On effective σ-boundedness and σ-compactness

Vladimir Kanovei*

September 19, 2018

Abstract

Different generalizations of a known theorem by Kechris, saying that any Σ^1_1 set A of the Baire space either is effectively sigma-bounded (that is, covered by a countable union of compact Δ^1_1 sets), or it contains a superperfect subset, are obtained, in particular, 1) with covering by compact sets and equivalence classes of a given finite collection of Δ^1_1 equivalence relations, 2) generalizations to Σ^1_2 sets, 3) generalizations true in the Solovay model.

A generalization to Σ^1_1 sets A, of a theorem by Louveau, saying that any Δ^1_1 set A of the Baire space either is effectively σ-compact (that is, is equal to a countable union of compact Δ^1_1 sets), or it contains a relatively closed superperfect subset, is obtained as well.

*Contact author, kanovei@mccme.ru

Introduction

Effective descriptive set theory appeared in the 1950s as a useful technique of simplification and clarification of constructions of classical descriptive set theory (see *e.g.* [1], [23], or [12]). Yet it has become clear that development of effective descriptive set theory also leads to results having no direct analogies in classical descriptive set theory. As an example we recall the following well-known *basis theorem*: any countable $\varDelta^1_1$ set A of the Baire space $\mathscr{N} = \omega^\omega$ consists of $\varDelta^1_1$ points. Its remote predecessor in classical descriptive set theory is the Luzin – Novikov theorem on splitting of Borel sets with countable cross-sections into countable unions of uniform Borel sets.

In this paper, we focus on effectivity aspects of the properties of σ-compactness and σ-boundedness of pointsets. Our starting point will be a pair of classical dichotomy theorems on pointsets, together with their effective versions obtained in the end of 1970s.

The first of them deals with the property of σ-boundedness. Recall that a pointset is σ*-bounded* iff it is a subset of a σ-compact set.[1] Saint Raymond [21] proved that if X is a $\mathbf{\Sigma}^1_1$ set then one and only one of the following two (obviously incompatible) conditions holds:

(I) the set X is σ-bounded;

(II) there is a superperfect set $Y \subseteq X$.

Recall that a *superperfect* set is a closed set homeomorphic to $\mathscr{N}$.

An effective version of this result (Theorem 4.1 below), by Kechris [15], says that if X is a $\varSigma^1_1$ set then condition (I) can be strengthened to a $\varDelta^1_1$-effective σ-boundedness (so that a given set X is covered by a $\varDelta^1_1$ sequence of compact sets). The proof in [15] uses the determinacy-style technique. A different proof of this result, based rather on methods of effective descriptive set theory, will be presented in Section 4, in particular, as a foundation for a more general dichotomy theorem in Section 11.

The other background result, an immediate concequence of a theorem by Hurewicz [7], deals with the property of σ-compactness instead of σ-boundedness. It says that if X is a $\mathbf{\Sigma}^1_1$ set then one and only one of the following two (clearly incompatible as well) conditions (I), (II) holds:

(I) the set X is σ-compact;

(II) there is a set $Y \subseteq X$ homeomorphic to $\mathscr{N}$ and relatively closed in X.

[1] For subsets of the Baire space $\mathscr{N} = \omega^\omega$, the property of σ-boundedness is equivalent to being bounded in $\mathscr{N}$ with the *eventual domination* order, while the compactness is equivalent to being bounded in $\mathscr{N}$ with the *termwise domination* order.

An effective version of this theorem (Theorem 5.1 below), essentially by Louveau [17] (see also 4F.18 in [20] which the author of [20] credits to Louveau), shows that if X is a lightface $\varDelta^1_1$ set then condition (I) can be strengthened to a $\varDelta^1_1$-effective σ-compactness (so that a given set is equal to the union of a $\varDelta^1_1$ sequence of compact sets). We present here a somewhat different proof of this result in Section 5, in particular, as a base for the proof of a similar but more complicated dichotomy theorem on $\varSigma^1_1$ sets in Section 6.

Some well-known classical results related to Theorems 4.1 and 5.1 are discussed in Section 7. We outline several counterexamples with sets more complicated than $\varSigma^1_1$ in Section 8.

Sections 9, 10, 11 contain a generalization of Theorem 4.1 (Theorem 11.1) which replaces σ-bounded sets by $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded sets, where F_1 $,\dots,\mathsf{F}_n$ are given $\varDelta^1_1$ equivalence relations and being $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded means being covered by the union of a σ-bounded set and countably many equivalence classes of $\mathsf{F}_1,\dots,\mathsf{F}_n$. Accordingly the condition of existence of a superperfect set strengthens by the requirement that the superperfect set is pairwise F_i-inequivalent for $i=1,\dots,n$. Section 9 develops a necessary technique while the proof of the generalized dichotomy is presented in Section 11. In the classical form, the case of a single equivalence relation F in this dichotomy was earlier obtained by Zapletal, see [14].

In parallel to this, we prove in Section 10 that a σ-bounded set and a countable union of equivalence classes as above can be defined so that they depend only on a given set X (and the collection of equivalence relations F_j), but are independent of the choice of a parameter p such that X is $\varSigma^1_1(p)$ and the relations are $\varDelta^1_1(p)$.

In the remaining parts of the paper, we prove, in Sections 12, 13, 14, a generalization, along the same lines, of another Kechris' result of [15], related to $\varSigma^1_2$ sets, which by necessity involves uncountable unions of equivalence classes and σ-bounded sets as well as coding by uncountable constructible Borel codes. In the course of the proof of this generalized theorem (Theorem 13.1), it will be shown (Theorem 14.1) that if a countable union of equivalence classes of a $\varDelta^1_1$ equivalence relation is $\varDelta^1_1(\xi)$, where $\xi<\omega_1$, then all classes in this union admit Borel coding by constructible (not necessarily countable) codes.

In the final Sections 15, 16 we present generalizations of some of the abovementioned theorems to ordinal definable pointsets in the Solovay model. Some questions here remain open.

We thank Alekos Kechris, Ben Miller, Marcin Sabok, and Jindra Zapletal for valuable remarks and suggestions.

1 Preliminaries

We use standard notation Σ^1_1, Π^1_1, Δ^1_1 for effective classes of points and pointsets in $\mathscr{N}$, as well as $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$ for corresponding projective classes.

Let $\omega^{<\omega}$ be the set of all finite strings of natural numbers, including the empty string Λ. If $u, v \in \omega^{<\omega}$ then $\mathtt{lh}\, u$ is the *length* of u, and $u \subset v$ means that v is a *proper extension* of u. If $u \in \omega^{<\omega}$ and $n \in \omega$ then $u^\wedge n$ is the string obtained by adding n to u as the rightmost term. Let, for $u \in \omega^{<\omega}$,

$$\mathscr{N}_u = \{x \in \mathscr{N} : u \subset x\} \quad (\text{a \textit{Baire interval} in } \mathscr{N}) .$$

If a set $X \subseteq \mathscr{N}$ contains at least two elements then there is a longest string $u = \mathtt{stem}(X)$ such that $X \subseteq \mathscr{N}_u$. We put $\mathtt{diam}(X) = \frac{1}{1+\mathtt{stem}(X)}$ in this case, and additionally $\mathtt{diam}(X) = 0$ whenever X has at most one element.

A set $T \subseteq \omega^{<\omega}$ is a *tree* if $u \in T$ holds whenever $u^\wedge n \in T$ for at least one n, and a *pruned* tree iff $u \in T$ implies $u^\wedge n \in T$ for at least one n. Any non-empty tree contains Λ. A string $u \in T$ is a *branching point* of T if there are $k \neq n$ such that $u^\wedge k \in T$ and $u^\wedge n \in T$; let $\mathbf{bran}(T)$ be the set of all branching points of T. The *branching height* $\mathbf{BH}_T(u)$ of a string $u \in T$ in a tree T is equal to the number of strings $v \in \mathbf{bran}(T)$, $v \subset u$. For instance, if $T = \omega^{<\omega}$ then $\mathbf{BH}_{\omega^{<\omega}}(u) = \mathtt{lh}\, u$ for any string u.

A tree $T \subseteq \omega^{<\omega}$ is *compact*, if it is pruned and has *finite branchings*, that is, if $u \in \mathbf{bran}(T)$ then $u^\wedge n \in T$ holds for finitely many n. Then

$$[T] = \{x \in \mathscr{N} : \forall\, m\, (x \restriction m \in T)\},$$

the *body* of T, is a compact set. Conversely, if $X \subseteq \mathscr{N}$ is compact then

$$\mathtt{tree}(X) = \{x \restriction n : x \in X \wedge n \in \omega\}$$

is a compact tree. Let $\mathbf{CT}$ be the Δ^1_1 set of all non-empty compact trees.

A pruned tree $T \subseteq \omega^{<\omega}$ is *perfect*, if for each $u \in T$ there is a string $v \in \mathbf{bran}(T)$ with $u \subset v$. Then $[T]$ is a perfect set. A perfect tree T is *superperfect*, if for each $u \in \mathbf{bran}(T)$ there are infinitely many numbers n such that $u^\wedge n \in T$. Then $[T]$ is a superperfect set. Conversely, if $X \subseteq \mathscr{N}$ is a perfect set then $\mathtt{tree}(X)$ is a perfect tree, while for any superperfect set $X \subseteq \mathscr{N}$ there is a superperfect tree $T \subseteq \mathtt{tree}(X)$.

If $\mathbb{X}, \mathbb{Y}$ are any sets and $P \subseteq \mathbb{X} \times \mathbb{Y}$ then

$$\mathtt{proj}\, P = \{x \in \mathbb{X} : \exists\, y\, (\langle x, y\rangle \in P)\} \quad \text{and} \quad (P)_x = \{y \in \mathbb{Y} : \langle x, y\rangle \in P\}$$

are, resp., the *projection* of P to $\mathbb{X}$, and the *cross-section* of P defined by $x \in \mathbb{X}$. A set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is *uniform* if every cross-section $(P)_x$ $(x \in \mathbb{X})$ contains at most one element.

