For any $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$, if D' is a Borel equivalence relation on a set of the form $\prod_k X'_k$ with finite nonempty factors X'_k, and $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ then D' itself is a $\mathbf{c}_0$-equality.*

Proof. Let a sequence $0 = n_0 < n_1 < n_2 < \dots$ and a collection of maps $H_i : X'_i \to \prod_{n_i \le k < n_{i+1}} X_k$ witness $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$. For $x', y' \in X'_i$ put

$$d'_i(x', y') = \max_{n_i \le k < n_{i+1}} d_k(H_i(x')_k, H_i(y')_k)\,.$$

Then easily $\mathsf{D}' = \mathsf{D}(\langle X'_k; d'_k\rangle_{k\in\mathbb{N}})$. □

It is perhaps not true that $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$ implies $\mathsf{D} \le_{\mathrm{A}} \mathsf{D}'$ for any pair of $\mathbf{c}_0$-equalities. Yet a somewhat weaker statement holds by the next theorem of Farah [9].

Theorem 14.6. *If $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ and $\mathsf{D}' = \mathsf{D}(\langle X'_k; d'_k\rangle_{k\in\mathbb{N}})$ are $\mathbf{c}_0$-equalities and $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$ then there is an infinite set $A \subseteq \mathbb{N}$ such that the $\mathbf{c}_0$-equality $\mathsf{D}_A = \mathsf{D}(\langle X_k; d_k\rangle_{k\in A})$ satisfies $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}'$.*

Proof. Define $X_C = \prod_{k\in C} X_k$ and $X'_C = \prod_{k\in C} X'_k$ for any set $C \subseteq \mathbb{N}$, and $d'_C(x,y) = \sup_{k\in C} d'_k(x(k), y(k))$ for all $x, y \in X'$. Suppose that

$$\vartheta : X = \textstyle\prod_{k\in\mathbb{N}} X_k \to X' = \prod_{k\in\mathbb{N}} X'_k$$

is a Borel reduction of D to D'. Then there exists an infinite set $A' \subseteq \mathbb{N}$ such that $\mathsf{D}(\langle X_k; d_k\rangle_{k\in A'}) \le_{\mathrm{C}} \mathsf{D}'$ (via a continuous reduction) — this can be proved

[3] See Section 4b on $\le_{\mathrm{A}}$ and the associated relations $<_{\mathrm{A}}$ and $\sim_{\mathrm{A}}$.

analogously to the second claim of Lemma 4.3. Thus it can be accumed from the beginning that ϑ is a continuous reduction of D to D'.

To extract an additive reduction, we employ a version of the stabilizers construction used in the proof of Theorem 5.8(i). In fact our task here is somewhat simpler because the given countinuity of ϑ allows us to avoid the Cohen genericity arguments.

Put $[s] = \{x \in X : x \restriction u = s\}$ for any $u \subseteq \mathbb{N}$ and $s \in X_u$. Consider the closed set $W = \bigcap_{i\in\mathbb{N}} [s_i]$ of all points $x \in X$ such that $x \restriction (n_i, n_{i+1}) = s_i$ for all i. Arguing approximately as in the proof of Theorem 5.8(i), we can define an increasing sequence $0 = k_0 = n_0 < k_1 < n_1 < k_2 < n_2 < \dots$ and elements $s_i \in X_{(n_i,n_{i+1})}$ such that for all $u, v \in X_{[0,n_i]}$ and all $x, y \in X_{[n_{i+1},\infty)}$ satisfying $x \restriction (n_j, n_{j+1}) = y \restriction (n_j, n_{j+1}) = s_j$ for all indices $j > i$ and $u \restriction (n_j, n_{j+1}) = v \restriction (n_j, n_{j+1}) = s_j$ for all indices $j < i$, [4] the following holds:

(a) $\vartheta(u \cup s_i \cup x) \restriction [0, k_{i+1}) = \vartheta(u \cup s_i \cup y) \restriction [0, k_{i+1})$, and

(b) $d_{[k_{i+1},\infty)}(\vartheta(u \cup s_i \cup x), \vartheta(v \cup s_i \cup x)) < \frac{1}{i}$.

Put $A = \{n_i : i \in \mathbb{N}\}$ and fix any $z \in X_A$. For any i, if $\xi \in X_{n_i}$ then define $z^{i\xi} \in W$ so that $z^{i\xi}(n_i) = \xi$, $z^{i\xi}(n_j) = z(n_j)$ for all $j \neq i$, and $z^{i\xi} \restriction (n_j, n_{j+1}) = s_j$ for all j. If $x \in X_A$ then define $H(x) \in X'$ as follows:

$$H(x) \restriction [k_i, k_{i+1}) = \vartheta(z^{i,x(n_i)}) \restriction [k_i, k_{i+1}) \quad \text{for every} \quad i \in \mathbb{N}. \tag{1}$$

Clearly H is a continuous map from X_A to X' (in the sense of the Polish product topologies). Moreover for any i the value $H(x) \restriction [k_i, k_{i+1})$ obviously depends only on $x(n_i)$. Thus to accomplish the proof of the theorem we need only to prove that H is a reduction of D_A to D'.

For any $x \in X_A$ define $f(x) \in W$ so that $f(x) \restriction A = x$ and $f(x) \restriction (n_j, n_{j+1}) = s_j$ for all j. Then f is a reduction of D_A to D, therefore it suffices to prove that $\vartheta(f(x))\ \mathsf{D}'\ H(x)$ for every $x \in X_A$. For an arbitrary $i \geq 1$, let us show that

$$d'_{[k_i,k_{i+1})}(\vartheta(f(x)), H(x)) \leq 1/i\,. \tag{2}$$

The key fact is that by the construction the elements $a = f(x)$ and $b = z^{i,x(n_i)}$ of W satisfy $a \restriction (n_j, n_{j+1}) = b \restriction (n_j, n_{j+1}) = s_j$ for all j and in addition $a(n_i) = b(n_i) = x(n_i)$. Define an auxiliary element $c \in W$ by

$$c \restriction [0, n_i] = a \restriction [0, n_i] \quad \text{and} \quad c \restriction [n_{i+1}, \infty) = b \restriction [n_{i+1}, \infty).$$

Then $d'_{[k_i,k_{i+1})}(\vartheta(b), \vartheta(c)) \leq \frac{1}{i}$ by (b), and $\vartheta(a) \restriction [k_i, k_{i+1}) = \vartheta(c) \restriction [k_i, k_{i+1})$ by (a). (Note that (b) is applied in fact for the value $i-1$ instead of i.) It follows that $d'_{[k_i,k_{i+1})}(\vartheta(a), \vartheta(b)) \leq \frac{1}{i}$. However $H(x) \restriction [k_i, k_{i+1}) = \vartheta(b) \restriction [k_i, k_{i+1})$ by (1). This proves (2) as required. □

[4] Note that under this assumption the points $u \cup s_i \cup x$, $u \cup s_i \cup y$, $v \cup s_i \cup x$ mentioned in (a), (b), belong to W.

14c A maximal $\mathbf{c_0}$-equality

We define $\mathsf{c}_{\mathtt{max}} = \mathsf{D}(X_k\,; d_k)$, where $X_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ and d_k is the distance on X_k inherited from the real line $\mathbb{R}$. The next theorem says that $\mathsf{c}_{\mathtt{max}}$ is $\le_{\mathrm{B}}$-maximal among all $\mathbf{c_0}$-equalities. The proof will show that in fact $\mathsf{D} \le_{\mathrm{A}} \mathsf{c}_{\mathtt{max}}$ in (ii), in the sense of the additive reducibility.

Theorem 14.7 (Farah [9] with a reference to Oliver**).** (i) $\mathsf{c}_{\mathtt{max}} \sim_{\mathrm{B}} \mathbf{c_0}$;

(ii) *if* D *is a* $\mathbf{c_0}$*-equality then* $\mathsf{D} \le_{\mathrm{B}} \mathsf{c}_{\mathtt{max}}$.

It follows from (i) and Lemma 5.6 that $\mathsf{c}_{\mathtt{max}} \sim_{\mathrm{B}} \mathsf{Z_0}$.

Proof. (i) It is clear that $\mathsf{c}_{\mathtt{max}}$ is the same as $\mathbf{c_0} \restriction \mathbb{X}$, where $\mathbb{X} \subseteq \mathbb{R}^{\mathbb{N}}$ is defined as in the proof of Lemma 5.6, where it is also shown that $\mathbf{c_0} \sim_{\mathrm{B}} \mathbf{c_0} \restriction \mathbb{X}$.

(ii) To prove $\mathsf{D} \le_{\mathrm{B}} \mathsf{c}_{\mathtt{max}}$, it suffices by (i) to show that $\mathsf{D} \le_{\mathrm{B}} \mathbf{c_0}$. The proof is based on the following:

Claim 14.8. *Any finite n-element metric space $\langle X\,; d\rangle$ is isometric to an n-element subset of $\langle \mathbb{R}^n\,; \rho_n\rangle$, where ρ_n is the distance on $\mathbb{R}^n$ defined by* $\rho_n(x,y) = \mathtt{max}_{i<n} |x(i) - y(i)|$.

Proof of the claim. Let $X = \{x_1, \dots, x_n\}$. It suffices to prove that for any $k \ne l$ there is a set of reals $\{r_1, \dots, r_n\}$ such that $|r_k - r_l| = d(x_k, x_l)$ and

(∗) $|r_i - r_j| \le d_{ij} = d(x_i, x_j)$ for all i, j .

We can assume that $k = 1$ and $l = n$.

Step 1. There is a least number $h_1 \ge 0$ such that (∗) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-1 \text{ times}}, h\}$ for any $0 \le h \le h_1$. Then, for some index k, $1 \le k < n$, we have $h_1 - 0 = d_{kn}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-1$.

Step 2. Similarly, there is a least number $h_2 \ge 0$ such that (∗) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-2 \text{ times}}, h, h_1 + h\}$ for any $0 \le h \le h_2$. (For example, $h_2 = 0$ in the case when on step 1 we have one more index $k' \ne k$ such that $h_1 = d_{k'n}$.) Then, for some k, ν, $1 \le k < n-1 \le \nu \le n$, we have $h_2 - 0 = d_{k\nu}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-2$.

Step 3. Similarly, there is a least number $h_3 \ge 0$ such that (∗) holds for the reals $\{r_i\} = \{\underbrace{0, 0, \dots, 0}_{n-3 \text{ times}}, h, h_2 + h, h_1 + h_2 + h\}$ for any $0 \le h \le h_3$. Then again, for some k, ν, $1 \le k < n-2 \le \nu \le n$, we have $h_3 - 0 = d_{k\nu}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-3$.

Et cetera.

This process ends, after a number m $(m < n)$ steps, in such a way that the index k obtained at the final step is equal to 1. Then $(*)$ holds for the numbers $\{\underbrace{0,0,\dots,0}_{n-m \text{ times}}, r_{n-m+1}, r_{n-m+1}, \dots, r_n\}$, where $r_{n-m+j} = h_m + h_{m-1} + \dots + h_{m-j+1}$ for each $j = 1, \dots, m$. Moreover it follows from the construction that there is a decreasing sequence $n = k_0 > k_1 > k_2 > \dots > k_\mu = 1$ $(\mu \le m)$ such that $r_{k_i} - r_{k_{i+1}} = d_{k_{i+1},k_i}$ exactly for any i. Then $d_{1n} \le \sum_i r_{k_i} - r_{k_{i+1}}$ by the triangle inequality. But the right-hand side is a part of the sum $r_n = h_1 + \dots + h_m$, and hence $r_n \ge d_{1n}$. On the other hand we have $r_n \le d_{1n}$ by $*$. We conclude that $r_n = d_{1n}$, as required. □ (*Claim*)

We come back to the proof of (ii), that is, $\mathsf{D} \le_{\mathrm{B}} \mathbf{c}_0$ for an arbitrary $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(X_k; d_k)$ on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$, where each $\langle X_k\,; d_k\rangle$ is a finite metric space. Let n_k be the number of elements in X_k. Let, by the claim, $\eta_k : X_k \to \mathbb{R}^{n_k}$ be an isometric embedding of $\langle X_k\,; d_k\rangle$ into $\langle \mathbb{R}^{n_k}\,; \rho_{n_k}\rangle$. It easily follows that the map $\vartheta(x) = \eta_0(x_0)^\wedge\eta_1(x_1)^\wedge\eta_2(x_2)^\wedge\dots$ (from $\mathbb{X}$ to $\mathbb{R}^{\mathbb{N}}$) reduces D to $\mathbf{c}_0$. □ (*Theorem 14.7*)

14d Classification

Recall that for a metric space $\langle A\,; d\rangle$, a rational $q > 0$, and $a \in A$, the galaxy $\mathtt{Gal}^q_A(a)$ is the set of all $b \in A$ which can be connected with a by a finite chain $a = a_0, a_1, \dots, a_n = b$ with $d(a_i, a_{i+1}) < q$ for all i. Define, for $r > 0$,

$$\delta(r, A) = \mathtt{inf}\,\{q \in \mathbb{Q}^+ : \exists\, a \in A\,(\mathtt{diam}\,(\mathtt{Gal}^q_A(a)) \ge r)\}$$

(with the understanding that here $\mathtt{inf}\,\varnothing = +\infty$), and

$$\Delta(A) = \{d(a,b) : a \ne b \in A\}\,, \quad \text{so that} \quad \mathtt{diam}\,A = \mathtt{sup}(\Delta(A) \cup \{0\})\,.$$

Now suppose that $\mathsf{D} = \mathsf{D}(X_k; d_k)$ is a $\mathbf{c}_0$-equality on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$. The next theorem of Farah [9] shows that basic properties of D in the $\le_{\mathrm{B}}$-structure of Borel ERs are determined by the following two conditions:

(co1) $\mathtt{lim\,inf}_{k\to\infty}\,\delta(r, X_k) = 0$ for some $r > 0$.

(co2) $\forall\,\varepsilon > 0\;\exists\,\varepsilon' \in (0,\varepsilon)\;\exists^\infty k\,\big(\Delta(X_k) \cap [\varepsilon', \varepsilon) \ne \varnothing\big)$.

Clearly (co1) implies both the nontriviality of $\mathsf{D}(X_k; d_k)$ and (co2).

Theorem 14.9. *Let* $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle_{k\in\mathbb{N}})$ *be a nontrivial* $\mathbf{c}_0$*-equality. Then:*

(i) *if* (co2) *fails* (then (co1) also fails) *then* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E}_0$;

(ii) *if* (co1) *fails but* (co2) *holds then* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E}_3$;

(iii) *if* (co1) *holds* (then (co2) also holds) *then there exists a turbulent* $\mathbf{c}_0$-*equality* D' *satisfying* $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{D}'$ *and* $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$.

Thus any nontrivial $\mathbf{c}_0$-equality D $\le_{\mathrm{B}}$-contains a turbulent $\mathbf{c}_0$-equality D' with $\mathsf{E}_3 <_{\mathrm{B}} \mathsf{D}'$ unless D is $\sim_{\mathrm{B}}$-equivalent to either E_0 or E_3. In addition, (co1) is necessary for the turbulence of D itself and sufficient for a turbulent $\mathbf{c}_0$-equality $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$ to exist. The proof will show that in fact $\le_{\mathrm{B}}$ can be improved to $\le_{\mathrm{A}}$ in the theorem.

Proof. (i) To show that $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{D}$ note that, by the nontriviality of D, there exist: a number $p > 0$, an increasing sequence $0 = n_0 < n_1 < n_2 < \dots$, and, for any i, a pair of elements $x_{n_i}, y_{n_i} \in X_{n_i}$ with $d_{n_i}(x_{n_i}, y_{n_i}) \ge p$. For n not of the form n_i fix an arbitrary $z_n \in X_n$. Now, if $a \in 2^{\mathbb{N}}$, then define $\vartheta(a) \in \prod_k X_k$ so that $\vartheta(a)(n) = z_n$ for n not of the form n_i, while $\vartheta(a)(n_i) = x_{n_i}$ or $= y_{n_i}$ if resp. $a_i = 0$ or $= 1$. This map ϑ witnesses $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{D}$.

