On effective compactness and sigma-compactness

Vladimir Kanovei*

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Abstract

Using the Gandy – Harrington topology and other methods of effective descriptive set theory, we prove several theorems on compact and σ-compact pointsets. In particular we show that any Σ^1_1 set A of the Baire space $\mathscr{N}$ either is covered by a countable union of compact Δ^1_1 sets, or A contains a subset closed in $\mathscr{N}$ and homeomorphic to $\mathscr{N}$ (and then A is not covered by a σ-compact set, of course).

1 Introduction

Effective descriptive set theory appeared in the 1950s as a useful technique of simplification and clarification of constructions of classical descriptive set theory (see *e.g.* [12] or [7]). Yet it had soon become clear that development of effective descriptive set theory leads to results having no direct analogies in classical descriptive set theory. As an example we recall the following *basis theorem*: any countable Δ^1_1 set A of the Baire space $\mathscr{N} = \mathbb{N}^{\mathbb{N}}$ consists of Δ^1_1 points. Its remote predecessor in classical descriptive set theory is the Luzin – Novikov theorem on Borel sets with countable cross-sections.

In this note, methods of effective descriptive set theory are applied to the properties of compactness and σ-compactness of pointsets. The following theorem is our main result.

Recall that $[T] = \{x \in \mathscr{N} : \forall\, m\, (x \restriction m \in T\}$ for any tree $T \subseteq \mathbb{N}^{<\omega}$.

Theorem 1. *If* $A \subseteq \mathscr{N}$ *is a* Σ^1_1 *set then one and only one of the following two claims holds*: $\overleftarrow{\text{mt'}}$

(I) A *is covered by the union* U *of all sets of the form* $[T]$, *where* $T \subseteq \mathbb{N}^{<\omega}$ *is a compact* Δ^1_1 *tree — and moreover there is a* Δ^1_1 *sequence* $\{T_n\}_{n\in\mathbb{N}}$ *of compact trees* $T_n \subseteq \mathbb{N}^{<\omega}$ *such that* $A \subseteq \bigcup_n [T_n]$;

*Contact author, kanovei@mccme.ru

(II) *there is a set* $Y \subseteq A$ *homeomorphic to* $\mathscr{N}$ *and closed in* $\mathscr{N}$.

Here conditions (I) and (II) are incompatible: if Y is a set as is (II) then Y cannot be covered by a σ-compact set U as in (I).

In parallel to Theorem 1 and using basically the same technique, we prove the following similar theorem, which is, on the other hand, a direct corollary of some well-known results in this field.

Theorem 2. *If* $A \subseteq \mathscr{N}$ *is a* Δ^1_1 *set then one and only one of the following two claims holds*: $\longleftarrow$ mt

(I) A *is equal to the union* U *of all sets of the form* $[T]$, *where* $T \subseteq \mathbb{N}^{<\omega}$ *is a compact* Δ^1_1 *tree and* $[T] \subseteq A$ *— and moreover there is a* Δ^1_1 *sequence* $\{T_n\}_{n\in\mathbb{N}}$ *of compact trees* $T_n \subseteq \mathbb{N}^{<\omega}$ *such that* $A = \bigcup_n [T_n]$;

(II) *there is a set* $Y \subseteq A$ *homeomorphic to* $\mathscr{N}$ *and relatively closed in* A.

Conditions (I) and (II) of the theorem are incompatible since A is σ-compact provided (I) holds, so that any relatively closed subset of A is σ-compact itself, while the space $\mathscr{N}$ is not σ-compact, of course.

Theorem 2 has strong connections with 4F.18 in [10] which the author of [9] credits to Louveau. It is clear from 4F.18 that if A is a Δ^1_1 subset of $\mathscr{N}$ and σ-compact then it is equal to the union of compact Δ^1_1 sets $A' \subseteq A$. On the other hand, it follows from 4F.14 in [10] that if A is a compact Δ^1_1 subset of $\mathscr{N}$ then there is a compact Δ^1_1 tree $T \subseteq \mathbb{N}^{<\omega}$ such that $A = [T]$. To conclude, if A is a σ-compact Δ^1_1 subset of $\mathscr{N}$ then condition (I) of Theorem 2 is true. This allows to derive directly Theorem 2. Indeed if $A \subseteq \mathscr{N}$ is a Δ^1_1 set and it does not satisfy condition (I) of Theorem 2 then the set A is not σ-compact by the above, and so from the theorem of Hurewicz (see Theorem 21) the set A satisfies (II) of Theorem 2.

Nevertheless we present here a new proof of Theorem 2, in particular, as a base for the proof of *a similar but more complicated dichotomy theorem on* Σ^1_1 *sets* (Theorem 19), where, unfortunately, the level of effectivity of the covering by σ-compact sets in (I) will be less definite.

In addition, we'll prove *a generalization of Theorem 1* (Theorem 17) which deals, instead of compact sets, with closed sets whose trees contain branchings small in the sense of a chosen ideal on $\mathbb{N}$.

As usual, the theorems remain true in the relativized form, *i.e.* when classes Δ^1_1 and Σ^1_1 are replaced by $\Delta^1_1(p)$ and $\Sigma^1_1(p)$, where $p \in \mathscr{N}$ is a fixed parameter, with basically the same proofs.

Some well-known classical results related to the theorems above are discussed in the last section.

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2 Preliminaries

We use standard notation Σ^1_1, Π^1_1, Δ^1_1 for effective classes of points and pointsets in $\mathscr{N}$, as well as $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$ for corresponding projective classes.

Let $\mathbb{N}^{<\omega}$ be the set of all finite strings of natural numbers, including the empty string Λ. If $s, t \in \mathbb{N}^{<\omega}$ then $\mathtt{lh}\, s$ is the *length* of s, and $s \subset t$ means that t is a *proper extension* of s. If $s \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$ then $s^\wedge n$ is the string obtained by adding n to s as the rightmost term. Let, for $s \in \mathbb{N}^{<\omega}$,

$$\mathscr{N}_s = \{x \in \mathscr{N} : s \subset x\} \quad (\textit{a Baire interval} \text{ in } \mathscr{N}).$$

If a set $X \subseteq \mathscr{N}$ contains at least two elements then there is a longest string $s = \mathtt{stem}(X)$ such that $X \subseteq \mathscr{N}_s$. We put $\mathtt{diam}(X) = \frac{1}{1+\mathtt{stem}(X)}$ in this case, and additionally $\mathtt{diam}(X) = 0$ whenever X has at most one element.

A set $T \subseteq \mathbb{N}^{<\omega}$ is a *tree* if $s \in T$ holds whenever $s^\wedge n \in T$ for at least one n, and a *pruned* tree iff $s \in T$ implies $s^\wedge n \in T$ for at least one n. Any non-empty tree contains Λ. A string $s \in T$ is a *branching point* of T if there are $k \neq n$ such that $s^\wedge k \in T$ and $s^\wedge n \in T$; let $\mathbf{bran}(T)$ be the set of all branching points of T. The *branching height* $\mathtt{BH}_T(s)$ of a string $s \in T$ in a tree T is equal to the number of strings $t \in \mathbf{bran}(T)$, $t \subset s$. For instance, if $T = \mathbb{N}^{<\omega}$ then $\mathtt{BH}_{\mathbb{N}^{<\omega}}(s) = \mathtt{lh}\, s$ for any string s. A tree T is *perfect* iff for any $s \in T$ there is a string $t \in \mathbf{bran}(T)$ such that $s \subset t$.