2 Some basic facts

We'll make use of several known results of effective descriptive set theory. They are listed below, with a few proofs (of claims which are not in common use in this area) attached to make the text self-contained.

Definition 2.1. A *product space* is any finite product of factors $\omega, \omega^{<\omega}$, $\mathscr{N}$, $\mathscr{P}(\omega^{<\omega})$. A *discrete* product space is a finite product of $\omega, \omega^{<\omega}$. $\square$

Fact 2.2 (Kreisel selection, 4B.5 in [20]). *If $\mathbb{X}$ is a discrete product space, $P \subseteq \mathscr{N} \times \mathbb{X}$ is a Π^1_1 set, and $A \subseteq \mathtt{proj}\, P$ is a Σ^1_1 set, then there is a Δ^1_1 map $f : \mathscr{N} \to \mathbb{X}$ such that $\langle x, f(x)\rangle \in P$ for all $x \in A$.* $\square$

Fact 2.3 (4D.3 in [20]). *If $P(x, y, z, \dots)$ is a Π^1_1 relation on a product space then the following derived relations are Π^1_1, too*:

$$\exists\, x \in \Delta^1_1\, P(x, y, z, \dots) \quad \textit{and} \quad \exists\, x \in \Delta^1_1(y)\, P(x, y, z, \dots)\,. \qquad \square$$

Fact 2.4 (4D.14 in [20]). *If $\mathbb{X}$ is a product space then the following two sets are Π^1_1 : $D = \{x \in \mathbb{X} : x \text{ is } \Delta^1_1\}$ and*

$$\{\langle p, x\rangle : p \in \mathscr{N} \wedge x \in \mathbb{X} \wedge x \text{ is } \Delta^1_1(p)\}. \qquad \square$$

For instance, $x \in D \iff \exists\, y \in \Delta^1_1\, (x = y)$; then apply Fact 2.3.

Fact 2.5 (enumeration of Δ^1_1, 4D.2 in [20]). *Let $\mathbb{X}$ be a product space. There exist Π^1_1 sets $E \subseteq \omega$ and $W \subseteq \omega \times \mathbb{X}$, and a Σ^1_1 set $W' \subseteq \omega \times \mathbb{X}$ such that*

(i) *if $e \in E$ then $(W)_e = (W')_e$* (where $(W)_e = \{x \in \mathbb{X} : \langle e, x\rangle \in W\}$);

(ii) *a set $X \subseteq \mathbb{X}$ is Δ^1_1 iff there is $e \in E$ such that $X = (W)_e$.* $\square$

There is a useful uniform version of Fact 2.5.

Fact 2.6. *Let $\mathbb{X}$ be a product space. There exist Π^1_1 sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W} \subseteq \mathscr{N} \times \omega \times \mathbb{X}$, and a Σ^1_1 set $\mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathbb{X}$ such that*

(i) *if $\langle p, e\rangle \in \mathbf{E}$ then $(\mathbf{W})_{pe} = (\mathbf{W}')_{pe}$* (where, as above, $(\mathbf{W})_{pe} = \{x \in \mathbb{X} : \langle p, e, x\rangle \in \mathbf{W}\}$);

(ii) *if $p \in \mathscr{N}$ then a set $X \subseteq \mathbb{X}$ is $\Delta^1_1(p)$ iff there is a number $e \in E$ such that $T = (\mathbf{W})_{pe} = (\mathbf{W}')_{pe}$.* $\square$

This result implies the following stronger version of Fact 2.2.

Fact 2.7 (4D.6 in [20]). *Suppose that* $\mathbb{X}$ *is a product space,* $Q \subseteq \mathscr{N} \times \mathbb{X}$ *is* Π^1_1*,* $A \subseteq \mathtt{proj}\, Q$ *is* Σ^1_1*, and for each* $a \in A$ *there is a point* $x \in \Delta^1_1(a)$ *such that* $\langle a, x\rangle \in Q$*. Then there is a* Δ^1_1 *map* $f : \mathscr{N} \to \mathbb{X}$ *such that* $\langle a, f(a)\rangle \in Q$ *for all* $a \in A$*.*

Proof. Assume that $\mathbb{X} = \mathscr{N}$, for the sake of brevity. Then any $x \in \mathbb{X}$ satisfies $x \subseteq \mathbb{Y} = \omega \times \omega$. Making use of the sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W}, \mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathbb{Y}$ as in Fact 2.6, we let

$$P = \{\langle a, e\rangle \in \mathbf{E} : (\mathbf{W})_{ae} \in \mathscr{N} \wedge \langle a, (\mathbf{W})_{ae}\rangle \in Q\}.$$

Easily the set P and its projection $\mathtt{proj}\, P$ both are Π^1_1, and $A \subseteq \mathtt{proj}\, P$. By Fact 2.2, there is a Δ^1_1 map $f : \mathscr{N} \to \omega$ such that $\langle a, f(a)\rangle \in P$ for all $a \in A$. It remains to define $f(a) = (\mathbf{W})_{a,f(a)}$ for $a \in A$; to prove that f is Δ^1_1 use both sets $\mathbf{W}$ and $\mathbf{W}'$. $\square$

Fact 2.8 (4F.17 in [20]). *Let* $\mathbb{X}, \mathbb{Y}$ *be product spaces,* $P \subseteq \mathbb{X} \times \mathbb{Y}$ *be a* Δ^1_1 *set, and every cross-section* $(P)_x$ $(x \in \mathbb{X})$ *be at most countable. Then*

(i) $X = \mathtt{proj}\, P$ *is a* Δ^1_1 *set,*

(ii) *there is a* Δ^1_1 *set* $Q \subseteq \omega \times \mathbb{X} \times \mathbb{Y}$ *such that if* $n < \omega$ *then the set* $Q_n = \{\langle x, y\rangle : \langle n, x, y\rangle \in Q\}$ *is a uniform subset of* P*,* $\mathtt{proj}\, Q_n = X$*, and* $P = \bigcup_n Q_n$*, and hence*

(iii) P *is a countable union of* Δ^1_1 *sets each of which uniformizes* P*.* $\square$

Fact 2.9 (a corollary of 2.8). *If* $X \neq \varnothing$ *is a countable* Δ^1_1 *set then there is a* Δ^1_1 *map defined on* ω *such that* $X = \{f(n) : n < \omega\}$*.* $\square$

Fact 2.10 (4F.14 in [20]). *If* $F \subseteq \mathscr{N}$ *is a closed* Δ^1_1 *set and* $X \subseteq F$ *is a compact* Σ^1_1 *set then there is a compact* Δ^1_1 *tree* $T \subseteq \omega^{<\omega}$ *such that* $X \subseteq [T] \subseteq F$*. In particular, in the case* $X = F$*, any compact* Δ^1_1 *set* $X \subseteq \mathscr{N}$ *has the form* $X = [T]$ *for some compact* Δ^1_1 *tree* $T \subseteq \omega^{<\omega}$*.* $\square$

Facts 2.2, 2.3, 2.4 (the first set), 2.5, 2.7, 2.8, 2.10 remain true for relativized lightface classes $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$, where $p \in \mathscr{N}$ is an arbitrary fixed parameter. Therefore Facts 2.2, 2.8 also hold with lightface classes replaced by boldface projective classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_1$.

3 The Gandy – Harrington topology

The *Gandy – Harrington topology* on the Baire space $\mathscr{N}$ consists of all unions of Σ^1_1 sets $S \subseteq \mathscr{N}$. This topology includes the Polish topology on $\mathscr{N}$ but is not Polish. Nevertheless the Gandy – Harrington topology satisfies a condition typical for Polish spaces.

Definition 3.1. Let $\mathscr{F}$ be any family of sets, *e.g.* sets in a given background space $\mathbb{X}$. A set $\mathscr{D} \subseteq \mathscr{F}$ is *open dense* iff $\forall F \in \mathscr{F} \, \exists D \in \mathscr{D} \, (D \subseteq F)$, and

$$\forall F \in \mathscr{F} \, \forall D \in \mathscr{D} \, (F \subseteq D \implies F \in \mathscr{D}) .$$

Sets $\mathscr{D}$ satisfying only the first requirement are called *dense*. If $\mathscr{D} \subseteq \mathscr{F}$ is dense then the set $\mathscr{D}' = \{F \in \mathscr{F} : \exists D \in \mathscr{D} \, (F \subseteq D)\}$ is open dense. The notions of *open* and *dense* are related to a certain topology which we'll not discuss, but not necessarily with the topology of the background space $\mathbb{X}$.

A *Polish net* for $\mathscr{F}$ is any collection $\{\mathscr{D}_n : n \in \omega\}$ of open dense sets $\mathscr{D}_n \subseteq \mathscr{F}$ such that we have $\bigcap_n F_n \neq \varnothing$ for every sequence of sets $F_n \in \mathscr{D}_n$ satisfying the finite intersection property (*i.e.* $\bigcap_{k \le n} F_k \neq \varnothing$ for all n). $\square$

For instance the family of all non-empty closed sets of a complete metric space $\mathbb{X}$ admits a Polish net: let $\mathscr{D}_n$ contain all closed sets of diameter $\le n^{-1}$ in $\mathbb{X}$. The next theorem is less elementary. This theorem and the following corollary are well-known, see *e.g.* [5, 6, 10, 13].

Theorem 3.2. *The collection $\mathbb{P}$ of all non-empty Σ^1_1 sets in $\mathscr{N}$ admits a Polish net.* $\square$

4 Effective σ-boundedness dichotomy for Σ^1_1 sets

Here we present a proof of the following theorem by methods of effective descriptive set theory (including the Gandy – Harrington topology). The original proof in [15] was based rather on determinacy ideals.