Now prove that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}_0$. As (co2) fails, there is $\varepsilon > 0$ such that for each ε' with $0 < \varepsilon' < \varepsilon$ we have only finitely many k with the propery that $\varepsilon' \le d_k(\xi, \eta) < \varepsilon$ for some $\xi, \eta \in X_k$. Let G_k be the (finite) set of all $\frac{\varepsilon}{2}$-galaxies in X_k, and let $\vartheta : \mathbb{X} = \prod_k X_k \to G = \prod_k G_k$ be defined as follows: for every k, $\vartheta(x)(k)$ is that galaxy in G_k to which $x(k)$ belongs. Let E be the G-version of E_0, that is, if $g, h \in G$ then $g \mathrel{\mathsf{E}} h$ iff $g(k) = h(k)$ for all but finite k. As easily $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$, it suffices to demonstrate that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}$ via ϑ.

Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$ and prove $x \mathrel{\mathsf{D}} y$ (the nontrivial direction). Suppose towards the contrary that $x \not\mathrel{\mathsf{D}} y$, so that there is a number $p > 0$ with $d_k(x(k), y(k)) > p$ for infinitely many k. We can assume that $p < \frac{\varepsilon}{2}$. On the other hand, as $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, there is k_0 such that $x(k)$ and $y(k)$ belong to one and the same $\frac{\varepsilon}{2}$-galaxy in X_k for all $k > k_0$. Then, for any $k > k_0$ with $d_k(x(k), y(k)) > p$ (and hence for infinitely many indices k) there exists an element $z_k \in X_k$ in the same galaxy such that $p < d_k(x(k), z_k) < \varepsilon$, but this is a contradiction to the choice of ε (indeed, take $\varepsilon' = p$).

(ii) First prove that if (co2) holds then $\mathsf{E}_3 \le_{\mathrm{B}} \mathsf{D}$. It follows from (co2) that there exist: an infinite sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots > 0$, for any i an infinite set $J_i \subseteq \mathbb{N}$, and for any $j \in J_i$ a pair of elements $x_{ij}, y_{ij} \in X_j$ with $d_j(x_{ij}, y_{ij}) \in [\varepsilon_{i+1}, \varepsilon_i)$. We may assume that the sets J_i are pairwise disjoint. Then the $\mathbf{c}_0$-equality $\mathsf{D}' = \mathsf{D}(\langle \{x_{ij}, y_{ij}\}; d_j \rangle_{i \in \mathbb{N},\, j \in J_i})$ satisfies both $\mathsf{D}' \le_{\mathrm{B}} \mathsf{D}$ and $\mathsf{D}' \cong \mathsf{E}_3$ (isomorphism via a bijection between the underlying sets).

Now, assuming that, in addition, (co1) fails, we show that $\mathsf{D} \le_{\mathrm{B}} \mathsf{E}_3$. For all k, $n \in \mathbb{N}$ let G_{kn} be the (finite) set of all $\frac{1}{n}$-galaxies in X_k. For any $x \in \mathbb{X} = \prod_i X_i$ define $\vartheta(x) \in G = \prod_{k,n} G_{kn}$ so that for any k, n $\vartheta(x)(k, n)$ is that $\frac{1}{n}$-galaxy in G_{kn} to which $x(k)$ belongs (for all k, n). The equivalence relation E on G, defined so that

$$g \mathrel{\mathsf{E}} h \quad \text{iff} \quad \forall n\, \forall^\infty k\, (g(k, n) = h(k, n)) \qquad (g, h \in G)$$

is obviously $\le_{\rm B} \mathsf{E}_3$, so it suffices to show that $\mathsf{D} \le_{\rm B} \mathsf{E}$ via ϑ. Suppose that x, $y \in \mathbb{X}$ and $\vartheta(x) \mathsf{E} \vartheta(y)$ and prove $x \mathsf{D} y$ (the nontrivial direction). Otherwise there is some $r > 0$ with $d_k(x(k), y(k)) > r$ for infinitely many indices k. As (co1) fails for this r, there is n big enough for $\delta(r, X_k) > \frac{1}{n}$ to hold for almost all k. Then, by the choice of r, we have $\vartheta(x)(k,n) \ne \vartheta(y)(k,n)$ for infinitely many k, hence, $\vartheta(x) \not\mathsf{E} \vartheta(y)$, contradiction.

(iii) Fix $r > 0$ with $\mathtt{liminf}_{k\to\infty} \delta(r, X_k) = 0$. For any increasing sequence $n_0 < n_1 < n_2 < \dots$ we have $\mathsf{D}(\langle X_{n_k}; d_{n_k}\rangle_{k\in\mathbb{N}}) \le_{\rm B} \mathsf{D}$. Therefore it can be assumed that $\mathtt{lim}_k\, \delta(r, X_k) = 0$, and further that $\delta(r, X_k) < \frac{1}{k}$ for all k. (Otherwise choose an appropriate subsequence.) Then every set X_k contains a $\frac{1}{k}$-galaxy $Y_k \subseteq X_k$ such that $\mathtt{diam}\, Y_k \ge r$. As easily $\mathsf{D}(Y_k; d_k) \le_{\rm B} \mathsf{D}$, the following lemma suffices to prove (iii).

Lemma 14.10. *Suppose that $r > 0$ and each X_k is a $\frac{1}{k}$-galaxy and $\mathtt{diam}(X_k) \ge r$. Then the $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(\langle X_k; d_k\rangle_{k\in\mathbb{N}})$ is turbulent and satisfies $\mathsf{E}_3 \le_{\rm B} \mathsf{D}$.*

Proof. We know from the proof of (iii) above that $\mathsf{E}_3 \le_{\rm B} \mathsf{D}$. Now prove that the natural action of the Polish group $\mathbb{G}$ defined as in the proof of Lemma 14.3 is turbulent under the assumptions of the lemma.

That every D-class is dense in $\mathbb{X} = \prod_k X_k$ (with the product topology on $\mathbb{X}$) is an easy exercise. To see that every D-class $[x]_{\mathsf{D}}$ also is meager in $\mathbb{X}$, note that by the assumptions of the lemma any X_k contains a pair of elements x'_k, x''_k with $d_k(x'_k, x''_k) \ge r$. Let y_k be one of x'_k, x''_k which is d_k-fahrer than $\frac{r}{2}$ from x_k. The set $Z = \{z \in \mathbb{X} : \exists^\infty k\, (z(k) = y_k)\}$ is comeager in $\mathbb{X}$ and disjoint from $[x]_{\mathsf{D}}$.

It remains to prove that local orbits are somewhere dense. Let G be an open nbhd of the neutral element in $\mathbb{G}$ and $\varnothing \ne X \subseteq \mathbb{X}$ be open in $\mathbb{X}$. We can assume that, for some n, G is the $\frac{1}{n}$-ball around the neutral element in $\mathbb{G}$ while $X = \{x \in \mathbb{X} : \forall\, k < n\, (x(k) = \xi_k)\}$, where elements $\xi_k \in X_k$, $k < n$, are fixed. It is enough to prove that all local orbits, *i.e.* equivalence classes of $\sim^G_X$, are dense subsets of X. Consider an open set $Y = \{y \in \mathbb{X} : \forall\, k < m\, (y(k) = \xi_k)\} \subseteq X$, where $m > n$ and elements $\xi_k \in X_k$, $n \le k < m$, are fixed in addition to the above.

Let $x \in X$. Then $x(k) = \xi_k$ for all $k < n$. Let $n \le k < m$. The elements ξ_k and $x(k)$ belong to X_k, which is a $\frac{1}{k}$-galaxy, therefore, there is a chain, of a length $\ell(k)$, of elements of X_k, which connects $x(k)$ to ξ_k so that every step within the chain has d_k-length $< \frac{1}{k}$. Then there is a permutation g_k of X_k such that $g_k^{\ell(k)}(x(k)) = \xi_k$, $g_k(\xi_k) = x(k)$, and $d_k(\xi, g_k(\xi)) < \frac{1}{k}$ for all $\xi \in X_k$.

In addition let g_k be the identity on X_k whenever $k < n$ or $k \ge m$. This defines an element $g \in \mathbb{G}$ which obviously belongs to G. Moreover, the set X is g-invariant and $g^\ell(x) \in U$, where $\ell = \prod_{k=n}^{m-1} \ell(k)$. It follows that $x \sim^G_X g(x)$, as required. □ (*Lemma*)

□ (*Theorem 14.9*)

14e LV-equalities

By Farah, an LV-*equality* is a $\mathbf{c}_0$-equality $\mathsf{D} = \mathsf{D}(\langle X_k ; d_k \rangle_{k\in\mathbb{N}})$ satisfying

$$\forall m\, \forall \varepsilon > 0\, \forall^\infty k\, \forall x_0, \dots, x_m \in X_k\ \big(d_k(x_0, x_m) \le \varepsilon + \max_{j<m} d_k(x_j, x_{j+1})\big). \quad (*)$$

In other words, the metrics involved are postulated to be *asymptotically close* to ultrametrics. This sort of $\mathbf{c}_0$-equalities was first considered by Louveau and Velickovic [43].

Exercise 14.11. Put $X_k = \{1, 2, \dots, 2^k\}$ and $d_k(m,n) = \frac{\log_2(|m-n|+1)}{k}$ for all k and $1 \le m, n \le 2^k$. Prove that $\mathsf{D}(X_k ; d_k)$ is an LV-equality and satisfies (co1) of Section 14d. □

The next theorem of Louveau and Velickovic [43] is a major application of $\mathbf{c}_0$-equalities. One of its corollaries is that there exist big families of mutually irreducible Borel equivalence relations, see below.

Theorem 14.12. *Let* $\mathsf{D} = \mathsf{D}(\langle X_k ; d_k \rangle_{k\in\mathbb{N}})$ *be an* LV-*equality satisfying* (co1) *of Section 14d. Then we can associate, with each infinite set* $A \subseteq \mathbb{N}$, *an* LV-*equality* $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}$ *such that for all* $A, B \subseteq \mathbb{N}$ *the following are equivalent*:

(i) $A \subseteq^* B$ (that is, $A \smallsetminus B$ is finite);

(ii) $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ (the additive reducibility);

(iii) $\mathsf{D}_A \le_{\mathrm{B}} \mathsf{D}_B$.

Proof. Since D is turbulent, the necessary turbulence condition (co1) of Section 14d holds. Moreover, as in the proof of Theorem 14.9 (part (iii)), we can assume that it takes the following special form for some $r > 0$:

(1) Each X_k is a $\mathtt{min}\{\frac{r}{2}, \frac{1}{k+1}\}$-galaxy and $\mathtt{diam}(X_k) \ge 4r$.

The intended transformations (reduction to a certain infinite subsequence of spaces $\langle X_k ; d_k \rangle$, and then of each X_k to a suitable galaxy $Y_k \subseteq X_k$) preserve the LV-condition $(*)$, of course. Moreover, we can assume that $(*)$ holds in the following special form:

(2) $d_k(x_0, x_{\mu_k}) \le \frac{1}{k+1} + \mathtt{max}_{i<\mu_k} d_k(x_i, x_{i+1})$ whenever $x_0, \dots, x_{\mu_k} \in X_k$, where $\mu_k = \prod_{j=0}^{k-1} \#(X_j)$ and $\#X$ is the number of elements in a finite set X.

(For if not then take a suitable subsequence once again.)

We can derive the following important consequence:

(3) For any k there is a set $Y_k \subseteq X_k$ having exactly $\#(Y_k) = \mu_k$ elements and such that $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k.

To prove this note that by (1) there is a set $\{x_0, \dots, x_m\} \subseteq X_k$ such that $d_k(x_0, x_m) \geq 4r$ but $d_k(x_i, x_{i+1}) < r$ for all i. We may assume that m is the least possible length of such a sequence $\{x_i\}$. Define a subsequence $\{y_0, y_1, \dots, y_n\}$ of $\{x_i\}$, the number $n \leq m$ will be specified in the course of the construction.

a) Put $y_0 = x_0$.

b) If $y_j = x_{i(j)}$ has been defined, and there is an index $l > i(j)$, $l \leq m$, such that $d_k(y_j, x_l) \geq r$, then let $y_{j+1} = x_l$ for the least such l.

   Note that in this case $d_k(y_j, y_{j+1}) < 2r$, for otherwise $d_k(y_j, x_{l-1}) > r$ because $d_k(x_{l-1}, x_l) < r$.

c) Otherwise put $n = j$ and stop the construction.

By definition $d_k(y_j, y_{j+1}) \geq r$ for all $j < n$, moreover, $d_k(y_{j'}, y_{j+1}) \geq r$ for any $j' < j$ by the minimality of m. Thus $Y_k = \{y_j : j \leq n\}$ satisfies $d_k(x, y) \geq r$ for all $x \neq y$ in Y_k. It remains to prove that $n \geq \mu_k$. Suppose otherwise. Add $y_{n+1} = x_m$ as an extra term. Then $d_k(x_0, x_m) = d_k(y_0, y_{n+1}) \leq 3r$ by (2) because $d_k(y_j, y_{j+1}) < 2r$ (see above). However we know that $d_k(x_0, x_m) \geq 4r$, contradiction. This proves (3).

In continuation of the proof of the theorem, define $\mathsf{D}_A = \mathsf{D}(\langle X_k; d_k \rangle_{k \in A})$ for any $A \subseteq \mathbb{N}$. Thus D_A is essentially a $\mathbf{c}_0$-equality on $\prod_{k \in A} X_k$. The direction (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is routine. Thus it remains to prove (iii) $\Longrightarrow$ (i). In view of Theorem 14.6, it is enough to prove the following lemma.

Lemma 14.13. *If $A, B \subseteq \mathbb{N}$ are infinite and disjoint then $\mathsf{D}_A \leq_{\mathrm{A}} \mathsf{D}_B$ fails.*

Proof. Suppose, towards the contrary, that $\mathsf{D}_A \leq_{\mathrm{A}} \mathsf{D}_B$ holds, and let this be witnessed by a reduction Ψ defined (as in Section 4b) from an increasing sequence $\mathtt{min}\, B = n_0 < n_1 < n_2 < \dots$ of numbers $n_k \in B$ and a collection of maps $H_k : X_k \to \prod_{m \in [n_k, n_{k+1}) \cap B} X_m$, $k \in A$. We put

$$f_k(\delta) = \max_{\xi, \eta \in X_k,\, d_k(\xi, \eta) < \delta} \quad \max_{m \in [n_k, n_{k+1}) \cap B} \quad d_m(H_k(\xi)(m), H_k(\eta)(m)),$$

for $k \in \mathbb{N}$ and $\delta > 0$ (with the understanding that $\mathtt{max}\, \varnothing = 0$ if applicable). Then $f(\delta) = \mathtt{sup}_{k \in A} f_k(\delta)$ is a nondecreasing map $\mathbb{R}^+ \to [0, \infty)$.

We claim that $\mathtt{lim}_{\delta \to 0} f(\delta) = 0$. Indeed otherwise there is $\varepsilon > 0$ such that $f(\delta) \geq \varepsilon$ for all δ. Then the numbers

$$s_k = \mathtt{min}_{\xi, \eta \in X_k,\, \xi \neq \eta}\, d_k(\xi, \eta) \quad \text{(all of them are } > 0\text{)}$$

must satisfy $\mathtt{inf}_{k \in A}\, s_k = 0$. This allows us to define a sequence $k_0 < k_1 < k_2 < \dots$ of numbers $k_i \in A$, and, for any k_i, a pair of elements $\xi_i, \eta_i \in X_{k_i}$ with $d_{k_i}(\xi_i, \eta_i) \to 0$, and also a number $m_i \in [n_{k_i}, n_{k_i+1}) \cap B$ such that $d_{m_i}(H_{k_i}(\xi_i)(m_i), H_{k_i}(\eta_i)(m_i)) \geq \varepsilon$. Let $x, y \in \prod_{k \in A} X_k$ satisfy $x(k_i) = \xi_i$ and

$y(k_i) = \eta_i$ for all i and $x(k) = y(k)$ for all $k \in A$ not of the form k_i. Then easily $x \mathsf{D}_A y$ holds but $\Psi(x) \mathsf{D}_B \Psi(y)$ fails, which is a contradiction. Thus in fact $\mathtt{lim}_{\delta\to 0} f(\delta) = 0$.

