On automorphisms behind the Gitik – Koepke model for violation of the Singular Cardinals Hypothesis w/o large cardinals

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Abstract

It is known that the assumption that "GCH first fails at $\aleph_\omega$" leads to large cardinals in **ZFC**. Gitik and Koepke [2] demonstrated that this is not so in **ZF**: namely there is a generic cardinal-preserving extension of **L** (or any universe of **ZFC** + GCH) in which all **ZF** axioms hold, the axiom of choice fails, $\mathtt{card}\,2^{\aleph_n} = \aleph_{n+1}$ for all natural n, but there is a surjection from $2^{\aleph_\omega}$ onto λ, where $\lambda > \aleph_{\omega+1}$ is any previously chosen cardinal in **L**, for instance, $\aleph_{\omega+17}$. In other words, in such an extension GCH holds in proper sense for all cardinals $\aleph_n$ but fails at $\aleph_\omega$ in Hartogs' sense.

The goal of this note is to analyse the system of automorphisms involved in the Gitik – Koepke construction.

It is known (see [1]) that the consistency of the statement "GCH first fails at $\aleph_\omega$" with **ZFC** definitely requires a large cardinal. Gitik and Koepke [2] demonstrated that picture changes in the absense of the axiom of choice, if one agrees to treat the violation of GCH in Hartogs' sense. Namely there is a generic cardinal-preserving extension of **L** (or any universe of **ZFC** + GCH) in which all **ZF** axioms hold, the axiom of choice fails, $\mathtt{card}\,2^{\aleph_n} = \aleph_{n+1}$ for all natural n, but there is a surjection from $2^{\aleph_\omega}$ onto λ, where $\lambda > \aleph_{\omega+1}$ is any previously chosen cardinal in **L**, for instance, $\aleph_{\omega+17}$. Thus in such an extension GCH holds in proper sense for all cardinals $\aleph_n$ but fails at $\aleph_\omega$ in Hartogs' sense.

For the sake of convenience we formulate the main result as follows.

Theorem 1 (Gitik – Koepke [2])**.** *Let $\lambda > \aleph_{\omega+1}$ be a cardinal in **L**, the constructible universe. There is a set-generic extension $\mathbf{L}[G]$ of **L** and a symmetric cardinal-preserving subextension $\mathbf{L}_{\mathtt{sym}}[G] \subseteq \mathbf{L}[G]$, such that the following is true in $\mathbf{L}_{\mathtt{sym}}[G]$:*

(i) *all axioms of **ZF**;*

(ii) $\mathtt{card}\,2^{\aleph_n} = \aleph_{n+1}$ *for all natural n;*

(iii) *there is a surjection from $2^{\aleph_\omega}$ onto λ.*

The goal of this note is to analyse the system of automorphisms (which turns out to consist of three different subsistems) involved in the Gitik – Koepke proof of this

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theorem in [2].[1] On the base of our analysis, we present the proof in a somewhat more pedestrian way than in [2].

1 Basic definitions and the forcing

After an array of auxiliary definitions, we'll introduce the forcing.

λ is a fixed cardinal everywhere; $\lambda > \aleph_\omega$.

1 Basic definitions

We define:

$\mathbb{D}[n] = $ all sets $d \subseteq [\aleph_n, \aleph_{n+1})$ such that $\mathtt{card}\, d \leqslant \aleph_n$

$\mathbb{P}^+[n] = $ all functions $p : \mathtt{dom}\, p \to 2$, such that $\varnothing \neq \mathtt{dom}\, p \subseteq [\aleph_n, \aleph_{n+1})$,

$\mathbb{P}[n] = $ all functions $p \in \mathbb{P}^+[n]$, such that $\mathtt{dom}\, p \in \mathbb{D}[n]$,

$\mathbb{D} = $ all sets $d \subseteq [\omega, \aleph_\omega)$ such that $d \cap [\aleph_n, \aleph_{n+1}) \in \mathbb{D}[n]$ for all n,

$\mathbb{D}^* = $ all sets $d \subseteq [\omega, \aleph_\omega)$ such that there is $n_0 \in \omega$ such that $d \cap [\aleph_n, \aleph_{n+1}) \in \mathbb{D}[n]$ for all $n \geq n_0$,

$\mathbb{P}^+ = $ all functions $p : \mathtt{dom}\, p \to 2$ such that $\mathtt{dom}\, p \subseteq [\omega, \aleph_\omega)$,

$\mathbb{P} = $ all functions $p \in \mathbb{P}^+$ such that $\mathtt{dom}\, p \in \mathbb{D}$.

If $n \in \omega$ then we let $d[n] = d \cap [\aleph_n, \aleph_{n+1})$ and $p[n] = p {\restriction} [\aleph_n, \aleph_{n+1})$ for all $d \in \mathbb{D}$ and $p \in \mathbb{P}^+$. Thus $d \in \mathbb{D}$ iff $d[n] \in \mathbb{D}[n]$ for all n, and $p \in \mathbb{P}$ iff $p[n] \in \mathbb{P}[n]$ for all n.

We order $\mathbb{P}$ so that $p \leqslant q$ iff $\mathtt{dom}\, q \subseteq \mathtt{dom}\, p$ and $q = p {\restriction} \mathtt{dom}\, q$.

Note that if $m \neq n$ then $\mathbb{P}[n] \cap \mathbb{P}[m] = \varnothing$.

2 Assignments

An *assignment* will be any function a such that

(a1) $\mathtt{dom}\, a = \mathtt{bas}\, a \times |a|$, where $\mathtt{bas}\, a \subseteq \omega$ and $|a| \subseteq \lambda$ are finite sets, and

(a2) if $\langle n, \gamma \rangle \in \mathtt{dom}\, a$ then $a(n, \gamma) \in [\aleph_n, \aleph_{n+1})$.

In particular, $\varnothing$ (the empty assignment) belongs to $\mathbb{A}$.[2]

If $n \in \mathtt{bas}\, a$ then define a map $a[n]$ on the set $|a|$ by $a[n](\gamma) = a(n, \gamma)$.

The set $\mathbb{A}$ of all assignments is *ordered* so that $a \leqslant b$ (a is stronger) iff

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[1] The author learned the description of the Gitik – Koepke model in the course of his visit to Bonn in the Winter of 2009/2010.

[2] We suppose that $\mathtt{bas}\, \varnothing = |\varnothing| = \varnothing$, but it can be consistently assumed that either $\mathtt{bas}\, \varnothing = \varnothing$ and $|\varnothing| = \Gamma \subseteq \lambda$ is any finite set, or $|\varnothing| = \varnothing$ and $\mathtt{bas}\, \varnothing = N \subseteq \omega$ is any finite set, depending on the context. Any assignment $a \neq \varnothing$ has definite values of $|a|$ and $\mathtt{bas}\, a$.

(a3) $\mathtt{bas}\,b \subseteq \mathtt{bas}\,a$ and $|b| \subseteq |a|$, and

(a4) if $n \in \mathtt{bas}\,a \smallsetminus \mathtt{bas}\,b$ and $\gamma \neq \delta$ belong to $|b|$ then $a(n,\gamma) \neq a(n,\delta)$.

Clearly $\varnothing$ is the $\leqslant$-largest element in $\mathbb{A}$.

Assignments a, b are *coherent* iff $\mathtt{dom}\,a = \mathtt{dom}\,b$, and for any $n \in \mathtt{bas}\,a = \mathtt{bas}\,b$ and $\gamma, \delta \in |a| = |b|$ we have: $a(n,\gamma) = a(n,\delta)$ iff $b(n,\gamma) = b(n,\delta)$.

If $a \in \mathbb{A}$ and $\Delta \subseteq |a|$ then let $a \,\|\, \Delta$ be the restriction $a{\upharpoonright}(\mathtt{bas}\,a \times \Delta)$.

3 Narrow subconditions

Let $\mathbb{H}^+$ consist of all indexed sets $h = \{h_\xi\}_{\xi \in |h|}$, where $|h| \subseteq [\omega, \aleph_\omega)$ and $h_\xi \in \mathbb{P}^+[n]$ for all n and $\xi \in |h| \cap [\aleph_n, \aleph_{n+1})$.

We put $h[n] = h{\upharpoonright}[\aleph_n, \aleph_{n+1})$ (restriction) for $h \in \mathbb{H}^+$ and any n. Thus still $h[n] \in \mathbb{H}^+$ and $|h[n]| = |h| \cap [\aleph_n, \aleph_{n+1})$.

Let $\mathbb{H}$ consist of all $h \in \mathbb{H}^+$ such that

(h1) $\mathtt{card}\,|h[n]| \leqslant [\aleph_n, \aleph_{n+1})$ for all n,

(h2) the set $\mathtt{bas}\,h = \{n : h[n] \neq \varnothing\}$ is finite,

(h3) $h_\xi \in \mathbb{P}[n]$ for all n and $\xi \in |h| \cap [\aleph_n, \aleph_{n+1})$.

We say that a condition $h \in \mathbb{H}$ is

regular at some $n \in \mathtt{bas}\,h$, iff for every $\xi \in |h| \cap [\aleph_n, \aleph_{n+1})$ the set $\{\eta \in |h| \cap [\aleph_n, \aleph_{n+1}) : h_\eta = h_\xi\}$ has cardinality exactly $\aleph_n$,

stronger than another condition $g \in \mathbb{H}$, symbolically $h \leqslant g$, iff $|g| \subseteq |h|$, and $h_\xi \leqslant g_\xi$ for all $\xi \in |g|$.

The empty condition $\varnothing \in \mathbb{H}$ ($|\varnothing| = \varnothing$) is $\leqslant$-largest in $\mathbb{H}$.

We further define $\mathbb{H}[n] = \{h \in \mathbb{H} : |h| \subseteq [\aleph_n, \aleph_{n+1})\}$; thus $\mathbb{H}[n]$ consists of all indexed sets $h = \{h_\xi\}_{\xi \in |h|}$, where $|h| \in \mathbb{D}[n]$ (that is, $|h| \subseteq [\aleph_n, \aleph_{n+1})$ and $\mathtt{card}\,|h| \leqslant \aleph_n$), and $h_\xi \in \mathbb{P}[n]$ for all $\xi \in |h|$.

It is clear that $h \in \mathbb{H}$ iff $h[n] \in \mathbb{H}[n]$ for all n and the set $\mathtt{bas}\,h$ is finite.

4 Wide subconditions

Let $\mathbb{Q}^+$ consist of all indexed sets $q = \{q_\gamma\}_{\gamma \in |q|}$, where $|q| \subseteq \lambda$ and $q_\gamma \in \mathbb{P}^+$ for all $\gamma \in |q|$. We define

$\mathbb{Q}^* = $ all $q \in \mathbb{Q}^+$ such that $|q|$ is finite,

$\mathbb{Q} = $ all $q \in \mathbb{Q}^+$ such that $|q|$ is finite and $q_\gamma \in \mathbb{P}$ for all $\gamma \in |q|$.

We say that a condition $q \in \mathbb{Q}^+$ is:

uniform, if $\mathtt{dom}\,q_\gamma[n] = \mathtt{dom}\,q_\delta[n]$ for all $\gamma, \delta \in |q|$ and $n \in \omega$,

compatible with an assignment $a \in \mathbb{A}$, iff we have $q_\gamma[n] = q_\delta[n]$ whenever $\gamma, \delta \in |q| \cap |a|$, $n \in \mathtt{bas}\,a$, and $a(n,\gamma) = a(n,\delta)$.

equally shaped with another condition $p \in \mathbb{Q}^+$, iff $|p| = |q|$, and we have $\mathtt{dom}\, p_\gamma[n] = \mathtt{dom}\, q_\gamma[n]$ holds for all $\gamma \in |p|$ and $n \in \omega$.

stronger than another condition $p \in \mathbb{Q}^+$, symbolically $q \leqslant p$, iff $|p| \subseteq |q|$, and $p_\gamma \leqslant q_\gamma$ in $\mathbb{P}$ for all $\gamma \in |p|$.

Once again, the empty condition $\varnothing \in \mathbb{Q}$ $(|\varnothing| = \varnothing)$ is $\leqslant$-largest in $\mathbb{Q}$.

5 Conditions

Let $\mathbb{T}$, **the forcing**, consist of all triples of the form $t = \langle q^t, a^t, h^t \rangle$, where $q^t \in \mathbb{Q}$, $a^t \in \mathbb{A}$, $h^t \in \mathbb{H}$, and

(t1) $|a^t| = |q^t|$ and $\mathtt{bas}\, a^t = \mathtt{bas}\, h^t$ — we put $|t| := |a^t|$ and $\mathtt{bas}\, t := \mathtt{bas}\, a^t$,

(t2) $\mathtt{ran}\, a^t \subseteq |h^t|$ and we have $h^t_{a^t(n,\gamma)} = q^t_\gamma[n]$ for all $n \in \mathtt{bas}\, t$ and $\gamma \in |t|$.