Theorem 4.1 (Kechris [15], p. 198)**.** *If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims* (I), (II) *holds*:

(I) *A is Δ^1_1-effectively σ-bounded, so that there is a Δ^1_1 sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ such that $A \subseteq \bigcup_n [T_n]$;*

(II) *there is a superperfect set $Y \subseteq A$.*

Corollary 4.2. *If $A \subseteq \mathscr{N}$ is a σ-bounded Σ^1_1 set then it is Δ^1_1-effectively σ-bounded in the sense of* (I) *of Theorem 4.1.* $\square$

Proof (theorem)**.** Recall that $\mathbf{CT}$ is the set of all compact trees $\varnothing \neq T \subseteq \omega^{<\omega}$; $\mathbf{CT}$ is Δ^1_1, of course. Let U be the union of all sets of the form $[T]$, where $T \subseteq \omega^{<\omega}$ is a compact tree. Formally,

$$x \in U \iff \exists T \in \Delta^1_1 \, (T \in \mathbf{CT} \wedge x \in [T]) ,$$

and hence U is Π^1_1 by Fact 2.3, and the difference $A' = A \smallsetminus U$ is a Σ^1_1 set.

Lemma 4.3. *Under the conditions of Theorem 4.1, if $Y \subseteq A'$ is a non-empty Σ^1_1 set then its topological closure $\overline{Y}$ in $\mathscr{N}$ is not compact, i.e., the tree $\mathtt{tree}(Y) = \{y \restriction n : y \in Y \wedge n \in \omega\}$ has at least one infinite branching.*

Proof. Suppose otherwise: $\overline{Y}$ is compact. Then by Fact 2.10 (with $F = \mathscr{N}$) there is a compact Δ^1_1 tree T such that $\overline{Y} \subseteq [T]$. Therefore $Y \subseteq \overline{Y} \subseteq [T] \subseteq U$, and this contradicts to the assumption $\varnothing \neq Y \subseteq A'$. □ (Lemma)

Case 1: $A' = \varnothing$, that is, $A \subseteq U$. To prove (I) of Theorem 4.1, note that

$$Q = \{\langle x, T\rangle : x \in \mathscr{N} \wedge T \in \mathbf{CT} \cap \Delta^1_1 \wedge x \in [T]\}$$

is a Π^1_1 set by Facts 2.3 and 2.4, and obviously $U = \mathtt{proj}\, Q$. By Σ^1_1 separation there is a Δ^1_1 set X such that $A \subseteq X \subseteq U$. Then

$$P = \{\langle x, T\rangle \in Q : x \in X\}$$

is still a Π^1_1 set, and $\mathtt{proj}\, P = X$ is a Δ^1_1 set. Therefore by Fact 2.7 there is a Δ^1_1 function $\tau : X \to \mathbf{CT}$ such that $\langle x, \tau(x)\rangle \in Q$ for all $x \in X$.

Note that $\tau(x) \in \mathbf{CT} \cap \Delta^1_1$ and $x \in [\tau(x)]$ for all $x \in A$ by the construction. Thus the full image $R = \{\tau(x) : x \in A\}$ is a Σ^1_1 subset of the Π^1_1 set $\mathbf{CT} \cap \Delta^1_1$, and hence there is a Δ^1_1 set D such that $R \subseteq D \subseteq \mathbf{CT} \cap \Delta^1_1$. By Fact 2.8, there is a Δ^1_1 map $\delta : \omega \xrightarrow{\text{onto}} D$. Now put $T_n = \delta(n)$ for all n, getting (I) of Theorem 4.1.

Case 2: $A' = A \smallsetminus U \neq \varnothing$. To prove that (II) of Theorem 4.1 holds, we'll define a system of Σ^1_1 sets $\varnothing \neq Y_u \subseteq A'$ satisfying the following conditions:

(1) if $u \in \omega^{<\omega}$ and $i \in \omega$ then $Y_{u^\wedge i} \subseteq Y_u$;

(2) $\mathtt{diam}(Y_u) \leq 2^{-\mathtt{lh}\, u}$;

(3) if $u \in \omega^{<\omega}$ and $k \neq n$ then $Y_{u^\wedge k} \cap Y_{u^\wedge n} = \varnothing$, and moreover, sets $Y_{u^\wedge k}$ are covered by pairwise disjoint (clopen) Baire intervals $J_{u^\wedge k}$;

(4) $Y_s \in \mathscr{D}_{\mathtt{lh}\, u}$, where by Theorem 3.2 $\{\mathscr{D}_n : n \in \omega\}$ is a fixed Polish net for the family $\mathbb{P}$ of all non-empty Σ^1_1 sets $Y \subseteq \mathscr{N}$;

(5) if $u \in \omega^{<\omega}$ and $x_k \in Y_{u^\wedge k}$ for all $k \in \omega$ then the sequence of points x_k does not have convergent subsequences in $\mathscr{N}$.

If such a construction is accomplished then (4) implies that $\bigcap_m Y_{a \restriction m} \neq \varnothing$ for each $a \in \mathscr{N}$. On the other hand by (2) every such an intersection contains a single point, which we denote by $f(a)$, and the map $f : \mathscr{N} \xrightarrow{\text{onto}} Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\}$ is a homeomorphism by clear reasons.

Prove that Y is closed in $\mathscr{N}$. Consider an arbitrary sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of points $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$. If the sequence $\{a_n\}_{n\in\omega}$ contains a subsequence of points $b_k = a_{n(k)}$ convergent to some $b \in \mathscr{N}$ then quite obviously the sequence of points $z_k = f(b_k)$ (a subsequence of $\{y_n\}_{n\in\omega}$) converges to $z = f(b) \in Y$, as required. Thus suppose that the sequence $\{a_n\}_{n\in\omega}$ has no convergent subsequences. Then it cannot be covered by a compact set, and it easily follows that there is a string $u \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and for each $k \in K$ — a number $n(k)$ such that $u^\wedge k \subset a_{n(k)}$. But then $y_{n(k)} \in Y_{u^\wedge k}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k\in\omega}$ diverges by (5), which is a contradiction.

Thus Y is closed, and hence we have (II) of Theorem 4.1.

As for the construction of sets Y_u, if a Σ^1_1 set $Y_u \subseteq A'$ is defined then by Lemma 4.3 there is a string $t \in T(Y_u)$ such that $t^\wedge k \in T(Y_u)$ for all k in an infinite set $K_u \subseteq \omega$. This allows us to define a sequence of pairwise different points $y_k \in Y_u$ $(k \in \omega)$ having no convergent subsequences. We cover these points by Baire intervals U_k small enough for (5) to be true for the Σ^1_1 sets $Y_{u^\wedge i} = Y_u \cap U_i$, and then shrink these sets if necessary to fulfill (2) and (4).

$\square$ (Theorem 4.1)

5 Effective σ-compactness dichotomy for Δ^1_1 sets

Here we present a proof of the following result.

Theorem 5.1 (essentially Louveau [17]). *If $A \subseteq \mathscr{N}$ is a Δ^1_1 set then one and only one of the next two claims holds*:

(I) *A is Δ^1_1-effectively σ-compact, so that there is a Δ^1_1 sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ such that $A = \bigcup_n [T_n]$;*

(II) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and relatively closed in A.*

Corollary 5.2. *If $A \subseteq \mathscr{N}$ is a σ-compact Δ^1_1 set then it is Δ^1_1-effectively σ-compact in the sense of* (I) *of Theorem 5.1.* $\square$

Proof (theorem). By Theorem 4.1, we can w.l.o.g. assume that A is σ-bounded, and hence if $F \subseteq A$ is a closed set then F is σ-compact. Further, the union U of all sets $[T] \subseteq A$, where T is a compact Δ^1_1 tree, is Π^1_1:

$$x \in U \iff \exists T \in \Delta^1_1 \, (T \text{ is a compact tree and } x \in [T] \subseteq A),$$

and the result follows from Fact 2.3. We conclude that $A' = A \smallsetminus U$ is Σ^1_1.

Lemma 5.3. *If $F \subseteq A'$ is a non-empty Σ^1_1 set then $\overline{F} \not\subseteq A$.*

Proof. We first prove that if $X \subseteq A$ is a compact Σ^1_1 set then $A' \cap X = \varnothing$. Suppose towards the contrary that $A' \cap X$ is non-empty. We are going to find a closed Δ^1_1 set F satisfying $X \subseteq F \subseteq A$ — this would imply $X \subseteq U$ by Fact 2.10, which is a contradiction.

Since the complementary Π^1_1 set $C = \mathscr{N} \smallsetminus X$ is open, the set

$$H = \{\langle x, u\rangle : u \in \omega^{<\omega} \wedge x \in C \cap \mathscr{N}_u \wedge \mathscr{N}_u \cap X = \varnothing\}$$

is Π^1_1 and $\mathtt{proj}\, H = C$. Thus the Δ^1_1 set $D = \mathscr{N} \smallsetminus A$ is included in $\mathtt{proj}\, H$. By Fact 2.2, there is a Δ^1_1 map $\nu : D \to \omega^{<\omega}$ such that $x \in D \implies \langle x, \nu(x)\rangle \in H$, or equivalently, $x \in \mathscr{N}_{\nu(x)} \subseteq C$ for all $x \in D$. Then the set $\Sigma = \mathtt{ran}\, \nu = \{\nu(x) : x \in D\} \subseteq \omega^{<\omega}$ is Σ^1_1 and $D \subseteq \bigcup_{u\in\Sigma} \mathscr{N}_u \subseteq C$.

But $\Pi = \{u \in \omega^{<\omega} : \mathscr{N}_u \subseteq C\}$ is a Π^1_1 set and $\Sigma \subseteq \Pi$. It follows that there exists a Δ^1_1 set Δ such that $\Sigma \subseteq \Delta \subseteq \Pi$. Then still $D \subseteq \bigcup_{s\in\Delta} \mathscr{N}_s \subseteq C$, and hence the closed set $F = \mathscr{N} \smallsetminus \bigcup_{u\in\Delta} \mathscr{N}_u$ satisfies $X \subseteq F \subseteq A$. But $x \in F$ is equivalent to $\forall\, u\, (u \in \Delta \implies x \notin \mathscr{N}_u)$, thus F is Δ^1_1, as required.