Let $k \in A$, and let $Y_k \subseteq X_k$ be as in (3). Then there exist elements $x_k \neq y_k$ in Y_k such that $H_k(x_k) \restriction k = H_k(y_k) \restriction k$. By (1) there is a chain $x_k = \xi_0, \xi_1, \dots, \xi_n = y_k$ of elements $\xi_i \in X_k$ with $d_k(\xi_i, \xi_{i+1}) \le \frac{1}{k+1}$ for all $i < n$. Now $H_k(\xi_i) \in \prod_{m\in[n_k,n_{k+1})\cap B} X_m$ for each $i \le n$.

Suppose that $m \in [n_k, n_{k+1}) \cap B$, and hence $m \ge n_k \ge k$. The elements $y_i^m = H_k(\xi_i)(m)$, $i \le n$, satisfy $d_m(y_i^m, y_{i+1}^m) \le f_k(\frac{1}{k+1})$. Note that $m \neq k$ because $k \in A$ while $m \in B$. Thus we have $m > k$ strictly. It follows that $n \le \mu_m$, therefore, by (2),

(4) $d_m(H_k(x_k)(m), H_k(y_k)(m)) \le f_k(\frac{1}{k+1}) + \frac{1}{m+1} \le f(\frac{1}{k+1}) + \frac{1}{k+1}$

for all $m \in [n_k, n_{k+1}) \cap B$.

Both $x = \{x_k\}_{k\in A}$ and $y = \{y_k\}_{k\in A}$ are elements of $\prod_{k\in A} X_k$, and $x \mathsf{D}_A y$ fails because $d_k(x_k, y_k) \ge r$ for all k. On the other hand, we have $\Psi(x) \mathsf{D}_B \Psi(y)$ by (4), because $\mathtt{lim}_{\delta\to 0} f(\delta) = 0$. This is a contradiction to the assumption that Ψ reduces D_A to D_B. □ (*Lemma 14.13*)

□ (*Theorem 14.12*)

14f Non-σ-compact case

For any metric space $\mathbb{X} = \langle X ; d\rangle$, let $\mathsf{D}(\mathbb{X})$ denote the equivalence relation $\mathsf{D}(X_k; d_k)$ on $X^{\mathbb{N}}$, where $\langle X_k ; d_k\rangle = \langle \mathbb{X} ; d\rangle$ for all k. Thus $\mathbf{c}_0$ is equal to $\mathsf{D}(\mathbb{R})$. One may ask what is the place of equivalence relations of the form $\mathsf{D}(\mathbb{X})$, where $\mathbb{X}$ is a Polish space, in the global $\le_{\mathrm{B}}$-structure of Borel equivalence relations?

The case of σ-compact Polish spaces here can be reduced to the case of finite spaces, *i.e.* to $\mathbf{c}_0$-equalities, by Lemma 14.4. Thus in this case we obtain a family of Borel ERs situated $\le_{\mathrm{B}}$-between the relations E_3 and $\mathbf{c}_0$ by Theorems 14.9 and 14.7, and this family has a rather rich $\le_{\mathrm{B}}$-structure by Theorem 14.12.

The case of non-σ-compact spaces is much less studied.

Example 14.14. Let $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$ be the Baire space, with the standard distance $d(a,b) = \frac{1}{m(a,b)+1}$, where $m(a,b)$ (for $a \neq b \in \mathbb{N}^{\mathbb{N}}$) is the largest integer m such that $a \restriction m = b \restriction m$. [5] If $x \in \mathbb{N}^{\mathbb{N}}$ and $n, k \in \mathbb{N}$ then $x(n) \restriction k$ is a finite sequence of k integers. It follows from the fact that $\mathbb{N}^{\mathbb{N}}$ is 0-dimensional that $x \,\mathsf{D}(\mathbb{N}^{\mathbb{N}})\, y$ is equivalent to

$$\forall n\, \exists k_0\, \forall k \ge k_0\; (x(n) \restriction k = y(n) \restriction k)\,.$$

for any $x, y \in \mathbb{N}^{\mathbb{N}}$. **Exercise:** use this to show that $\mathsf{D}(\mathbb{N}^{\mathbb{N}}) \sim_{\mathrm{B}} \mathsf{E}_3$. □

[5] Note that the relation $\mathsf{D}(\mathbb{X})$ depends on the metric rather than topological structure of a space $\mathbb{X}$, and hence it is, generally speaking, essential to specify a concrete distance compatible with the given topology.

Question 14.15. Now let $\mathbb{X}$ be the Polish space $C[0,1]$ of all continuous maps $f:[0,1]\to\mathbb{R}$, with the distance $d(f,g)=\mathtt{max}_{0\le x\le 1}|f(x)-g(x)|$. (This space is not σ-compact, of course. What is the position of $\mathsf{D}(C[0,1])$ in the global $\le_{\mathrm{B}}$-structure of Borel equivalence relations and what are its $\le_{\mathrm{B}}$-connections with such better known equivalence relations as E_i, $i=0,1,2,3$, and $\boldsymbol{\ell}^p$, $\mathbf{c}_0$? □

This question (see, *e.g.*, Su Gao [14]) remains open. The question is also connected with $\mathbf{c}_0$-equalities, in particular, with $\mathbf{c}_0$ itself from another side. Let us consider the following continual version C_0 of the equivalence relation $\mathbf{c}_0$. If f,g are continuous maps from $[0,+\infty)$ to $\mathbb{R}$ then we define $x\,\mathsf{C}_0\,y$ iff $\mathtt{lim}_{x\to+\infty}|f(x)-g(x)|=0$.

It is clear that any continuous map $f:[0,+\infty)\to\mathbb{R}$ can be identified with the sequence of its restrictions to intervals of the form $[n_n,n_{n+1})\tilde{t}n\in\mathbb{N}$, that is, with a certain point of the Polish product space $C[0,1]^{\mathbb{N}}$. With such an identification, the domain of C_0 is naturally identified with a certain Borel set in $C[0,1]^{\mathbb{N}}$, while C_0 itself is identified with a Borel equivalence relation, equal to $\mathsf{D}(C[0,1])$ on that set. (The domain of $\mathsf{D}(C[0,1])$ is the whole space $C[0,1]^{\mathbb{N}}$.) Question 14.15 also can be addressed to C_0.

Su Gao proved in [14] that C_0 (there defined as E_K) satisfies $\mathsf{C}_0\le_{\mathrm{B}}\mathbf{u}_0^*$, where $\mathbf{u}_0^*$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ defined as follows:

$$x\,\mathbf{u}_0^*\,y \qquad \text{iff} \qquad \forall\,\varepsilon>0\,\exists\,m_0\,\forall\,m\ge m_0\,\forall\,n\,(|x(m,n)-y(m,n|<\varepsilon)\,.$$

In addition, a more complicated Borel ER $\mathbf{u}_0$ on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ is defined in [14] such that $\mathsf{C}_0\sim_{\mathrm{B}}\mathbf{u}_0^*$. Investigations of $\mathbf{u}_0$, $\mathbf{u}_0^*$, C_0, $\mathsf{D}(C[0,1])$ remain work in progress.

Chapter 15

Pinned equivalence relations

In this Chapter we consider a class of equivalence relations E characterized by the property that if E has an equivalence class in a generic extension $\mathbb{V}^+$ of the ground set universe $\mathbb{V}$, definable in $\mathbb{V}^+$ in certain way in terms of sets in $\mathbb{V}$ as parameters then this equivalence class contains an element in $\mathbb{V}$. We call them *pinned* ERs.

The main goal will be to prove that certain families of Borel ERs are pinned, while on the other hand the equivalence relation T_2 of equality of countable sets of the reals is not pinned, and hence not Borel reducible to any pinned equivalence relation. The class of pinned ERs includes, for instance, continuous actions of CLI groups and some ideals, not necessarily Polishable, and is closed under the Fubini product modulo $\mathtt{Fin}$.

Recall that T_2 is defined on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ as follows: $x \mathrel{\mathsf{T}_2} y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$.

Definition 15.1. $\mathbb{V}$ will denote the ground set universe. In this Chapter we'll consider forcing extensions of $\mathbb{V}$. [1]

Suppose that X is $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$ in the universe $\mathbb{V}$, and an extension $\mathbb{V}^+$ of $\mathbb{V}$ is considered. In this case, let $X^\#$ denote what results by the definition of X applied in $\mathbb{V}^+$. There is no ambiguity here by the Shoenfield absoluteness theorem, and easily $X = X^\# \cap \mathbb{V}$. □

15a The definition of pinned equivalence relations

For instance, if, in the universe $\mathbb{V}$, E is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on a fixed polish space $\mathbb{X}$, then, still by the Shoenfield absoluteness, $\mathsf{E}^\#$ is a $\boldsymbol{\Sigma}^1_1$ ER on $\mathbb{X}^\#$. If now $x \in \mathbb{X}$ (hence, $x \in \mathbb{V}$) then the E-class $[x]_{\mathsf{E}} \subseteq \mathbb{X}$ of x (defined in $\mathbb{V}$) is included in a unique $\mathsf{E}^\#$-class $[x]_{\mathsf{E}^\#} \subseteq \mathbb{X}^\#$ (in $\mathbb{V}^+$). Classes of the form $[x]_{\mathsf{E}^\#}$, $x \in \mathbb{X}$, belong to a wider category of $\mathsf{E}^\#$-classes which admit a description from the point of view of the ground universe $\mathbb{V}$.

[1] Basically, a more rigorous treatment would be either to consider boolean-valued extensions of the universe, or to to assume that in fact $\mathbb{V}$ is a countable model in a wider universe.

Definition 15.2 (based on an argument in Hjorth [18]). Assume that E is a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space $\mathbb{X}$ and $\mathbb{P}$ is a notion of forcing in $\mathbb{V}$. A *virtual* E*-class* is any $\mathbb{P}$-term $\boldsymbol{\xi}$ such that $\mathbb{P}$ forces $\boldsymbol{\xi} \in \mathbb{X}^{\#}$ and $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\mathtt{left}} \, \mathsf{E}^{\#} \, \boldsymbol{\xi}_{\mathtt{right}}$. [2]

A virtual class is *pinned* if there is, in $\mathbb{V}$, a point $x \in \mathbb{X}$ which pins it, in the sense that $\mathbb{P}$ forces $x \, \mathsf{E}^{\#} \, \boldsymbol{\xi}$. Finally, E is *pinned* if, for any forcing notion $\mathbb{P} \in \mathbb{V}$, all virtual E-classes are pinned. □

If $\boldsymbol{\xi}$ is a virtual E-class then, in any extension $\mathbb{V}^+$ of $\mathbb{V}$, if U and V are generic subsets of $\mathbb{P}$ then $x = \boldsymbol{\xi}[U]$ and $y = \boldsymbol{\xi}[V]$ belong to $\mathbb{X}^{\#}$ and satisfy $x \, \mathsf{E}^{\#} \, y$, hence $\boldsymbol{\xi}$ induces a $\mathsf{E}^{\#}$-class in the extension. If $\boldsymbol{\xi}$ is pinned then this class contains an element in the ground universe $\mathbb{V}$ — in other words, pinned virtual classes induce $\mathsf{E}^{\#}$-equivalence classes of the form $[x]_{\mathsf{E}^{\#}}$, $x \in \mathbb{V}$, in the extensions of the universe $\mathbb{V}$.

The following theorem (originally [31, 30]) is the main result in this Chapter. Part (ii) here is from [18]. Part (iii) also belongs to Hjorth and is published with his permission.

Theorem 15.3. *The class of all pinned* $\mathbf{\Sigma}^1_1$ *equivalence relations*:

(i) *is closed under Fubini products modulo* Fin ;

and contains the following equivalences:

(ii) *all orbit ERs of Polish actions of (Polish)* CLI *groups on a Polish space*; [3]

(iii) *all Borel ERs, all of whose equivalence classes are* $\mathbf{G}_{\delta\sigma}$;

(iv) *all ERs of the form* $\mathtt{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}$, *where* φ_i *are lower semicontinuous (*LSC*) submeasures on* $\mathbb{N}$ *.*

On the other hand, T_2 *is not pinned and hence* T_2 *in Borel irreducible to any pinned equivalence relation.*

Quite recently, Thompson [62] proved that for a Polish group $\mathbb{G}$ to be CLI it is not only necessary (which is by (ii)) but also sufficient that all orbit equivalence relations of Polish actions of $\mathbb{G}$ are pinned.

15b T_2 is not pinned

Here we prove the last claim of Theorem 15.3.

[2] $\boldsymbol{\xi}_{\mathtt{left}}$ and $\boldsymbol{\xi}_{\mathtt{right}}$ are $\mathbb{P} \times \mathbb{P}$-terms meaning $\boldsymbol{\xi}$ associated with the resp. left and right factors $\mathbb{P}$ in the product forcing. Formally, $\boldsymbol{\xi}_{\mathtt{left}}[U \times V] = \boldsymbol{\xi}[U]$ and $\boldsymbol{\xi}_{\mathtt{right}}[U \times V] = \boldsymbol{\xi}[V]$ for any $\mathbb{P} \times \mathbb{P}$-generic set $U \times V$, where $\boldsymbol{\xi}[U]$ is the interpretation of a term $\boldsymbol{\xi}$ via a generic set U.

[3] Recall that a Polish group $\mathbb{G}$ is *complete left-invariant*, CLI for brevity, if $\mathbb{G}$ admits a compatible left-invariant complete metric.

Claim 15.4. T_2 *is not pinned.*

Proof. To prove that T_2 is not pinned, consider, in $\mathbb{V}$, the forcing notion $\mathbb{P} = \text{Coll}(\mathbb{N}, 2^{\mathbb{N}})$ to produce a generic map $f : \mathbb{N} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. ($\mathbb{P}$ consists of all functions $p : u \to 2^{\mathbb{N}}$ where $u \subseteq \mathbb{N}$ is finite.) The $\mathbb{P}$-term $\boldsymbol{\xi}$ for the set $\mathtt{ran}\, f = \{f(n) : n \in \mathbb{N}\}$ is obviously a virtual T_2-class, but it is not pinned because $2^{\mathbb{N}}$ is uncountable in the ground universe $\mathbb{V}$. □

Lemma 15.5. *If* E, F *are* $\boldsymbol{\Sigma}^1_1$ *ERs,* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$*, and* F *is pinned, then so is* E*.*

Proof. Suppose that, in $\mathbb{V}$, $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel reduction of E to F, where $\mathbb{X} = \mathtt{dom}\,\mathsf{E}$ and $\mathbb{Y} = \mathtt{dom}\,\mathsf{F}$. We can assume that $\mathbb{X}$ and $\mathbb{Y}$ are just two copies of $2^{\mathbb{N}}$. Let $\mathbb{P}$ be a forcing notion and a $\mathbb{P}$-term $\boldsymbol{\xi}$ be a virtual E-class. By the Shoenfield absoluteness, $\vartheta^{\#}$ is a reduction of $\mathsf{E}^{\#}$ to $\mathsf{F}^{\#}$ in any extension of $\mathbb{V}$, hence, $\boldsymbol{\sigma}$, a $\mathbb{P}$-term for $\vartheta^{\#}(\boldsymbol{\xi})$, is also a virtual F-class. Since F is pinned, there is $y \in \mathbb{Y}$ such that $\mathbb{P}$ forces $y\,\mathsf{F}^{\#}\,\boldsymbol{\sigma}$. Note that it is true in the $\mathbb{P}$-extension that $y\,\mathsf{F}^{\#}\,\vartheta^{\#}(x)$ for some $x \in \mathbb{X}^{\#}$, hence, by the Shoenfield theorem, in the ground universe there is $x \in \mathbb{X}$ with $y\,\mathsf{F}\,\vartheta(x)$. Clearly $\mathbb{P}$ forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$. □

15c Fubini product of pinned ERs is pinned

Here we prove part (i) of Theorem 15.3. Recall that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ of ERs E_k on $\mathbb{N}^{\mathbb{N}}$ modulo $\mathtt{Fin}$ is an equivalence relation on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $x\,\mathsf{E}\,y$ if $x(k)\,\mathsf{E}_k\,y(k)$ for all but finite k.