(t3) therefore q^t is compatible with a^t in the sense above, that is, if $\gamma, \delta \in |t|$, $n \in \mathtt{bas}\, t$, and $a^t(n,\gamma) = a^t(n,\delta)$, then $q^t_\gamma[n] = q^t_\delta[n]$.

The set $\mathbb{T}$ is ordered componentwise: a condition $t \in \mathbb{T}$ is *stronger than* $s \in \mathbb{T}$, symbolically $t \leqslant s$, iff $q^t \leqslant q^s$ in $\mathbb{Q}$, $a^t \leqslant a^s$ in $\mathbb{A}$, $h^t \leqslant h^s$ in $\mathbb{H}$. Clearly $t = \langle \varnothing, \varnothing, \varnothing \rangle$ is the largest condition in $\mathbb{T}$.

A condition $t \in \mathbb{T}$ is *uniform*, symbolically $t \in \mathbb{T}^{\mathtt{uni}}$, iff q^t is uniform.

2 Permutations

In this section and the following two sections we consider three groups of full or partial order-preserving transformations of conditions.

Let $\Pi_{\mathtt{fin}}$ be the group of all permutations of the set $[\omega, \aleph_\omega)$ such that

(A) for any n, the restriction $\pi[n] = \pi \restriction [\aleph_n, \aleph_{n+1})$ is a permutation of the set $[\aleph_n, \aleph_{n+1})$,

(B) the set $\mathtt{bas}\, \pi = \{n : \pi[n] \neq \text{ the identity}\}$ is finite.

Let $\Pi_{\mathtt{fin}}[n]$ consist of all $\pi \in \Pi_{\mathtt{fin}}$ equal to the identity outside of $[\aleph_n, \aleph_{n+1})$. Any $\pi \in \Pi_{\mathtt{fin}}[n]$ is naturally identified with $\pi[n]$.

There are two types of induced action of transformations $\pi \in \Pi_{\mathtt{fin}}$, namely:

(I) if f is a function such that $\mathtt{ran}\, f \subseteq [\omega, \aleph_\omega)$ then $f' = \pi \cdot f$ is a function with the same domain and $f'(x) = \pi(f(x))$ for all $x \in \mathtt{dom}\, f = \mathtt{dom}\, f'$;

(II) if f is a function such that $\mathtt{dom}\, f \subseteq [\omega, \aleph_\omega)$ then $f' = \pi \cdot f$ is a function, $\mathtt{dom}\, f' = \{\pi(\xi) : \xi \in \mathtt{dom}\, f\}$, and $f'(\pi(x)) = f(x)$ for all $\xi \in \mathtt{dom}\, f$.[3]

Accordingly, we define that any $\pi \in \Pi_{\mathtt{fin}}$:

(1) acts on $\mathbb{A}$ by (I), so that if $a \in \mathbb{A}$ then $a' = \pi \cdot a \in \mathbb{A}$, $\mathtt{dom}\, a' = \mathtt{dom}\, a$, and $a'(n,\gamma) = \pi(a(n,\gamma))$ for all $\langle n,\gamma \rangle \in \mathtt{dom}\, a$;

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[3] We ignore the conflicting case when both $\mathtt{ran}\, f \subseteq [\omega, \aleph_\omega)$ and $\mathtt{dom}\, f \subseteq [\omega, \aleph_\omega)$ as it will never happen in the domains of action of transformations $\pi \in \Pi_{\mathtt{fin}}$ considered below.

(2) acts on $\mathbb{H}^+$ (and on $\mathbb{H} \subseteq \mathbb{H}^+$) by (II), so that if $h \in \mathbb{H}^+$ then $h' = \pi \cdot h \in \mathbb{H}^+$, $|h'| = \{\pi(\xi) : \xi \in |h|\}$, and $h'_{\pi(\xi)} = h_\xi$ for all $\xi \in |h|$.

Finally if $t = \langle q^t, a^t, h^t \rangle \in \mathbb{T}$ then put $\pi \cdot t = \langle q^t, \pi \cdot a^t, \pi \cdot h^t \rangle$.

The following lemma is rather obvious.

Lemma 2. *Any $\pi \in \Pi_{\mathtt{fin}}$ is an order-preserving automorphism of the ordered sets $\mathbb{A}$, $\mathbb{H}$, and $\mathbb{T}$. Moreover if $a \in \mathbb{A}$ and $n \in \mathtt{bas}\, a \smallsetminus \mathtt{bas}\, \pi$ then $(\pi \cdot a)\,[n] = a\,[n]$, and accordingly if $h \in \mathbb{H}$ and $n \notin \mathtt{bas}\, \pi$ then $(\pi \cdot h)\,[n] = h\,[n]$.* $\quad\square$

3 Swaps

Suppose that $a, b \in \mathbb{A}$, $\mathtt{dom}\, a = \mathtt{dom}\, b = D$, and $\mathtt{ran}\, a = \mathtt{ran}\, b$. Such a pair of assignments induces a *swap transformation* $\mathbf{S}_{ab}$, acting:

$$
\begin{aligned}
\text{from} \quad & \mathbb{A}_a = \{c \in \mathbb{A} : c \leqslant a\} && \text{to} \quad \mathbb{A}_b\,, \\
\text{from} \quad & \mathbb{Q}_a^+ = \{q \in \mathbb{Q}^+ : |a| \subseteq |q| \wedge q \text{ is compatible with } a\} && \text{to} \quad \mathbb{Q}_b^+\,, \\
\text{from} \quad & \mathbb{Q}_a = \{q \in \mathbb{Q} : |a| \subseteq |q| \wedge q \text{ is compatible with } a\} && \text{to} \quad \mathbb{Q}_b\,.
\end{aligned}
$$

Recall that $q \in \mathbb{Q}^+$ is compatible with $a \in \mathbb{A}$ iff $q_\gamma\,[n] = q_\delta\,[n]$ holds whenever $\gamma, \delta \in |a| \cap |q|$, $n \in \mathtt{bas}\, a$, and $a(n, \gamma) = a(n, \delta)$. Obviously $\mathbb{Q}_a = \mathbb{Q}_a^+ \cap \mathbb{Q}$.

The action of $\mathbf{S}_{ab}$ on $\mathbb{A}_a$ is defined as follows:

(1) if $c \in \mathbb{A}_a$ then $c' = \mathbf{S}_{ab} \cdot c \in \mathbb{A}$, $\mathtt{dom}\, c' = \mathtt{dom}\, c$, $c' {\restriction} D = b$ (where $D = \mathtt{dom}\, a = \mathtt{dom}\, b$), and $c' {\restriction} (\mathtt{dom}\, c \smallsetminus D) = c {\restriction} (\mathtt{dom}\, c \smallsetminus D)$.

The action of $\mathbf{S}_{ab}$ on $\mathbb{Q}_a^+$ is defined as follows. First of all, if $n \in \mathtt{bas}\, a$ and $\gamma \in |a|$ then let $\mathbf{s}_{ab}^n(\gamma)$ be the least $\vartheta \in |a|$ satisfying $a(n, \vartheta) = b(n, \gamma)$; such ordinals ϑ exist because $\mathtt{ran}\, a = \mathtt{ran}\, b$. Thus $\mathbf{s}_{ab}^n : |a| \to |a|$. Then:

(2) if $q \in \mathbb{Q}_a^+$ then $q' = \mathbf{S}_{ab} \cdot q \in \mathbb{Q}^+$, $|q'| = |q|$, and for all $n \in \omega$ and $\gamma \in |q|$:

    (a) if $\gamma \in |a|$ and $n \in \mathtt{bas}\, a$ then $q'_\gamma\,[n] = q_\vartheta\,[n]$, where $\vartheta = \mathbf{s}_{ab}^n(\gamma)$,

    (b) if either $\gamma \notin |a|$ or $n \notin \mathtt{bas}\, a$ then $q'_\gamma\,[n] = q_\gamma\,[n]$.

Finally if $t \in \mathbb{T}_a = \{t \in \mathbb{T} : a^t \leqslant a\}$ (then $a^t \in \mathbb{A}_a$ and $q^t \in \mathbb{Q}_a$) then put

$$\mathbf{S}_{ab} \cdot t = \langle \mathbf{S}_{ab} \cdot q^t, \mathbf{S}_{ab} \cdot a^t, h^t \rangle.$$

Lemma 3. *Assume that $a, b \in \mathbb{A}$, $\mathtt{bas}\, a = \mathtt{bas}\, b = B$, $|a| = |b| = \Delta$, and $\mathtt{ran}\, a = \mathtt{ran}\, b$. Then $\mathbf{S}_{ab}$ is an order-preserving bijection $\mathbb{A}_a \xrightarrow{\text{onto}} \mathbb{A}_b$, $\mathbb{Q}_a \xrightarrow{\text{onto}} \mathbb{Q}_b$, $\mathbb{T}_a \xrightarrow{\text{onto}} \mathbb{T}_b$ and $\mathbf{S}_{ba}$ is the inverse in each of the three cases.*

Lett $t \in \mathbb{T}_a$. Then $t' = \mathbf{S}_{ab} \cdot t \in \mathbb{T}_b$, $|t| = |t'|$, $\mathtt{bas}\, t = \mathtt{bas}\, t'$, and:

(i) *if t is uniform, then so is t' and $q^t, q^{t'}$ are equally shaped;*

(ii) *if $n \in B$, $\gamma \in \Delta$, and $a(n, \gamma) = b(n, \gamma)$ then $a^t(n, \gamma) = a^{t'}(n, \gamma) = a(n, \gamma) = b(n, \gamma)$ and $q_\gamma^{t'}\,[n] = q_\gamma^t\,[n]$;*

(iii) *if* $n \in |t|$ *then* $\{q_\gamma^{t'}[n] : \gamma \in |t'|\} = \{q_\gamma^t[n] : \gamma \in |t|\}$.

Proof. The first essential part of the lemma is to show that if $t \in \mathbb{T}_a$ then $t' = \mathbf{S}_{ab} \cdot t \in \mathbb{T}_b$. Basically it's enough to show that $t' \in \mathbb{T}$. And here the only notable task is to prove (t2) of Section 5, that is, $q_\gamma^{t'}[n] = h_{a^{t'}(n,\gamma)}^{t'}$ for all $n \in \mathtt{bas}\, t'$ and $\gamma \in |t'|$.

We can assume that $n \in \mathtt{bas}\, a$ and $\gamma \in |a|$, simply because $\mathbf{S}_{ab}$ is the identity outside of $\mathtt{dom}\, a = \mathtt{bas}\, a \times |a|$. We have $a^{t'}(n, \gamma) = b(n, \gamma)$ within this narrower domain, hence the result to prove is $q_\gamma^{t'}[n] = h_{b(n,\gamma)}^t$ for all $n \in \mathtt{bas}\, a$ and $\gamma \in |a|$. (Recall that $\mathbf{S}_{ab}$ does not change h^t, so that $h^{t'} = h^t$.)

However $q_\gamma^{t'}[n] = q_\vartheta^t[n]$ by (2)a, where $\vartheta = \mathbf{s}_{ab}^n(\gamma)$, so that, in particular, $a(n, \vartheta) = b(n, \gamma)$. Thus the equality required turns out to be $q_\vartheta^t[n] = h_{a(n,\vartheta)}^t$, which is true since t is a condition.

The other essential claim is that the action of $\mathbf{S}_{ba}$ is the inverse of the action of $\mathbf{S}_{ab}$. Suppose that $t \in \mathbb{T}_a$ and let $t' = \mathbf{S}_{ab} \cdot t$; $t \in \mathbb{T}_b$. Put $s = \mathbf{S}_{ba} \cdot t'$; $s \in \mathbb{T}_a$ once again. We have to show that $s = t$. The key fact is $q_\gamma^s[n] = q_\gamma^t[n]$ for all $n \in \mathtt{bas}\, a$ and $\gamma \in |a|$. By definition $q_\gamma^s[n] = q_\zeta^{t'}[n]$, where $\zeta = \mathbf{s}_{ba}^n$, in particular, $b(n, \zeta) = a(n, \gamma)$. Still by definition, $q_\zeta^{t'}[n] = q_\vartheta^t[n]$, where $\vartheta = \mathbf{s}_{ab}^n(\zeta)$, so that $a(n, \vartheta) = b(n, \zeta)$. To conclude, $q_\gamma^s[n] = q_\vartheta^t[n]$, where $a(n, \gamma) = a(n, \vartheta)$. But then $q_\gamma^t[n] = q_\vartheta^t[n]$ by (t3) of Section 5, and hence we have $q_\gamma^s[n] = q_\gamma^t[n]$, as required.