Now suppose towards the contrary that $\varnothing \neq F \subseteq A'$ is a Σ^1_1 set but $\overline{F} \subseteq A$. By the w.l.o.g. assumption above, $\overline{F} = \bigcup_n F_n$ is σ-compact, where all F_n are compact. There is a Baire interval $\mathscr{N}_u$ such that the set $X = \mathscr{N}_u \cap \overline{F}$ is non-empty and $X \subseteq F_n$ for some n. Thus $X \subseteq A$ is a non-empty compact Σ^1_1 set, hence $X \cap A' = \varnothing$ by the first part of the proof. In other words, $\mathscr{N}_u \cap \overline{F} \cap A' = \varnothing$. It follows that $\mathscr{N}_u \cap F = \varnothing$ (because $F \subseteq A'$), contrary to $X = \mathscr{N}_u \cap \overline{F} \neq \varnothing$. □ (Lemma)

We return to the proof of the theorem.

Case 1: $A' = \varnothing$, that is, $A = U$. This implies (I) of Theorem 5.1, exactly as in the proof of Theorem 4.1 above.

Case 2: $A' = A \smallsetminus U \neq \varnothing$. To get a set $Y \subseteq A'$, *relatively* closed in A and homeomorphic to $\mathscr{N}$, as in (II) of Theorem 5.1, we'll define a system of non-empty Σ^1_1 sets $Y_u \subseteq A'$ satisfying conditions (1), (2), (3), (4) of Section 4, along with the next requirement instead of (5):

(5′) if $u \in \omega^{<\omega}$ then there is a point $y_u \in \overline{Y_u} \smallsetminus A$ such that any sequence of points $x_k \in Y_{u^\wedge k}$ $(k \in \omega)$ converges to y_u.

If we have defined such a system of sets, then the associated map $f : \mathscr{N} \to A'$ is $1-1$ and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A'$, as in the proof of Theorem 4.1.

Let's prove that Y is relatively closed in A. Consider a sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of $y_n = f(a_n) \in Y$

converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence convergent to $b \in \mathscr{N}$ then, as in the proof of Theorem 4.1, $\{y_n\}$ converges to $f(b) \in Y$. If the sequence $\{a_n\}$ has no convergent subsequences, then there exist a string $u \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and for each $k \in K$ — a number $n(k)$, such that $u^\wedge k \subset a_{n(k)}$. But then $y_{n(k)} \in Y_{u^\wedge k}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k \in \omega}$ converges to a point $y_u \notin A$ by (5′), as required.

Finally on the construction of sets Y_s.

Suppose that a Σ^1_1 set $\varnothing \neq Y_u \subseteq A'$ is defined. Then its closure $\overline{Y_u}$ is a Σ^1_1 set, too, therefore $\overline{Y_u} \not\subseteq A$ by Lemma 5.3. There is a sequence of pairwise different points $x_n \in Y_u$ which converges to a point $y_u \in \overline{Y_u} \smallsetminus A$. Let U_n be a neighbourhood of x_n (a Baire interval) of diameter less than $\frac{1}{3}$ of the least distance from x_n to the points x_k, $k \neq n$. Put $Y_{u^\wedge n} = Y_u \cap U_n$, and shrink the sets $Y_{u^\wedge n}$ so that they satisfy (2) and (4).

□ (Theorem 5.1)

6 Effective σ-compactness dichotomy: generalization to Σ^1_1

There is a difference between Theorem 4.1 and Theorem 5.1: the first theorem deals with Σ^1_1 sets A while the other one — with Δ^1_1 sets only. The proof of Theorem 5.1 in Section 5 does not work in the case when A is a Σ^1_1 set. Indeed then A' is a set in $\mathbf{\Sigma}^1_1$ and Σ^1_2, but, generally speaking, it cannot be expected to be a Σ^1_1 set, so the rest of the proof does not go through.

As a matter of fact, Theorem 5.1 *per se* fails for Σ^1_1 sets A, as the following counterexample shows.

Example 6.1. Let $\{y\}$ be a Π^1_1 singleton such that $y \in 2^\omega$ is not Δ^1_1. The set $A = 2^\omega \smallsetminus \{y\}$ is then Σ^1_1 and an open subset of 2^ω, hence, σ-compact. Suppose towards the contrary that Theorem 5.1 holds for A. Then (I) of Theorem 5.1 must be true. Let $\{T_n\}_{n \in \omega}$ be a Δ^1_1 sequence of compact trees such that $A = \bigcup_n [T_n]$. Therefore y is Δ^1_1, as the only point in 2^ω which does not belong to $\bigcup_n [T_n]$, a contradiction. □

Our best result in the direction of Theorem 5.1 for Σ^1_1 sets with still some effectivity in (I) is the following theorem:

Theorem 6.2. *If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims holds*:

(I) *A is Δ^1_3-effectively σ-compact, so that there exists a Δ^1_3 sequence $\{T^n\}_{n<\omega}$ of compact Δ^1_3 trees $T^n \subseteq \omega^{<\omega}$ such that $A = \bigcup_{n<\omega} [T^n]$;*

(II) *there is a set* $Y \subseteq A$ *homeomorphic to* $\mathscr{N}$ *and relatively closed in* A.

Proof. Given a tree $S \subseteq (\omega \times \omega)^{<\omega}$, define a *derived tree* $S' \subseteq S$ so that

($*$) S' consists of all nodes $\langle u, v \rangle \in S$ such that $\overline{\texttt{proj}\,[S \restriction \langle u, v \rangle]} \not\subseteq A$, where $S \restriction \langle u, v \rangle = \{\langle u', v' \rangle \in S : (u \subset u' \land v \subset v') \lor (u' \subseteq u \land v' \subseteq v)\}$.

Note that S' can contain maximal nodes even if S contains no maximal nodes. Yet if $\langle u, v \rangle$ is a maximal node in S, or generally a note in the well-founded part of S (so $[S \restriction \langle u, v \rangle] = \varnothing$), then definitely $\langle u, v \rangle \notin S'$.

Lemma 6.3. *The set* $\{\langle S, u, v \rangle : \langle u, v \rangle \in S'\}$ *is* Σ^1_2.
In addition, $S' \subseteq S$, *and if* $S \subseteq T$ *then* $S' \subseteq T'$.
Moreover, if $\mathfrak{M}$ *is a countable transitive model of a large enough fragment of* **ZFC** *and* $S \in \mathfrak{M}$ *then* $(S')^{\mathfrak{M}} \subseteq S'$.

Proof. As A is Σ^1_1, the key condition $\overline{\texttt{proj}\,[S \restriction \langle u, v \rangle]} \not\subseteq A$ is Σ^1_2. $\square$

Beginning the proof of Theorem 6.2, we w.l.o.g. assume, by Theorem 4.1, that A, the given set, is σ-bounded, and hence if $F \subseteq A$ is a closed set then F is σ-compact. Let $P \subseteq \mathscr{N} \times \mathscr{N}$ be a Π^0_1 set such that $A = \texttt{proj}\, P$. Let

$$S = \{\langle x \restriction n, y \restriction n \rangle : n \in \omega \land \langle x, y \rangle \in P\} \subseteq \omega^{<\omega} \times \omega^{<\omega},$$

so that $P = [S]$. A decreasing sequence of derived trees $S^{(\alpha)}$, $\alpha \in \mathbf{Ord}$, is defined by transfinite induction so that $S^{(0)} = S$, if λ is a limit ordinal then naturally $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$, and $S^{(\alpha+1)} = (S^{(\alpha)})'$ for any α.

Obviously there is a countable ordinal λ such that $S^{(\lambda+1)} = S^{(\lambda)}$.

Case 1: $S^{(\lambda)} = \varnothing$. Then, if $x \in A = \texttt{proj}\, P$ then by construction there exist an ordinal $\alpha < \lambda$ and a node $\langle u, v \rangle \in S^{(\alpha)}$ such that

$$x \in A^{(\alpha)}_{uv} \subseteq \overline{A^{(\alpha)}_{uv}} \subseteq A, \quad \text{where} \quad A^{(\alpha)}_{uv} = \texttt{proj}\,[S^{(\alpha)} \restriction \langle u, v \rangle]\,,$$

and hence A is a countable union of sets $F \subseteq A$ of the form $\overline{A^{(\alpha)}_{uv}}$, where $\alpha < \lambda$ and $\langle u, v \rangle \in S^{(\alpha)}$, closed, therefore σ-compact by the above.

Let us show how this leads to (I) of the theorem.

It easily follows from Lemma 6.3 that both the ordinal λ, and each ordinal $\alpha < \lambda$, and the sequence $\{S^{(\alpha)}\}_{\alpha<\lambda}$ itself, are Δ^1_3. Therefore there is a Δ^1_3 sequence $\{U^{(n)}\}_{n<\omega}$ of the same trees, that is,

$$\{S^{(\alpha)} : \alpha < \lambda\} = \{U^{(n)} : n < \omega\}.$$

Each tree $U^{(n)}$, $n < \omega$, is Δ^1_3 either, as well as all restricted subtrees of the form $U^{(n)} \restriction \langle u, v \rangle$ (where $\langle u, v \rangle \in U^{(n)}$) and their "projections"

$$T^{(n)}_{uv} = \{u : \exists\, v\, (\langle u, v \rangle \in U^{(n)} \restriction \langle u, v \rangle)\} \subseteq \omega^{<\omega}.$$

On the other hand, if $\alpha < \lambda$ and $\langle u, v \rangle \in S^{(\alpha)}$ then we have $\overline{A^{(\alpha)}_{uv}} = [T^{(n)}_{uv}]$ for some $n = n(\alpha)$ by construction.