A tree $T \subseteq \mathbb{N}^{<\omega}$ is *compact*, if it is pruned and has *finite branchings*, that is, if $s \in T$ then $s^\wedge n \in T$ holds for at most finitely many n. Then

$$[T] = \{x \in \mathscr{N} : \forall m\, (x \restriction m \in T)\}$$

is a compact set. Conversely, if $X \subseteq \mathscr{N}$ is a compact set then

$$\mathtt{tree}(X) = \{x \restriction n : x \in X \wedge n \in \mathbb{N}\}$$

is a compact tree. Let $\mathbf{CT}$ be the Δ^1_1 set of all non-empty compact trees.

If $\mathbb{X}, \mathbb{Y}$ are any sets and $P \subseteq \mathbb{X} \times \mathbb{Y}$ then

$$\mathtt{proj}\, P = \{x \in \mathbb{X} : \exists y\, (\langle x, y\rangle \in P)\} \quad \text{and} \quad (P)_x = \{y \in \mathbb{Y} : \langle x, y\rangle \in P\}$$

are, resp., the *projection* of P to $\mathbb{X}$, and the *cross-section* of P defined by $x \in \mathbb{X}$. A set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is *uniform* if every cross-section $(P)_x$ ($x \in \mathbb{X}$) contains at most one element. If $P \subseteq Q \subseteq \mathbb{X} \times \mathbb{Y}$, P is uniform, and $\mathtt{proj}\, P = \mathtt{proj}\, Q$, then they say that P *uniformizes* Q.

3 Some facts of effective descriptive set theory

We'll make use of the following well-known results.

Fact 3 (Σ^1_1 Separation)**.** *If $X, Y \subseteq \mathscr{N}$ are disjoint Σ^1_1 sets then there is a Δ^1_1 set $Z \subseteq \mathscr{N}$ such that $X \subseteq Z$ and $Y \cap Z = \varnothing$.* □ ← 21

Fact 4 (Kreisel selection, 4B.5 in [10])**.** *If $P \subseteq \mathscr{N} \times \mathbb{N}$ is a Π^1_1 set and the projection $\mathtt{proj}\, P$ is a Δ^1_1 set then there is a Δ^1_1 map $f : \mathtt{proj}\, P \to \mathbb{N}$ such that $\langle x, f(x) \rangle \in P$ for all $x \in \mathtt{proj}\, P$.* □ ← 22

Fact 5 (4D.3 in [10])**.** *If $P(x, y, z, \dots)$ is a Π^1_1 relation* (where the domain of each argument can be $\mathscr{N}$, $\mathscr{P}(\mathbb{N}^{<\omega})$, the set of all compact trees in $\mathbb{N}^{<\omega}$, or any other similar Polish space) *then the relations $\exists\, x \in \Delta^1_1\, P(x, y, z, \dots)$ and $\exists\, x \in \Delta^1_1(y)\, P(x, y, z, \dots)$ are Π^1_1, too.* □ ← BQ

Fact 6 (4D.14 in [10])**.** *The set $D = \{T \subseteq \mathbb{N}^{<\omega} : T$ is $\Delta^1_1\}$ is Π^1_1.* ← dp

The set $\{\langle p, T \rangle : p \in \mathscr{N} \wedge T \subseteq \mathbb{N}^{<\omega} \wedge T$ is $\Delta^1_1(p)\}$ is Π^1_1 as well. □

To prove the first claim, note that $T \in D \iff \exists\, T' \in \Delta^1_1\, (T = T')$; then the result follows from Fact 5.

Fact 7 (Enumeration of Δ^1_1 trees, 4D.2 in [10])**.** *There exist Π^1_1 sets $E \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$, and a Σ^1_1 set $W' \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ such that* ← 23

(i) *$(W)_e = (W')_e$ for any $e \in E$* (where $(W)_e = \{s \in \mathbb{N}^{<\omega} : \langle e, s \rangle \in W\}$);

(ii) *a set $T \subseteq \mathbb{N}^{<\omega}$ is Δ^1_1 iff there is a number $e \in E$ such that $T = (W)_e = (W')_e$.* □

Fact 8 (4F.17 in [10])**.** *If $P \subseteq \mathscr{N} \times \mathscr{N}$ is a Δ^1_1 set and every cross-section $(P)_x$ ($x \in \mathscr{N}$) is at most countable then $\mathtt{proj}\, P$ is a Δ^1_1 set, and P is a countable union of uniform Δ^1_1 sets each of which uniformizes P.* □ ← 25

Fact 9 (4F.14 in [10])**.** *If $F \subseteq \mathscr{N}$ is a closed Δ^1_1 set and $X \subseteq F$ is a compact Σ^1_1 set then there is a compact Δ^1_1 tree $T \subseteq \mathbb{N}^{<\omega}$ such that $X \subseteq [T] \subseteq F$. In particular, in the case $X = F$, any compact Δ^1_1 set $X \subseteq \mathscr{N}$ has the form $X = [T]$ for some compact Δ^1_1 tree $T \subseteq \mathbb{N}^{<\omega}$.* ← ks=t

Fact 10 (4F.11 in [10])**.** *Any compact Δ^1_1 set $\varnothing \neq A \subseteq \mathscr{N}$ contains a Δ^1_1 element $x \in A$.* □ ← DinD

There is a useful uniform version of Fact 7.

Fact 11 (Uniform enumeration)**.** *There exist Π^1_1 sets $\mathbf{E} \subseteq \mathscr{N} \times \mathbb{N}$ and $\mathbf{W} \subseteq \mathscr{N} \times \mathbb{N} \times \mathbb{N}^{<\omega}$, and a Σ^1_1 set $\mathbf{W}' \subseteq \mathscr{N} \times \mathbb{N} \times \mathbb{N}^{<\omega}$ such that* ← 23+

(i) $(\mathbf{W})_{xe} = (\mathbf{W}')_{xe}$ *for any* $\langle x, e\rangle \in \mathbf{E}$ (where $(\mathbf{W})_{xe} = \{s \in \mathbb{N}^{<\omega} : \langle x, e, s\rangle \in \mathbf{W}\}$);

(ii) *if* $x \in \mathscr{N}$ *then a set* $T \subseteq \mathbb{N}^{<\omega}$ *is* $\Delta^1_1(x)$ *iff there is a number* $e \in E$ *such that* $T = (\mathbf{W})_{xe} = (\mathbf{W}')_{xe}$. □

This result allows us to prove the following generalization of Fact 4, also well-known in effective descriptive set theory.