Claims (i), (ii) are rather obvious.

It follows from (2)b that claim (iii) is trivial for $n \in |t| \smallsetminus B$, while in the case $n \in B$ it suffices to prove $\{q_\gamma^{t'}[n] : \gamma \in B\} = \{q_\gamma^t[n] : \gamma \in B\}$. The inclusion $\subseteq$ holds because $q_\gamma^{t'}[n] = q_\vartheta^t[n]$ by (2)a, where $\vartheta = \mathbf{s}_{ab}^n(\gamma)$. The inclusion $\supseteq$ holds by the same reason with respect to the inverse swap $\mathbf{S}_{ba}$. $\square$

4   Rotations

This is a more complicated type of transformations, and we have to define it by extension beginning from most elementary conditions.

1   Simple rotations

If $d \in \mathbb{D}$ and $p \in \mathbb{P}$, or generally even $d \in \mathbb{D}^*$ and $p \in \mathbb{P}^+$, then define $d \cdot p = p' : \mathtt{dom}\, p' \to 2$ so that $\mathtt{dom}\, p = \mathtt{dom}\, p'$ and

$$p'(\alpha) = \begin{cases} p(\alpha) & \text{whenever} \quad \alpha \in (\mathtt{dom}\, p) \smallsetminus d, \\ 1 - p(\alpha) & \text{whenever} \quad \alpha \in d \cap \mathtt{dom}\, p. \end{cases}$$

Clearly $p \mapsto d \cdot p$ is an order-preserving automorphism of $\mathbb{P}$ and of $\mathbb{P}^+$.

Transformations of this type, as well as those based on them and defined below, will be called *rotations*.

2 Rotations for narrow subconditions

We define product rotations which fit to conditions in $\mathbb{H}^+$ and $\mathbb{H} \subseteq \mathbb{H}^+$. Let Ψ consist of all indexed sets $\psi = \{\psi_\xi\}_{\xi \in |\psi|}$, where $|\psi| \subseteq [\omega, \aleph_\omega)$ is a finite set, and $\psi_\xi \in \mathbb{D}[n]$ for all $n \in \omega$ and $\xi \in |\psi| \cap [\aleph_n, \aleph_{n+1})$. If $\psi \in \Psi$ and $h \in \mathbb{H}^+$ then define $h' = \psi \cdot h \in \mathbb{H}^+$ so that $|h'| = |h|$ and for all ξ:

$$
h'_\xi = \begin{cases} h_\xi & \text{whenever} \quad \xi \in |h| \smallsetminus |\psi|, \\ \psi_\xi \cdot h_\xi & \text{whenever} \quad \xi \in |h| \cap |\psi|. \end{cases}
$$

Let $\Psi[n] = \{\psi \in \Psi : |\psi| \subseteq [\aleph_n, \aleph_{n+1})\}$; and accordingly if $\psi \in \Psi$ then let $\psi[n] = \psi{\restriction}[\aleph_n, \aleph_{n+1})$; then $\psi[n] \in \Psi[n]$. The next lemma is obvious.

Lemma 4. *If* $\psi \in \Psi$ *then the map* $h \mapsto \psi \cdot h$ *is an order-preserving action* $\mathbb{H}^+ \xrightarrow{\text{onto}} \mathbb{H}^+$ *and* $\mathbb{H} \xrightarrow{\text{onto}} \mathbb{H}$. $\qquad\square$

3 Rotations for wide subconditions

Now define product rotations which fit to conditions in $\mathbb{Q}^+$ and $\mathbb{Q} \subseteq \mathbb{Q}^+$. Let Φ consist of all indexed sets $\varphi = \{\varphi_\xi\}_{\xi \in |\varphi|}$, where $|\varphi| \subseteq \lambda$ is a finite set and $\varphi_\gamma \in \mathbb{D}$ for all $\gamma \in |\varphi|$. If $\varphi \in \Phi$ and $q \in \mathbb{Q}^+$ then define $q' = \varphi \cdot q \in \mathbb{Q}^+$ so that $|q'| = |q|$ and for all γ:

$$
q'_\gamma = \begin{cases} q_\gamma & \text{whenever} \quad \gamma \in |q| \smallsetminus |\varphi|, \\ \varphi_\gamma \cdot q_\gamma & \text{whenever} \quad \xi \in |\varphi| \cap |q|. \end{cases}
$$

The lext elementary lemma is left to the reader.

Lemma 5. *If* $\varphi \in \Phi$ *then the map* $q \mapsto \varphi \cdot q$ *is an order-preserving action* $\mathbb{Q}^+ \xrightarrow{\text{onto}} \mathbb{Q}^+$ *and* $\mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}$. *If* $q \in \mathbb{Q}^+$ *then* q *and* $\varphi \cdot q$ *are equally shaped.* $\qquad\square$

As above, say that $\varphi \in \Phi$ is *compatible* with an assignment $a \in \mathbb{A}$, in symbol $\varphi \in \Phi_a$, iff $\varphi_\gamma[n] = \varphi_\delta[n]$ holds whenever $\gamma, \delta \in |\varphi| \cap |a|$, $n \in \text{bas}\, a$, and $a(n, \gamma) = a(n, \delta)$. In this case, if in addition $|\varphi| \subseteq |a|$ then we define:

(1) a rotation $\psi = \varphi \downarrow a \in \Psi$ (*a-projection*) so that

$$
|\psi| = \{a(n, \gamma) : n \in \text{bas}\, a \wedge \gamma \in |\varphi|\}
$$

and if $n \in \text{bas}\, a$, $\gamma \in |\varphi|$, and $\xi = a(n, \gamma)$ then $\psi_\xi = \varphi_\gamma[n]$;

(2) a rotation $\varepsilon = \varphi \vec{\,} a \in \Phi$ (*a-extension*) so that $|\varepsilon| = |a|$, $\varepsilon_\delta = \varphi_\delta$ for all $\delta \in |\varphi|$, and the following holds for all $\gamma \in |a| \smallsetminus |\varphi|$ and $n \in \omega$:

$$
\varepsilon_\gamma[n] = \begin{cases} \varphi_\delta[n] & \text{iff} \quad n \in \text{bas}\, a \wedge \delta \in |\varphi| \wedge a(n, \gamma) = a(n, \delta), \\ \varnothing & \text{iff} \quad n \notin \text{bas}\, a \vee \neg \exists \delta \in |\varphi|\, (a(n, \gamma) = a(n, \delta)). \end{cases}
$$

The consistency of both (1) and (2) follows from the compatibility assumption.

7

4 Rotations for conditions

Finally we define how any $\varphi \in \Phi$ acts on the set

$$\mathbb{T}_\varphi = \{t \in \mathbb{T} : |\varphi| \subseteq |t| \ \wedge \ \varphi \text{ is compatible with } a^t\}.$$

If $t \in \mathbb{T}_\varphi$ then let $\varphi \cdot t = t'$, where $q^{t'} = (\varphi^\rightarrow a^t) \cdot q^t$, $a^{t'} = a^t$, $h^{t'} = (\varphi \downarrow a^t) \cdot h^t$.

Lemma 6. *Suppose that $\varphi \in \Phi$. Then the map $t \mapsto \varphi \cdot t$ is an order-preserving action $\mathbb{T}_\varphi \overset{\text{onto}}{\longrightarrow} \mathbb{T}_\varphi$, with $t \mapsto \varphi^{-1} \cdot t$ being the inverse.*
If $t \in \mathbb{T}_\varphi$ is uniform then so is $t' = \varphi \cdot t$, and $q^t, q^{t'}$ are equally shaped.

Proof. Assume that $t \in \mathbb{T}_\varphi$ and prove that $t' = \varphi \cdot t$ belongs to $\mathbb{T}_\varphi$ as well; this is the only part of the lemma not entirely trivial. We have to check (t2) of Section 5, that is, $h^{t'}_{a^{t'}(n,\gamma)} = q^{t'}_\gamma[n]$ for all $n \in \mathtt{bas}\, t'$ and $\gamma \in |t'|$. By definition $a^{t'} = a^t$, $\mathtt{bas}\, t' = \mathtt{bas}\, t$, and $|t'| = |t|$, hence we have to prove $q^{t'}_\gamma[n] = h^{t'}_{a^t(n,\gamma)}$, for all $n \in \mathtt{bas}\, t = \mathtt{bas}\, t'$, $\gamma \in |t| = |t'|$.

Note that $q^{t'} = (\varphi^\rightarrow a^t) \cdot q^t$ and $h^{t'} = \psi \cdot h^t$, where $\psi = \varphi \downarrow a^t \in \Psi$.

Case 1: $\gamma \in |\varphi|$. Then $q^{t'}_\gamma[n] = \varphi_\gamma[n] \cdot q^t_\gamma[n]$. Let $\xi = a^t(n,\gamma)$. By definition $h^{t'}_\xi = \psi_\xi \cdot h^t_\xi$. On the other hand, $\psi_\xi = \varphi_\gamma[n]$ and $h^t_\xi = q^t_\gamma[n]$. Therefore $h^{t'}_\xi = \varphi_\gamma[n] \cdot q^t_\gamma[n] = q^{t'}_\gamma[n]$, as required.

Case 2: $\gamma \notin |\varphi|$, and there is an ordinal $\delta \in |\varphi|$ such that $a^t(n,\gamma) = a^t(n,\delta)$. Then the extended rotation $\varepsilon = \varphi^\rightarrow a^t$ satisfies $\varepsilon_\gamma[n] = \varphi_\delta[n]$, and hence $q^{t'}_\gamma[n] = \varepsilon_\gamma[n] \cdot q^t_\gamma[n] = \varphi_\delta[n] \cdot q^t_\delta[n] = q^{t'}_\delta[n] = h^{t'}_\xi$, where $\xi = a^t(n,\gamma) = a^t(n,\delta)$ (we refer to Case 1), as required.

Case 3: $\gamma \notin |\varphi|$, but there is no ordinal $\delta \in |\varphi|$ such that $a^t(n,\gamma) = a^t(n,\delta)$. The extended rotation $\varepsilon = \varphi^\rightarrow a^t$ satisfies $\varepsilon_\gamma[n] = \varnothing$ in this case, and hence $q^{t'}_\gamma[n] = q^t_\gamma[n]$. Moreover, the Case 3 assumption means that $\xi = a^t(n,\gamma) \notin |\psi|$, and hence $h^{t'}_\xi = h^t_\xi$, and we are done. $\qquad\square$

5 The symmetry lemma

We begin with auxiliary definitions. If $u \in \mathbb{T}$ then let

$$\mathbb{T}_{\leqslant u} = \{u' \in \mathbb{T} : u' \leqslant u\}.$$

Definition 7. Suppose that $N \subseteq \omega$ and $\Gamma \subseteq \lambda$ are finite sets. Conditions $s, t \in \mathbb{T}$ are *similar on $N \times \Gamma$* iff

(a) $\Gamma \subseteq |s| = |t|$, $N \subseteq \mathtt{bas}\, s = \mathtt{bas}\, t$,

(b) $q^s \restriction \Gamma = q^t \restriction \Gamma$ and the restricted assignments $a^s \Vert \Gamma$ and $a^t \Vert \Gamma$ are coherent (see Section 2),

(c) if $n \in N$ then $h^s[n] = h^t[n]$, and $a^s(n,\gamma) = a^t(n,\gamma)$ for all $\gamma \in \Gamma$,

and *strongly similar on $N \times \Gamma$* if in addition

(d) s, t are uniform conditions, and q^s, q^t are equally shaped (see Section 4),

(e) $\mathtt{ran}\, a^s = \mathtt{ran}\, a^t$ and $|h^s| = |h^t|$,

(f) conditions h^s and h^t are regular at every $n \in \mathtt{bas}\, s \smallsetminus N$ (Section 3),

(g) $\{h^s_\xi : \xi \in |h^s|\} = \{h^t_\xi : \xi \in |h^t|\}$ — then easily
$\{h^s_\xi : \xi \in |h^s| \cap [\aleph_n, \aleph_{n+1})\} = \{h^t_\xi : \xi \in |h^t| \cap [\aleph_n, \aleph_{n+1})\}$ for all n. $\qquad\square$

Theorem 8 (the symmetry lemma)**.** *Suppose that $N \subseteq \omega$, $\Gamma \subseteq \lambda$ are finite sets, conditions $s, t \in \mathbb{T}$ are strongly similar on $N \times \Gamma$, $B = \mathtt{bas}\, s = \mathtt{bas}\, t$, $\Delta = |s| = |t|$. Then:*

(i) *there exists a transformation $\pi \in \Pi_{\mathtt{fin}}$ such that $\pi[n]$ is the identity for all $n \in N$, condition $u = \pi \cdot s$ is strongly similar to t on $N \times \Gamma$, and moreover $\pi \cdot h^s = h^u = h^t$, and $a^u \,\|\, \Gamma = a^t \,\|\, \Gamma$;*

(ii) *condition $v = \mathbf{S}_{a^u a^t} \cdot u$ is strongly similar to t on $N \times \Gamma$, and moreover $h^v = h^u$ and $a^v = a^t$;*

(iii) *there is a rotation $\varphi \in \Phi_{a^v}$ (i.e., compatible with a^v) such that $|\varphi| = \Delta$, $\varphi_\gamma[n] = \varnothing$ for all $n \in B$ and $\gamma \in \Delta$,[4] and moreover $t = \varphi \cdot v$;*

(iv) *$\tau = \varphi \circ \mathbf{S}_{a^u a^t} \circ \pi$ is an order preserving bijection from $\mathbb{T}_{\leqslant s}$ onto $\mathbb{T}_{\leqslant t}$;*

(v) *any condition $s' \in \mathbb{T}_{\leqslant s}$ is similar to $t' = \tau \cdot s'$ on $N \times \Gamma$.*

Proof. (i) Let $\Xi = |h^s| = |h^t|$. Under our assumptions, obviously there is a transformation $\pi \in \Pi_{\mathtt{fin}}$ such that

(1) $\mathtt{bas}\, \pi = B$ and if $n \in N$ then $\pi[n]$ is the identity;

(2) $\pi(a^s(n, \gamma)) = a^t(n, \gamma)$ [5] for all $n \in B$ and $\gamma \in \Gamma$;

(3) π maps the set Ξ onto itself, and π is the identity outside of Ξ,

(4) if $\xi \in \Xi = |h^s|$ then $h^s_\xi = h^t_{\pi(\xi)}$.