Suppose that $\boldsymbol{\Sigma}^1_1$ equivalence relations E_k on Polish spaces $\mathbb{X}_k$ are pinned. Prove that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ is a pinned ER (on the Polish space $\mathbb{X} = \prod_k \mathbb{X}_k$). Consider a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-term $\boldsymbol{\xi}$. Assume that $\boldsymbol{\xi}$ is a virtual E-class. There is a number k_0 and conditions $p, q \in \mathbb{P}$ such that $\langle p, q\rangle$ $\mathbb{P} \times \mathbb{P}$-forces $\boldsymbol{\xi}_{\mathtt{left}}(k)\,\mathsf{E}_k{}^{\#}\,\boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. As all E_k are ERs, we conclude that the condition $\langle p, p\rangle$ also forces $\boldsymbol{\xi}_{\mathtt{left}}(k)\,\mathsf{E}_k{}^{\#}\,\boldsymbol{\xi}_{\mathtt{right}}(k)$ for all $k \ge k_0$. Therefore, since E_k are pinned, there is in $\mathbb{V}$ a sequence of points $x_k \in \mathbb{X}_k$ such that p $\mathbb{P}$-forces $x_k\,\mathsf{E}_k{}^{\#}\,\boldsymbol{\xi}(k)$ for any $k \ge k_0$. Let $x \in \mathbb{X}$ satisfy $x(k) = x_k$ for all $k \ge k_0$. (The values $x(k) \in \mathbb{X}_k$ for $k < k_0$ can be arbitrary.) Then p obviously $\mathbb{P}$-forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$.

It remains to show that just every $q \in \mathbb{P}$ also forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$. Suppose otherwise, that is, some $q \in \mathbb{P}$ forces that $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$ fails. Consider the pair $\langle p, q\rangle$ as a condition in $\mathbb{P} \times \mathbb{P}$: it forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\mathtt{left}}$ and $\neg\, x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\mathtt{right}}$, as well as $\boldsymbol{\xi}_{\mathtt{left}}\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\mathtt{right}}$ by the choice of E and $\boldsymbol{\xi}$, which is a contradiction.

15d Complete left-invariant actions induce pinned ERs

Here we prove part (ii) of Theorem 15.3. Suppose that $\mathbb{G}$ is a Polish CLI group continuously acting on a Polish space $\mathbb{X}$. By definition $\mathbb{G}$ admits a compatible left-invariant complete metric. Then easily $\mathbb{G}$ also admits a compatible **right**-invariant complete metric, which will be practically used.

Let $\mathbb{P}$ be a forcing notion and $\boldsymbol{\xi}$ be a virtual E-class. Let $\le$ denote the partial order of $\mathbb{P}$; we assume, as usual, that $p \le q$ means that p is a stronger condition. Let us fix a compatible complete right-invariant metric ρ on $\mathbb{G}$. For any $\varepsilon > 0$, put $G_\varepsilon = \{g \in \mathbb{G} : \rho(g, 1_{\mathbb{G}}) < \varepsilon\}$. Say that $q \in \mathbb{P}$ *is of size* $\le \varepsilon$ if $\langle q, q\rangle$ $\mathbb{P} \times \mathbb{P}$-forces the existence of $g \in {G_\varepsilon}^{\#}$ such that $\boldsymbol{\xi}_{\mathtt{left}} = g \cdot \boldsymbol{\xi}_{\mathtt{right}}$.

Lemma 15.6. *If* $q \in \mathbb{P}$ *and* $\varepsilon > 0$, *then there is a condition* $r \in \mathbb{P}$, $r \le q$, *of size* $\le \varepsilon$.

Proof. Otherwise for any $r \in \mathbb{P}$, $r \le q$, there is a pair of conditions $r', r'' \in \mathbb{P}$ stronger than r and such that $\langle r', r''\rangle$ $\mathbb{P} \times \mathbb{P}$-forces that there is no $g \in {G_\varepsilon}^{\#}$ with $\boldsymbol{\xi}_{\mathtt{left}} = g \cdot \boldsymbol{\xi}_{\mathtt{right}}$. Applying an ordinary splitting construction in such a generic extension $\mathbb{V}^+$ of $\mathbb{V}$ where $\mathscr{P}(\mathbb{P}) \cap \mathbb{V}$ is countable, we find an uncountable set $\mathscr{U}$ of generic sets $U \subseteq \mathbb{P}$ with $q \in U$ such that any pair $\langle U, V\rangle$ with $U \ne V$ in $\mathscr{U}$ is $\mathbb{P} \times \mathbb{P}$-generic (over $\mathbb{V}$), hence, there is no $g \in {G_\varepsilon}^{\#}$ with $\boldsymbol{\xi}[U] = g \cdot \boldsymbol{\xi}[V]$.[4] Fix $U_0 \in \mathscr{U}$. We can associate in $\mathbb{V}^+$ with each $U \in \mathscr{U}$, an element $g_U \in G^{\#}$ such that $\boldsymbol{\xi}[U] = g_U \cdot \boldsymbol{\xi}[U_0]$; then $g_U \notin {G_\varepsilon}^{\#}$ by the above. Moreover, we have $g_V g_U^{-1} \cdot \boldsymbol{\xi}[U] = \boldsymbol{\xi}[V]$ for all $U, V \in \mathscr{U}$, hence $g_V g_U^{-1} \notin {G_\varepsilon}^{\#}$ whenever $U \ne V$, which implies $\rho(g_U, g_V) \ge \varepsilon$ by the right invariance. But this contradicts the separability of G. □ (*Lemma*)

Coming back to the proof of (iii) of Theorem 15.3, suppose towards the contrary that a condition $p \in \mathbb{P}$ forces that there is no $x \in \mathbb{X}$ (in the ground universe $\mathbb{V}$) satisfying $x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$. According to Lemma 15.6, there is, in $\mathbb{V}$, a sequence of conditions $p_n \in \mathbb{P}$ of size $\le 2^{-n}$, and closed sets $X_n \subseteq \mathbb{X}$ with $\mathbb{X}$-diameter $\le 2^{-n}$, such that $p_0 \le p$, $p_{n+1} \le p_n$, $X_{n+1} \subseteq X_n$, and p_n forces $\boldsymbol{\xi} \in {X_n}^{\#}$ for any n. Let x be the common point of the sets X_n in $\mathbb{V}$. We claim that p_0 forces $x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$.

Indeed, otherwise there is $q \in \mathbb{P}$, $q \le p_0$, which forces $\neg\, x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$. Consider an extension $\mathbb{V}^+$ of $\mathbb{V}$ rich enough to contain, for any n, a generic set $U_n \subseteq \mathbb{P}$ with $p_n \in U_n$ such that each pair $\langle U_n, U_{n+1}\rangle$ is $\mathbb{P} \times \mathbb{P}$-generic (over $\mathbb{V}$), and, in addition, $q \in U_0$. Let $x_n = \boldsymbol{\xi}[U_n]$ (an element of $\mathbb{X}^{\#}$), then $\{x_n\} \to x$. Moreover, for any n, both U_n and U_{n+1} contain p_n, hence, as p_n has size $\le 2^{-n-1}$, there is $g_{n+1} \in {\mathbb{G}_\varepsilon}^{\#}$ with $x_{n+1} = g_{n+1} \cdot x_n$. Thus, $x_n = h_n \cdot x_0$, where $h_n = g_n \dots g_1$. However $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \le 2^{-n+1}$ by the right-invariance of the metric, thus, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^{\#}$. Let $h = \mathtt{lim}_{n\to\infty} h_n \in \mathbb{G}^{\#}$ be its limit. As the action considered is continuous, we have $x = \mathtt{lim}_n x_n = h \cdot x_0$. It follows that $x \mathrel{\mathsf{E}^{\#}} x_0$ holds in $\mathbb{V}^+$, hence also in $\mathbb{V}[U_0]$. However $x_0 = \boldsymbol{\xi}[U_0]$ while $q \in U_0$ forces $\neg\, x \mathrel{\mathsf{E}^{\#}} \boldsymbol{\xi}$, which is a contradiction.

[4] $\boldsymbol{\xi}[U]$ is the interpretation of the $\mathbb{P}$-term $\boldsymbol{\xi}$ obtained by taking U as the generic set.

Thus p_0 $\mathbb{P}$-forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$. Then any $r\in\mathbb{P}$ also forces $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$: indeed, if some $r\in\mathbb{P}$ forces $\neg\,x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}$ then the pair $\langle p_0,r\rangle$ $\mathbb{P}\times\mathbb{P}$-forces that $x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\texttt{left}}$ and $\neg\,x\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\texttt{right}}$, which contradicts the fact that $\mathbb{P}\times\mathbb{P}$ forces $\boldsymbol{\xi}_{\texttt{left}}\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\texttt{right}}$.

15e All ERs with $\mathbf{G}_{\delta\sigma}$ classes are pinned

Here we prove part (iii) of Theorem 15.3. Suppose that E is a Borel equivalence relation on $\mathbb{N}^{\mathbb{N}}$ and all E-equivalence classes are $\mathbf{G}_{\delta\sigma}$. Prove that then E is pinned.

It follows from a theorem of Louveau [40] that there is a Borel map γ, defined on $\mathbb{N}^{\mathbb{N}}$, so that $\gamma(x)$ is a $\mathbf{G}_{\delta\sigma}$-code of $[x]_{\mathsf{E}}$ for any $x\in\mathbb{N}^{\mathbb{N}}$, that is, for instance, $\gamma(x)\subseteq\mathbb{N}^2\times\mathbb{N}^{<\omega}$ and

$$[x]_{\mathsf{E}}=\bigcup_i\bigcap_j\bigcup_{\langle i,j,s\rangle\in\gamma(x)}B_s,\quad\text{where}\quad B_s=\{a\in\mathbb{N}^{\mathbb{N}}:s\subset a\}\ \text{ for all }\ s\in\mathbb{N}^{<\omega}.$$

We consider a forcing notion $\mathbb{P}=\langle\mathbb{P}\,;\leq\rangle$ and a virtual E-class $\boldsymbol{\xi}$. Then $\mathbb{P}\times\mathbb{P}$ forces $\boldsymbol{\xi}_{\texttt{left}}\,\mathsf{E}^{\#}\,\boldsymbol{\xi}_{\texttt{right}}$; hence there is a number i_0 and a condition $\langle p_0,q_0\rangle\in\mathbb{P}\times\mathbb{P}$ which forces $\boldsymbol{\xi}_{\texttt{left}}\in\vartheta^{\#}(\boldsymbol{\xi}_{\texttt{right}})$, where $\vartheta(x)=\bigcap_j\bigcup_{\langle i_0,j,s\rangle\in\gamma(x)}B_s$ for all $x\in\mathbb{N}^{\mathbb{N}}$.

The key idea of the proof is to substitute $\mathbb{P}$ by the Cohen forcing. Let $\mathbb{S}$ denote the set of all $s\in\mathbb{N}^{<\omega}$ such that p_0 does <u>not</u> $\mathbb{P}$-force that $s\not\subset\boldsymbol{\xi}$. We consider $\mathbb{S}$ as a forcing, and $s\subseteq t$ (that is, t is an extension of s) means that t is a stronger condition; Λ, the empty sequence, is the weakest condition in $\mathbb{S}$. If $s\in\mathbb{S}$ then obviously there is at least one n such that $s^\wedge n\in\mathbb{S}$; hence $\mathbb{S}$ forces an element of $\mathbb{N}^{\mathbb{N}}$, whose $\mathbb{S}$-name will be $\mathbf{a}$.

Lemma 15.7. *The pair* $\langle\Lambda,q_0\rangle$ $\mathbb{S}\times\mathbb{P}$*-forces* $\mathbf{a}\in\vartheta^{\#}(\boldsymbol{\xi})$.

Proof. Otherwise some condition $\langle s_0,q\rangle\in\mathbb{S}\times\mathbb{P}$ with $q\leq q_0$ forces $\mathbf{a}\notin\vartheta^{\#}(\boldsymbol{\xi})$. By the definition of ϑ we can assume that

$$\langle s_0,q\rangle\quad\mathbb{S}\times\mathbb{P}\text{-forces}\quad\neg\,\exists\,s\,(\langle i_0,j_0,s\rangle\in\gamma(\boldsymbol{\xi})\wedge s\subset\mathbf{a})\qquad(*)$$

for some j_0. Since $s_0\in\mathbb{S}$, there is a condition $p'\in\mathbb{P}$, $p'\leq p_0$, which $\mathbb{P}$-forces $s_0\subset\boldsymbol{\xi}$. By the choice of $\langle p_0,q_0\rangle$ we can assume that

$$\langle p',q'\rangle\quad\mathbb{P}\times\mathbb{P}\text{-forces}\quad\langle i_0,j_0,s\rangle\in\gamma(\boldsymbol{\xi}_{\texttt{right}})\wedge s\subset\boldsymbol{\xi}_{\texttt{left}}\,.$$

for suitable $s\in\mathbb{S}$ and $q'\in\mathbb{P}$, $q'\leq q$. This means that 1) p' $\mathbb{P}$-forces $s\subset\boldsymbol{\xi}$ and 2) q' $\mathbb{P}$-forces $\langle i_0,j_0,s\rangle\in\gamma(\boldsymbol{\xi})$. In particular, by the above, p' forces both $s_0\subset\boldsymbol{\xi}$ and $s\subset\boldsymbol{\xi}$, therefore, either $s\subseteq s_0$ – then let $s'=s_0$, or $s_0\subset s$ – then let $s'=s$. In both cases, $\langle s',q'\rangle$ $\mathbb{S}\times\mathbb{P}$-forces $\langle i_0,j_0,s\rangle\in\gamma(\boldsymbol{\xi})$ and $s\subset\mathbf{a}$, contradiction to $(*)$. □ (*Lemma*)

Note that $\mathbb{S}$ is a subforcing of the Cohen forcing $\mathbb{C} = \mathbb{N}^{<\omega}$, therefore, by Lemma 15.7, there is a $\mathbb{C}$-term $\boldsymbol{\sigma}$ such that $\langle \Lambda, q_0 \rangle$ $\mathbb{C} \times \mathbb{P}$-forces $\boldsymbol{\sigma} \in \vartheta^{\#}(\boldsymbol{\xi})$, hence, forces $\boldsymbol{\sigma} \, \mathsf{E}^{\#} \, \boldsymbol{\xi}$. It follows, by consideration of the forcing $\mathbb{C} \times \mathbb{P} \times \mathbb{P}$, that generally $\mathbb{C} \times \mathbb{P}$ forces $\boldsymbol{\sigma} \, \mathsf{E}^{\#} \, \boldsymbol{\xi}$. Therefore, by ordinary arguments, first, $\mathbb{C} \times \mathbb{C}$ forces $\boldsymbol{\sigma}_{\texttt{left}} \, \mathsf{E}^{\#} \, \boldsymbol{\sigma}_{\texttt{right}}$, and second, to prove the theorem it suffices now to find $x \in \mathbb{N}^{\mathbb{N}}$ in $\mathbb{V}$ such that $\mathbb{C}$ forces $x \, \mathsf{E}^{\#} \, \boldsymbol{\sigma}$. This is our next goal.