Fact 12 (4D.6 in [10])**.** *Suppose that* $Q \subseteq \mathscr{N} \times \mathscr{P}(\mathbb{N}^{<\omega})$ *is* Π^1_1*, the projection* $\mathtt{proj}\, Q$ *onto* $\mathscr{N}$ *is* Δ^1_1*, and for each* $x \in \mathtt{proj}\, Q$ *there exists a set* $T \in \Delta^1_1(x)$ *such that* $\langle x, T\rangle \in Q$*. Then there is a* Δ^1_1 *map* $\tau : \mathtt{proj}\, Q \to \mathscr{P}(\mathbb{N}^{<\omega})$ *such that* $\langle x, \tau(x)\rangle \in Q$ *for all* $x \in \mathtt{proj}\, Q$*.* ⟵ 22+

Proof. Making use of sets $\mathbf{E}, \mathbf{W}, \mathbf{W}'$ as in Fact 11, we let

$$P = \{\langle x, e\rangle \in \mathbf{E} : \langle x, (\mathbf{W})_{xe}\rangle \in Q\}.$$

Immediately the set P is Π^1_1 and $\mathtt{proj}\, P = \mathtt{proj}\, Q$ is a Δ^1_1 subset of $\mathscr{N}$. By Fact 4, there is a Δ^1_1 map $f : \mathtt{proj}\, P \to \mathbb{N}$ such that $\langle x, f(x)\rangle \in P$ for all $x \in \mathtt{proj}\, P$. It remains to define $\tau(x) = (\mathbf{W})_{xf(x)}$ for all $x \in \mathtt{proj}\, Q$; to prove that τ is Δ^1_1 use both $\mathbf{W}$ and $\mathbf{W}'$. □

Facts 3, 4, 5, 7, 6 (the first claim), 8, 9, 11 , 12 remain true for relativized lightface classes $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$, where $p \in \mathscr{N}$ is an arbitrary fixed parameter. Therefore Facts 3, 4, 8 also hold with lightface classes replaced by boldface projective classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$.

4 The Gandy – Harrington topology

The *Gandy – Harrington topology* on the Baire space $\mathscr{N}$ consists of all unions of Σ^1_1 sets $S \subseteq \mathscr{N}$. This topology includes the Polish topology on $\mathscr{N}$ but is not Polish. Nevertheless the Gandy – Harrington topology satisfies a condition typical for Polish spaces.

Definition 13. Let $\mathscr{F}$ be any family of sets, *e.g.* sets in a given background space $\mathbb{X}$. A set $\mathscr{D} \subseteq \mathscr{F}$ is *open dense* iff $\forall F \in \mathscr{F}\, \exists D \in \mathscr{D}\, (D \subseteq F)$, and ⟵ genb'

$$\forall F \in \mathscr{F}\, \forall D \in \mathscr{D}\, (F \subseteq D \implies F \in \mathscr{D}).$$

Sets $\mathscr{D}$ satisfying only the first requirement are called *dense*. If $\mathscr{D} \subseteq \mathscr{F}$ is dense then the set $\mathscr{D}' = \{F \in \mathscr{F} : \exists D \in \mathscr{D}\, (F \subseteq D)\}$ is open dense. The notions of *open* and *dense* are related to a certain topology which we'll not discuss, but not necessarily with the topology of the background space $\mathbb{X}$.

A *Polish net* for $\mathscr{F}$ is any collection $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of open dense sets $\mathscr{D}_n \subseteq \mathscr{F}$ such that we have $\bigcap_n F_n \neq \varnothing$ for every sequence of sets $F_n \in \mathscr{D}_n$ satisfying the finite intersection property (*i.e.* $\bigcap_{k \le n} F_k \neq \varnothing$ for all n). □

For instance the family of all non-empty closed sets of a complete metric space $\mathbb{X}$ admits a Polish net: let $\mathscr{D}_n$ contain all closed sets of diameter $\le n^{-1}$ in $\mathbb{X}$. The next theorem is less elementary. This theorem and the following corollary are well-known, see *e.g.* [2, 3, 6, 8].

Theorem 14. *The collection $\mathbb{P}$ of all non-empty Σ^1_1 sets in $\mathscr{N}$ admits a Polish net.* □ ← s11p

5 The proof of Theorem 1

Recall that **CT** is the set of all compact trees $\varnothing \neq T \subseteq \mathbb{N}^{<\omega}$; **CT** is Δ^1_1, of course. Let U be the set as in (I) of the theorem. We claim that U is Π^1_1. Indeed, by definition

$$x \in U \iff \exists T \in \Delta^1_1 \, (T \in \mathbf{CT} \wedge x \in [T]),$$

and the result follows from Fact 5.

It follows that the difference $A' = A \smallsetminus U$ is a Σ^1_1 set.

Lemma 15. *Under the conditions of Theorem 1, if $Y \subseteq A'$ is a non-empty Σ^1_1 set then its topological closure $\overline{Y}$ in $\mathscr{N}$ is not compact, i.e., the tree $\mathtt{tree}(Y) = \{y \restriction n : y \in Y \wedge n \in \mathbb{N}\}$ has at least one infinite branching.* ← tkm*

Proof. Suppose otherwise: $\overline{Y}$ is compact. Then by Fact 9 (with $F = \mathscr{N}$) there is a compact Δ^1_1 tree T such that $\overline{Y} \subseteq [T]$. Therefore $Y \subseteq \overline{Y} \subseteq [T] \subseteq U$, and this contradicts to the assumption $\varnothing \neq Y \subseteq A'$. □ (Lemma)

Case 1: the set $A' = A \smallsetminus U$ is non-empty. We assert that then there is a system of non-empty Σ^1_1 sets $Y_s \subseteq A'$ satisfying the following conditions

(1) if $s \in \mathbb{N}^{<\omega}$ and $i \in \mathbb{N}$ then $Y_{s^\wedge i} \subseteq Y_s$;

(2) $\mathtt{diam}(Y_s) \le 2^{-\mathtt{lh}\, s}$;

(3) if $s \in \mathbb{N}^{<\omega}$ and $k \neq n$ then $Y_{s^\wedge k} \cap Y_{s^\wedge n} = \varnothing$, and moreover, sets $Y_{s^\wedge k}$ are covered by pairwise disjoint (clopen) Baire intervals $J_{s^\wedge k}$;

(4) $Y_s \in \mathscr{D}_{\mathtt{lh}\, s}$, where by Theorem 14 $\{\mathscr{D}_n : n \in \mathbb{N}\}$ is a Polish net for the family $\mathbb{P}$ of all non-empty Σ^1_1 sets $Y \subseteq \mathscr{N}$;

(5) if $s \in \mathbb{N}^{<\omega}$ and $x_k \in Y_{s^\wedge k}$ for all $k \in \mathbb{N}$ then the sequence of points x_k does not have convergent subsequences in $\mathscr{N}$.

If such a construction is accomplished then (4) implies that $\bigcap_m Y_{a\restriction m} \neq \varnothing$ for each $a \in \mathscr{N}$. On the other hand by (2) every such an intersection contains a single point, which we denote by $f(a)$, and the map $f : \mathscr{N} \xrightarrow{\text{onto}} Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\}$ is a homeomorphism by clear reasons.