The only point of contention is whether (2) does not contradict to (4). That is, we have to check that $h^s_{a^s(n,\gamma)} = h^t_{a^t(n,\gamma)}$. Note that $h^s_{a^s(n,\gamma)} = q^s_\gamma[n]$ and $h^t_{a^t(n,\gamma)} = q^t_\gamma[n]$ by (t2) of Section 1. On the other hand $q^s_\gamma = q^t_\gamma$ by (b) of Definition 7, as required.

Lemma 9. *The transformation π satisfies* (i) *of the theorem, and in addition if $s' \in \mathbb{T}_{\leqslant s}$ then s' is similar to $u' = \pi \cdot s'$ on $N \times \Gamma$.*

Proof (Lemma)**.** Prove that $h^u = \pi \cdot h^s$ is equal to h^t. (This is a fragment of (i).) We have $|h^u| = \{\pi(\xi) : \xi \in |h^s|\} = \Xi$ by (3), and $|h^t| = \Xi$ as well. Thus it remains to prove that $h^u_\eta = h^t_\eta$ for any $\eta = \pi(\xi) \in \Xi$, where $\xi \in \Xi$. Yet by definition (Section 2) $h^u_\eta = h^s_\xi$, and $h^t_\eta = h^s_\xi$ by (4).

The equality $a^u \,\|\, \Gamma = a^t \,\|\, \Gamma$ follows from (2) since $a^u(n, \gamma) = \pi(a^s(n, \gamma))$.

Prove that any $s' \in T$, $s' \leqslant s$, is similar to $u' = \pi \cdot s'$ on $N \times \Gamma$.

---

[4] Then obviously φ is compatible with each of the assignments a^s, a^t, a^u, a^v.

[5] As s, t are similar on Γ, here we avoid a contradiction related to the possibility of equalities $a^t(n, \gamma) = a^t(n, \gamma')$ for $\gamma \neq \gamma'$ in Γ.

Item (a) of Definition 7 holds for the pair of conditions s', u' simply because the action of any $\pi \in \Pi_{\mathtt{fin}}$ preserves $|\cdot|$ and $\mathtt{bas}$.

Prove (b). We have $q^{s'} = q^{u'}$ because the action of π does not change $q^{s'}$ at all. To show the coherence of $a^{s'} \Vert \Gamma$ and $a^{u'} \Vert \Gamma$ suppose that $\gamma, \delta \in \Gamma$, $n \in \omega$, and $a^{s'}(n, \gamma) = a^{s'}(n, \delta)$, and prove that $a^{u'}(n, \gamma) = a^{u'}(n, \delta)$. (The inverse implication can be checked pretty the same way.)

Suppose first that $n \in B$. Then $a^{s'}(n, \gamma) = a^s(n, \gamma)$ and $a^{s'}(n, \delta) = a^s(n, \gamma)$, therefore $a^s(n, \gamma) = a^s(n, \delta)$. It follows that $a^t(n, \gamma) = a^t(n, \delta)$ by the coherence in (b) for s, t, therefore $a^u(n, \gamma) = a^u(n, \delta)$ since $a^u \Vert \Gamma = a^t \Vert \Gamma$, and finally $a^{u'}(n, \gamma) = a^{u'}(n, \delta)$, as required.

Now suppose that $n \notin B$. Then the equality $a^{s'}(n, \gamma) = a^{s'}(n, \delta)$ implies $\gamma = \delta$ by (a4) of Section 1, so obviously $a^{u'}(n, \gamma) = a^{u'}(n, \delta)$.

To check (c), that is, $h^{u'}[n] = h^{s'}[n]$ and $a^{u'}(n, \gamma) = a^{s'}(n, \gamma)$ for all $\gamma \in \Gamma$ and $n \in N$, use the fact that $\pi[n]$ is the identity for any $n \in N$ by (1).

Prove that s is strongly similar to $u = \pi \cdot s$ on $N \times \gamma$. We have (d) of Definition 7 (for the pair of conditions s', u') by rather obvious reasons. The equalities $\mathtt{ran}\, a^{u'} = \mathtt{ran}\, a^{s'}$ and $|h^{u'}| = |h^{s'}|$ in (e) hold by (3) since $\mathtt{ran}\, a^{u'}$ is equal to the π-image of $\mathtt{ran}\, a^{s'}$. Finally the equality $\{h_\xi^{u'} : \xi \in |h^{u'}|\} = \{h_\xi^{s'} : \xi \in |h^{s'}|\}$ in (g) holds whenever $u' = \pi \cdot s'$ for some π. We conclude that conditions u and t are strongly similar on $N \times \Gamma$. $\square$ (*Lemma*)

(ii) Let $a = a^u$ and $b = a^t$. Thus $a, b \in \mathbb{A}$, $\mathtt{dom}\, a = \mathtt{dom}\, b = B \times \Delta$, $\mathtt{ran}\, a = \mathtt{ran}\, b$, and $a \Vert \Gamma = b \Vert \Gamma$ by the above. Thus, as obviously $u \in \mathbb{T}_a^{\mathtt{uni}}$, we define $v = \mathbf{S}_{ab} \cdot u \in \mathbb{T}_b^{\mathtt{uni}}$.

Lemma 10. *Condition* (ii) *of the theorem holds, and in addition if* $u' \in \mathbb{T}_{\leqslant u}$ *then* u' *is similar to* $v' = \mathbf{S}_{ab} \cdot u'$ *on* $N \times \Gamma$.

Proof (Lemma). That equalities $h^v = h^u$ and $a^v = a^t$ in (ii) hold is clear by definition: for instance swaps do not change h^u at all.

Prove that any $u' \in T$, $u' \leqslant u$, is similar to $v' = \mathbf{S}_{ab} \cdot u'$ on $N \times \Gamma$.

By definition (see Section 3) v' and u' are equal outside of the domain $N \times \Delta$, and $h^{v'} = h^{u'}$. Therefore we can w.l.o.g. assume that $|v'| = |u'| = \Delta$ and $\mathtt{bas}\, v' = \mathtt{bas}\, u' = B$. Then $a^{v'} = b = a^t$ and $a^{u'} = a = a^u$, thus the restricted assignments $a^{v'} \Vert \Gamma = b \Vert \Gamma$ and $a^{u'} \Vert \Gamma = a \Vert \Gamma$ are not merely coherent (as required by (b) of Definition 7) but just equal by the above. The equality $q^{v'} \upharpoonright \Gamma = q^{u'} \upharpoonright \Gamma$ in (b) follows from $a \Vert \Gamma = b \Vert \Gamma$ as well. And finally we have $h^{v'} = h^{u'}$ ($\mathbf{S}_{ab}$ does not change this component), proving (c).

Now prove that any u is strongly similar to $v = \mathbf{S}_{ab} \cdot u$ on $N \times \gamma$. We skip (d) of Definition 7 as clear and rather boring. Further, as $h^v = h^u$, we have $|h^v| = |h^u|$ in (e) and the whole of (g). It remains to show $\mathtt{ran}\, a^v = \mathtt{ran}\, a^u$ in (e). Recall that $a^v = a^t$ while conditions s, t, u are strongly similar, therefore $\mathtt{ran}\, a^v = \mathtt{ran}\, a^t = \mathtt{ran}\, a^s = \mathtt{ran}\, a^u$. We conclude that conditions v and t are strongly similar on $N \times \Gamma$. $\square$ (*Lemma*)

(iii) Thus v, t are uniform conditions, strongly similar on $N \times \Gamma$, and $a^v = a^t$. In particular q^v and q^t are equally shaped, that is, in this case, $|q^v| = |q^t| = \Delta$ and

$\operatorname{dom} q_\gamma^v[n] = \operatorname{dom} q_\gamma^t[n]$ holds for all $\gamma \in \Delta$ and $n \in \omega$. Define a rotation $\varphi \in \Phi$ so that still $|\varphi| = \Delta$, and

$$\varphi_\gamma[n] = \{\alpha \in \operatorname{dom} q_\gamma^v[n] = \operatorname{dom} q_\gamma^t[n] : q_\gamma^v(\alpha) \neq q_\gamma^t(\alpha)\}$$

for all $\gamma \in \Delta$ and $n \in \omega$. Then clearly $\varphi \cdot q^v = q^t$. Moreover φ is compatible with $a^v = a^t$, because so are q^t and q^v in the sense of (t3) of Section 1. Thus conditions v and t belong to $\mathbb{T}_\varphi$, so $\varphi \cdot v$ makes sense.

Lemma 11. *Condition* (iii) *of the theorem holds, and in addition if $v' \in \mathbb{T}_{\leqslant v}$ then v' is similar to $t' = \varphi \cdot v'$ on $N \times \Gamma$.*

Proof (Lemma). Recall that $a^v = a^t$ and $h^v = h^u = h^t$ by (i), (ii). It follows by (t2) of Section 1 that $q_\gamma^v[n] = q_\gamma^t[n]$, and hence $\varphi_\gamma[n] = \varnothing$, whenever $\gamma \in \Delta$ and $n \in B$. To accomplish the proof of (iii) check that $\varphi \cdot v = t$. Indeed $a^v = a^t$ since φ does not change this component. Further, $q^t = \varphi \cdot q^v$ simply by the choice of φ. Let us show that $h^t = h^v$ as well. Indeed, since by definition $\operatorname{bas} h^v = B = \operatorname{bas} t$, any change in h^v by the action of φ can be only due to a component $\varphi_\gamma[n]$ for some $\gamma \in \Delta$ and $n \in B$ — but this is the identity since $\varphi_\gamma[n] = \varnothing$ in this case.

Now prove that any $v' \in T$, $v' \leqslant v$, is similar to $t' = \varphi \cdot v'$ on $N \times \Gamma$. By definition $a^{v'} = a^{q'}$, covering the coherence in (b) of Definition 7. Further the extended rotation $\varphi' = \varphi \,{}^\frown a^{v'}$ obviously satisfies the same property $\varphi_\gamma'[n] = \varnothing$ for all $n \in B$ and $\gamma \in \Delta' = |v'|$. This implies $h^{t'}[n] = h^{v'}[n]$ even for all $n \in B$, so that (c) holds for v', t' for all $n \in B$. It only remains to prove that $q^{t'} \upharpoonright \Gamma = q^{v'} \upharpoonright \Gamma$ in (b) of Definition 7, that is, $q_\gamma^{t'} = q_\gamma^{v'}$ for all $\gamma \in \Gamma$.