Let $\mathbf{a}$ be a $\mathbb{C}$-name of the Cohen generic element of $\mathbb{N}^{\mathbb{N}}$. The term $\boldsymbol{\sigma}$ can be of complicated nature, but we can substitute it by a term of the form $f^{\#}(\mathbf{a})$, where $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel map in the ground universe $\mathbb{V}$. It follows from the above that $f^{\#}(\mathbf{a}) \, \mathsf{E}^{\#} \, f^{\#}(\mathbf{b})$ for any $\mathbb{C} \times \mathbb{C}$-generic, over $\mathbb{V}$, pair $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. We conclude that $f^{\#}(\mathbf{a}) \, \mathsf{E}^{\#} \, f^{\#}(\mathbf{b})$ also holds even for any pair of separately Cohen generic $\mathbf{a}$, $\mathbf{b} \in \mathbb{N}^{\mathbb{N}}$. Thus, in a generic extension of $\mathbb{V}$, where there are comeager-many Cohen generic reals, there is a comeager $\mathbf{G}_\delta$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that $f^{\#}(a) \, \mathsf{E}^{\#} \, f^{\#}(b)$ for all $a, b \in X$. By the Shoenfield absoluteness theorem, the statement of existence of such a set X is true also in $\mathbb{V}$, hence, in $\mathbb{V}$, there is $x \in \mathbb{N}^{\mathbb{N}}$ such that we have $x \, \mathsf{E} \, f(a)$ for comeager-many $a \in \mathbb{N}^{\mathbb{N}}$. This is again a Shoenfield absolute property of x, hence, $\mathbb{C}$ forces $x \, \mathsf{E}^{\#} \, f^{\#}(\mathbf{a})$, as required.

15f A family of pinned ideals

Here we prove part (iv) of Theorem 15.3.

Let us say that a Borel ideal $\mathscr{I}$ is *pinned* if the induced ER $\mathsf{E}_{\mathscr{I}}$ is such. It follows from Theorem 15.3(ii) that any P-ideal is pinned because Borel P-ideals are polishable by Theorem 8.5 while all Polish abelian groups are CLI. Yet there are non-P pinned ideals.

Suppose that $\{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of lower semicontinuous (LSC) submeasures on $\mathbb{N}$. Define the exhaustive ideal of the sequence,

$$\texttt{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}, \quad \text{where} \quad \varphi_\infty(X) = \limsup_{i \to \infty} \varphi_i(X).$$

It follows from Theorem 8.5 that for any Borel P-ideal $\mathscr{I}$ there is a LSC submeasure φ such that $\mathscr{I} = \texttt{Exh}_{\{\varphi_i\}} = \texttt{Exh}_{\varphi}$, where $\varphi_i(x) = \varphi(x \cap [i, \infty))$. On the other hand, the non-polishable ideal $\mathscr{I}_1 = \mathtt{Fin} \times 0$ also is of the form $\texttt{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define $\varphi_i(x) = 0$ or 1 if resp. $x \subseteq$ or $\not\subseteq \{0, \dots, n-1\} \times \mathbb{N}$.

Thus suppose that φ_i is a LSC submeasure on $\mathbb{N}$ for each $i \in \mathbb{N}$. The goal is to prove that the ideal $\mathscr{I} = \texttt{Exh}_{\{\varphi_i\}}$ is pinned.

We can assume that the submeasures φ_i decrease, that is $\varphi_{i+1}(x) \le \varphi_i(x)$ for any x, for if not then consider the LSC submeasures $\varphi'_i(x) = \sup_{j \ge i} \varphi_j(x)$.

Suppose towards the contrary that the equivalence $\mathsf{E} = \mathsf{E}_{\mathscr{I}}$ is not pinned. Then there is a forcing notion $\mathbb{P}$, a virtual E-class $\boldsymbol{\xi}$, and a condition $p \in \mathbb{P}$ which $\mathbb{P}$-forces $\neg \, x \, \mathsf{E}^{\#} \, \boldsymbol{\xi}$ for any $x \in \mathscr{P}(\mathbb{N})$ in $\mathbb{V}$. By definition, for any $p' \in \mathbb{P}$ and $n \in \mathbb{N}$ there are $i \ge n$ and conditions $q, r \in \mathbb{P}$ with $q, r \le p'$, such that $\langle q, r \rangle$ $\mathbb{P} \times \mathbb{P}$-forces the inequality $\varphi_i(\boldsymbol{\xi}_{\texttt{left}} \, \Delta \, \boldsymbol{\xi}_{\texttt{right}}) \le 2^{-n-1}$, hence, $\langle q, q \rangle$

$\mathbb{P} \times \mathbb{P}$-forces $\varphi_i(\boldsymbol{\xi}_{\texttt{left}} \,\Delta\, \boldsymbol{\xi}_{\texttt{right}}) \le 2^{-n}$. It follows that, in $\mathbb{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < \dots$, and a sequence $p_0 \ge p_1 \ge p_2 \ge \dots$ of conditions in $\mathbb{P}$, and, for any n, a set $u_n \subseteq [0, n)$, such that $p_0 \le p$ and

(1) each p_n $\mathbb{P}$-forces $\boldsymbol{\xi} \cap [0, n) = u_n$;

(2) each $\langle p_n, p_n \rangle$ $\mathbb{P} \times \mathbb{P}$-forces $\varphi_{i_n}(\boldsymbol{\xi}_{\texttt{left}} \,\Delta\, \boldsymbol{\xi}_{\texttt{right}}) \le 2^{-n}$.

Arguing in the universe $\mathbb{V}$, put $a = \bigcup_n u_n$; then $a \cap [0, n) = u_n$ for all n. We claim that p_0 forces $a \,\mathsf{E}^{\#}\, \boldsymbol{\xi}$. This contradicts the assumption above, ending the proof of (iv) of Theorem 15.3.

To prove the claim, note that otherwise there is a condition $q_0 \le p_0$ which forces $\neg\, a \,\mathsf{E}^{\#} \boldsymbol{\xi}$. Consider a generic extension $\mathbb{V}^+$ of the universe, where there exists a sequence of $\mathbb{P}$-generic sets $U_n \subseteq \mathbb{P}$ such that for any n, the pair $\langle U_n, U_{n+1} \rangle$ is $\mathbb{P} \times \mathbb{P}$-generic, $p_n \in U_n$, and in addition $q_0 \in U_0$. Then, in $\mathbb{V}^+$, the sets $x_n = \boldsymbol{\xi}[U_n] \in \mathscr{P}(\mathbb{N})$ satisfy $\varphi_{i_n}(x_n \,\Delta\, x_m) \le 2^{-n}$ by (2), whenever $n \le m$. It follows that $\varphi_{i_n}(x_n \,\Delta\, a) \le 2^{-n}$, because $a = \texttt{lim}_m\, x_m$ by (1). However we assume that the submeasures φ_j decrease, therefore $\varphi_\infty(x_n \,\Delta\, a) \le 2^{-n}$. On the other hand, $\varphi_\infty(x_n \,\Delta\, x_0) = 0$ because $\boldsymbol{\xi}$ is a virtual E-class. We conclude that $\varphi_\infty(x_0 \,\Delta\, a) \le 2^{-n}$ for any n. In other words, $\varphi_\infty(x_0 \,\Delta\, a) = 0$, that is, $x_0 \,\mathsf{E}^{\#}\, a$, which is a contradiction with the choice of U_0 because $x_0 = \boldsymbol{\xi}[U_0]$ and $q_0 \in U_0$.

□ (*Theorem 15.3*)

One might ask whether all Borel ideals are pinned. This question answers in the negative. Indeed it will be proved in the next Chapter that for every Borel equivalence relation E there exists a Borel ideal $\mathscr{I}$ such that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$. In particular this is true for the ER T_2, non-pinned by Theorem 15.3. It follows, still by Theorem 15.3, that any Borel ideal $\mathscr{I}$ satisfying $\mathsf{T}_2 \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$ is non-pinned as well.

Question 15.8 (Kechris). Is it true that T_2 is the $\le_{\mathrm{B}}$-least non-pinned Borel equivalence relation ? □

Chapter 16

Reduction of Borel equivalences to Borel ideals

The main goal of this Chapter is to show that any Borel equivalence relation is Borel reducible to a relation of the form $\mathsf{E}_{\mathscr{I}}$ for some Borel ideal $\mathscr{I}$, and moreover, there is a $\le_{\rm B}$-cofinal ω_1-sequence of Borel ideals in the sense of the next theorem:

Theorem 16.1. *There is a $\subseteq$-decreasing sequence of Borel ideals $\mathscr{I}_\xi$ $(\xi < \omega_1)$ on $\mathbb{N}$, $\le_{\rm B}$-cofinal in the sense that every Borel equivalence relation is Borel reducible to one of the relations $\mathsf{E}_{\mathscr{I}_\xi}$.*

The proof (due to Rosental [52]) of this important result involves a universal analytic equivalence generated by an analytic ideal, followed by a well-known construction of upper Borel approximations of $\mathbf{\Sigma}^1_1$ sets. Note that this theorem, together with Corollary 11.19, accomplishes the proof of Theorem 4.10.

In the end we briefly outline the results of subsequent study [32]: the ideals $\mathscr{I}$ and the corresponding relations $\mathsf{E}_{\mathscr{I}_\xi}$ as above can be explicitly and meaningfully defined on the base of a certain game.

16a Trees

We begin with a review of basic notation related to trees of finite sequences. Recall that for any set X, X^n denotes the set of all sequences, of length n, of elements of X, and $X^{<\omega} = \bigcup_{n\in\mathbb{N}} X^n$ – the set of all finite sequences of elements of X. Regarding product sets, note that any $s \in (X_1 \times \cdots \times X_n)^{<\omega}$ is formally a finite sequence of n-tuples $\langle x_1, \dots, x_n\rangle$, where $x_i \in X_i$, $\forall\, i$. We identify such a sequence s with the n-tuple $\langle s_1, \dots, s_n\rangle$, where all $s_i \in {X_i}^{<\omega}$ have the same length as s itself, and $s(i) = \langle s_1(i), \dots, s_n(i)\rangle$ for all i.

$\mathtt{lh}\, s$ is the *length* of a sequence s. Λ, *the empty sequence*, is the only one of length 0. If s is a finite sequence and x any set then by $s^\wedge x$, resp., $x^\wedge s$ we denote the result of adjoining x as the new rightmost, resp, leftmost term to s. If s, t are sequences then $s \subseteq t$ means that t is an *extension* of s, that is, $s = t \restriction m$ for some $m \le \mathtt{lh}\, t$.

A *tree* on a set X is any subset $T \subseteq X^{<\omega}$ closed under restrictions — that is, if $t \in T$, $s \in X^{<\omega}$, and $s \subseteq t$, then $s \in T$. Note that Λ, the empty sequence, belongs to any tree $\varnothing \ne R \subseteq X^{<\omega}$. An *infinite branch* in a tree $T \subseteq X^{<\omega}$ is any infinite sequence $b \in X^\omega$ such that $b \restriction m \in T$, $\forall\, m$. A tree T is *well-founded* iff it has no infinite branches. Otherwise T is *ill-founded*.

The following transformations of trees on $\mathbb{N}$ preserve in this or another way the properties of well- and ill-foundednes.

Finite union. If S, T are trees then so is $W = S \cup T$, and clearly $S \cup T$ is ill-founded iff so is at least one of S, T.

Contraction. Let $S \subseteq 2^{<\omega}$ be a tree. Fix once and for all a bijection $b : \mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$. For any sequence $s = \langle k_0, k_1, \dots, k_n \rangle \in 2^{<\omega}$ with $\mathtt{lh}\, s = n + 1 \ge 2$ define a sequence $s^\downarrow = \langle b(k_0, k_1), k_2, \dots, k_n \rangle$ of length n. The *contracted tree*

$$S^\downarrow = \{\Lambda\} \cup \{\hat{s} : s \in S \wedge \mathtt{lh}\, s \ge 2\}$$

is ill-founded iff so is S itself.

Countable sum. Countable unions do not preserve well-foundedness. Yet there is another useful operation. For any sequence of trees $T_n \subseteq \mathbb{N}^{<\omega}$, we let $\sum_n^* T_n$ denote the tree $T = \{\Lambda\} \cup \{n^\wedge t : t \in T_n\}$. Clearly T is ill-founded iff so is **at least one** of the trees T_n.

Countable product. Let $\prod_n^* T_n$ denote the set T of all finite sequences of the form $t = \langle t_0, \dots, t_n \rangle$, where $t_k \in T_k$ and $\mathtt{lh}\, t_k = n$ for all $k \le n$. We put $\langle t_0, \dots, t_n \rangle \preccurlyeq \langle s_0, \dots, s_m \rangle$ iff $n \le m$ and $t_k \subseteq s_k$ (in $\mathbb{N}^{<\omega}$) for all $k \le n$. In addition, let Λ belong to T, with $\Lambda \preccurlyeq t$ for any $t \in T$. Obviously $\langle T\,; \preccurlyeq \rangle$ is an at most countable tree, order isomorphic to a tree in $\mathbb{N}^{<\omega}$. Moreover $T = \prod_n^* T_n$ is ill-founded iff so is **every** tree T_n.

Componentwise addition. This is a less trivial operation. First of all, if s, $t \in 2^{<\omega}$ then $s \le_{\mathtt{cw}} t$ (the *componentwise* ordering) means that $\mathtt{lh}\, s = \mathtt{lh}\, t$ and $s(i) \le t(i)$ for all $i < \mathtt{lh}\, s$. Similarly, then $s +_{\mathtt{cw}} t$ denotes the componentwise addition of finite sequences s, t of equal length. We now define

$$S +_{\mathtt{cw}} T = \{s +_{\mathtt{cw}} t : s \in S \wedge t \in T \wedge \mathtt{lh}\, s = \mathtt{lh}\, t\}$$

for any trees $S, T \subseteq \mathbb{N}^{<\omega}$. The following lemma shows that the componentwise addition of trees behaves somewhat like the "equal-length" cartesian product $S \times T = \{\langle s, t \rangle : s \in S \wedge t \in T \wedge \mathtt{lh}\, s = \mathtt{lh}\, t\}$.

Lemma 16.2. *Let $S, T \subseteq \mathbb{N}^{<\omega}$ be any trees. The tree $W = S +_{\mathtt{cw}} T$ is ill-founded iff so are both S and T.*

Proof. In the nontrivial direction, suppose that $\gamma \in \mathbb{N}^\omega$ is an infinite branch in W, *i.e.*, $\gamma \restriction n \in W$ for all n. Then, for each n, there exist $s_n \in S$ and $t_n \in T$ of length n such that $s_n +_{\mathtt{cw}} t_n = \gamma \restriction n$. The sequences s_n, t_n then belong to $\{t \in \mathbb{N}^{<\omega} : t \le_{\mathtt{cw}} \gamma \restriction \mathtt{lh}\, t\}$, a finite-branching tree. Therefore, by König's lemma, there exist infinite sequences $\alpha, \beta \in \mathbb{N}^\omega$ such that $\forall\, m\, \exists\, n \ge m\, (\alpha \restriction m = s_n \restriction m \wedge \beta \restriction m = t_n \restriction m)$. Then α, β are infinite brances in resp. S, T, as required. □

16b Louveau – Rosendal transform

Suppose that A is a $\mathbf{\Sigma}^1_1$ subset of $2^\omega \times 2^\omega$. It is known from elementary topology of Polish spaces that any $\mathbf{\Sigma}^1_1$ subset of a Polish space S is is equal to the projection of a closed subset of $S \times \mathbb{N}^\omega$ on S. Thus there exists a closed set $P \subseteq 2^\omega \times 2^\omega \times \mathbb{N}^\omega$ satisfying $A = \mathtt{dom}\, P = \{\langle x, y\rangle : \exists\, z\, P(x,y,z)\}$. Further, there is a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ (a tree on $2 \times 2 \times \mathbb{N}$) such that $P = [R] = \{\langle x, y, \gamma\rangle : \forall\, n\, R(x \restriction n, y \restriction n, \gamma \restriction n)\}$, and hence

$$\langle x, y\rangle \in A \iff R_{xy} = \{s \in \mathbb{N}^{<\omega} : R(x \restriction \mathtt{lh}\, s, y \restriction \mathtt{lh}\, s, s)\} \text{ is ill-founded.} \quad (1)$$

(Obviously R_{xy} is a tree in $\mathbb{N}^{<\omega}$.) If A is an arbitrary $\mathbf{\Sigma}^1_1$ set then, perhaps, not much can be established regarding the structure of a tree R which generates A in the sense of (1). However, assuming that $A = \mathsf{E}$ is an equivalence relation on 2^ω, we can expect a nicer behaviour of R. This is indeed the case.

The following key definition goes back to [42, 52].