Prove that Y is closed in $\mathscr{N}$. Consider an arbitrary sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of points $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$. If the sequence $\{a_n\}_{n\in\mathbb{N}}$ contains a subsequence of points $b_k = a_{n(k)}$ convergent to some $b \in \mathscr{N}$ then quite obviously the sequence of points $z_k = f(b_k)$ (a subsequence of $\{y_n\}_{n\in\mathbb{N}}$) converges to $z = f(b) \in Y$, as required. Thus suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ has no convergent subsequences. Then it cannot be covered by a compact set, and it easily follows that there is a string $s \in \mathbb{N}^{<\omega}$, an infinite set $K \subseteq \mathbb{N}$, and for each $k \in K$ — a number $n(k)$ such that $s^\wedge k \subset a_{n(k)}$. But then $y_{n(k)} \in Y_{s^\wedge k}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k\in\mathbb{N}}$ diverges by (5), which is a contradiction.

Thus Y is closed, and hence we have (II) of Theorem 1.

As for the construction of sets Y_s, if a Σ^1_1 set $Y_s \subseteq A'$ is defined then by Lemma 15 there is a string $t \in T(Y_s)$ such that $t^\wedge k \in T(Y_s)$ for all k in an infinite set $K_s \subseteq \mathbb{N}$. This allows us to define a sequence of pairwise different points $y_k \in Y_s$ ($k \in \mathbb{N}$) having no convergent subsequences. We cover these points by Baire intervals U_k small enough for (5) to be true for the Σ^1_1 sets $Y_{s^\wedge i} = Y_s \cap U_i$, and then shrink these sets if necessary to satisfy (2) and (4).

Case 2: $U = \varnothing$, that is, $A \subseteq U$. Recall that $\mathbf{CT}$ is the Δ^1_1 set of all compact trees $T \subseteq \mathbb{N}^{<\omega}$. The sets

$$Q = \{\langle x, T\rangle : x \in \mathscr{N} \wedge T \in \mathbf{CT} \cap \Delta^1_1 \wedge x \in [T]\}, \qquad \text{and}$$
$$Z = \{x \in \mathscr{N} : \exists T \in \Delta^1_1 \, (T \in \mathbf{CT} \wedge x \in [T])\} \qquad = \mathtt{proj}\, Q$$

are Π^1_1 by Facts 5 and 6. Moreover, $A \subseteq U$ implies $A \subseteq Z$, and hence by Fact 3 there is a Δ^1_1 set X such that $A \subseteq X \subseteq Z$. Then $P = \{\langle x, n\rangle \in Q : x \in X\}$ is still a Π^1_1 set, and $\mathtt{proj}\, P = X$ is a Δ^1_1 set. Therefore by Fact 12 there is a Δ^1_1 function $\tau : X \to \mathbf{CT}$ such that $\langle x, \tau(x)\rangle \in Q$ for all $x \in X$.

Note that $\tau(x) \in \mathbf{CT} \cap \Delta^1_1$ and $x \in [\tau(x)]$ for all $x \in A$ by the construction. Thus the full image $R = \{\tau(x) : x \in A\}$ is a Σ^1_1 subset of the Π^1_1 set $\mathbf{CT} \cap \Delta^1_1$, and hence there is a Δ^1_1 set D such that $R \subseteq D \subseteq \mathbf{CT} \cap \Delta^1_1$. But countable Δ^1_1 sets are known to admit a Δ^1_1 enumeration, so there is a Δ^1_1 map $\delta : \mathbb{N} \xrightarrow{\text{onto}} D$. Now let $T_n = \delta(n)$ for for all n.

□ (Theorem 1)

6 The proof of Theorem 2

By Theorem 1, we can w.l.o.g. assume that A is covered by a σ-compact set, and hence if $F \subseteq A$ is a closed set then F is σ-compact. Further, the set U in (I) of Theorem 2 (the union of all sets $[T] \subseteq A$, where T is a compact Δ^1_1 tree) is Π^1_1. Indeed,

$$x \in U \iff \exists T \in \Delta^1_1 \, (T \text{ is a compact tree and } x \in [T] \subseteq A),$$

and the result follows from Fact 5 since the property of "being a compact tree" can be straightforwardly expressed by an arithmetic formula, while $[T] \subseteq A$ can be expressed by a Π^1_1 formula.

We conclude that $A' = A \smallsetminus U$ is Σ^1_1.

Lemma 16. *If $F \subseteq A'$ is a non-empty Σ^1_1 set then $\overline{F} \not\subseteq A$.* $\overleftarrow{\text{tkm-l}}$

Proof. We first prove that if $X \subseteq A$ is a compact Σ^1_1 set then $A' \cap X = \varnothing$. Suppose towards the contrary that $A' \cap X$ is non-empty. We are going to find a closed Δ^1_1 set F satisfying $X \subseteq F \subseteq A$ — this would imply $X \subseteq U$ by Fact 9, which is a contradiction.

Since the complementary Π^1_1 set $C = \mathscr{N} \smallsetminus X$ is open, the set

$$H = \{\langle x, s\rangle : s \in \mathbb{N}^{<\omega} \wedge x \in C \cap \mathscr{N}_s \wedge \mathscr{N}_s \cap X = \varnothing\}$$

is Π^1_1 and $\mathtt{proj}\, H = C$. Thus the Δ^1_1 set $D = \mathscr{N} \smallsetminus A$ is included in $\mathtt{proj}\, H$. By Fact 4, there is a Δ^1_1 map $\nu : D \to \mathbb{N}^{<\omega}$ such that $x \in D \Longrightarrow \langle x, \nu(x)\rangle \in H$, or equivalently, $x \in \mathscr{N}_{\nu(x)} \subseteq C$ for all $x \in D$. Then the set $\Sigma = \mathtt{ran}\, \nu = \{\nu(x) : x \in D\} \subseteq \mathbb{N}^{<\omega}$ is Σ^1_1 and $D \subseteq \bigcup_{s \in \Sigma} \mathscr{N}_s \subseteq C$.

But $\Pi = \{s \in \mathbb{N}^{<\omega} : \mathscr{N}_s \subseteq C\}$ is a Π^1_1 set and $\Sigma \subseteq \Pi$. It follows that there exists a Δ^1_1 set Δ such that $\Sigma \subseteq \Delta \subseteq \Pi$. Then still $D \subseteq \bigcup_{s \in \Delta} \mathscr{N}_s \subseteq C$, and hence the closed set $F = \mathscr{N} \smallsetminus \bigcup_{s \in \Delta} \mathscr{N}_s$ satisfies $X \subseteq F \subseteq A$. But $x \in F$ is equivalent to $\forall s \, (s \in \Delta \Longrightarrow x \notin \mathscr{N}_s)$, thus F is Δ^1_1, as required.