To conclude, if $x \in A$ then there is a Δ^1_3 tree $T^{(n)}_{uv} \subseteq \omega^{<\omega}$ such that $x \in [T^{(n)}_{uv}] \subseteq A$ — and $[T^{(n)}_{uv}]$ is σ-compact in this case. Then by Theorem 5.1 (relativized version) there is a $\Delta^1_1(T^{(n)}_{uv})$ sequence of compact trees $T^{(n)}_{uv}(k)$ such that $[T^{(n)}_{uv}] = \bigcup_k [T^{(n)}_{uv}(k)]$. This easily leads to (I) of the theorem.[2]

Case 2: $S^{(\lambda)} \neq \varnothing$, and then $S^{(\lambda)} \subseteq S$ is a pruned tree.

Lemma 6.4. *If $\langle u, v \rangle \in S^{(\lambda)}$, $u' \in \omega^{<\omega}$, $u \subset u'$, and $A^{(\lambda)}_{uv} \cap \mathscr{N}_{u'} \neq \varnothing$ then there is a string $v' \in \omega^{<\omega}$ such that $v \subset v'$ and $\langle u', v' \rangle \in S^{(\lambda)}$.* □

We'll define a pair $\langle u(t), v(t) \rangle \in S^{(\lambda)}$ for each $t \in \omega^{<\omega}$, such that

(1) if $t \in \omega^{<\omega}$ then $t \subseteq u(t)$;

(2) if $s, t \in \omega^{<\omega}$ and $s \subseteq t$ then $u(s) \subseteq u(t)$ and $v(s) \subseteq v(t)$;

(3) if $t \in \omega^{<\omega}$ and $k \neq n$ then $u(t^\wedge k)$ and $u(t^\wedge n)$ are $\subseteq$-incomparable;

(4) if $s \in \omega^{<\omega}$ then there exists a point $y_s \in \overline{A^{(\lambda)}_{u(s)v(s)}} \smallsetminus A$ such that any sequence of points $x_k \in A^{(\lambda)}_{u(s^\wedge k)v(s^\wedge k)}$ converges to y_s.

Suppose that such a system of sets is defined. Then the associated map $f(a) = \bigcup_n u(a \restriction n) : \mathscr{N} \to A$ is $1-1$ and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A$.

Let's prove that Y is relatively closed in A. Consider a sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of points $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence convergent to $b \in \mathscr{N}$ then $\{y_n\}$ converges to $f(b) \in Y$. So suppose that the sequence $\{a_n\}$ has no convergent subsequences. Then there exist a string $s \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and for each $k \in K$ — a number $n(k)$, such that $s^\wedge k \subset a_{n(k)}$. Then

[2] Class Δ^1_3 in (I) of the theorem looks too weird. One may want to improve it to Δ^1_2 at least. This would be the case if the ordinal λ in the argument of Case 1 could be shown to be Δ^1_2. Yet by Martin [19] closure ordinals of inductive constructions of this sort may exceed the domain of Δ^1_2 ordinals.

$y_{n(k)} \in A^{(\lambda)}_{u(s^\wedge k)v(s^\wedge k)}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k\in\omega}$ converges to a point $y_s \notin A$ by (4), as required.

Finally on the construction of sets Y_s.

Suppose that a pair $\langle u(t), v(t)\rangle \in S^{(\lambda)}$ is defined. Then $\overline{A^{(\lambda)}_{u(t)v(t)}} \not\subseteq A$ by the choice of λ. There is a sequence of pairwise different points $x_n \in A^{(\lambda)}_{u(t)v(t)}$ which converges to a point $y_s \in \overline{A^{(\lambda)}_{u(t)v(t)}} \smallsetminus A$. We can associate a string $u_n \in \omega^{<\omega}$ with each x_n such that $u(t) \subset u_n \subset x_n$, the strings u_n are pairwise $\subseteq$-incompatible, and $\mathtt{lh}\, u_n \to \infty$. Then, by Lemma 6.4, for each n there is a matching string v_n such that $v(t) \subset v_n$ and $\langle u_n, v_n\rangle \in S^{(\lambda)}$. Put $u(t^\wedge n) = u_n$ and $v(t^\wedge n) = v_n$ for all n.

□ (Theorem 6.2)

7 Related classical results

The "effective" results presented above can be compared with some known theorems of classical descriptive set theory, including the following two.

Theorem 7.1 (Saint Raymond [21] or 21.23 in [16]). *If A is a $\mathbf{\Sigma}^1_1$ set in a Polish space then either A is σ-bounded or there is a superperfect set $P \subseteq A$.* □

Theorem 7.2 (Hurewicz [7]). *If A is a $\mathbf{\Sigma}^1_1$ set in a Polish space then either A is σ-compact or there is a subset $Y \subseteq A$ homeomorphic to the Baire space $\mathscr{N}$ and relatively closed in A.* □

Arguments in [16] show that it's sufficient to prove each of these theorems in the case $\mathbb{X} = \mathscr{N}$; then the results can be generalized to an arbitrary Polish space $\mathbb{X}$ by purely topological methods. In the case $\mathbb{X} = \mathscr{N}$, Theorem 7.1 immediately follows from our Theorem 4.1 (in relativized form, *i.e.*, for classes $\varSigma^1_1(p)$, where $p \in \mathscr{N}$ is arbitrary), while Theorem 7.2 follows from Theorem 6.2 (relativized). On the other hand, Theorem 7.2 also follows from Theorem 5.1 (relativized) for sets A in $\mathbf{\Delta}^1_1$ (that is, Borel sets).[3]

Theorem 5.1 implies yet another theorem, which combines several classical results of descriptive set theory by Arsenin, Kunugui, Saint Raymond, Shegolkov, see references in [16] or in [9, § 4].

Theorem 7.3 (compare with Fact 2.8). *Suppose that $\mathbb{X}, \mathbb{Y}$ are Polish spaces, $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a $\mathbf{\Delta}^1_1$ set, and all cross-sections $(P)_x = \{y : \langle x, y\rangle \in P\}$ $(x \in \mathbb{X})$ are σ-compact. Then*

[3] See [16, 18] for another modern approach to those classical theorems, based mainly on infinite games rather than methods of effective descriptive theory.

(i) *the projection* $\mathtt{proj}\, P$ *is a* $\mathbf{\Delta}^1_1$ *set*;

(ii) P *is a countable union of* $\mathbf{\Delta}^1_1$ *sets with compact cross-sections*;

(iii) P *can be uniformized by a* $\mathbf{\Delta}^1_1$ *set.*

Proof (a sketch for the case $\mathbb{X} = \mathbb{Y} = \mathscr{N}$). (i) Assume, for the sake of simplicity, that $P \subseteq \mathscr{N} \times \mathscr{N}$ is a $\varDelta^1_1$ set. The set

$$H = \{\langle x, T\rangle : x \in \mathscr{N} \wedge T \in \mathbf{CT} \wedge T \in \varDelta^1_1(x) \wedge [T] \subseteq (P)_x\}$$

is $\varPi^1_1$ by Fact 2.4. It follows from Theorem 5.1 that if $\langle x, y\rangle \in P$ then there is a tree T such that $\langle x, T\rangle \in H$ and $y \in [T]$. Therefore the $\varPi^1_1$ set

$$E = \{\langle x, y, T\rangle : \langle x, y\rangle \in P \wedge \langle x, T\rangle \in H \wedge y \in [T]\} \subseteq \mathscr{N} \times \mathscr{N} \times 2^{(\omega^{<\omega})}$$

satisfies $\mathtt{proj}_{xy}\, E = P$, that is, if $\langle x, y\rangle \in P$ then there is a tree T such that $\langle x, y, T\rangle \in E$. There is a uniform $\varPi^1_1$ set $U \subseteq E$ which *uniformizes* E, *i.e.*, if $\langle x, y\rangle \in P$ then there is a unique T such that $\langle x, y, T\rangle \in U$. Yet U is $\varSigma^1_1$ as well by Fact 2.3, since $\langle x, y, T\rangle \in U$ is equivalent to:

$$\langle x, y\rangle \in P \wedge y \in [T] \wedge \forall T' \in \varDelta^1_1(x)\, (\langle x, y, T'\rangle \in U \implies T = T')\,.$$

Thus the $\varSigma^1_1$ set $F = \{\langle x, T\rangle : \exists\, y\, (\langle x, y, T\rangle \in U)\}$ is a subset of the $\varPi^1_1$ set H. By $\varSigma^1_1$ separation, there is a $\varDelta^1_1$ set V such that $F \subseteq V \subseteq H$. Then

$$\langle x, y\rangle \in P \iff \exists\, T\, (\langle x, T\rangle \in V \wedge y \in [T])$$

by definition. Finally all cross-sections of V are at most countable: indeed if $\langle x, T\rangle \in V$ then $T \in \varDelta^1_1(x)$ (since $V \subseteq H$). Note that $\mathtt{proj}\, P = \mathtt{proj}\, V$, and hence the projection $D = \mathtt{proj}\, P$ is $\varDelta^1_1$ (hence Borel) by Fact 2.8.

(ii) It follows from Fact 2.8 that V is equal to a union $V = \bigcup_n V_n$ of uniform $\varDelta^1_1$ sets V_n, and then each projection $D_n = \mathtt{proj}\, V_n \subseteq D$ is $\varDelta^1_1$. Each V_n is basically the graph of a $\varDelta^1_1$ map $\tau_n : D_n \to \mathbf{CT}$, and $(P)_x = \bigcup_{x \in D_n} [\tau_n(x)]$. If $n \in \omega$ then we put

$$P_n = \{\langle x, y\rangle : x \in D_n \wedge y \in [\tau_n(x)]\}\,.$$

Then $P = \bigcup_n P_n$ by the above, each set P_n has only compact cross-sections, and each P_n is a $\varDelta^1_1$ set, since the sets D_n and maps τ_n belong to $\varDelta^1_1$.