Definition 16.3. A tree T on a set of the form $X \times \mathbb{N}$ is *normal* if for any $u \in X^{<\omega}$ and $s, t \in \mathbb{N}^{<\omega}$ such that $\mathtt{lh}\, u = \mathtt{lh}\, s = \mathtt{lh}\, t$ and $s \le_{\mathtt{cw}} t$, we have $\langle u, s\rangle \in T \Longrightarrow \langle u, t\rangle \in T$. □

Thus normality means that the tree is $\le_{\mathtt{cw}}$-closed upwards w.r.t. the second component. $X = 2 \times 2$ in the next theorem, and the case $X = 2 = \{0,1\}$ will also be considered. But in all cases $(X \times \mathbb{N})^{<\omega}$ itself is a normal tree.

Theorem 16.4. *Suppose that $Q \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ is a tree and the set*

$$\mathsf{E} = \{\langle x, y\rangle \in 2^\omega \times 2^\omega : Q_{xy} \text{ is ill-founded}\} \quad (2)$$

is an equivalence relation on 2^ω. Then there is a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ satisfying the following requirements (i) – (v):

(i) *symmetry*: $R(u,v,s) \Longleftrightarrow R(v,u,s)$, *hence* $R_{xy} = R_{yx}$ *for all* x, y;

(ii) *if* $u \in 2^\omega$, $s \in \mathbb{N}^\omega$, $\mathtt{lh}\, s = \mathtt{lh}\, u$ *then* $R(u,u,s)$;

(iii) *normality*: *if* $R(u,v,s)$, $t \in \mathbb{N}^\omega$, *and* $s \le_{\mathtt{cw}} t$, *then* $R(u,v,t)$;

(iv) *transitivity*: *if* $R(u,v,s)$ *and* $R(v,w,t)$ *then* $R(u,w,s +_{\mathtt{cw}} t)$;

(v) *for any* $x, y \in 2^\omega$, R_{xy} *is ill-founded iff so is* Q_{xy} *— and hence* (2) *holds for the tree* R *instead of* Q;

This theorem is equal to Theorem 4 in [42].

Proof. *Part 1.* We observe that the tree

$$\widehat{Q} = Q \cup \{\langle u, u, s\rangle : u \in 2^\omega \wedge s \in \mathbb{N}^\omega \wedge \mathtt{lh}\, s = \mathtt{lh}\, u\} \cup \{\langle u, v, s\rangle : Q(v, u, s)\}.$$

satisfies $\widehat{Q}_{xy} = Q_{xy} \cup Q_{yx} \cup D_{xy}$, where $D_{xy} = \mathbb{N}^{<\omega}$ provided $x = y$ and $D_{xy} = \varnothing$ otherwise. It easily follows that (2) still holds for $\widehat{Q}$. In addition, $\widehat{Q}$ obviously satisfies both (i) and (ii). Thus *we can assume, from the beginning, that* Q *satisfies both* (i) *and* (ii).

Part 2. In this assumption, to fulfill (iii), we define

$$\widehat{Q} = \{\langle u, v, t\rangle \in (2 \times 2 \times \mathbb{N})^{<\omega} : \exists\, \langle u, v, s\rangle \in Q\ (s \le_{\mathtt{cw}} t)\}.$$

This is still a tree on $2 \times 2 \times \mathbb{N}$, containing Q and satisfying (i), (ii), (iii). In addition, we have $\widehat{Q}_{xy} = Q_{xy} +_{\mathtt{cw}} 2^{<\omega}$ for any $x, y \in 2^\omega$, therefore the trees Q_{xy} and $\widehat{Q}_{xy}$ are ill-founded simultaneously by Lemma 16.2. It follows that (2) still holds for $\widehat{Q}$. Thus, *we can assume that* Q *itself satisfies* (i), (ii), (iii).

Part 3. It is somewhat more difficult to fulfill (iv). A straightforward plan would be to define a new tree R containing all triples of the form $\langle u_0, u_{n+1}, s_0 +_{\mathtt{cw}} \cdots +_{\mathtt{cw}} s_k\rangle$, where $\langle u_i, u_{i+1}, s_i\rangle \in Q$ for all $i = 0, 1, \ldots, k$. However, to work properly, such a construction has to be equipped with a kind of counter for the number k of steps in the finite chain. This idea can be realized as follows.

Working in the assumption that Q satisfies (i), (ii), (iii) (see Part 2), we define a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ as follows. Suppose that $n \in \mathbb{N}$, $u, v \in 2^n$, $s \in \mathbb{N}^n$, $k \in \mathbb{N}$, and $i, j \in 2 = \{0, 1\}$. We put $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle \in R$ iff

$$\exists\, u_0, u_1, \ldots, u_k \in 2^n\ (u_0 = u \wedge u_k = v \wedge \forall\, \ell < k\, Q(u_\ell, u_{\ell+1}, s)). \tag{3}$$

In addition, we put $\langle \Lambda, \Lambda, \Lambda\rangle \in R$, of course. ($\Lambda$ is the empty sequence.) Note that R is a tree on $2 \times 2 \times \mathbb{N}$ because so is Q.

We claim that, in our assumptions, the tree R satisfies all of (i) – (v).

(i) If $u_0, \ldots, u_k$ witness $R(u^\wedge i, v^\wedge j, k^\wedge s)$ then the reversed sequence $u_k, \ldots, u_0$ witnesses $R(v^\wedge j, u^\wedge i, k^\wedge s)$ in the sense of (3), because the tree Q satisfies (i).

(iii) Suppose that $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle \in R$, and let $u_0, \ldots, u_k$ witness (3). Let $n = \mathtt{lh}\, u = \mathtt{lh}\, v = \mathtt{lh}\, s = \mathtt{lh}\, u_\ell$, $\forall\, \ell$. Suppose that $k \le k'$ and $s \le_{\mathtt{cw}} s'$ (still $\mathtt{lh}\, s' = n$). Put $u_\ell = v$ whenever $k < \ell \le k'$. Note that $Q(u_\ell, u_{\ell+1}, s)$ also holds for $k < \ell < k'$ by (ii) for Q. (Indeed, in this case $u_\ell = u_{\ell+1}$.) Thus, $Q(u_\ell, u_{\ell+1}, s')$ holds for all $\ell < k'$ by (iii) for Q. By definition, this witnesses $\langle u^\wedge i, v^\wedge j, k'^\wedge s'\rangle \in R$, as required.

(ii) If $k = 0$ and $u = v$ then (5.1) obviously holds (with the empty list of intermediate sequences $u_1, \dots, u_{k-1}$), and hence $R(u^\wedge i, u^\wedge j, 0^\wedge s)$ holds for all $u \in 2^\omega$, $s \in \mathbb{N}^\omega$ of equal length, in particular, $R(u, u, 0^n)$ for all n and $u \in \mathbb{N}^\omega$ with $\mathtt{lh}\, u = n$. It remains to apply property (iii) just proved.

(iv) Suppose that the triples $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle$ and $\langle v^\wedge j, w^\wedge \rho, \kappa^\wedge \sigma\rangle$ belong to R, and n is the length of all sequences u, v, s, w, t. Let $R(u^\wedge i, v^\wedge j, k^\wedge s)$ be witnessed, in the sense of (3), by $u_0, \dots, u_k$ and, accordingly, $R(v^\wedge j, w^\wedge \rho, \kappa^\wedge \sigma)$ be witnessed by $v_0, \dots, v_\kappa$. (All u_ℓ and v_ℓ belong to 2^n.) Since Q satisfies (iii), the same sequences also witness $R(u^\wedge i, v^\wedge j, k^\wedge t)$ and $R(v^\wedge j, w^\wedge \rho, \kappa^\wedge t)$, where $t = s +_{\mathtt{cw}} \sigma$ (componentwise). It easily follows that the concatenated complex $u_0, \dots, u_{k-1}, u_k = v_0, v_1, \dots, v_\kappa$ witnesses $R(u^\wedge i, w^\wedge \rho, (k+\kappa)^\wedge t)$, as required.

(v) We observe that, by definition, $Q(u, v, s) \Longrightarrow R(u^\wedge i, v^\wedge j, 1^\wedge s)$ for any $i, j = 0, 1$. It follows that, for any $x, y \in 2^\omega$, $s \in Q_{xy} \Longrightarrow 1^\wedge s \in R_{xy}$, and hence R_{xy} is ill-founded provided so is Q_{xy}. The inverse implication in (v) needs more work. This argument belongs to Louveau and Rosendal [42]. Assume that R_{xy} is ill-founded, that is, there exists an infinite sequence $\delta \in \mathbb{N}^\omega$ such that $\forall\, n\, R(x \restriction n, y \restriction n, \delta \restriction n)$. Let $k = \delta(0)$ and $\gamma(m) = \delta(m+1)$ for all m, so that $\delta = k^\wedge \gamma$. By definition, for any n there exist sequences $u_0^n, \dots, u_k^n \in 2^n$ such that $u_0^n = x \restriction n$, $u_k^n = y \restriction n$, and $Q(u_\ell^n, u_{\ell+1}^n, \gamma \restriction n)$ for all $\ell < k$. Each $k+1$-tuple $\langle u_0^n, \dots, u_k^n\rangle \in (2^n)^{k+1}$ can be considered as an n-tuple in $(2^{k+1})^n$. By König's lemma, there exist infinite sequences $x_0, \dots, x_k \in 2^\omega$ such that for any m there is a number $n \ge m$ with $x_\ell \restriction m = u_\ell^n \restriction m$ for all $\ell \le k$. It follows that $x_0 = x$, $x_k = y$, and, as Q is a tree, $Q(x_\ell \restriction m, x_{\ell+1} \restriction m, \gamma \restriction m)$ holds for all $\ell < k$ and all m. We conclude that $x_\ell \mathsf{E}\, x_{\ell+1}$ for all $\ell < k$ by (2) for Q, therefore, $x \mathsf{E}\, y$ because E is an equivalence. Finally, Q_{xy} is ill-founded still by (2) for Q. □

16c Embedding and equivalence of normal trees

Let **NT** denote the set of all non-empty normal trees $T \subseteq (2 \times \mathbb{N})^{<\omega}$. Suppose that $S, T \in \mathbf{NT}$. The set of all finite sequences $f \in \mathbb{N}^{<\omega}$ such that $\langle u, s\rangle \in S \Longrightarrow \langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle \in T$ for all $n \le \mathtt{lh}\, f$ and $u \in 2^n$, $s \in \mathbb{N}^n$, will be denoted by $\textsc{Emb}(S, T)$. Obviously $\textsc{Emb}(S, T)$ is a tree in $\mathbb{N}^{<\omega}$ containing Λ.

We proceed with the following key definition of [42].

Definition 16.5. Define $S \le_{\text{NT}} T$ iff the tree $\textsc{Emb}(S, T)$ is ill-founded, that is,

$$\exists\, \gamma \in \mathbb{N}^\omega\, \forall\, n\, \forall\, u \in 2^n\, \forall\, s \in \mathbb{N}^n\, (\langle u, s\rangle \in S \Longrightarrow \langle u, s +_{\mathtt{cw}} \gamma \restriction n\rangle \in T).$$

Define $S\, \mathsf{E}_{\text{NT}}\, T$ iff $S \le_{\text{NT}} T$ and $T \le_{\text{NT}} S$. [1] □

Thus $S \le_{\text{NT}} T$ indicates the existence of a certain shift-type embedding of S into T. The relation $\le_{\text{NT}}$ is a partial order on the set **NT**, To check that $\le_{\text{NT}}$

[1] $\le_{\text{NT}}$ and E_{NT} are denoted in [52] by, resp., $\le^*_{\max}$ and $E^*_{\max}$.

is transitive, suppose that $R \le_{\rm NT} S$ and $S \le_{\rm NT} T$, where R, S, T are normal trees in $(2 \times \mathbb{N})^{<\omega}$. Then the trees $U = \textsc{Emb}(R, S)$ and $V = \textsc{Emb}(S, T)$ (trees in $\mathbb{N}^{<\omega}$) are ill-founded, and hence so is $W = U +_{\tt cw} V$ by Lemma 16.2. On the other hand, easy verification shows that $W \subseteq \textsc{Emb}(R, T)$. Thus $\textsc{Emb}(R, T)$ is ill-founded, as required. It follows that $\mathsf{E}_{\rm NT}$ is an equivalence relation on $\mathbf{NT}$.

Moreover, applying the componentwise addition to the sequences γ that witness $\le_{\rm NT}$, one proves that $S \, \mathsf{E}_{\rm NT} \, T$ is equivalent to the existence of $\gamma \in \mathbb{N}^\omega$ such that for all n and all $u \in 2^n$, $s \in \mathbb{N}^n$ the following holds simultaneously:

$$\langle u, s\rangle \in S \Longrightarrow \langle u, s +_{\tt cw} \gamma \restriction n\rangle \in T \text{ and } \langle u, s\rangle \in T \Longrightarrow \langle u, s +_{\tt cw} \gamma \restriction n\rangle \in S.$$

Corollary 16.6. *If* $S, T \in \mathbf{NT}$ *then* $S \, \mathsf{E}_{\rm NT} T$ *iff the tree* $\textsc{Emb}(S, T) \cap \textsc{Emb}(T, S)$ *is ill-founded.* □

Note that any tree $T \in \mathbf{NT}$ is, by definition, a subset of the countable set $(2 \times \mathbb{N})^{<\omega}$, Thus, $\mathbf{NT}$ is a subset of the Polish space $\mathscr{P}((2 \times \mathbb{N})^{<\omega})$, identified, as usual, with the product space $2^{(2\times\mathbb{N})^{<\omega}}$. (Elementary computations show that in fact $\mathbf{NT}$ is a closed set.) Therefore, the relations $\le_{\rm NT}$ and $\mathsf{E}_{\rm NT}$ are, formally, subsets of $\mathscr{P}((2 \times \mathbb{N})^{<\omega}) \times \mathscr{P}((2 \times \mathbb{N})^{<\omega})$.

Lemma 16.7. $\le_{\rm NT}$ *and* $\mathsf{E}_{\rm NT}$ *are* $\mathbf{\Sigma}^1_1$ *relations.*

Proof. Straightforward estimations. The principal quantifier expresses the existence of $\gamma \in \mathbb{N}^\omega$ with certain properties. □

It occurs that $\mathsf{E}_{\rm NT}$ belongs to a special type of $\mathbf{\Sigma}^1_1$ equivalence relations.

Definition 16.8. An $\mathbf{\Sigma}^1_1$ equivalence relation U is *universal*, or *complete*, if and only if $\mathsf{F} \le_{\rm B} \mathsf{U}$ holds for any other $\mathbf{\Sigma}^1_1$ equivalence relation F. □

There is a simple construction that yields a universal $\mathbf{\Sigma}^1_1$ equivalence relation.

Example 16.9. We begin with a $\mathbf{\Sigma}^1_1$ set $U \subseteq (\mathbb{N}^\mathbb{N})^3$, universal in the sense that for any $\mathbf{\Sigma}^1_1$ set $P \subseteq (\mathbb{N}^\mathbb{N})^2$ there is an index $x \in \mathbb{N}^\mathbb{N}$ such that P is equal to the cross-section $U_x = \{\langle y, z\rangle : \langle x, y, z\rangle \in U\}$. Define a set $P \subseteq (\mathbb{N}^\mathbb{N})^3$ so that every cross-section P_x is equal to the *equivalence hull* of U_x, that is, to the least equivalence relation containing U_x. Formally, $\langle y, z\rangle \in P_x$ iff there is a finite chain $y = y_0, y_1, y_2, \dots y_n, y_{n+1} = z$ such that, for any $k \le n$, either $\langle y_k, y_{k+1}\rangle$ belongs to U_x, or $\langle y_{k+1}, y_k\rangle$ belongs to U_x, or just $y_k = y_{k+1}$.