Now suppose towards the contrary that $\varnothing \neq F \subseteq A'$ is a Σ^1_1 set but $\overline{F} \subseteq A$. By the w.l.o.g. assumption above, $\overline{F} = \bigcup_n F_n$ is σ-compact, where all F_n are compact. There is a Baire interval $\mathscr{N}_s$ such that the set $X = \mathscr{N}_s \cap \overline{F}$ is non-empty and $X \subseteq F_n$ for some n. Thus $X \subseteq A$ is a non-empty compact Σ^1_1 set, hence $X \cap A' = \varnothing$ by the first part of the proof. In other words, $\mathscr{N}_s \cap \overline{F} \cap A' = \varnothing$. It follows that $\mathscr{N}_s \cap F = \varnothing$ (because $F \subseteq A'$), contrary to $X = \mathscr{N}_s \cap \overline{F} \neq \varnothing$. □ (Lemma)

Case 1: the Σ^1_1 set $A' \subseteq A$ is non-empty. To get a set $Y \subseteq A'$, *relatively* closed in A and homeomorphic to $\mathscr{N}$, as in (II) of the theorem, we'll define a system of non-empty Σ^1_1 sets $Y_s \subseteq A'$ satisfying conditions (1), (2), (3), (4) of Section 5, along with the next requirement instead of (5):

(5′) if $s \in \mathbb{N}^{<\omega}$ then there is a point $y_s \in \overline{Y_s} \smallsetminus A$ such that any sequence of points $x_k \in Y_{s^\wedge k}$ $(k \in \mathbb{N})$ converges to y_s.

If we have defined such a system of sets, then the associated map $f : \mathscr{N} \to A'$ is $1-1$ and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A'$, as in the proof of Theorem 1.

Let's prove that Y is relatively closed in A. Consider a sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence convergent to $b \in \mathscr{N}$ then, as in the proof of Theorem 1, $\{y_n\}$ converges to $f(b) \in Y$. If the sequence $\{a_n\}$ has no convergent subsequences, then there exist a string $s \in \mathbb{N}^{<\omega}$, an infinite set $K \subseteq \mathbb{N}$, and for each $k \in K$ — a number $n(k)$, such that $s^\wedge k \subset a_{n(k)}$. But then $y_{n(k)} \in Y_{s^\wedge k}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k \in \mathbb{N}}$ converges to a point $y_s \notin A$ by (5), as required.

Finally on the construction of sets Y_s.

Suppose that a Σ^1_1 set $\varnothing \neq Y_s \subseteq A'$ is defined. Then its closure $\overline{Y_s}$ is a Σ^1_1 set, too, therefore $\overline{Y_s} \not\subseteq A$ by Lemma 16. There is a sequence of pairwise different points $x_n \in Y_s$ which converges to a point $y_s \in \overline{Y_s} \smallsetminus A$. Let U_n be a neighbourhood of x_n (a Baire interval) of diameter less than $\frac{1}{3}$ of the least distance from x_n to the points x_k, $k \neq n$. Put $Y_{s^\wedge n} = Y_s \cap U_n$, and shrink the sets $Y_{s^\wedge n}$ so that they satisfy (2) and (4).

Case 2: $A' = \varnothing$, that is, $A = U$. This implies (I) of the theorem, exactly as in the proof of Theorem 1 above.

□ (Theorem 2)

7 A generalization of Theorem 1

Let $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ be an ideal on $\mathbb{N}$. A tree $T \subseteq \mathbb{N}^{<\omega}$ is:

$\mathscr{I}$-small, if for any $s \in T$ the set $\mathtt{Succ}_T(s) = \{n : s^\wedge n \in T\}$ belongs to $\mathscr{I}$;

$\mathscr{I}$-positive, if 1) it is perfect, and 2) if $s \in \mathbf{bran}(T)$ then the set $\mathtt{Succ}_T(s)$ does *not* belong to $\mathscr{I}$.

Accordingly, a set $X \subseteq \mathscr{N}$ is:

$\mathscr{I}$-small, if $\mathtt{tree}(X) = \{x \restriction n : n \in \mathbb{N} \wedge x \in X\}$ is an $\mathscr{I}$-small tree;

σ-$\mathscr{I}$-small, if it is a countable union of $\mathscr{I}$-small sets;

$\mathscr{I}$-positive, if it contains a subset of the form $[T]$, where $T \subseteq \mathbb{N}^{<\omega}$ is an $\mathscr{I}$-positive tree.

For instance, if $\mathscr{I} = \mathbf{Fin}$ is the Frechet ideal of all finite sets $x \subseteq \mathbb{N}$ then $\mathscr{I}$-small trees and sets are exactly compact trees, resp., sets, σ-$\mathscr{I}$-small sets are σ-compact sets, while $\mathscr{I}$-positive trees are perfect trees with infinite branchings. Moreover if T is such a **Fin**-positive tree then the set $[T]$ is closed and homeomorphic to $\mathscr{N}$, hence, non-σ-compact. Thus condition (II) of Theorem 1 can be reformulated as follows: *A is a* ***Fin****-positive set.*

Here we prove the following theorem (compare with Theorem 1).

Theorem 17. *Let $\mathscr{I}$ be a Π^1_1 ideal on $\mathbb{N}$. If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims holds*: $\longleftarrow$ nt

(I) *A is σ-$\mathscr{I}$-small*;

(II) *A is an $\mathscr{I}$-positive set.*

Condition (I) of this theorem is notably weaker than a true generalization of Theorem 1 would require: *A is covered by the union of all sets $[T]$, where $T \subseteq \mathbb{N}^{<\omega}$ is an $\mathscr{I}$-small Δ^1_1 tree.* Unfortunately such a stronger version is not accessible so far. The key element in the proof of Theorem 1, which allows to strengthen (I) from Σ^1_1 to Δ^1_1, is Lemma 15 based on Fact 9. We don't know whether the latter is true in the context of Theorem 17, *e.g.*, at least in the form: *any $\mathscr{I}$-small Σ^1_1 set is covered by a $\mathscr{I}$-small Δ^1_1 set.* It would be sufficient to assume that $\mathscr{I}$ satisfies the following: *if $p \in \mathscr{N}$ and $x \in \mathscr{I}$ is a $\Sigma^1_1(p)$ set then there is a $\Delta^1_1(p)$ set $y \in \mathscr{I}$ such that $x \subseteq y$.*

Proof. As covering of small Σ^1_1 sets by small Δ^1_1 sets is not available, we'll follow a line of arguments which differ from the proof of Theorem 1 above. First of all, $A = \mathtt{proj}\, P = \{x \in \mathscr{N} : \exists y\, P(x,y)\}$, where $P \subseteq \mathscr{N} \times \mathscr{N}$ is a Π^0_1 set. Consider the tree

$$S = \{\langle x \restriction n, y \restriction n\rangle : n \in \mathbb{N} \wedge \langle x,y\rangle \in P\} \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega},$$

so that $P = [S] = \{\langle x,y\rangle \in \mathscr{N}^2 : \forall n\, (\langle x \restriction n, y \restriction n\rangle \in S)\}$. If $u, v \in \mathbb{N}^{<\omega}$ then let $P_{uv} = \{\langle x,y\rangle \in P : u \subset x \wedge v \subset y\}$ and $A_{uv} = \mathtt{proj}\, P_{uv}$; thus, in particular, $P_{\Lambda\Lambda} = P$ and $A_{\Lambda\Lambda} = A$. If the subtree

$$S' = \{\langle u,v\rangle \in S : A_{uv} \text{ is not } \sigma\text{-}\mathscr{I}\text{-small}\}$$

of S is empty then $A = A_{\Lambda\Lambda}$ is σ-$\mathscr{I}$-small, getting (I) of the theorem. Therefore we assume that $S' \neq \varnothing$, and the goal is to get (II) of the theorem.