By definition it suffices to show that $\varphi_\gamma[n] = \varnothing$ for all $\gamma \in \Gamma$ and $n \in \omega$, or equivalently, $q^v \upharpoonright \Gamma = q^t \upharpoonright \Gamma$ — yet this is the case since v and t are similar on $N \times \Gamma$ by the above. $\qquad \square$ (*Lemma*)

Finally, (iv) of the theorem is a consequence of lemmas 2, 3, 6, while (v) is a corollary of lemmas 9, 10, 11.

$$\square \; (Theorem)$$

6 The extension

Let a set $G \subseteq \mathbb{T}$ be $\mathbb{T}$-generic over $\mathbf{L}$. It naturally produces:

- for any n and $\xi \in [\aleph_n, \aleph_{n+1})$, $\mathbf{x}_\xi^G = \bigcup_{t \in G} h_\xi^t \in 2^{[\aleph_n, \aleph_{n+1})}$,

- for every n, $\mathbf{x}^G[n] = \{\mathbf{x}_\xi^G\}_{\xi \in [\aleph_n, \aleph_{n+1})} : [\aleph_n, \aleph_{n+1}) \to 2^{[\aleph_n, \aleph_{n+1})}$,

- for any $\gamma < \lambda$, $\mathbf{y}_\gamma^G = \bigcup_{t \in G} q_\gamma^t \in 2^{[\omega, \aleph_\omega)}$,

- for any $\gamma < \lambda$ and n, $\mathbf{y}_\gamma^G[n] = \mathbf{y}_\gamma^G \upharpoonright [\aleph_n, \aleph_{n+1}) \in 2^{[\aleph_n, \aleph_{n+1})}$,

- $\vec{\mathbf{y}}[G] = \{\mathbf{y}_\gamma^G\}_{\gamma < \lambda}$, a map $\lambda \to 2^{[\omega, \aleph_\omega)}$,

- a map $\mathbf{a}^G = \bigcup_{t \in G} a^t : \omega \times \lambda \to [\omega, \aleph_\omega)$ such that $\mathbf{a}^G(n, \gamma) \in [\aleph_n, \aleph_{n+1})$ for all n and γ.

Lemma 12. *If a set $G \subseteq \mathbb{T}$ is $\mathbb{T}$-generic over $\mathbf{L}$ then*

(i) *if $n < \omega$, $\gamma < \lambda$, and $\mathbf{a}^G(n, \gamma) = \xi$ then $\mathbf{y}_\gamma^G[n] = \mathbf{x}_\xi^G$;*

(ii) *if $n < \omega$, $\gamma, \delta < \lambda$, and $\mathbf{a}^G(n, \gamma) \neq \mathbf{a}^G(n, \delta)$ then $\mathbf{y}_\gamma^G[n] \neq \mathbf{y}_\delta^G[n]$;*

(iii) *if $\gamma \neq \delta < \lambda$ then there is a number $n_0 = n_0(\gamma, \delta)$ such that $\mathbf{a}^G(n, \gamma) \neq \mathbf{a}^G(n, \delta)$ for all $n \geq n_0$.*

Proof. (i) is obvious.

(ii) Suppose that a condition $t \in G$ forces otherwise, and $\gamma, \delta \in |t|$, $n \in \mathtt{bas}\, t$. Then $\xi = a^t(n, \gamma) \neq a^t(n, \delta) = \eta$; ξ, η are ordinals in $[\aleph_n, \aleph_{n+1})$. Note that h_ξ^t and h_η^t are conditions in $\mathbb{P}[n]$. Let $w_\xi \leqslant h_\xi^t$ and $w_\eta \leqslant h_\eta^t$ be any pair of **incompatible** conditions in $\mathbb{P}[n]$. Let $t' \in T$ be a condition which differs from t only in the following: $q_\gamma^{t'}[n] = h_\xi^{t'} = w_\xi$ and $q_\delta^{t'}[n] = h_\eta^{t'} = w_\eta$. Obviously $t' \leqslant t$, and t' forces that $\mathbf{y}_\gamma^G[n] \neq \mathbf{y}_\delta^G[n]$.

(iii) Definitely there is a condition $t \in G$ such that $|t|$ contains both γ and δ. Let $B = \mathtt{bas}\, t$ (a finite subset of ω) and let n_0 be bigger than $\mathtt{max}\, B$. Now if $s \in G$, $s \leqslant t$, and $n \in \mathtt{bas}\, s$, $n \geq n_0$, then $a^s \leqslant a^t$, and hence $a^s(n, \gamma) \neq a^s(n, \delta)$. This implies $\mathbf{a}^G(n, \gamma) \neq \mathbf{a}^G(n, \delta)$. $\quad\square$

Now let us define a **symmetric subextension** of $\mathbf{L}[G]$, on the base of certain symmetric hulls of sets $\mathbf{x}^G[n]$ and $\mathbf{y}_\gamma^G$.

Blanket agreement 13. Below, $\Pi_{\mathtt{fin}}$, Φ, Ψ, $\mathbb{D}$, $\mathbb{D}[n]$, mean objects defined in $\mathbf{L}$ as in Sections 1 — 4. Thus in particular $\Pi_{\mathtt{fin}} \in \mathbf{L}$ and all elements of $\Pi_{\mathtt{fin}}$ belong to $\mathbf{L}$ either. $\quad\square$

In $\mathbf{L}[G]$, put

– for every n, $\mathbf{X}^G[n] =$ the $(\Pi_{\mathtt{fin}}, \Psi)$-hull of $\mathbf{x}^G[n]$. Thus the set $\mathbf{X}^G[n]$ consists of elements of the form $\pi \cdot (\psi \cdot \mathbf{x}^G[n])$, where $\pi \in \Pi_{\mathtt{fin}}$ and $\psi \in \Psi$.

– $\vec{\mathbf{X}}[G] = \{\mathbf{X}^G[n]\}_{n < \omega}$.

The actions of $\pi \in \Pi_{\mathtt{fin}}$ and $\psi \in \Psi$ are defined as in sections 2 and 4 above. In particular $\psi \cdot \mathbf{x}^G[n]$ and $\pi \cdot (\psi \cdot \mathbf{x}^G[n])$ are maps $[\aleph_n, \aleph_{n+1}) \to 2^{[\aleph_n, \aleph_{n+1})}$ in $\mathbf{L}[\mathbf{x}^G[n]]$. It is clear that $\mathbf{X}^G[n]$ is closed under further application of transformations in $\Pi_{\mathtt{fin}}$ and Ψ, so there is no need to consider iterated actions.

It takes more time to define suitable hulls of elements $\mathbf{y}_\gamma^G$. First of all, put

– for any n and $\gamma < \lambda$, $\mathbf{Y}_\gamma^G[n] = \{d \cdot \mathbf{y}_\gamma^G[n] : d \in \mathbb{D}[n]\} \subseteq 2^{[\aleph_n, \aleph_{n+1})}$;

– for any n, $\mathbf{Y}^G[n] = \bigcup_{\gamma < \lambda} \mathbf{Y}_\gamma^G[n]$ — still $\mathbf{Y}^G[n] \subseteq 2^{[\aleph_n, \aleph_{n+1})}$, and obviously $\mathbf{Y}^G[n]$ is the $\mathbb{D}[n]$-hull of $\{\mathbf{y}_\gamma^G[n] : \gamma < \lambda\}$.

Finally, if $\gamma < \lambda$ then we let $\mathbf{Y}_\gamma^G$ be the set of all $z \in 2^{[\omega, \aleph_\omega)}$ in $\mathbf{L}[G]$ such that there exist a set $d \in \mathbb{D}$ and a number n_0 satisfying:

1) $z[n] = d[n] \cdot \mathbf{y}_\gamma^G[n]$ for all $n \geq n_0$;[6]

---

[6] Regarding the action of $d \in \mathbb{D}$ see Section 1.

2) $z[n] \in \mathbf{Y}^G[n]$ for all $n < n_0$.

In other words, to obtain $\mathbf{Y}_\gamma^G$ we first define the $\mathbb{D}$-hull $\mathbb{D} \cdot \mathbf{y}_\gamma^G = \{d \cdot \mathbf{y}_\gamma^G : d \in D\}$ of $\mathbf{y}_\gamma^G$, and then allow to substitute sets in $\mathbf{Y}^G[n]$ for $y[n]$ for any $y \in \mathbb{D} \cdot \mathbf{y}_\gamma^G$ and finitely many n, so that

($\star$) $\mathbf{Y}_\gamma^G$ is the set of all $z \in 2^{[\omega, \aleph_\omega)}$ (in $\mathbf{L}[G]$) such that there exist an element $y \in \mathbb{D} \cdot \mathbf{y}_\gamma^G$ and a number n_0 satisfying: $z[n] = y[n]$ for all $n \geq n_0$, and $z[n] \in \mathbf{Y}^G[n]$ for all $n < n_0$.

Lemma 14. *If $\gamma \neq \delta$ then $\mathbf{Y}_\gamma^G \cap \mathbf{Y}_\delta^G = \varnothing$.*

Proof. Suppose towards the contrary that $z \in \mathbf{Y}_\gamma^G \cap \mathbf{Y}_\delta^G$. Then by ($\star$) there exist rotations $d', d'' \in \mathbb{D}$ and a number n_0 such that the elements $y' = d' \cdot \mathbf{y}_\gamma^G$ and $y'' = d'' \cdot \mathbf{y}_\delta^G$ satisfy $y[n] = y'[n]$ for all $n \geq n_0$. In other words, $\mathbf{y}_\gamma^G[n] = (d \cdot \mathbf{y}_\delta^G)[n]$ for all $n \geq n_0$, where $d = d' \bigtriangleup d'' \in \mathbb{D}$ (symmetric difference). Now use Lemma 12(iii) to obtain a number $n \geq n_0$ such that $\mathbf{a}^G(n, \gamma) \neq \mathbf{a}^G(n, \delta)$; still we have $\mathbf{y}_\gamma^G[n] = d[n] \cdot \mathbf{y}_\delta^G[n]$. But this yields a contradiction similarly to the proof of Lemma 12(ii). $\square$

Now we let, in $\mathbf{L}[G]$, $\vec{\mathbf{Y}}[G] = \{\mathbf{Y}_\gamma^G\}_{\gamma < \lambda}$, a function defined on λ.
We finally define

$$W[G] = \bigcup_n \mathbf{X}^G[n] \cup \bigcup_{\gamma < \lambda} \mathbf{Y}_\gamma^G \cup \{\vec{\mathbf{X}}[G], \vec{\mathbf{Y}}[G]\}.$$

Definition 15. $\mathbf{L}_{\mathtt{sym}}[G] = \mathbf{L}(W[G]) = \mathrm{HOD}$ over $W[G]$ in $\mathbf{L}[G]$. $\square$

Thus by definition every set in $\mathbf{L}_{\mathtt{sym}}G$ is definable in $\mathbf{L}[G]$ by a formula with parameters in $\mathbf{L}$, two special parameters $\vec{\mathbf{X}}[G]$ and $\vec{\mathbf{Y}}[G]$, and finally parameters which belong to the sets $\mathbf{X}^G[n]$ and $\mathbf{Y}_\gamma^G$ for various $n < \omega$ and $\gamma < \lambda$. The next lemma allows to reduce the last category of parameters, basically, to those in $\{\mathbf{x}^G[n] : n < \omega\} \cup \{\mathbf{y}_\gamma^G : \gamma < \lambda\}$.

Lemma 16. *If $n < \omega$ then every $x \in \mathbf{X}^G[n]$ belongs to $\mathbf{L}[\mathbf{x}^G[n]]$. If $\gamma < \lambda$ and $z \in \mathbf{Y}_\gamma^G$ then there is a finite set $\Delta \subseteq \lambda$ such that $z \in \mathbf{L}[\{\mathbf{y}_\delta^G : \delta \in \Delta\}]$.*

Proof. By definition x belongs to the $(\Pi_{\mathtt{fin}}, \Psi)$-hull of $\mathbf{x}^G[n]$. But $\Pi_{\mathtt{fin}}$ and Ψ belong to $\mathbf{L}$ (see Blanket Agreement 13). Regarding the claim for $z \in \mathbf{Y}_\gamma^G$, come back to ($\star$). Note that y as in ($\star$) belongs to $\mathbf{L}[\mathbf{y}_\gamma^G]$ (since $\mathbb{D} \in \mathbf{L}$). Then to obtain z from y we replace a finite number of intervals $y[n]$ in y by elements of sets $\mathbf{Y}^G[n]$. Thus suppose that $n < \omega$ and $w \in \mathbf{Y}^G[n]$, that is, $w \in \mathbf{Y}_\delta^G[n]$, where $\delta < \lambda$. But then $w \in \mathbf{L}[\mathbf{y}_\delta^G]$ (since $\mathbb{D}[n] \in \mathbf{L}$), so that it suffices to define Δ as the (finite) set of all ordinals δ which appear in this argument for all intervals $y[n]$ to be replaced. $\square$

7 Definability lemma

The next theorem plays key role in the analysis of the abovedefined symmetric subextension.