(iii) Still by Fact 2.8, the set V can be uniformized by a uniform $\varDelta^1_1$ set, that is, there exists a $\varDelta^1_1$ map $\tau : D \to \mathbf{CT}$ such that $\langle x, \tau(x)\rangle \in V$ for all $x \in D$. To uniformize the original set P, let Q consist of all pairs

$\langle x, y\rangle \in P$ such that y is the lexicographically leftmost point in the compact set $[\tau(x)]$. Clearly Q uniformizes P. To check that Q is $\varDelta^1_1$, note that "y is a the lexicographically leftmost point in $[T]$" is an arithmetic relation in the assumption that $T \in \mathbf{CT}$. □

Similar arguments, this time based on Theorem 4.1, also lead to an alternative proof of the following known result.

Theorem 7.4 (Burgess, Hillard, 35.43 in [16])**.** *If P is a $\mathbf{\Sigma}^1_1$ set in the product $\mathbb{X}\times\mathbb{Y}$ of two Polish spaces $\mathbb{X}$, and every section $(P)_x$ is a σ-bounded set, then there is a sequence of Borel sets $P_n \subseteq \mathbb{X}\times\mathbb{Y}$ with compact sections $(P_n)_x$ such that $P \subseteq \bigcup_n P_n$.* □

But at the moment it seems that no conclusive theory of $\mathbf{\Sigma}^1_1$ sets with σ-compact sections (as opposed to those with σ-bounded sections) is known. For instance what about effective decompositions of such sets into countable unions of definable sets with compact sections? Our Theorem 6.2 can be used to show that such a decomposition is possible, but the decomposing sets with compact sections appear to be excessively complicated (3rd projective level by rough estimation). It is an interesting **problem** to improve this result to something more reasonable like Borel combinations of $\mathbf{\Sigma}^1_1$ sets.

On the other hand, it is known from [26, 28] that $\mathbf{\Sigma}^1_1$ sets with σ-compact sections are not necessarily decomposable into countably many $\mathbf{\Sigma}^1_1$ sets with compact sections.

8 Counterexamples above $\varSigma^1_1$

Here we outline several counterexamples to Theorems 4.1 and 5.1 with sets A more complicated than $\varSigma^1_1$.

Example 8.1. Suppose that the universe is a Cohen real extension $\mathbf{L}[a]$ of the constructible universe $\mathbf{L}$. The set $A = \mathscr{N} \cap \mathbf{L}$ is $\varSigma^1_2$ and it is not σ-bounded in $\mathbf{L}[a]$. On the other hand, it is known from [4] that A has no perfect subsets, let alone superperfect ones. Thus A is a $\varSigma^1_2$ counterexample to both Theorem 4.1 and Theorem 5.1 in $\mathbf{L}[a]$. We then immediately obtain a similar $\varPi^1_1$ counterexample, using the $\varPi^1_1$ uniformization theorem. □

Example 8.2. Suppose that the universe is a dominating real extension $\mathbf{L}[d]$ of $\mathbf{L}$. The set $A = \mathscr{N} \cap \mathbf{L}$ is then σ-bounded in $\mathbf{L}[d]$. The dominating forcing is homogeneous enough for any OD (ordinal-definable) real in $\mathbf{L}[d]$ to be constructible, and hence it is true in $\mathbf{L}[d]$ that A cannot be covered by a countable union of OD compact sets in $\mathbf{L}[d]$. Thus A is a $\varSigma^1_2$ counterexample to Corollary 4.2. □

Yet it is not clear how a similar $\mathit{\Pi}^1_1$ counterexample, or even a $\mathit{\Sigma}^1_2$ counterexample to Corollary 5.2, can be produced.

Example 8.3. Let $A = \{y\}$ be a $\mathit{\Pi}^1_1$ singleton such that y is not a $\mathit{\Delta}^1_1$ real. Then conditions (I), (II) of Theorem 4.1 obviously fail for A.

The same for Theorem 5.1.

Moreover A is a $\mathit{\Pi}^1_1$ counterexample to Corollary 4.2 as well, although not as strong as those given in Example 8.2. $\square$

It is known that there is a countable $\mathit{\Pi}^1_1$ set $A \subseteq \mathscr{N}$ containing at least one non-$\mathit{\Delta}^1_2$ element. Can it serve as a more profound $\mathit{\Pi}^1_1$ counterexample than the singleton A of Example 8.3 ?

9 Generalizing the σ-bounded dichotomy: preliminaries

Below in Section 11, we establish a generalization of Theorem 4.1 for a certain system of pointset ideals which include the ideal of σ-bounded sets along with equivalence classes of a given finite or countable family of equivalence relations. The next definition introduces a necessary framework.

Definition 9.1. Let $\mathscr{F}$ be a family of equivalence relations on a set $X_0 \subseteq \mathscr{N}$. A set $X \subseteq X_0$ is *$\mathscr{F}$-σ-bounded*, iff it is covered by a union of the form $B \cup \bigcup_{n\in\omega} Y_n$, where B is a σ-bounded set and each Y_n is an F-equivalence class for an equivalence relation $\mathsf{F} = \mathsf{F}(n) \in \mathscr{F}$ which depends on n.

A set $X \subseteq X_0$ is *$\mathscr{F}$-superperfect*, if it is a superperfect pairwise F-inequivalent set (*i.e.*, a partial F-transversal) for every $\mathsf{F} \in \mathscr{F}$. $\square$

Clearly $\mathscr{F}$-σ-bounded sets form a σ-ideal containing all σ-bounded sets, and no $\mathscr{F}$-σ-bounded set can be $\mathscr{F}$-superperfect. What are properties of these ideals? Do they have some semblance of the superperfect ideal itself? We begin with a lemma and a corollary afterwards, which show that this is indeed the case w.r.t. the property of being $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The lemma is a generalization of Corollary 4.2, of course.

Lemma 9.2. *Suppose that $\{\mathsf{F}_n\}_{n<\omega}$ is a $\mathit{\Delta}^1_1$ sequence of equivalence relations on $\mathscr{N}$, and a $\mathit{\Sigma}^1_1$ set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded. Then X is $\mathit{\Delta}^1_1$-effectively $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded, in the sense that there exist*:

(1) *a $\mathit{\Delta}^1_1$ sequence of compact trees T_k,*

(2) *a $\mathit{\Delta}^1_1$ sequence of numbers n_k, and*

(3) *a $\mathit{\Delta}^1_1$ set $H \subseteq \omega \times \mathscr{N}$*

such that, for every $k<\omega$ the cross-section $(H)_k=\{a:\langle k,a\rangle\in H\}$ is an F_{n_k}-equivalence class and $X\subseteq\bigcup_k[T_k]\cup\bigcup_k(H)_k$.

In particular, if a Σ^1_1 set $X\subseteq\mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then X is covered by the union of all Δ^1_1 F_0-classes, all Δ^1_1 F_1-classes, all Δ^1_1 F_2-classes, et cetera, and all Δ^1_1 compact sets.

Proof. The set $C=\mathbf{CT}\cap\Delta^1_1$ of all Δ^1_1 compact trees is Π^1_1, and hence so is $K=\bigcup_{T\in C}[T]$. If $n<\omega$ then let U_n be the union of all Δ^1_1 F_n-classes. Let's show that $U=\bigcup_n U_n$ is Π^1_1 either. We make use of sets $E\subseteq\omega$ and $W,W'\subseteq\omega\times\mathscr{N}$ as in Fact 2.5. The Π^1_1 formula $\varphi(e,n)$:=

$$e\in E\ \wedge\ \forall y,z\in(W')_e\,(y\ \mathsf{F}_n\ z)\ \wedge\ \forall y\in(W')_e\,\forall z\,(y\ \mathsf{F}_n\ z\Longrightarrow z\in(W)_e)$$

says that $e\in E$ and $(W')_e=(W)_e$ is a F_n-equivalence class. Moreover

$$x\in U\iff\exists n\,\exists e\,(\varphi(e,n)\wedge x\in(W)_e).$$

Case 1: $X\subseteq K\cup U$. Then the set S of all pairs $\langle x,h\rangle$ such that

- either $h=T\in C$ and $x\in[T]$,
- or $h=\langle e,n\rangle\in\Phi=\{\langle e,n\rangle\in E\times\omega:\varphi(e,n)\}$ and $x\in(W)_e$,

is a Π^1_1 set satisfying $X\subseteq\mathtt{proj}\,S$. By Fact 2.7 there is a Δ^1_1 map f defined on $\mathscr{N}$ and such that $\langle a,f(a)\rangle\in S$ for each $a\in X$. The sets

$$X'=\{x\in X:f(x)\in\mathbf{CT}\}\quad\text{and}\quad X''=\{x\in X:f(x)\in\Phi\}$$

are Σ^1_1 as well as their images

$$R'=\{f(x):x\in X'\}\subseteq C\quad\text{and}\quad R''=\{f(x):x\in X''\}\subseteq\Phi\,,$$

and $X'\cup X''=X$, $R'\cup R''=\{f(x):x\in X\}$. By the Σ^1_1 Separation theorem there is a Δ^1_1 set τ such that $R'\subseteq\tau\subseteq C$, and by Fact 2.9 we have $\tau=\{T_k:k<\omega\}$, where $k\longmapsto T_k$ is a Δ^1_1 map. By similar reasons, there is a Δ^1_1 map $k\longmapsto\langle e_k,n_k\rangle$ such that $R''\subseteq\rho=\{\langle e_k,n_k\rangle:k<\omega\}\subseteq\Phi$. To finish the proof in Case 1, it remains to define

$$H=\{\langle k,x\rangle\in\omega\times\mathscr{N}:x\in(W)_{e_k}\}=\{\langle k,x\rangle\in\omega\times\mathscr{N}:x\in(W')_{e_k}\}\,.$$

Case 2: $A=X\smallsetminus(K\cup U)\neq\varnothing$. Then A is a non-empty Σ^1_1 set. We are going to derive a contradiction. By definition, we have $X\subseteq\bigcup_k C_k\cup\bigcup_n\bigcup_k E_{nk}$, where each C_k is compact and each E_{nk} is an F_n-class. Let

M be a countable elementary substructure of a sufficiently large structure, containing, in particular, the whole sequence of covering sets C_k and E_{nk}. Below "generic" will mean Gandy – Harrington generic over M.