Note that $P_{uv} = \bigcup_{k,n} P_{u^\wedge k, v^\wedge n}$, and hence the tree S' has no maximal nodes: if $\langle u,v\rangle \in S'$ then $\langle u^\wedge k, v^\wedge n\rangle \in S'$ for some k, n. We consider the corresponding closed set

$$P' = [S'] = \{\langle x,y\rangle \in \mathscr{N}^2 : \forall n\, (\langle x \restriction n, y \restriction n\rangle \in S')\}$$

and the $\mathbf{\Sigma}^1_1$ set $A' = \mathtt{proj}\, P'$. If $\langle u, v\rangle \in S'$ then let

$$P'_{uv} = \{\langle x, y\rangle \in P' : u \subset x \wedge v \subset y\} \quad \text{and} \quad A'_{uv} = \mathtt{proj}\, P'_{uv}\,,$$

so that A'_{uv} is a non-empty $\mathbf{\Sigma}^1_1$ subset of A', not σ-$\mathscr{I}$-small by the definition of S'. The next lemma is quite obvious.

Lemma 18. *If $\langle u, v\rangle \in S'$, $u' \in \mathbb{N}^{<\omega}$, $u \subset u'$, and $A'_{uv} \cap \mathscr{N}_{u'} \neq \varnothing$ then there is a string $v' \in \mathbb{N}^{<\omega}$ such that $v \subset v'$ and $\langle u', v'\rangle \in S'$.* □ ← tt'

We are going to define a pruned tree $T \subseteq \mathbb{N}^{<\omega}$ and a string $v(t) \in \mathbb{N}^{<\omega}$ for all $t \in T$, such that

(1) if $t \in T$ then $\langle t, v(t)\rangle \in S'$;

(2) if $s, t \in T$ and $s \subseteq t$ then $v(s) \subseteq v(t)$;

(3) if $s \in T$ then there exists a string $t \in T$ such that $s \subset t$ and the set $\{k : t^\wedge k \in T\}$ does not belong to $\mathscr{I}$.

If such construction is accomplished then T is an $\mathscr{I}$-positive tree by (3), and on the other hand $[T] \subseteq A' \subseteq A$, so that (II) of the theorem holds.

Thus it remains to carry out the construction.

To begin with we define $\Lambda \in T$, of course, and let $v(\Lambda) = \Lambda$.

Suppose that $t \in T$, so that $\langle t, v(t)\rangle \in S'$ and the set $A'_{t,v(t)}$ is not σ-$\mathscr{I}$-small, in particular, not $\mathscr{I}$-small, hence the tree $\mathtt{tree}(A'_{t,v(t)})$ is not $\mathscr{I}$-small. We conclude that there is a string $s \in \mathbb{N}^{<\omega}$ such that $t \subseteq s$ and the set $K = \{k : \exists\, a \in A'_{t,v(t)}\, (s^\wedge k \subset a)\}$ does not belong to $\mathscr{I}$.

We let every string t' with $t \subset t' \subseteq s$ belong to T, and choose $v(t')$ for any such t' so that (1) and (2) hold, using Lemma 18. Then let every string $s^\wedge k$, $k \in K$, belong to T, and let $v(s^\wedge k) = v$, where v is any string such that $v(s) \subseteq v$ and $\langle s^\wedge k, v\rangle \in S'$. (The existence of at least one such string v follows from Lemma 18.)

□ (Theorem 17)

8 Theorem 2 for Σ^1_1 sets

There is a difference between Theorem 1 and Theorem 2: the first theorem deals with Σ^1_1 sets while the other one — with Δ^1_1 sets. We don't know whether Theorem 2 holds for all Σ^1_1 sets, but it is quite clear where the proof in Section 6 fails. Indeed if A is a Σ^1_1 set then A' turns out to be a set in $\mathbf{\Sigma}^1_1$ and Σ^1_2, but not Σ^1_1, so the rest of the proof just does not work. Nevertheless we can prove the following essentially weaker result.

Theorem 19. *If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims holds*: $\longleftarrow$ pt

(I) *there exist*: *a countable ordinal λ and an effectively defined sequence $\{T^\alpha\}_{\alpha<\lambda}$ of compact Δ^1_3 trees $T^\alpha \subseteq \mathbb{N}^{<\omega}$ such that $A = \bigcup_{\alpha<\lambda}[T^\alpha]$* — then A is σ-compact, of course;

(II) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and relatively closed in A.*

We'll not try to estimate the level and character of the effectivity condition in (I), since we don't think that our construction gives a result even close to optimal. But it will be quite clear from the construction that it is absolute for all transitive models containing the true ω_1, and lies within the projective hierarchy and probably within Δ^1_3. It is still an interesting **problem** to prove Theorem 2, as it stands, for Σ^1_1 sets.

Proof. By Theorem 1, we can w.l.o.g. assume that A is covered by a σ-compact set, and hence if $F \subseteq A$ is a closed set then F is σ-compact. Let $P \subseteq \mathscr{N} \times \mathscr{N}$ be a Π^0_1 set such that $A = \mathtt{proj}\, P$, and

$$S = \{\langle x \restriction n, y \restriction n\rangle : n \in \mathbb{N} \wedge \langle x, y\rangle \in P\} \subseteq \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega},$$

so that $P = [S]$. A decreasing sequence of derived trees $S^{(\alpha)}$, $\alpha \in \mathbf{Ord}$, is defined by induction so that $S^{(0)} = S$, if λ is limit then $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$, and for any α:

(A) we let $S^{(\alpha)}_*$ consist of all nodes $\langle u, v\rangle \in S^{(\alpha)}$ such that $\overline{A^{(\alpha)}_{uv}} \not\subseteq A$, where $A^{(\alpha)}_{uv} = \mathtt{proj}\, P^{(\alpha)}_{uv}$, $P^{(\alpha)}_{uv} = [S^{(\alpha)}_{uv}]$, and

$$S^{(\alpha)}_{uv} = \{\langle s, t\rangle \in S^{(\alpha)} : (u \subset s \wedge v \subset t) \vee (s \subseteq u \wedge t \subseteq v)\}\,;$$

(B) we let $S^{(\alpha+1)}$ be the *pruning* of $S^{(\alpha)}_*$, that is, $S^{(\alpha+1)}$ consists of all nodes $\langle u, v\rangle \in S^{(\alpha)}_*$ such that there is an infinite branch $\langle x, y\rangle \in [S^{(\alpha)}_*]$ satisfying $u \subset x$ and $v \subset y$.

Obviously there is a countable ordinal λ such that $S^{(\lambda+1)} = S^{(\lambda)}$.