Theorem 17 (the definability lemma). *Suppose that a set $G \subseteq \mathbb{T}$ is $\mathbb{T}$-generic over $\mathbf{L}$, and $N \subseteq \omega$, $\Gamma \subseteq \lambda$ are finite sets. Let $Z \in \mathbf{L}[G]$, $Z \subseteq \mathbf{L}$, be a set definable in $\mathbf{L}[G]$ by a formula with parameters in $\mathbf{L}$ and those in the list*

$$\{\vec{\mathbf{X}}[G], \vec{\mathbf{Y}}[G]\} \cup \{\mathbf{x}^G[n] : n \in N\} \cup \{\mathbf{y}^G_\gamma : \gamma \in \Gamma\}.$$

Then $Z \in \mathbf{L}[\{\mathbf{x}^G[n] : n \in N\}, \{\mathbf{y}^G_\gamma : \gamma \in \Gamma\}]$.

Beginning the **proof of Theorem 17**, we put $\vec{\mathbf{x}}_N[G] = \{\mathbf{x}^G[n]\}_{n \in N}$ and $\vec{\mathbf{y}}_\Gamma[G] = \{\mathbf{y}^G_\gamma\}_{\gamma \in \Gamma}$, and let

$$\vartheta(z) := \vartheta(z, \vec{\mathbf{X}}[G], \vec{\mathbf{Y}}[G], \vec{\mathbf{x}}_N[G], \vec{\mathbf{y}}_\Gamma[G])$$

be a formula such that $Z = \{z : \vartheta(z)\}$ in $\mathbf{L}[G]$. By Lemma 12(iii) there is n_0 such that $\mathbf{a}^G(n, \gamma) \neq \mathbf{a}^G(n, \delta)$ whenever $n > n_0$ and $\gamma \neq \delta$ belong to Γ.

Let $M = N \cup \{n : n \leqslant n_0\}$. Say that a condition $t \in \mathbb{T}$ *complies with* $\vec{\mathbf{x}}_N[G]$, $\vec{\mathbf{y}}_\Gamma[G]$ if $M \subseteq \mathtt{bas}\, t$, $\Gamma \subseteq |t|$, and

(I) if $n \in N$ and $\xi \in |h^t| \cap [\aleph_n, \aleph_{n+1})$ then $h^t_\xi \subset \mathbf{x}^G_\xi$,

(II) if $\gamma \in \Gamma$ then $q^t_\gamma \subset \mathbf{y}^G_\gamma$,

(III) if $n \in \mathtt{bas}\, t$ and $\gamma \in \Gamma$ then $a^t(n, \gamma) = \mathbf{a}^G(n, \gamma)$.

For instance any condition $t \in G$ with $M \subseteq \mathtt{bas}\, t$, $\Gamma \subseteq |t|$ complies with $\vec{\mathbf{x}}_N[G]$, $\vec{\mathbf{y}}_\Gamma[G]$ by obvious reasons.

It is quite clear that the set $\mathbb{T}_{N\Gamma}[G]$ of all conditions $t \in \mathbb{T}$ which comply with $\vec{\mathbf{x}}_N[G]$, $\vec{\mathbf{y}}_\Gamma[G]$ belongs to $\mathbf{L}[\vec{\mathbf{x}}_N[G], \vec{\mathbf{y}}_\Gamma[G]]$. Therefore to prove the theorem it suffices to verify the following assertion:

if $z \in \mathbf{L}$, $s, t \in \mathbb{T}_{N\Gamma}[G]$, and s forces $\vartheta(z)$, then t does not force $\neg\, \vartheta(z)$.

Suppose towards the contrary that this fails, so that

$(*)$ $z \in \mathbf{L}$, $s, t \in \mathbb{T}_{N\Gamma}[G]$, condition s forces $\vartheta(z)$, while t forces $\neg\, \vartheta(z)$.

The proof of Theorem 17 continues in Sections 8 and 9.

8   Proof of the definability lemma, part 1

Working towards the symmetry lemma. Our goal is now to strengthen s, t towards the requirements of Theorem 8.

Lemma 18. *There exists a condition $s' \in \mathbb{T}_{N\Gamma}[G]$ such that $|s'| = |s| \cup |t|$, $\mathtt{bas}\, s' = \mathtt{bas}\, s \cup \mathtt{bas}\, t$, and $s' \leqslant s$. Accordingly there is a condition $t' \in \mathbb{T}_{N\Gamma}[G]$ such that $|t'| = |s| \cup |t|$, $\mathtt{bas}\, t' = \mathtt{bas}\, s \cup \mathtt{bas}\, t$, and $t' \leqslant t$.*

Proof (Lemma). We define $a^{s'}$. This takes some time.

Domain $\mathtt{bas}\, s \times |s|$. If $n \in \mathtt{bas}\, s$ and $\gamma \in |s|$ then put $a^{s'}(n, \gamma) = a^s(n, \gamma)$ and $q^{s'}(n, \gamma) = q^s(n, \gamma)$.

Domain $(\mathtt{bas}\, t \smallsetminus \mathtt{bas}\, s) \times \Gamma$. If $n \in \mathtt{bas}\, t \smallsetminus \mathtt{bas}\, s$ and $\gamma \in \Gamma$ then put $a^{s'}(n, \gamma) = \mathbf{a}^G(n, \gamma)$, and $q^{s'}(n, \gamma) = q^s(n, \gamma)$, as above.

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Domain $(\mathtt{bas}\,t \smallsetminus \mathtt{bas}\,s) \times (|s| \smallsetminus \Gamma)$. For any $n \in \mathtt{bas}\,t \smallsetminus \mathtt{bas}\,s$ fix a bijection $\delta \longmapsto \xi_\delta^n$ from $|s| \smallsetminus \Gamma$ to $[\aleph_n, \aleph_{n+1}) \smallsetminus \{a^{s'}(n,\gamma) : \gamma \in \Gamma\}$. If now $\delta \in |s| \smallsetminus \Gamma$ then put $a^{s'}(n,\delta) = \xi_\delta^n$ and $q^{s'}(n,\delta) = \varnothing$.

Domain $(\mathtt{bas}\,t \cup \mathtt{bas}\,s) \times (|t| \smallsetminus |s|)$. Fix an ordinal $\delta^* \in |s|$. If $n \in \mathtt{bas}\,t \cup \mathtt{bas}\,s$ and $\delta \in |t| \smallsetminus |s|$ then put $a^{s'}(n,\delta) = a^{s'}(n,\delta^*)$ and $q^{s'}(n,\delta) = q^{s'}(n,\delta^*)$.

Domain $\big(\omega \smallsetminus (\mathtt{bas}\,t \cup \mathtt{bas}\,s)\big) \times (|t| \smallsetminus |s|)$. If $n \notin \mathtt{bas}\,t \cup \mathtt{bas}\,s$ and $\delta \in |t| \cup |s|$ then put $q^{s'}(n,\delta) = \varnothing$ and keep $a^{s'}(n,\delta)$ undefined.

On the top of the above definition, define $h^{s'}$ so that

$$|h^{s'}| = |h^s| \cup \{\xi_\delta^n : n \in \mathtt{bas}\,t \smallsetminus \mathtt{bas}\,s \wedge \delta \in |s| \smallsetminus \Gamma\},$$

$h_\xi^{s'} = h_\xi^s$ for all $\xi \in |h^s|$, and $h_{\xi_\delta^n}^{s'} = \varnothing$ for all $n \in \mathtt{bas}\,t \cup \mathtt{bas}\,s$ and $\delta \in |t| \smallsetminus |s|$.

We claim that s' is as required.

The key issue is to prove $a^{s'} \leqslant a^s$, in particular, (a4) of Section 1 for $a = a^{s'}$, $b = a^s$. Note that if $\gamma \neq \delta$ belong to Γ and $n \notin \mathtt{bas}\,s$ then $\mathbf{a}^G(n,\gamma) \neq \mathbf{a}^G(n,\delta)$ by the choice of M and because $M \subseteq \mathtt{bas}\,s$. Therefore if $n \in \mathtt{bas}\,t \smallsetminus \mathtt{bas}\,s$ and γ, δ as indicated then by definition $a^{s'}(n,\gamma) \neq a^{s'}(n,\delta)$, as required.

We have (I), (II), (III) by obvious reasons: in particular, $q_\gamma^{s'} = q_\gamma^s$ for all $\gamma \in \Gamma$, and if $n \in N$ then $n \in |s|$ and hence by construction $|h^{s'}| \cap [\aleph_n, \aleph_{n+1}) = |h^s| \cap [\aleph_n, \aleph_{n+1})$ and $h_\xi^{s'} = h_\xi^s$ for all $\xi \in |h^{s'}| \cap [\aleph_n, \aleph_{n+1})$. $\square$ *(Lemma)*

It follows from the lemma that we can w.l.o.g. assume in **??**hat

(1) conditions s, t satisfy $|s| = |t|$ and $\mathtt{bas}\,s = \mathtt{bas}\,t$.

Moreover we can w.l.o.g. assume that in addition to **??**nd (1):

(2) $|h^s| = |h^t|$, and if $n \in \mathtt{bas}\,s = \mathtt{bas}\,t$ then the set $|h^s| \cap [\aleph_n, \aleph_{n+1}) = |h^t| \cap [\aleph_n, \aleph_{n+1})$ is infinite.

This is rather elementary. If say $\xi \in |h^s| \smallsetminus |h^t|$ then simply add ξ to $|h^t|$ and define $h_\xi^t = \varnothing$.

Further, we can w.l.o.g. assume that, in addition to **??** (1), (2):

(3) conditions s, t satisfy $\mathtt{ran}\,a^s = \mathtt{ran}\,a^t$.

Suppose that $n \in \mathtt{bas}\,s$ and, say, $\xi \in (\mathtt{ran}\,a^t \smallsetminus \mathtt{ran}\,a^s) \cap [\aleph_n, \aleph_{n+1})$. Put $\xi_n = \xi$ and for any $m \in \mathtt{bas}\,s$, $m \neq n$ pick an ordinal $\xi_m \in |h^s| \cap [\aleph_m, \aleph_{m+1})$, $\xi_m \notin \mathtt{ran}\,a^s \cup \mathtt{ran}\,a^t$ (this is possible by (2)). Add an ordinal $\gamma \notin |s| = |t|$ to $|s|$ and to $|t|$. If $m \in \mathtt{bas}\,s = \mathtt{bas}\,t$ then put $a^s(m,\gamma) = a^t(m,\gamma) = \xi_m$ and $q_\gamma^s[m] = q_\gamma^t[m] = h_{\xi_m}^t$, and in addition define $q_\gamma^s[m] = q_\gamma^t[m] = \varnothing$ for all $m \notin \mathtt{bas}\,s = \mathtt{bas}\,t$. Conditions s, t extended this way still satisfy **??** (1), (2), but now $\xi \in \mathtt{ran}\,a^s$. One has to maintain such extension for all indices ξ in $\mathtt{ran}\,a^t \smallsetminus \mathtt{ran}\,a^s$ and $\mathtt{ran}\,a^s \smallsetminus \mathtt{ran}\,a^t$ one by one; the details are left to the reader.

Remark 19. After this step, the sets $\Delta = |s| = |t|$ and $B = \mathtt{bas}\,s = \mathtt{bas}\,t$ (finite subsets of resp. λ and ω) will not be changed, as well as the assignments $a = a^s$ and $b = a^t$ ($\mathtt{dom}\,a = \mathtt{dom}\,b = B \times \Delta$). Put $\Xi = |h^s| = |h^t|$. $\square$

Further we can w.l.o.g. assume that in addition to **??** (1), (2), (3):

(4) subconditions q^s, q^t are uniform and equally shaped.

It suffices to define a pair of stronger conditions $s', t' \in \mathbb{T}_{N\Gamma}[G]$ such that

$$|s'| = |t'| = \Delta, \ |h^{s'}| = |h^{t'}| = \Xi, \ \mathtt{bas}\, s' = \mathtt{bas}\, t' = B, \ a^{s'} = a, \ a^{t'} = b,$$

and in addition $q^{s'}, q^{t'}$ are uniform and equally shaped.

Consider any $n \in \omega$. Put $d[n] = \bigcup_{\delta \in \Delta}(\mathtt{dom}\, q^s_\delta[n] \cup \mathtt{dom}\, q^t_\delta[n])$, a set in $\mathbb{D}[n]$. If $\delta \in \Delta$ then define extensions $q^{s'}_\delta[n], q^{t'}_\delta[n] \in \mathbb{P}[n]$ of resp. $q^s_\delta[n], q^t_\delta[n]$ so that

(i) $\mathtt{dom}\, q^{s'}_\delta[n] = \mathtt{dom}\, q^{t'}_\delta[n] = d[n]$,

(ii) if $n \in N$ and $\delta \in \Gamma$ then simply $q^{s'}_\delta[n] = q^{t'}_\delta[n] = \mathbf{y}^G_\delta \restriction d[n]$,

(iii) if $n \in B$ and $\gamma, \delta \in \Delta$ then: if $a(n, \delta) = a(n, \gamma)$ then $q^{s'}_\delta[n] = q^{s'}_\gamma[n]$, and if $b(n, \delta) = b(n, \gamma)$ then $q^{t'}_\delta[n] = q^{t'}_\gamma[n]$.