Clearly P is still a $\mathbf{\Sigma}^1_1$ subset of $(\mathbb{N}^\mathbb{N})^3$, with each P_x being a $\mathbf{\Sigma}^1_1$ equivalence relation. Moreover, if U_x is an equivalence relation then $P_x = U_x$. Thus the family of all cross-sections P_x, $x \in \mathbb{N}^\mathbb{N}$, is equal to the family of all $\mathbf{\Sigma}^1_1$ equivalence relations on $\mathbb{N}^\mathbb{N}$. We claim that the equivalence relation U on $(\mathbb{N}^\mathbb{N})^2$, defined so that $\langle x, y\rangle \, \mathsf{U} \, \langle x', y'\rangle$ iff $x = x'$ and $\langle y, y'\rangle \in P_x$, is universal. For take any $\mathbf{\Sigma}^1_1$ equivalence F on $\mathbb{N}^\mathbb{N}$. Then $\mathsf{F} = P_x$ for some x by the above, therefore, the map $\vartheta(y) = \langle x, y\rangle$ is a continuous reduction of F to U, as required. □

Theorem 16.10 (Theorem 5 in [42]). $\mathsf{E}_{\rm NT}$ *is a universal* $\mathbf{\Sigma}^1_1$ *equivalence on* **NT**.

Proof. Consider any $\mathbf{\Sigma}^1_1$ equivalence relation E on 2^ω. Then E is a $\mathbf{\Sigma}^1_1$ subset of $2^\omega \times 2^\omega$, and hence there is a tree $Q \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ (a tree on $2 \times 2 \times \mathbb{N}$) such that, for all $x, y \in 2^\omega$,

$$x \,\mathsf{E}\, y \iff \text{the cross-section tree } Q_{xy} \text{ is ill-founded.} \tag{4}$$

It follows from Theorem 16.4 that it can be assumed that Q satisfies requirements (i) – (v) of Theorem 16.4. We claim that the map

$$x \longmapsto \vartheta(x) = \{\langle u, s\rangle \in (2 \times \mathbb{N})^{<\omega} : Q(u, x \restriction \mathtt{lh}\, u, s)\} \qquad (x \in 2^\omega) \tag{5}$$

is a Borel reduction of E to $\mathsf{E}_{\rm NT}$. That ϑ is a Borel, even continuous map, is rather easy. That $\vartheta(x) \in \mathbf{NT}$ immediately follows from (iii). The reduction property follows from the next lemma.

Lemma 16.11. *If a tree* $Q \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ *satisfies requirements* (i) – (iv) *of Theorem 16.4, and* $x, y \in 2^\omega$, *then* $\textsc{Emb}(\vartheta(x), \vartheta(y)) = Q_{xy}$.

Proof. Suppose that $f \in \textsc{Emb}(\vartheta(x), \vartheta(y))$, $m = \mathtt{lh}\, f$. Then, by definition, we have $Q(u, x \restriction m, s) \Longrightarrow R(u, y \restriction m, s +_{\mathtt{cw}} f)$ for all $u \in 2^m$ and $s \in \mathbb{N}^m$. Take here $u = x \restriction m$ and $s = 0^m$ (the sequence of m 0s); then $Q(x \restriction m, x \restriction m, 0^n) \Longrightarrow Q(x \restriction m, y \restriction m, f)$. Yet the left-hand side holds by (ii). Therefore, the right-hand side holds, thus $f \in Q_{xy}$.

To prove the converse let $f \in Q_{xy}$, that is, $Q(x \restriction m, y \restriction m, f)$, where $m = \mathtt{lh}\, f$ — and hence $Q(x \restriction n, y \restriction n, f \restriction n)$ for any $n \le m$ as Q is a tree. Assume that $n \le m$ and $u \in 2^n$, $s \in \mathbb{N}^n$. We have to prove $Q(u, x \restriction n, s) \Longrightarrow Q(u, y \restriction n, s +_{\mathtt{cw}} (f \restriction n))$. So suppose $Q(u, x \restriction n, s)$. In addition, $Q(x \restriction n, y \restriction n, f \restriction n)$ holds by the above. Then $Q(u, y \restriction n, s +_{\mathtt{cw}} (f \restriction n))$ holds by (iv), as required. □ (*Lemma*)

To accomplish the proof of Theorem 16.10, suppose that $x, y \in 2^\omega$. Then $x \mathsf{E} y$ iff the tree R_{xy} is ill-founded, iff (by the lemma) $\textsc{Emb}(\vartheta(y), \vartheta(x))$ is ill-founded, iff $\vartheta(x) \,\mathsf{E}_{\rm NT}\, \vartheta(y)$ (by Definition 16.5).

□ (*Theorem 16.10*)

16d Reduction to Borel ideals: first approach

We present two different proofs of Theorem 16.1. The first one, due to Rosendal [52], involves the ideal $\mathscr{I}_{\rm NT}$ on $(2 \times \mathbb{N})^{<\omega}$ finitely generated by all sets of the form $S \,\Delta\, T$, where $S, T \subseteq (2 \times \mathbb{N})^{<\omega}$ are normal trees and $S \,\mathsf{E}_{\rm NT}\, T$. Thus $\mathscr{I}_{\rm NT}$ consists of all subsets of $(2 \times \mathbb{N})^{<\omega}$, covered by unions of finitely many symmetric differences $S \,\Delta\, T$ of the type just indicated.

Theorem 16.12. *The ideal* $\mathscr{I}_{\rm NT}$ *is* $\mathbf{\Sigma}^1_1$ *as a subset of the Polish space* $\mathscr{P}((2\times\mathbb{N})^{<\omega})$. *Furthermore, the equivalence relation* $\mathsf{E}_{\rm NT}$ *is equal to* $\mathsf{E}_{\mathscr{I}} \restriction \mathbf{NT}$ *— this means that for any* $S,T\in\mathbf{NT}$, *the following holds*: $S\,\mathsf{E}_{\rm NT}\,T$ *if ant only if* $S\,\Delta\,T\in\mathscr{I}_{\rm NT}$.

Proof. That $\mathscr{I}_{\rm NT}$ is $\mathbf{\Sigma}^1_1$ is quite clear: the principal quantifier expresses the existence of a finite collection of elements of **NT**, whose properties are expressible still by a $\mathbf{\Sigma}^1_1$ relation because $\mathsf{E}_{\rm NT}$ is $\mathbf{\Sigma}^1_1$.

Suppose that $S\,\Delta\,T\in\mathscr{I}_{\rm NT}$, and prove $S\,\mathsf{E}_{\rm NT}\,T$ (the nontrivial direction). By definition $S\,\Delta\,T\subseteq\bigcup_{i=1}^k(S_i\,\Delta\,T_i)$, where $S_i,T_i\in\mathbf{NT}$ and $S_i\,\mathsf{E}_{\rm NT}\,T_i$. Then the trees $R_i=\textsc{Emb}(S_i,T_i)\cap\textsc{Emb}(T_i,S_i)$ are ill-founded by Corollary 16.6. We have to prove that $\textsc{Emb}(S,T)$ and $\textsc{Emb}(T,S)$ are ill-founded trees, too. To check the ill-foundedness of $\textsc{Emb}(S,T)$, note that the tree $R=R_1+_{\tt cw}\cdots+_{\tt cw}R_k$ is ill-founded by Lemma 16.2. Thus it remains to prove that $R\subseteq\textsc{Emb}(S,T)$.

Consider any $r=r_1+_{\tt cw}\cdots+_{\tt cw}r_k\in R$, where all sequences $r_i\in R_i$, $i=1,\dots,k$, have one and the same length, say m. Suppose towards the contrary that $r\notin\textsc{Emb}(S,T)$, *i.e.* there exists a pair $\langle u,s\rangle\in S$ such that $\langle u,s+_{\tt cw}(r\restriction n)\rangle\notin T$, where $n={\tt lh}\,u={\tt lh}\,s\le m$. Then $(*)$ $\langle u,s+_{\tt cw}r'\rangle\notin T$ whenever $r'\in 2^n$, $r'\le_{\tt cw}r\restriction n$. In particular, $\langle u,s\rangle\notin T$ by the normality, and hence $\langle u,s\rangle\in S\,\Delta\,T$, thus $\langle u,s\rangle\in S_{i_1}\,\Delta\,T_{i_1}$ for some $1\le i_1\le k$. This implies $\langle u,s_1\rangle\in S_{i_1}\cap T_{i_1}$, where $s_1=s+_{\tt cw}(r_{i_1}\restriction n)$. (Indeed we have $\langle u,s\rangle\in S_{i_1}\cup T_{i_1}$ by the choice of i_1. If say $\langle u,s\rangle\in S_{i_1}$ then $\langle u,s_1\rangle\in T_{i_1}$ because $r_{i_1}\in R_{i_1}\subseteq\textsc{Emb}(S_{i_1},T_{i_1})$. In addition $\langle u,s_1\rangle\in S_{i_1}$ by the normality of S_{i_1}.)

Once again, $\langle u,s_1\rangle\in S\smallsetminus T$ by $(*)$ above. It follows that $\langle u,s_1\rangle\in S_{i_2}\,\Delta\,T_{i_2}$ for some $1\le i_2\le k$ by the same argument. This implies $\langle u,s_2\rangle\in S_{i_2}\cap T_{i_2}$, where $s_2=s_1+_{\tt cw}(r_{i_2}\restriction n)$, because r_{i_2} belongs to R_{i_2}. Note that $i_2\ne i_1$ as $\langle u,s_1\rangle\in S_{i_1}\cap T_{i_1}$, and still $\langle u,s_2\rangle\in S_{i_1}\cap T_{i_1}$ since S_i and T_i are normal trees.

After k steps of this construction, all indices $1\le i\le k$ will be considered, and the final sequence $s_k=s+_{\tt cw}(r\restriction n)$ will satisfy $\langle u,s_k\rangle\in S_i\cap T_i$ for all $i=1,\dots,k$. It follows that $\langle u,s_k\rangle\notin S\,\Delta\,T$. However $\langle u,s_k\rangle\in S$ since $\langle u,s\rangle\in S$ and S is a normal tree. Thus $\langle u,s_k\rangle$ belongs to T, contrary to the above. □

Theorems 16.12 and 16.10 imply

Corollary 16.13. $\mathsf{E}_{\mathscr{I}_{\rm NT}}$ *is a universal* $\mathbf{\Sigma}^1_1$ *equivalence relation.* □

Let us show now that these properties of $\mathscr{I}_{\rm NT}$ suffice to prove Theorem 16.1.

We begin with a very general fact of basic descriptive set theory: as any $\mathbf{\Sigma}^1_1$ set, $\mathscr{I}_{\rm NT}$ can be presented in the form $\mathscr{I}_{\rm NT}=\bigcap_{\xi<\omega_1}\mathscr{I}^\xi_{\rm NT}$, where $\mathscr{I}^\xi_{\rm NT}$ are Borel subsets of $\mathscr{P}((2\times\mathbb{N})^{<\omega})$, $\xi<\eta\Longrightarrow\mathscr{I}^\eta_{\rm NT}\subseteq\mathscr{I}^\xi_{\rm NT}$, and for any $\mathbf{\Pi}^1_1$ set X in the same Polish space containing $\mathscr{I}_{\rm NT}$ there is an ordinal $\xi<\omega_1$ such that $\mathscr{I}^\xi_{\rm NT}\subseteq X$. [2] The sets $\mathscr{I}^\xi_{\rm NT}$ are called *(upper) Borel approximations* of $\mathscr{I}_{\rm NT}$.

The following lemma is the key fact.

[2] This *index restriction* property was first established by Lusin and Sierpiński [44], essentially in the dual form saying that the canonical representation of any $\mathbf{\Pi}^1_1$ set C in the form $C=$

Lemma 16.14. *For any $\xi < \omega_1$ there exists an ordinal ν, $\xi < \nu < \omega_1$, such that the Borel approximation $\mathscr{I}_{\rm NT}^{\nu}$ is still an ideal.*

Proof. *Step 1*: we claim that for any $\xi < \omega_1$ there is an ordinal $\eta = \eta(\xi)$, $\xi < \eta < \omega_1$, such that $y \subseteq x \in \mathscr{I}_{\rm NT}^{\eta} \Longrightarrow y \in \mathscr{I}_{\rm NT}^{\xi}$. Indeed the set $P = \{x \in \mathscr{I}_{\rm NT}^{\xi} : \forall\, y \subseteq x\, (y \in \mathscr{I}_{\rm NT}^{\xi})\}$ is a $\mathbf{\Pi}^1_1$ superset of $\mathscr{I}_{\rm NT}$ (since $\mathscr{I}_{\rm NT}$ is an ideal). It follows that there is an ordinal $\eta > \xi$ with $\mathscr{I}_{\rm NT}^{\eta} \subseteq P$.

Step 2: we claim that for any $\xi < \omega_1$ there is an ordinal $\zeta = \zeta(\xi)$, $\xi < \zeta < \omega_1$, such that $x, y \in \mathscr{I}_{\rm NT}^{\zeta} \Longrightarrow x \cup y \in \mathscr{I}_{\rm NT}^{\xi}$. The argument contains two substeps. First, the set $X = \{x \in \mathscr{I}_{\rm NT}^{\xi} : \forall\, y \in \mathscr{I}_{\rm NT}\, (x \cup y \in \mathscr{I}_{\rm NT}^{\xi})\}$ is a $\mathbf{\Pi}^1_1$ superset of $\mathscr{I}_{\rm NT}$ since $\mathscr{I}_{\rm NT}$ is an ideal. Thus there is an ordinal $\alpha > \xi$ with $\mathscr{I}_{\rm NT}^{\alpha} \subseteq X$. Then we have $x \cup y \in \mathscr{I}_{\rm NT}^{\xi}$ whenever $x \in \mathscr{I}_{\rm NT}^{\alpha}$ and $y \in \mathscr{I}_{\rm NT}$. It follows that the $\mathbf{\Pi}^1_1$ set $Y = \{y \in \mathscr{I}_{\rm NT}^{\alpha} : \forall\, x \in \mathscr{I}_{\rm NT}^{\alpha}\, (x \cup y \in \mathscr{I}_{\rm NT}^{\xi})\}$ is a superset of $\mathscr{I}_{\rm NT}$, and hence there is an an ordinal $\eta > \alpha$ such that $\mathscr{I}_{\rm NT}^{\eta} \subseteq Y$. Obviously η is as required.

Final argument. Put $\xi_0 = \xi$ and $\xi_{n+1} = \eta(\zeta(\xi_n))$ for all n. The ordinal $\nu = \sup_n \xi_n$ is as required. □

It follows that the set $\Xi = \{\xi < \omega_1 : \mathscr{I}_{\rm NT}^{\xi}$ is an ideal$\}$ is unbounded in ω_1. We also note that $\mathsf{E}_{\mathscr{I}_{\rm NT}^{\xi}}$ is a Borel equivalence relation on $\mathscr{P}((2 \times \mathbb{N})^{<\omega})$ for any $\xi \in \Xi$, and the sequence of these equivalences is $\subseteq$-decreasing and satisfies $\mathsf{E}_{\mathscr{I}_{\rm NT}} = \bigcap_{\xi \in \Xi} \mathsf{E}_{\mathscr{I}_{\rm NT}^{\xi}}$. The proof of Theorem 16.1, our main result here, is accomplished with the following lemma.

Lemma 16.15. *If E is a Borel equivalence relation on a Polish space X then there is an ordinal $\xi \in \Xi$ such that $\mathsf{E} \le_{\rm B} \mathsf{E}_{\mathscr{I}_{\rm NT}^{\xi}}$.*

Proof. It follows from Corollary 16.13 that $\mathsf{E} \le_{\rm B} \mathsf{E}_{\mathscr{I}_{\rm NT}}$, that is, there exists a Borel map $\vartheta : X \to \mathscr{P}((2 \times \mathbb{N})^{<\omega})$ such that $x \,\mathsf{E}\, y \iff \vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{I}_{\rm NT}$. Thus the full ϑ-image ϑ"P of the set $P = (X \times X) \smallsetminus \mathsf{E}$ is a $\mathbf{\Sigma}^1_1$ set disjoint from $\mathscr{I}_{\rm NT}$. Then by Lemma 16.14 there is an ordinal $\xi \in \Xi$ such that ϑ"P does not intersect $\mathscr{I}_{\rm NT}^{\xi}$, too. Thus ϑ reduces E not only to $\mathsf{E}_{\mathscr{I}_{\rm NT}}$ but also to the approximating Borel equivalence relation $\mathsf{E}_{\mathscr{I}_{\rm NT}^{\xi}}$. □

□ (*Theorem 16.1, first proof*)

16e Reduction to Borel ideals: second approach

Is there any method to prove Theorem 16.1 by a sequence of more "effective" and mathematically meaningful upper Borel approximations of a $\le_{\rm B}$-maximal analytic ideal? Paper [32] suggested a suitable definition.