Case 1: $S^{(\lambda)} = \varnothing$. Then, if $x \in A = \mathtt{proj}\, P$ then by construction there exist an ordinal $\alpha < \lambda$ and a node $\langle u, v\rangle \in S^{(\alpha)}$ such that

$$x \in A^{(\alpha)}_{uv} \subseteq \overline{A^{(\alpha)}_{uv}} \subseteq A\,,$$

and hence A is a countable union of sets $F \subseteq A$ of the form $\overline{A^{(\alpha)}_{uv}}$, where $\alpha < \lambda$ and $\langle u, v\rangle \in S^{(\alpha)}$, closed, therefore σ-compact by the above.

Let us show how this leads to (I) of the theorem.

First of all, quite obviously there is a certain Σ^1_2 formula $\varphi(\cdot,\cdot,\cdot)$ such that we have $S^{(\alpha+1)} = \{\langle u,v\rangle : \varphi(S^{(\alpha)},u,v)\}$ for all α. It follows by Shoenfield that the construction is absolute for every transitive model containing all countable ordinals, in particular, for $\mathbf{L}$, the class of Gödel constructible sets. Thus we can assume it from the beginning that *we argue in* $\mathbf{L}$.

Another consequence of the existence of φ is that both the ordinal λ and the sequence $\{\langle \alpha, S^{(\alpha)}\rangle : \alpha < \lambda\}$ are Δ^1_3. It follows (here we use the assumption that we argue in $\mathbf{L}$) that each ordinal $\alpha < \lambda$ is Δ^1_3 and each tree $S^{(\alpha)}$, $\alpha < \lambda$, is Δ^1_3 either, as well as all subtrees of the form $S^{(\alpha)}_{uv}$ (where $\langle u,v\rangle \in S^{(\alpha)}$) and their "projections" $T^{(\alpha)}_{uv} = \{u : \exists\, v\,(\langle u,v\rangle \in S^{(\alpha)}_{uv})\} \subseteq \mathbb{N}^{<\omega}$.

On the other hand, $\overline{A^{(\alpha)}_{uv}} = [T^{(\alpha)}_{uv}]$ holds by construction.

To conclude, if $x \in A$ then there is a pruned Δ^1_3 tree $T \subseteq \mathbb{N}^{<\omega}$ (of the form $T^{(\alpha)}_{uv}$) such that $x \in [T] \subseteq A$ — and $[T]$ is σ-compact in this case.

It remains to note that if $T \subseteq \mathbb{N}^{<\omega}$ is a pruned Δ^1_3 tree and the set $[T]$ is σ-compact then by Theorem 2 (relativized version) there is a sequence of compact $\Delta^1_1(T)$ trees T_n such that $[T] = \bigcup_n [T_n]$. But each T_n then is Δ^1_3 as so is T itself.

Case 2: $S^{(\lambda)} \neq \varnothing$, and then $S^{(\lambda)} \subseteq S$ is a pruned tree.

Lemma 20. *If* $\langle u,v\rangle \in S^{(\lambda)}$, $u' \in \mathbb{N}^{<\omega}$, $u \subset u'$, *and* $A^{(\lambda)}_{uv} \cap \mathscr{N}_{u'} \neq \varnothing$ *then there is a string* $v' \in \mathbb{N}^{<\omega}$ *such that* $v \subset v'$ *and* $\langle u',v'\rangle \in S^{(\lambda)}$. □ ⟵ tt”

We'll define a pair $\langle u(t), v(t)\rangle \in S^{(\lambda)}$ for each $t \in \mathbb{N}^{<\omega}$, such that

(1) if $t \in \mathbb{N}^{<\omega}$ then $t \subseteq u(t)$;

(2) if $s,t \in \mathbb{N}^{<\omega}$ and $s \subseteq t$ then $u(s) \subseteq u(t)$ and $v(s) \subseteq v(t)$;

(3) if $t \in \mathbb{N}^{<\omega}$ and $k \neq n$ then $u(t^\wedge k)$ and $u(t^\wedge n)$ are $\subseteq$-incomparable;

(4) if $s \in \mathbb{N}^{<\omega}$ then there exists a point $y_s \in \overline{A^{(\lambda)}_{u(s)v(s)}} \smallsetminus A$ such that any sequence of points $x_k \in A^{(\lambda)}_{u(s^\wedge k)v(s^\wedge k)}$ converges to y_s.

Suppose that such a system of sets is defined. Then the associated map $f(a) = \bigcup_n u(a \restriction n) : \mathscr{N} \to A$ is $1-1$ and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A$.

Let's prove that Y is relatively closed in A. Consider a sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of points $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence convergent to $b \in \mathscr{N}$ then

$\{y_n\}$ converges to $f(b) \in Y$. So suppose that the sequence $\{a_n\}$ has no convergent subsequences. Then there exist a string $s \in \mathbb{N}^{<\omega}$, an infinite set $K \subseteq \mathbb{N}$, and for each $k \in K$ — a number $n(k)$, such that $s^\wedge k \subset a_{n(k)}$. Then $y_{n(k)} \in A^{(\lambda)}_{u(s^\wedge k)v(s^\wedge k)}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k\in\mathbb{N}}$ converges to a point $y_s \notin A$ by (4), as required.

Finally on the construction of sets Y_s.

Suppose that a pair $\langle u(t), v(t)\rangle \in S^{(\lambda)}$ is defined. Then $\overline{A^{(\lambda)}_{u(t)v(t)}} \not\subseteq A$ by the choice of λ. There is a sequence of pairwise different points $x_n \in A^{(\lambda)}_{u(t)v(t)}$ which converges to a point $y_s \in \overline{A^{(\lambda)}_{u(t)v(t)}} \smallsetminus A$. We can associate a string $u_n \in \mathbb{N}^{<\omega}$ with each x_n such that $u(t) \subset u_n \subset x_n$, the strings u_n are pairwise $\subseteq$-incompatible, and $\mathtt{lh}\, u_n \to \infty$. Then, by Lemma 20, for each n there is a matching string v_n such that $v(t) \subset v_n$ and $\langle u_n, v_n\rangle \in S^{(\lambda)}$. Put $u(t^\wedge n) = u_n$ and $v(t^\wedge n) = v_n$ for all n.

□ (Theorem 19)

9 Remarks

The main results of this note can be compared with the following theorems of classical descriptive set theory.

Theorem 21 (Hurewicz [4])**.** *If a $\mathbf{\Sigma}^1_1$ set A in a Polish space $\mathbb{X}$ is not σ-compact then there is a subset $Y \subseteq A$ homeomorphic to the Baire space $\mathscr{N}$ and relatively closed in A.* □ ←hur

Theorem 22 (Saint Raymond [11], see also 21.23 in [9])**.** *If a $\mathbf{\Sigma}^1_1$ set A in a Polish space $\mathbb{X}$ cannot be covered by a σ-compact set $Z \subseteq \mathbb{X}$ then there is a set $P \subseteq A$, homeomorphic to $\mathscr{N}$ and closed in $\mathbb{X}$.* □ ←hur2

Arguments in [9] show that it's sufficient to prove either of these theorems in the case $\mathbb{X} = \mathscr{N}$; then the results can be generalized to an arbitrary Polish space $\mathbb{X}$ by purely topological methods. In the case $\mathbb{X} = \mathscr{N}$, Theorem 22 immediately follows from our Theorem 1 (in relativized form, *i.e.*, for classes $\Sigma^1_1(p)$, where $p \in \mathscr{N}$ is arbitrary), while Theorem 21 follows from Theorem 19 (relativized). On the other hand, Theorem 21 also follows from Theorem 2 (relativized) for sets A in $\mathbf{\Delta}^1_1$ (that is, Borel sets).