On the top of this, define $h^{s'}_{a(n,\delta)} = q^{s'}_\delta[n]$ and $h^{t'}_{b(n,\delta)} = q^{t'}_\delta[n]$ for all $n \in B$ and $\delta \in \Delta$. In the rest, put $|h^{s'}| = |h^{t'}| = \Xi$ (recall that $\Xi = |h^s| = |h^t|$), and $h^{s'}_\xi = h^s_\xi$, $h^{t'}_\xi = h^t_\xi$ for all $\xi \in \Xi$ **not** in $\mathtt{ran}\, a = \mathtt{ran}\, b$.

Further, we can w.l.o.g. assume that, in addition to **??**nd (1) — (4):

(5) conditions s, t coincide on the domain $N \times \Gamma$, so that

(a) if $\gamma \in \Gamma$ then $q^s_\gamma = q^t_\gamma$,

(b) if $n \in N$ then $h^s[n] = h^t[n]$, that is, $h^s_\xi = h^t_\xi$ for all $\xi \in |h^s| \cap [\aleph_n, \aleph_{n+1}) = |h^t| \cap [\aleph_n, \aleph_{n+1})$, and

(c) if $n \in N$ and $\gamma \in \Gamma$ then $a^s(n, \gamma) = a^t(n, \gamma) = \mathbf{a}^G(n, \gamma)$ — but this already follows from the compliance assumption.

Regarding (5)a, note that this is already done. Indeed, q^s, q^t are equally shaped by (4), and satisfy $q^s_\gamma \subset \mathbf{y}^G_\gamma$ and $q^t_\gamma \subset \mathbf{y}^G_\gamma$ by **??** therefore $q^s_\gamma = q^t_\gamma$.

Now consider (5)b; suppose that $n \in N$. Let $\xi \in |h^s| \cap [\aleph_n, \aleph_{n+1})$.

If $\xi \in \mathtt{ran}\, a^s = \mathtt{ran}\, a^t$ then $\xi = a^s(n, \gamma) = a^t(n, \delta)$ for some $\gamma, \delta \in \Delta$, and then $h^s_\xi = q^s_\gamma[n]$ and $h^t_\xi = q^t_\gamma[n]$. It follows that $\mathtt{dom}\, h^s_\xi = \mathtt{dom}\, h^t_\xi$, by (4). Therefore $h^s_\xi = h^t_\xi$, because we have $h^s_\xi \subset \mathbf{x}^G_\xi$ and $h^t_\xi \subset \mathbf{x}^G_\xi$.

If $\xi \in \mathtt{ran}\, a^s = \mathtt{ran}\, a^t$ then still $h^s_\xi \subset \mathbf{x}^G_\xi$ and $h^t_\xi \subset \mathbf{x}^G_\xi$, thus h^s_ξ and h^t_ξ are compatible as conditions in $\mathbb{P}$, and we simply replace either of them by $h^s_\xi \cup h^t_\xi$.

And finally, we can w.l.o.g. assume that, in addition to **??**nd (1) — (5):

(6) we have $\{h^s_\xi : \xi \in |h^s|\} = \{h^t_\xi : \xi \in |h^t|\}$ as in (g) of Definition 7, and subconditions h^s, h^t are regular on every $n \in B \smallsetminus N$ (Subsection 3).

The equality $\{h^s_\xi : \xi \in |h^s| \cap [\aleph_n, \aleph_{n+1})\} = \{h^t_\xi : \xi \in |h^t| \cap [\aleph_n, \aleph_{n+1})\}$ holds already for all $n \in N$ by (5).

Now suppose that $n \in B \smallsetminus N$. The requirement of compliance with $\vec{\mathbf{x}}_N[G]$, $\vec{\mathbf{y}}_\Gamma[G]$ is void for $n \notin N$, therefore we can simply extend $h^s[n]$ and $h^t[n]$ to a

bigger domain and appropriately define h_ξ^s and h_ξ^t for all "new" elements ξ in these extended domains so that (6) holds, without changing q^s, q^t and a^s, a^t.

To conclude, we can w.l.o.g. assume in **??**hat (1) — (6) hold, that is, in other words, conditions $s, t \in \mathbb{T}_{N\Gamma}[G]$ are strongly similar on $N \times \Gamma$ in the sense of Definition 7.

9   Proof of the definability lemma, part 2

We continue the proof of Theorem 17. Our intermediate result and the starting point of the final part of the proof is the contrary assumption **??**ith the additional assumption that conditions $s, t \in \mathbb{T}_{N\Gamma}[G]$ in **??**re strongly similar on $N \times \Gamma$, and to complete the proof of the theorem it suffices to derive a contradiction. This will be obtained by means of Theorem 8.

In accordance with Theorem 8, let $B = \mathtt{bas}\, s = \mathtt{bas}\, t$, $\Delta = |s| = |t|$, and let transformations π, $\mathbf{S}_{a^u a^t}$, φ and $\tau = \varphi \circ \mathbf{S}_{a^u a^t} \circ \pi$, and conditions $v, u \in \mathbb{T}$ satisfy $\mathtt{bas}\, u = \mathtt{bas}\, v = B$, $|u| = |v| = \Delta$, and

(i) $\pi \in \Pi_{\mathtt{fin}}$, $\pi[n]$ is the identity for all $n \in N$, $u = \pi \cdot s$, u is strongly similar to t on $N \times \Gamma$, and moreover $\pi \cdot h^s = h^u = h^t$, and $a^u \upharpoonright \Gamma = a^t \upharpoonright \Gamma$;

(ii) $v = \mathbf{S}_{a^u a^t} \cdot u$, v is strongly similar to t on $N \times \Gamma$, $h^v = h^u$, $a^v = a^t$;

(iii) $\varphi \in \Phi_{a^v}$, $|\varphi| = \Delta$, $\varphi_\gamma[n] = \varnothing$ for all $n \in B$ and $\gamma \in \Delta$, and $t = \varphi \cdot v$;

(iv) $\tau = \varphi \circ \mathbf{S}_{a^u a^t} \circ \pi$ is an order preserving bijection from $\mathbb{T}_{\leqslant s}$ onto $\mathbb{T}_{\leqslant t}$;

(v) any condition $s' \in \mathbb{T}_{\leqslant s}$ is similar to $t' = \tau \cdot s'$ on $N \times \Gamma$.

(= items (i) — (v) of Theorem 8).

Consider a set $G \subseteq \mathbb{T}$ generic over $\mathbf{L}$ and containing s. We assume that s is the largest (= weakest) condition in G. Then, by (v), $H = \{\tau \cdot s' : s' \in G\} \subseteq \mathbb{T}$ is generic over $\mathbf{L}$ either, and $\mathbf{L}[H] = \mathbf{L}[G]$. Moreover $t = \tau \cdot s \in H$. Therefore it follows from **??**hat

(†) $\vartheta(z, \vec{\mathbf{X}}[G], \vec{\mathbf{Y}}[G], \vec{\mathbf{x}}_N[G], \vec{\mathbf{y}}_\Gamma[G])$ is true in $\mathbf{L}[G]$, but

$\vartheta(z, \vec{\mathbf{X}}[H], \vec{\mathbf{Y}}[H], \vec{\mathbf{x}}_N[H], \vec{\mathbf{y}}_\Gamma[H])$ is false in $\mathbf{L}[H] = \mathbf{L}[G]$.

Our strategy to derive a contradiction will be to show that the parameters in the formulas are pairwise equal, and hence one and the same formula is simultaneously true and false in one and the same class. This is the content of the following lemma.

Lemma 20.   (i) $\vec{\mathbf{y}}_\Gamma[G] = \vec{\mathbf{y}}_\Gamma[H]$, that is, if $\gamma \in \Gamma$ then $\mathbf{y}_\gamma^G = \mathbf{y}_\gamma^H$;

(ii) $\vec{\mathbf{x}}_N[G] = \vec{\mathbf{x}}_N[H]$, that is, if $n \in N$ and $\xi \in [\aleph_n, \aleph_{n+1})$ then $\mathbf{x}_\xi^G = \mathbf{x}_\xi^H$;

(iii) $\vec{\mathbf{X}}[G] = \vec{\mathbf{X}}[H]$, that is, $\mathbf{X}^G[n] = \mathbf{X}^H[n]$ for all $n \in \omega$;

(iv) $\vec{\mathbf{Y}}[G] = \vec{\mathbf{Y}}[H]$, that is, $\mathbf{Y}_\gamma^G = \mathbf{Y}_\gamma^H$ for all $\gamma < \lambda$.

Proof. (i) If $\gamma \in \Gamma$ then by definition $\mathbf{y}_\gamma^G = \bigcup_{s' \in G} q_\gamma^{s'}$ and $\mathbf{y}_\gamma^H = \bigcup_{t' \in H} q_\gamma^{t'} = \bigcup_{s' \in G} q_\gamma^{(\tau \cdot s')}$. Yet if $s' \in G$ then condition $t' = \tau \cdot s'$ satisfies $q_\gamma^{t'} = q_\gamma^{s'}$ by (v).

(ii) A similar argument. Suppose that $n \in N$ and $\xi \in [\aleph_n, \aleph_{n+1})$. By definition, $\mathbf{x}_\xi^G = \bigcup_{s' \in G} h_\xi^{s'}$ and $\mathbf{x}_\xi^H = \bigcup_{t' \in H} h_\xi^{t'} = \bigcup_{s' \in G} h_\xi^{(\tau \cdot s')}$. However if $s' \in G$ then condition $t' = \tau \cdot s'$ satisfies $h_\xi^{t'} = h_\xi^{s'}$ still by (v).

(iii) By definition, $\mathbf{X}^G[n]$ and $\mathbf{X}^H[n]$ are the $(\Pi_{\mathtt{fin}}, \Psi)$-hulls of resp.

$$\mathbf{x}^G[n] = \{\mathbf{x}_\xi^G\}_{\xi \in [\aleph_n, \aleph_{n+1})} \quad \text{and} \quad \mathbf{x}^H[n] = \{\mathbf{x}_\xi^H\}_{\xi \in [\aleph_n, \aleph_{n+1})} \,.$$

Thus it remains to prove that $\mathbf{x}^H[n]$ belongs to the $(\Pi_{\mathtt{fin}}, \Psi)$-hull of $\mathbf{x}^G[n]$, and vice versa.

Let $\psi = \varphi \downarrow a^v$ (a rotation in Ψ, see Section 4). By definition, if $s' \in G$ and $t' = \tau \cdot s'$, then the subconditions $h^{s'}$ and $h^{t'}$ satisfy $h^{t'} = \psi \cdot (\pi \cdot h^{s'})$. (The middle transformation $\mathbf{S}_{a^u a^t}$ does not act on the h-components). It easily follows that $\mathbf{x}^H[n] = \psi \cdot (\pi \cdot \mathbf{x}^G[n])$, as required.

(iv) Note that the sequences $\vec{\mathbf{y}}_\Gamma[G] = \{\mathbf{y}_\gamma^G\}_{\gamma < \lambda}$ and $\vec{\mathbf{y}}_\Gamma[H] = \{\mathbf{y}_\gamma^H\}_{\gamma < \lambda}$ satisfy $\vec{\mathbf{y}}_\Gamma[H] = \varphi \cdot (\mathbf{S}_{a^u a^t} \cdot \vec{\mathbf{y}}_\Gamma[G])$. (Permutation π does not act on wide subconditions.) That is, the construction of $\vec{\mathbf{y}}_\Gamma[H]$ from $\vec{\mathbf{y}}_\Gamma[G]$) goes in two steps.

Step 1: we define $\vec{\mathbf{r}} = \{\mathbf{r}_\gamma\}_{\gamma < \lambda}$ by $\vec{\mathbf{r}} = \mathbf{S}_{a^u a^t} \cdot \vec{\mathbf{y}}_\Gamma[G]$. Thus by definition

1) $\mathbf{r}_\gamma \in 2^{[\omega, \aleph_\omega)}$ for all γ,

2) if $n \notin B$ or $\gamma \notin \Delta$ then $\mathbf{r}_\gamma[n] = \mathbf{y}_\gamma^G[n]$, and

3) if $n \in B$ and $\gamma \in \Delta$ then $\mathbf{r}_\gamma[n] = \mathbf{y}_\vartheta^G[n]$, where $\vartheta = \mathbf{s}_{a^u a^t}^n(\gamma)$.