$\bigcap_{\xi < \omega_1} C_\xi$ of a union of $\subseteq$-increasing Borel approximations has the property that for any $\mathbf{\Sigma}^1_1$ set $X \subseteq C$ there is an index $\xi < \omega_1$ with $X \subseteq C_\xi$. The shortest proof consists of observation that otherwise the relation $x \preccurlyeq y$ iff x appears in sets C_ξ not later than y on X is a $\mathbf{\Sigma}^1_1$ prevellordering of uncountable length, contrary to the Kunen – Martin prewellordering theorem (see, *e.g.*, [50, 2G.2]).

First of all recall that any tree $T \subseteq X^{<\omega}$ admits *the rank function*, a unique map $\mathtt{rnk}_R : R \to \mathtt{Ord} \cup \{\infty\}$, where ∞ denote a formal element bigger than any ordinal, satisfying the following requirements:

(a) $\mathtt{rnk}_R(r) = -1$ whenever $r \notin R$;

(b) $\mathtt{rnk}_R(r) = \sup_{r^\wedge n \in R} \mathtt{rnk}_R(r^\wedge n)$ for any $r \in R$. [3] In particular, $\mathtt{rnk}_R(r) = 0$ if and only if $r \in R$ is a $\subseteq$-maximal element of R;

(c) $\mathtt{rnk}_R(r) = \infty$ if and only if R has an infinite branch containing r, *i.e.*, there exists $\gamma \in X^\omega$ such that $\gamma \restriction n \in R$ for all n, and $\gamma \restriction \mathtt{lh}\, r = r$.

In addition, put $\mathtt{rnk}(\varnothing) = -1$ for the empty tree $\varnothing$, and $\mathtt{rnk}(R) = \mathtt{rnk}_R(\Lambda)$ for any non-empty tree R. (Λ, the empty sequence, belongs to any tree $\varnothing \neq R \subseteq X^{<\omega}$.) Obviously any tree R is well-founded iff $\mathtt{rnk}(R) < \infty$.

Definition 16.16. Suppose that $S, T \in \mathbf{NT}$ and $\xi < \omega_1$.
Define $S \leq^\xi_{\rm NT} T$ iff the tree $\textsc{Emb}(S,T)$ satisfies $\mathtt{rnk}(\textsc{Emb}(S,T)) \geq \xi$. [4]
Define $S \,\mathsf{E}^\xi_{\rm NT}\, T$ iff both $S \leq^\xi_{\rm NT} T$ and $T \leq^\xi_{\rm NT} S$. □

It is demonstrated in [32] by simple and rather straightforward arguments that all relations $\mathsf{E}^\xi_{\rm NT}$ are Borel equivalence relations on $\mathbf{NT}$, of certain explicitly defined Borel ranks. A notable part of this result is the proof of transitivity of $\leq^\xi_{\rm NT}$ and $\mathsf{E}^\xi_{\rm NT}$, based on the following generalization of Lemma 16.2.

Lemma 16.17 (Lemma 4 in [32]**).** *We have* $\mathtt{rnk}(S +_{\mathtt{cw}} T) = \mathtt{min}\{\mathtt{rnk}(S), \mathtt{rnk}(T)\}$ *for any trees* $S, T \subseteq \mathbb{N}^{<\omega}$*, well- or ill-founded independently of each other.* □

In addition, $\mathsf{E}_{\rm NT} = \bigcap_{\xi<\omega_1} \mathsf{E}^\xi_{\rm NT}$, and this intersection has the same restriction property as above: if P is a $\mathbf{\Pi}^1_1$ subset of $\mathbf{NT} \times \mathbf{NT}$ containing $\mathsf{E}_{\rm NT}$ then there is an ordinal $\xi < \omega_1$ such that $\mathsf{E}^\xi_{\rm NT} \subseteq P$.

It follows, essentially by the same arguments as above, that the sequence of Borel relations $\mathsf{E}^\xi_{\rm NT}$ is $\leq_{\rm B}$-cofinal among all Borel equivalence relations.

The following construction of Borel ideals that generate the equivalence relations $\mathsf{E}^\xi_{\rm NT}$ is a modification of a construction in [32].

Consider a set $X \subseteq (2 \times \mathbb{N})^{<\omega}$. Suppose that $f \in \mathbb{N}^{<\omega}$, $u \in 2^{<\omega}$, $n = \mathtt{lh}\, u \leq \mathtt{lh}\, f$. Let $\mathbf{G}^u_f(X)$ be the game in which $\mathbf{I}$ plays $s_1, s_2, \dots \in \mathbb{N}^n$, $\mathbf{II}$ plays $t_1, t_2 \dots \in \mathbb{N}^n$ so that $t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} t_m \leq_{\mathtt{cw}} f \restriction n$ for all m, and $\mathbf{I}$ wins if and only if $\langle u, \widehat{s}_k \rangle \in X$ for all k, where $\widehat{s}_k = s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k-1} +_{\mathtt{cw}} t_{k-1} +_{\mathtt{cw}} s_k$.

Define $\textsc{Wid}(X)$ to be the tree of all $f \in \mathbb{N}^{<\omega}$ such that for any $n \leq \mathtt{lh}\, f$ and $u \in 2^n$ $\mathbf{II}$ has a winning strategy in $\mathbf{G}^u_f(X)$. Thus, informally, $f \in \textsc{Wid}(X)$

[3] We define $\sup \Omega$, for $\Omega \subseteq \mathtt{Ord}$, to be the least ordinal strictly bigger than all ordinals in Ω. We also define $\sup \Omega = \infty$ provided Ω contains ∞.

[4] The inequality $\mathtt{rnk}(\textsc{Emb}(S,T)) \geq \xi$ means that either $\textsc{Emb}(S,T)$ (a tree in $\mathbb{N}^{<\omega}$) is ill-founded (then $\mathtt{rnk}(\textsc{Emb}(S,T)) = \infty$) or it is well-founded and its rank is an ordinal $\geq \xi$.

can be seen as a statement of the possibility to leave X for good in finitely many steps, the $+_{\mathtt{cw}}$-total length of which is at most f. Let $\mathscr{I}_{\rm NT}$ be the collection of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ such that $\textsc{Wid}(X)$ is ill-founded. For $\xi < \omega_1$, let $\mathscr{I}^{\xi}_{\rm NT}$ be the collection of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ with $\mathtt{rnk}(\textsc{Wid}(X)) \geq \xi$.

Lemma 16.18. *$\mathscr{I}_{\rm NT}$ and all sets $\mathscr{I}^{\xi}_{\rm NT}$ are ideals on $(2 \times \mathbb{N})^{<\omega}$.*

Proof. Suppose that sets $X, Y \subseteq (2 \times \mathbb{N})^{<\omega}$ belong to $\mathscr{I}_{\rm NT}$, and hence the trees $F = \textsc{Wid}(X)$ and $G = \textsc{Wid}(Y)$ are ill-founded. Then the tree $F +_{\mathtt{cw}} G$ is ill-founded by Lemma 16.2 (to be replaced by Lemma 16.17 for the ideals $\mathscr{I}^{\xi}_{\rm NT}$), and hence it suffices to prove that $F +_{\mathtt{cw}} G \subseteq \textsc{Wid}(X \cup Y)$.

Take any $f \in F$ and $g \in G$ with $\mathtt{lh}\, f = \mathtt{lh}\, g$. To prove that $h = f +_{\mathtt{cw}} g$ belongs to $\textsc{Wid}(X \cup Y)$ fix any $u \in 2^n$, $n \leq \mathtt{lh}\, f$, and a pair of winning strategies ξ, η for **II** in games resp. $\mathbf{G}^u_f(X)$ and $\mathbf{G}^u_g(Y)$. To describe a winning strategy for **II** in $\mathbf{G}^u_h(X \cup Y)$, let $s_1, t_1, s_2, t_2, \dots$ be a full sequence of moves. Put $K = \{k : \widehat{s}_k \in X\}$ and $K' = \{k : \widehat{s}_k \in Y \smallsetminus X\}$. Let $K = \{k_1, k_2, \dots\}$ and $K' = \{k'_1, k'_2, \dots\}$, in the increasing order.

For any k, if $k = k_j \in K$ then **II** plays $t_k = \xi(\sigma_1, \tau_1, \dots, \sigma_{j-1}, \tau_{j-1}, \sigma_j)$, where $\tau_i = t_{k_i}$ and, for any $1 \leq i \leq j$,

$$\sigma_i = s_{k_{i-1}+1} +_{\mathtt{cw}} t_{k_{i-1}+1} +_{\mathtt{cw}} s_{k_{i-1}+2} +_{\mathtt{cw}} t_{k_{i-1}+2} +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k_i-1} +_{\mathtt{cw}} t_{k_i-1} +_{\mathtt{cw}} s_{k_i}.$$

Accordingly if $k = k'_j \in K'$ then $t_k = \eta(\sigma'_1, \tau'_1, \dots, \sigma'_{j-1}, \tau'_{j-1}, \sigma'_j)$, where

$$\sigma'_i = s_{k'_{i-1}+1} +_{\mathtt{cw}} t_{k'_{i-1}+1} +_{\mathtt{cw}} s_{k'_{i-1}+2} +_{\mathtt{cw}} t_{k'_{i-1}+2} +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k'_i-1} +_{\mathtt{cw}} t_{k'_i-1} +_{\mathtt{cw}} s_{k'_i}$$

and $\tau'_i = t'_{k_i}$ for any $1 \leq i \leq j$. If to the contrary **I** wins then $K \cup K' = \mathbb{N}$. Let, say, $K = \{k_1, k_2, \dots\}$ be infinite. Then **II** must win the auxiliary play $\sigma_1, \tau_1, \sigma_2, \tau_2, \dots$ in $\mathbf{G}^u_f(X)$, hence one of the finite sums $\widehat{\sigma}_j = \sigma_1 +_{\mathtt{cw}} \tau_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} \sigma_{j-1} +_{\mathtt{cw}} \tau_{j-1} +_{\mathtt{cw}} \sigma_j$ satisfies $\widehat{\sigma}_j \notin X$. But obviously $\widehat{\sigma}_j = \widehat{s}_{k_j}$, which is a contradiction with $k_j \in K$. □

Thus $\mathscr{I}_{\rm NT}$ is a $\mathbf{\Sigma}^1_1$ ideal while each $\mathscr{I}^{\xi}_{\rm NT}$ is a Borel ideal.

Theorem 16.19. *The equivalence relation $\mathsf{E}_{\rm NT}$ is equal to $\mathsf{E}_{\mathscr{I}_{\rm NT}} \restriction \mathbf{NT}$, while for any ξ, $\mathsf{E}^{\xi}_{\rm NT}$ is equal to $\mathsf{E}_{\mathscr{I}^{\xi}_{\rm NT}} \restriction \mathbf{NT}$.*

Proof. Consider any $S, T \in \mathbf{NT}$. Assume that $S \,\mathsf{E}_{\rm NT}\, T$. Then the trees $F = \textsc{Emb}(S, T)$ and $G = \textsc{Emb}(T, S)$ are ill-founded, and hence so is $H = F +_{\mathtt{cw}} G$ by Lemma 16.2. (Lemma 16.17 is used in the case of $\mathscr{I}^{\xi}_{\rm NT}$.) Note that $H \subseteq G \cap F$ since both S and T are $\leq_{\mathtt{cw}}$-transitive to the right. Thus it suffices to prove that $G \cap F \subseteq \textsc{Wid}(S \,\Delta\, T)$. Consider any $f \in G \cap F$. By definition, for any $\langle u, s \rangle \in S \cup T$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \leq \mathtt{lh}\, f$, we have $\langle u, s +_{\mathtt{cw}} (f \restriction n) \rangle \in S \cap T$. In particular, $\langle u, s \rangle \in S \,\Delta\, T \Longrightarrow \langle u, s +_{\mathtt{cw}} (f \restriction n) \rangle \notin S \,\Delta\, T$, and easily $f \in \textsc{Wid}(S \,\Delta\, T)$.

To prove the converse, suppose that $S \Delta T \in \mathscr{J}_{\rm NT}$, thus $\textsc{Wid}(S \Delta T)$ is ill-founded. It suffices to prove that $\textsc{Wid}(S \Delta T) \subseteq \textsc{Emb}(S,T)$. Suppose, towards the contrary, that $f \in \textsc{Wid}(S \Delta T)$ but $f \notin \textsc{Emb}(S,T)$. The latter means that there exists a pair $\langle u, s\rangle \in S$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \le \mathtt{lh}\, f$, such that $\langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle \notin T$. Then also $\langle u, s\rangle \notin T$, and hence both $\langle u, s\rangle$ and $\langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle$ belong to $S \smallsetminus T$. It follows that

$(*)$ $\langle u, s + g\rangle \in S \smallsetminus T$ for any $g \in \mathbb{N}^n$, $g \le_{\mathtt{cw}} (f \restriction n)$.

Now consider a play in $\mathbf{G}^u_f(S \Delta T)$ in which **II** follows its winning strategy (which exists because $f \in \textsc{Wid}(S \Delta T)$) while **I** plays $s_1 = s$ and $s_k = 0^n$ (the sequence of n zeros) on every move $k \ge 2$. Let $t_1, t_2, \dots$ be the sequence of **II**'s moves. Then $t_1 +_{\mathtt{cw}} \cdots +_{\mathtt{cw}} t_k \le_{\mathtt{cw}} (f \restriction n)$ for all k, and hence, by $(*)$, the sum $\widehat{s}_k = s +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \cdots +_{\mathtt{cw}} t_k$ satisfies $\langle u, \widehat{s}_k\rangle \in S \Delta T$, which contradicts the choice of the strategy. □

□ (*Theorem 16.1, second proof*)

16f Some questions

It can be reasonably conjectured that $\mathsf{E}^{\eta}_{\rm NT} <_{\rm B} \mathsf{E}^{\omega\nu}_{\rm NT} <_{\rm B} \mathsf{E}^{\omega\nu+n}_{\rm NT}$ whenever $\eta < \omega\nu$ and $n \ge 1$. The background idea here is that there is no $\le_{\rm B}$-largest Borel equivalence relation (noted in [22]), therefore, the sequence of equivalence relations $\mathsf{E}^{\xi}_{\rm NT}$ has uncountably many indices of $<_{\rm B}$-increase (in strict sense). On the other hand, it seems plausible that $\mathsf{E}^{\omega\nu+n}_{\rm NT} \sim_{\rm B} \mathsf{E}^{\omega\nu+n+1}_{\rm NT}$ provided $n \ge 1$.

Few more interesting questions.

Which Borel classes contain complete equivalence relations?

A related problem can be discussed here. It was once considered a viable conjecture (see, *e.g.*, [30]) that the equivalence relation T_2 called *the equality of countable sets of reals* [5] is not Borel reducible to any equivalence relation $\mathsf{E}_{\mathscr{I}}$ induced by a Borel ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$. It follows from Theorem 16.1 that this is not the case, in fact there is an ordinal $\xi < \omega_1$ such that $\mathsf{T}_2 \le_{\rm B} \mathsf{E}^{\xi}_{\rm NT}$. What is the least ordinal ξ satisfying this statement?

Finally, it should be stressed that all evaluations of Borel class of equivalence relations in this paper were related to the actual Borel class in Cantor's discontinuum-like spaces. A somewhat deeper approach of "potential" Borel classes of equivalence relations in [22] may require corresponding adjustment of arguments.

[5] T_2 is defined on $\mathbb{R}^\omega$, the set of countable sequences of reals, so that $x \mathbin{\mathsf{T}_2} y$ iff the sets $\{x(n) : n \in \omega\}$ and $\{y(n) : n \in \omega\}$ (countable sets of reals) are equal to each other.

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