As $A \neq \varnothing$ is Σ^1_1, there is a perfect set $P \subseteq A$ of points both generic and pairwise generic. It is known that then P is a pairwise F_n-inequivalent set for every n, hence, definitely a set not covered by a countable union of F_n-classes for all $n < \omega$. Thus to get a contradiction it suffices to prove that $P \cap C_k = \varnothing$ for all k. In other words, we have to prove that if $k < \omega$ and $x \in A$ is any generic real then $x \notin C_k$.

Suppose towards the contrary that a non-empty Σ^1_1 condition $Y \subseteq A$ forces that $\mathbf{a} \in C_k$, where $\mathbf{a}$ is a canonical name for the Gandy – Harrington generic real. We claim that Y is not σ-bounded. Indeed otherwise we have $Y \subseteq \bigcup_n [T_n]$ by Theorem 4.1, where all trees $T_n \subseteq \omega^{<\omega}$ are Δ^1_1 and compact, which contradicts the fact that A does not intersect any compact Δ^1_1 set.

Therefore $Y \not\subseteq C_k$. Then there is a point $x \in Y$ and a number m such that the set $I = \{y \in \mathscr{N} : y \restriction m = x \restriction m\}$ does not intersect C_k. But then the Σ^1_1 condition $Y' = Y \cap I$ forces that $\mathbf{a} \notin C_k$, a contradiction. □

Corollary 9.3. *If $\{\mathsf{F}_n\}_{n<\omega}$ is a Δ^1_1 sequence of equivalence relations on $\mathscr{N}$ then the ideal of $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded sets is Π^1_1 on Σ^1_1 and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

See [31, section 3.8] on Π^1_1 on Σ^1_1 and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideals.

Proof. Consider a Σ^1_1 set $P \subseteq \mathscr{N} \times \mathscr{N}$. We have to prove that

$$X = \{x \in \mathscr{N} : (P)_x = \{y : \langle x, y\rangle \in P\} \text{ is } \{\mathsf{F}_n\}_{n<\omega}\text{-}\sigma\text{-bounded}\}$$

is a Π^1_1 set. By the relativized version of Lemma 9.2, $x \in X$ iff

(∗) there exist $\Delta^1_1(x)$ sequences $\{T_k\}_{k<\omega}$ (of compact trees) and $\{n_k\}_{k<\omega}$ and a $\Delta^1_1(x)$ set $H \subseteq \omega \times \mathscr{N}$ such that, for every $k < \omega$ the cross-section $(H)_k$ is an F_{n_k}-equivalence class and $(P)_x \subseteq \bigcup_k [T_k] \cup \bigcup_k (H)_k$.

A routine analysis (as in the proof of Lemma 9.2) shows that this is a Π^1_1 description of the set X. □

10 Digression: another look on the effectivity

As usual, Lemma 9.2 and Corollary 9.3 remain true for relativized classes. In particular, if $p \in \mathscr{N}$, F_n are $\Delta^1_1(p)$ equivalence relations, and a $\Sigma^1_1(p)$ set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then X is covered by the union of all $\Delta^1_1(p)$ F_n-classes, $n = 0, 1, 2, \dots$, and all $\Delta^1_1(p)$ compact sets. If now $p \neq q \in \mathscr{N}$ is a different parameter, but still F_n are $\varDelta^1_1(q)$ and X is $\varSigma^1_1(q)$ and $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then accordingly X is covered by the union of all $\varDelta^1_1(q)$ F_n-classes, $n = 0, 1, 2, \dots$, and all $\varDelta^1_1(q)$ compact sets. Those two countable coverings of the same set X can be different, of course. This leads to the question: is there a covering of X of the type indicated, which depends on X and F_n themselves, but not on the choice of a parameter p such that X is $\varSigma^1_1(p)$ and F_n are $\varDelta^1_1(p)$. We are able to answer this question in the positive at least in the case of finitely many equivalence relations. The next theorem will be instrumental in the proof of a theorem in Section 13.

Theorem 10.1. *Suppose that $n \geq 1$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ are Borel equivalence relations on $\mathscr{N}$, and a $\boldsymbol{\Sigma}^1_1$ set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Then there exist Borel sets $Y_1, \dots, Y_n, X_{n+1} \subseteq \mathscr{N}$ such that*

(i) $X \subseteq Y_1 \cup \dots \cup Y_n \cup X_{n+1}$,

(ii) *each set Y_j is a countable union of F_j-equivalence classes while the set X_{n+1} is σ-bounded,*

(iii) *if $p \in \mathscr{N}$, X is $\varSigma^1_1(p)$, and all relations F_m are $\varDelta^1_1(p)$, then there is a parameter $\bar{p} \in \mathscr{N}$ in $\varDelta^1_2(p)$ such that both X_{n+1} and all sets Y_j are $\varDelta^1_1(\bar{p})$ — hence, $\varDelta^1_2(p)$.*

This, under the assumptions of the theorem, there is a Borel covering of X satisfying (i) and (ii), and effective as soon as X and F_j are granted some effectivity. It is a challenging problem to get rid of $\bar{p}$ in (iii) (so that X_{n+1} and all Y_j are just $\varDelta^1_1(p)$ with the same p), but this remains open.

Proof. We define sets $X = X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots \supseteq X_n \supseteq X_{n+1}$ so that $X_{j+1} = X_j \smallsetminus Y_j$, where by induction

$$Y_j = \{x \in \mathscr{N} : \text{the set } X_j \cap [x]_{\mathsf{F}_j} \text{ is not } \{\mathsf{F}_j, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\}. \quad (1)$$

In particular,

$$\begin{aligned}
Y_1 &= \{x \in \mathscr{N} : \text{the set } X_1 \cap [x]_{\mathsf{F}_1} \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\}, \\
Y_2 &= \{x \in \mathscr{N} : \text{the set } X_2 \cap [x]_{\mathsf{F}_2} \text{ is not } \{\mathsf{F}_3, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\}, \\
&\dots \\
Y_{n-1} &= \{x \in \mathscr{N} : \text{the set } X_{n-1} \cap [x]_{\mathsf{F}_{n-1}} \text{ is not } \{\mathsf{F}_n\}\text{-}\sigma\text{-bounded}\}, \\
Y_n &= \{x \in \mathscr{N} : \text{the set } X_n \cap [x]_{\mathsf{F}_n} \text{ is not } \varnothing\text{-}\sigma\text{-bounded}\},
\end{aligned}$$

where $\varnothing$-σ-bounded is the same as just σ-bounded.

Lemma 10.2. *If $1 \le j \le n$ then Y_j is a countable union of F_j-equivalence classes and the set $X_{j+1} = X_j \smallsetminus Y_j$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded.*

Proof. Let $\mathscr{Y}_j$ be the family of all sets Y such that Y is a union of at most countably many F_j-classes and $X_j \smallsetminus Y$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Note that $\mathscr{Y}_j$ is a non-empty (since X_j is $\{\mathsf{F}_j, \dots, \mathsf{F}_n\}$-$\sigma$-bounded by induction) σ-filter (since the collection of all $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded sets is a σ-ideal). Therefore $Y'_j = \bigcap \mathscr{Y}_j$ is a set in $\mathscr{Y}_j$, in fact, the $\subseteq$-least set in $\mathscr{Y}_j$.

It remains to show that $Y_j = Y'_j$. We claim that if C is an F_j-class then $C \subseteq Y'_j$ iff $C \subseteq Y'_j$. Indeed if $C \cap Y_j = \varnothing$ then $X_j \cap C$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, thus $Y'_j \smallsetminus C$ is still a set in $\mathscr{Y}_j$, therefore $C \cap Y'_j = \varnothing$. Conversely if $C \cap Y'_j = \varnothing$ then $(X_j \cap C) \subseteq (X_j \smallsetminus Y'_j)$, and hence $X_j \cap C$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, so $C \cap Y_j = \varnothing$, as required. $\square$ (Lemma)

Thus by the lemma the sets Y_j and X_{n+1} satisfy (i) and (ii) of the theorem. To verify (iii), assume that $p \in \mathscr{N}$, X is $\Sigma^1_1(p)$, and all F_m are $\Delta^1_1(p)$. The main issue is that the sets Y_j, albeit Borel (as countable unions of Borel equivalence classes) do not seem to be $\Delta^1_1(p)$, at least straightforwardly. For instance, Y_1 is $\Sigma^1_1(p)$ by Corollary 9.3 (relativized), and accordingly X_2 is $\Pi^1_1(p)$ (instead of $\Delta^1_1(p)$), which makes it very difficult to directly estimate the class of Y_2 at the nest step. This is where a new parameter appears.

We precede the last part of the proof of the theorem with the following auxiliary fact on equivalence relations, perhaps, already known.

Lemma 10.3. *Let E be a Δ^1_1 equivalence relation on $\mathscr{N}$, and $X \subseteq \mathscr{N}$ be a Σ^1_1 set which intersects only countably many E-classes.*

Then all E-classes $[x]_{\mathsf{E}}$, $x \in X$, are Δ^1_1 sets, and there is an E-invariant Δ^1_1 set $Y \subseteq \mathscr{N}$ such that $X \subseteq Y$ and all E-classes $[y]_{\mathsf{E}}$, $y \in Y$, are Δ^1_1 sets (therefore Y still contains only countably many E-classes).