Thus the difference between $\vec{\mathbf{r}}$ and $\vec{\mathbf{y}}_\Gamma[G]$ is located within the finite domain $B \times \Delta$. Moreover, as in Lemma 3(iii), we have

(‡) $\{\mathbf{r}_\gamma[n] : \gamma < \lambda\} = \{\mathbf{y}_\gamma^G[n] : \gamma < \lambda\}$ for every n.

Step 2: we define $\vec{\mathbf{y}}_\Gamma[H] = \varphi \cdot \vec{\mathbf{r}}$. Thus by definition

4) if $\gamma \in \Delta$ then directly $\mathbf{y}_\gamma^H = \varphi \cdot \mathbf{r}_\gamma$, that is, $\mathbf{y}_\gamma^H[n] = \varphi[n] \cdot \mathbf{r}_\gamma[n]$, $\forall n$;

5) if $\gamma \notin \Delta$ and $\exists \delta \in \Delta \, (\mathbf{a}^G(n, \gamma) = \mathbf{a}^G(n, \delta))$, then $\mathbf{y}_\gamma^H[n] = \varphi[n] \cdot \mathbf{r}_\gamma[n]$;

6) if $\gamma \notin \Delta$ but $\neg \exists \delta \in \Delta \, (\mathbf{a}^G(n, \gamma) = \mathbf{a}^G(n, \delta))$, then $\mathbf{y}_\gamma^H[n] = \mathbf{r}_\gamma[n]$.

Now it immediately follows from (‡) that $\mathbf{Y}^G[n] = \mathbf{Y}^H[n]$ for every n: both sets are equal to the $\mathbb{D}[n]$-hull of one and the same set mentioned in (‡).

We are ready to prove that $\mathbf{Y}_\gamma^G = \mathbf{Y}_\gamma^H$ for every $\gamma < \lambda$.

We start with a couple of definitions. If $y, y' \in 2^{[\aleph_n, \aleph_{n+1})}$ and there exists a set $d \in \mathbf{Y}^G[n] = \mathbf{Y}^H[n]$ such that $y' = d \cdot y$ then write $y \equiv_n y'$. If $y, y' \in 2^{[\omega, \aleph_\omega)}$ and there exists a number n_0 such that $y'[n] \equiv_n y[n]$ for all $n < n_0$ and $y'[n] = y[n]$ for all $n \geq n_0$ then write $y \equiv^* y'$.

Then by $(\star)$ in Section 6 we have:

$$\left. \begin{aligned} \mathbf{Y}_\gamma^G &= \{z \in 2^{[\omega, \aleph_\omega)} : \exists y \in \mathbb{D} \cdot \mathbf{y}_\gamma^G \, (z \equiv^* y)\}; \\ \mathbf{Y}_\gamma^H &= \{z \in 2^{[\omega, \aleph_\omega)} : \exists y \in \mathbb{D} \cdot \mathbf{y}_\gamma^H \, (z \equiv^* y)\}; \end{aligned} \right\} \qquad (**)$$

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and hence to prove $\mathbf{Y}_\gamma^G = \mathbf{Y}_\gamma^H$ it suffices to check that $\mathbf{y}_\gamma^H \in \mathbf{Y}_\gamma^G$ and $\mathbf{y}_\gamma^G \in \mathbf{Y}_\gamma^H$.

Case 1: $\gamma \in \Delta$. It follows from 2) and 3) that $\mathbf{r}_\gamma \equiv^* \mathbf{y}_\gamma^G$ and hence $\mathbf{y}_\gamma^H \equiv^* y$ by 4), where $y = \varphi \cdot \mathbf{y}_\gamma^G$. Thus $\mathbf{y}_\gamma^H \in \mathbf{Y}_\gamma^G$ by $(**)$, the first line. On the other hand, $\mathbf{r}_\gamma = \varphi^{-1} \cdot \mathbf{y}_\gamma^H$ still by 4), so that $\mathbf{y}_\gamma^G \in \mathbf{Y}_\gamma^H$ by $(**)$, the second line.

Case 2: $\gamma \notin \Delta$. Note that for a given γ 5) holds only for finitely many numbers n by Lemma 12(iii), so 6) holds for almost all n. Therefore $\mathbf{y}_\gamma^H \equiv^* \mathbf{r}_\gamma$. But $\mathbf{r}_\gamma = \mathbf{y}_\gamma^G$ in this case by 2). Thus $\mathbf{y}_\gamma^H \in \mathbf{Y}_\gamma^G$ by $(**)$, the first line (with $y = \mathbf{y}_\gamma^G$). And $\mathbf{y}_\gamma^G \in \mathbf{Y}_\gamma^H$ holds by a similar argument. $\square$ (*Lemma*)

$\square$ (*Theorem 17*)

10 The structure of the extension

Here we accomplish the proof of Theorem 1.

Blanket agreement 21. We fix a set $G \subseteq \mathbb{T}$, $\mathbb{T}$-generic over $\mathbf{L}$, during the course of this section.

It will be shown that the symmetric subextension $\mathbf{L}_{\mathtt{sym}}[G] = \mathbf{L}(W[G])$ (see Section 6) satisfies Theorem 1. The following is a key technical claim.

Theorem 22. *Suppose that* $\nu < \omega$, *and* $Z \in \mathbf{L}_{\mathtt{sym}}[G]$, $Z \subseteq [0, \aleph_{\nu+1})$. *Then* $Z \in \mathbf{L}[\{\mathbf{x}^G[n] : n \leqslant \nu\}]$.

Proof. It follows from Lemma 16 and Theorem 17 that there exist finite sets $N \subseteq \omega$ and $\Gamma \subseteq \lambda$ such that $Z \in \mathbf{L}[\{\mathbf{x}^G[n] : n \in N\}, \{\mathbf{y}_\gamma^G : \gamma \in \Gamma\}]$. We can assume that

(1) $N = \{0, 1, 2, \ldots, \kappa\}$ for some $\kappa < \omega$, $\kappa \geq \nu$;

(2) if $\gamma \neq \delta$ belong to Γ and $n < \omega$ satisfies $\mathbf{a}^G(n, \gamma) = \mathbf{a}^G(n, \delta)$ then $n \leqslant \kappa$.

(Lemma 12(iii) is used to justify (2).) Define, in $\mathbf{L}$,

$$\mathbb{T}[N, \Gamma] = \{s \in \mathbb{T} : \mathtt{bas}\, s = N \wedge |s| = \Gamma \wedge a^s = \mathbf{a}^G \restriction (N \times \Gamma)\}.$$

Lemma 23. *The set* $G[N, \Gamma] = G \cap \mathbb{T}[N, \Gamma]$ *is* $\mathbb{T}[N, \Gamma]$-*generic over* $\mathbf{L}$ *and* $Z \in \mathbf{L}[G[N, \Gamma]]$.

Proof (Lemma). Suppose that $t \in \mathbb{T}$, $N \subseteq \mathtt{bas}\, t$, $\Gamma \subseteq |t|$. Define the *projection* $s = t[N, \Gamma] \in \mathbb{T}[N, \Gamma]$ so that $q^s = q^t \restriction \Gamma$, $a^s = a^t \restriction (N \times \Gamma)$, and h^s is the restriction of h^t to the set $|h^t| \cap \bigcap_{n \leqslant \kappa} [\aleph_n, \aleph_{n+1})$. (It is not asserted that $t \leqslant s$.) Given a condition $s' \in \mathbb{T}[N, \Gamma]$, $s' \leqslant s$, we have to accordingly find a condition $t' \leqslant t$ such that $t'[N, \Gamma] = s'$.

Define t' as follows. First of all, $\mathtt{bas}\, t' = \mathtt{bas}\, t$, $|t'| = |t|$, $a^{t'} = a^t$.

Put $h^{t'}[n] = h^{s'}[n]$ for $n \in N$ but $h^{t'}[n] = h^t[n]$ for $n \in \mathtt{bas}\, t \smallsetminus N$.

If $n \notin N$ then put $q_\gamma^{t'}[n] = q_\gamma^{t'}[n]$ for all $\gamma \in |t'| = |t|$. If $n \in N$ and $\gamma \in |t'|$ then put $q_\gamma^{t'}[n] = h_\xi^{t'} = h_\xi^{s'}$, where $\xi = a^{t'}(n, \gamma)$. $\square$ (*Lemma*)

In continuation of the proof of the theorem, let us analyse $\mathbb{T}[N, \Gamma]$ as the forcing notion. It looks like the product $\prod_{n=0}^{\kappa} \mathbb{H}[n] \times \mathbb{P}^{\Gamma}$: indeed, if $s \in \mathbb{T}[N, \Gamma]$ then h^s can be seen as an element of $\prod_{n=0}^{\kappa} \mathbb{H}[n]$, q^s can be seen as an element of $\mathbb{P}^{\Gamma}$ (the product of $\mathtt{card}\,\Gamma$ copies of $\mathbb{P}$; $\mathtt{card}\,\Gamma < \omega$), while $a^s = \mathbf{a}^G{\restriction}(N \times \Gamma)$ is a constant. However if $n \in N$ and $\gamma \in \Gamma$ then $q_\gamma^s[n] = q_{a^s(n,\gamma)}^t$, hence in fact $\mathbb{T}[N, \Gamma]$ can be identified with

$$\textstyle\prod_{n=0}^{\kappa} \mathbb{H}[n] \times (\prod_{n=\kappa+1}^{\infty} \mathbb{P}[n])^{\Gamma} = \prod_{n=0}^{\kappa} \mathbb{H}[n] \times \prod_{n=\kappa+1}^{\infty}(\mathbb{P}[n]^{\Gamma}). \tag{3}$$

However the sets $\mathbb{P}[n]$ and $\mathbb{H}[n]$ as forcing notions are $\aleph_n^+$-closed, meaning that any decreasing sequence of length $\leqslant \aleph_n$ has a lower bound in the same set. Therefore if we present $\mathbb{T}[N, \Gamma]$ as

$$\textstyle\prod_{n=0}^{\nu} \mathbb{H}[n] \times \prod_{n=\nu+1}^{\kappa} \mathbb{H}[n] \times \prod_{n=\kappa+1}^{\infty}(\mathbb{P}[n]^{\Gamma}), \tag{4}$$

then it becomes clear that the second and third subproducts are $\aleph_{n+1}^+$-closed forcing notions. Hence, by basic results of forcing theory, the set $Z \subseteq [0, \aleph_{\nu+1})$ belongs to the subextension corresponding to the first subproduct $\prod_{n=0}^{\nu} \mathbb{H}[n]$. That is, $Z \in \mathbf{L}[\{\mathbf{x}^G[n] : n \leqslant \nu\}]$, as required. $\qed$

Corollary 24. *If $n < \omega$ then it is true in $\mathbf{L}_{\mathtt{sym}}[G]$ that $\aleph_n$ remains a cardinal, the power set $\mathscr{P}(\aleph_n)$ is wellorderable, and $\mathtt{card}(\mathscr{P}(\aleph_n)) = \aleph_{n+1}$.* $\qed$

Yet cardinal preservation holds for all cardinals!

Corollary 25. *Any cardinal in $\mathbf{L}$ remains a cardinal in $\mathbf{L}_{\mathtt{sym}}[G]$.*

Proof. Indeed we have established (see the proof of Theorem 22) that any set $Z \in \mathbf{L}_{\mathtt{sym}}G$, $Z \subseteq \mathbf{L}$, belongs to a generic extension of $\mathbf{L}$ via a forcing as in (3) in the proof of Theorem 22. However any such a forcing is cardinal-preserving by a simple cardinality argument. $\qed$

To accomplish the proof of Theorem 1, it remains to check that the symmetric subextension $\mathbf{L}_{\mathtt{sym}}[G]$ contains a surjection $\sigma : 2^{[\omega, \aleph_\omega)} \xrightarrow{\mathtt{onto}} \lambda$. We define σ in $\mathbf{L}_{\mathtt{sym}}[G]$ as follows. If $\gamma < \lambda$ and $z \in \mathbf{Y}_\gamma^G$ then put $\sigma(z) = \gamma$. (The definition is consistent by Lemma 14.) If $z \in 2^{[\omega, \aleph_\omega)}$ does not belong to $\bigcup_{\gamma < \lambda} \mathbf{Y}_\gamma^G$ then $\sigma(z) = 0$. As any set $\mathbf{Y}_\gamma^G$ definitely contains $\mathbf{y}_\gamma^G$, σ is a surjection onto λ, as required.

$\qed$ (*Theorem 1*)

References

[1] A reference to the fact that the consistency of the statement "GCH first fails at $\aleph_\omega$" with **ZFC** definitely requires a large cardinal.

[2] Moti Gitik and Peter Koepke, *Violating the Singular Cardinals Hypothesis without large cardinals*, August 3, 2010.