Theorem 2 implies yet another theorem, which combines several classical results of descriptive set theory by Arsenin, Kunugui, Saint Raymond, Shegolkov, see references in [9] or in [5, § 4].

Theorem 23 (compare with Fact 8)**.** *Suppose that $\mathbb{X}, \mathbb{Y}$ are Polish spaces,* ←tks

$P \subseteq \mathbb{X} \times \mathbb{Y}$ *is a* $\mathbf{\Delta}^1_1$ *set, and all cross-sections* $(P)_x = \{y : \langle x, y\rangle \in P\}$ $(x \in \mathbb{X})$ *are* σ*-compact. Then*

(i) *the projection* $\mathtt{proj}\, P$ *is a* $\mathbf{\Delta}^1_1$ *set*;

(ii) P *is a countable union of* $\mathbf{\Delta}^1_1$ *sets with compact cross-sections*;

(iii) P *can be uniformized by a* $\mathbf{\Delta}^1_1$ *set.*

Proof (a sketch for the case $\mathbb{X} = \mathbb{Y} = \mathscr{N}$)**.** (i) Assume, for the sake of simplicity, that $P \subseteq \mathscr{N} \times \mathscr{N}$ is a $\varDelta^1_1$ set. The set

$$H = \{\langle x, T\rangle : x \in \mathscr{N} \wedge T \in \mathbf{CT} \wedge T \in \varDelta^1_1(x) \wedge [T] \subseteq (P)_x\}$$

is $\varPi^1_1$ by Fact 6. It follows from Theorem 2 that if $\langle x, y\rangle \in P$ then there is a tree T such that $\langle x, T\rangle \in H$ and $y \in [T]$. Therefore the $\varPi^1_1$ set

$$E = \{\langle x, y, T\rangle : \langle x, y\rangle \in P \wedge \langle x, T\rangle \in H \wedge y \in [T]\} \subseteq \mathscr{N} \times \mathscr{N} \times 2^{(\mathbb{N}^{<\omega})}$$

satisfies $\mathtt{proj}_{xy}\, E = P$, that is, if $\langle x, y\rangle \in P$ then there is a tree T such that $\langle x, y, T\rangle \in E$. There is a uniform $\varPi^1_1$ set $U \subseteq E$ which *uniformizes* E, *i.e.*, if $\langle x, y\rangle \in P$ then there is a unique T such that $\langle x, y, T\rangle \in U$. Yet U is $\varSigma^1_1$ as well by Fact 5, since $\langle x, y, T\rangle \in U$ is equivalent to:

$$\langle x, y\rangle \in P \wedge y \in [T] \wedge \forall T' \in \varDelta^1_1(x)\, (\langle x, y, T'\rangle \in U \implies T = T')\,.$$

Thus the $\varSigma^1_1$ set $F = \{\langle x, T\rangle : \exists\, y\, (\langle x, y, T\rangle \in U)\}$ is a subset of the $\varPi^1_1$ set H. Fact 3 implies that there is a $\varDelta^1_1$ set V such that $F \subseteq V \subseteq H$. Then

$$\langle x, y\rangle \in P \iff \exists\, T\, (\langle x, T\rangle \in V \wedge y \in [T])$$

by definition. Finally all cross-sections of V are at most countable: indeed if $\langle x, T\rangle \in V$ then $T \in \varDelta^1_1(x)$ (since $V \subseteq H$). Note that $\mathtt{proj}\, P = \mathtt{proj}\, V$, and hence the projection $D = \mathtt{proj}\, P$ is $\varDelta^1_1$ (hence Borel) by Fact 8.

(ii) It follows from Fact 8 that V is equal to a union $V = \bigcup_n V_n$ of uniform $\varDelta^1_1$ sets V_n, and then each projection $D_n = \mathtt{proj}\, V_n \subseteq D$ is $\varDelta^1_1$. Each V_n is basically the graph of a $\varDelta^1_1$ map $\tau_n : D_n \to \mathbf{CT}$, and $(P)_x = \bigcup_{x \in D_n} [\tau_n(x)]$. If $n \in \mathbb{N}$ then we put

$$P_n = \{\langle x, y\rangle : x \in D_n \wedge y \in [\tau_n(x)]\}\,.$$

Then $P = \bigcup_n P_n$ by the above, each set P_n has only compact cross-sections, and each P_n is a $\varDelta^1_1$ set, since the sets D_n and maps τ_n belong to $\varDelta^1_1$.

(iii) Still by Fact 8, the set V can be uniformized by a uniform $\varDelta^1_1$ set, that is, there exists a $\varDelta^1_1$ map $\tau : D \to \mathbf{CT}$ such that $\langle x, \tau(x)\rangle \in V$ for all $x \in D$. To uniformize the original set P, let Q consist of all pairs $\langle x, y\rangle \in P$ such that y is the lexicographically leftmost point in the compact set $[\tau(x)]$. Clearly Q uniformizes P. To check that Q is $\varDelta^1_1$, note that "y is a the lexicographically leftmost point in $[T]$" is an arithmetic relation in the assumption that $T \in \mathbf{CT}$. □

Similar arguments, this time based on Theorem 1, also lead to an alternative proof of the following known result.

Theorem 24 (Burgess, Hillard, 35.43 in [9])**.** *If P is a $\mathbf{\Sigma}^1_1$ set in the product $\mathbb{X} \times \mathbb{Y}$ of two Polish spaces $\mathbb{X}$, and every section $(P)_x$ is covered by a σ-compact set, then there is a sequence of Borel sets $P_n \subseteq \mathbb{X} \times \mathbb{Y}$ with compact sections $(P_n)_x$ such that $P \subseteq \bigcup_n P_n$.* □

But at the moment it seems that no conclusive theory of $\mathbf{\Sigma}^1_1$ sets with σ-compact sections (as opposed to those with sections covered by σ-compact sets) is known. For instance what about effective decompositions of such sets into countable unions of definable sets with compact sections? Our Theorem 19 can be used to show that such a decomposition is possible, but the decomposing sets with compact sections appear to be excessively complicated (3rd projective level by rough estimation). It is an interesting **problem** to improve this result to something more reasonable like Borel combinations of $\mathbf{\Sigma}^1_1$ sets.

On the other hand, it is known from [13, 14] that $\mathbf{\Sigma}^1_1$ sets with σ-compact sections are not necessarily decomposable into countably many $\mathbf{\Sigma}^1_1$ sets with compact sections.

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