Proof. The union C of all Δ^1_1 E-classes is an E-invariant Π^1_1 set. (See, *e.g.*, 10.1.2 in [13].) Thus, if $X \not\subseteq C$ then $H = X \smallsetminus C$ is a non-empty Σ^1_1 set which does not intersect Δ^1_1 E-classes. Then (see, *e.g.*, Case 2 in the proof of Theorem 10.1.1 in [13]) H contains a perfect pairwise E-inequivalent set, which contradicts our assumptions. Therefore $X \subseteq C$, so indeed all E-classes $[x]_{\mathsf{E}}$, $x \in X$, are Δ^1_1. To prove the second claim apply the invariant Σ^1_1 separation theorem (see, *e.g.*, 10.4.2 in [13]), which yields an E-invariant Δ^1_1 set Y satisfying $X \subseteq [X]_{\mathsf{E}} \subseteq Y \subseteq C$. $\square$ (Lemma)

We continue the proof of Theorem 10.1. The next goal is to find a parameter $q_1 \in \mathscr{N}$ in $\Delta^1_2(p)$ such that the $\Sigma^1_1(p)$ set Y_1 is $\Delta^1_1(q_1)$. Let Π^1_1

sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W} \subseteq \mathscr{N} \times \omega \times \mathscr{N}$, and a Σ^1_1 set $\mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathscr{N}$ be as in Lemma 2.6. Let $E(p) = \{e : \langle p, e\rangle \in \mathbf{E}\}$ and, for all $e < \omega$,

$$W_e(p) = \{x : \langle p, e, x\rangle \in \mathbf{W}\}\,, \quad W'_e(p) = \{x : \langle p, e, x\rangle \in \mathbf{W}'\}\,,$$

so that $E(p)$ and all sets $W_e(p)$ are $\Pi^1_1(p)$ while all sets $W'_e(p)$ are $\Sigma^1_1(p)$. By Lemma 10.3 (relativized), a point $x \in \mathscr{N}$ belongs to Y_1 iff

$$\exists\, e\,\big(e \in E(p) \,\wedge\, x \in W_e(p) \,\wedge\, W_e(p) \text{ is an } \mathsf{F}_1\text{-class} \,\wedge \\ \wedge\, W'_e(p) \cap X_1 \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\big).$$

The first line is $\Pi^1_1(p)$. (Note that $W_e(p) = W'_e(p)$ for all $e \in E(p)$.) The second line is only $\Sigma^1_1(p)$ by Corollary 9.3. However the set

$$Q_1(p) = \{e \in E(p) : W'_e(p) \cap X_1 \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\} \subseteq \omega$$

is $\Delta^1_2(p)$ (more precisely, an intersection of $\Pi^1_1(p)$ and $\Sigma^1_1(p)$), and

$$x \in Y_1 \iff \exists\, e \in E(p) \cap Q_1(p)\,\big(x \in W_e(p) \,\wedge\, W_e(p) \text{ is an } \mathsf{F}_1\text{-class}\big).$$

We conclude that Y_1 is $\Delta^1_1(p, Q_1)$, hence, $\Delta^1_1(q_1)$, where $q_1 \in \mathscr{N}$ is a "concatenation" of p and Q_1 (so that q_1 is $\Delta^1_2(p)$).

Arguing the same way, we find parameters $q_2, q_3, \dots$ such that each Y_j is $\Delta^1_1(q_j)$ and each q_{j+1} is $\Delta^1_2(q_j)$, and hence $\Delta^1_2(p)$ by induction. Wrapping this construction up in a parameter $\bar{p}$ as in (iii) is a routine. $\square$

We don't know whether the theorem still holds for countably infinite sequences of equivalence relations. Yet the proof miserably fails in this case. Indeed, let, for any n, F_n be an equivalence relation on $\mathscr{N}$ whose classes are $I_k = \{x \in \mathscr{N} : x(0) = k\}$, $k = 0, 1, \dots, n$, and all singletons outside of these large classes. The whole space $\mathscr{N} = \bigcup_n I_n$ is $\{\mathsf{F}_0, \mathsf{F}_1, \mathsf{F}_2, \dots\}$-$\sigma$-bounded, of course. But running the construction as above, we'll obviously have $Y_0 = Y_1 = Y_2 = \dots = \varnothing$ (as each F_n-class is covered by an appropriate F_{n+1}-class), which results in nonsense.

There is another interesting problem. Under the assumptions of the theorem, the covering of X by sets $Y_1, \dots, Y_n, X_{n+1} \subseteq \mathscr{N}$ depends on X but is independent of the choice of a parameter p as in (iii). On the other hand, if such a parameter p, and accordingly $\bar{p}$ as in (iii), is given then not only each Y_j but also a representation of $Y_j = \bigcup_m Y_{jm}$ as a countable union of F_j-classes Y_{jm}, can be obtained in $\Delta^1_1(\bar{p})$ by Lemma 9.2. One may ask whether such a decomposition of each Y_j is available in a way independent of the choice of p (as the sets Y_j themselves). The answer in the negative is expected, but it may likely take a lot of work. On the other hand, Theorem 14.1 below will show that, under some restrictions, if a countable union of equivalence classes of a Δ^1_1 equivalence relation is $\Delta^1_1(\xi)$, where $\xi < \omega_1$, then all classes in this union admit constructible (not necessarily countable) Borel codes.

11 Generalizing the σ-bounded dichotomy: the theorem

Coming back to the content of Section 9, we'll prove the following theorem in this section.

Theorem 11.1 (common with Marcin Sabok and Jindra Zapletal)**.** *Suppose that $n < \omega$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ are Δ^1_1 equivalence relations on $\mathscr{N}$, and $A \subseteq \mathscr{N}$ is a Σ^1_1 set. Then one and only one of the following two claims holds*:

(I) *the set A is $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded — and therefore Δ^1_1-effectively $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded as in Lemma 9.2*;

(II) *there exists an $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-superperfect set $P \subseteq A$.*

If $n = 0$ then this theorem is equivalent to Theorem 4.1: indeed, if $\mathscr{F} = \varnothing$ then $\varnothing$-σ-bounded sets are just σ-bounded, while $\varnothing$-superperfect sets are just superperfect.

The following key result of Solecki – Spinas [25, Theorem 2.1 and Corollary 2.2] will be an essential pre-requisite in the proof of Theorem 11.1.

Theorem 11.2. *Suppose that $E \subseteq \mathscr{N} \times \mathscr{N}$ and (*) there is a decomposition $E = \bigcup_n E_n$ such that*

(i) *if $n < \omega$ and $U \subseteq \mathscr{N} \times \mathscr{N}$ is open then the projection* $\mathtt{proj}\,(E_n \cap U)$ *has the Baire property in $\mathscr{N}$*;

(ii) *if $n < \omega$ and $a \in \mathscr{N}$ then the cross-section $(E_n)_a = \{x : \langle a, x\rangle \in E_n\}$ is bounded* (= covered by a compact set).

*Then there is a superperfect set $P \subseteq \mathscr{N}$ **free for** E in the sense that if $x \neq y$ belong to P then $\langle x, y\rangle \notin E$.* $\square$

Corollary 11.3. *If $E \subseteq \mathscr{N} \times \mathscr{N}$ is a $\boldsymbol{\Sigma}^1_1$ set and each cross-section $(E)_a$, $a \in \mathscr{N}$, is σ-bounded, then there is a superperfect set $P \subseteq \mathscr{N}$ free for E. In particular, if E is a $\boldsymbol{\Sigma}^1_1$ equivalence relation on $\mathscr{N}$ with all E-equivalence classes σ-bounded then there is a superperfect pairwise E-inequivalent set.*

Proof (see [25])**.** By Theorem 7.4 E admits a decomposition satisfying (*) of Theorem 11.2. We also note that if E is an equivalence relation then a set free for E is the same as a pairwise E-inequivalent set. $\square$

Proof (Theorem 11.1)**.** We argue by induction on n. The case $n = 0$ (then $\{\mathsf{F}_1, \dots, \mathsf{F}_n\} = \varnothing$) is covered by Theorem 4.1. Now the step $n \to n+1$.

Let $\mathsf{F}_1, \dots, \mathsf{F}_n, \mathsf{F}_{n+1}$ be Δ^1_1 equivalence relations on $\mathscr{N}$, and $A \subseteq \mathscr{N}$ be a Σ^1_1 set. The set

$$\boldsymbol{U} = \{x \in A : [x]_{\mathsf{F}_1} \text{ is non-}\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}\text{-}\sigma\text{-bounded}\}$$

is Σ^1_1 by Corollary 9.3.

Case 1: the Σ^1_1 set $\boldsymbol{U}$ has only countably many F_1-classes. Then by Lemma 10.3, there is an F_1-invariant Δ^1_1 set D such that $\boldsymbol{U} \subseteq D$, D contains only countably many F_1-classes, and all of them are Δ^1_1.

Subcase 1.1: the complementary Σ^1_1 set $B = A \smallsetminus D$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. Then the whole domain $A = D \cup B$ is $\{\mathsf{F}_1, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded, hence we have (I) for $\mathsf{F}_1, \dots, \mathsf{F}_n, \mathsf{F}_{n+1}$.

Subcase 1.2: B is non-$\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. By the inductive hypothesis there is an $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect set $P \subseteq B$. Let $x \in P$. Then the class $[x]_{\mathsf{F}_1}$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. We claim that the set $P_x = [x]_{\mathsf{F}_1} \cap P$ is just σ-bounded. Indeed by definition $P_x \subseteq Y \cup \bigcup_k X_k$, where Y is σ-bounded while each X_k is an $\mathsf{F}_{n(k)}$-equivalence class for some $n(k) = 2, 3, \dots, n+1$. By construction P has at most one common point with each X_k. Therefore the set $P_x \smallsetminus Y$ is at most countable, hence, σ-bounded, and we are done.

Thus all F_1-classes inside P are σ-bounded. By Corollary 11.3, there is a superperfect pairwise F_1-inequivalent set $Q \subseteq P$ — then the set Q is $\{\mathsf{F}_1, \dots, \mathsf{F}_{n+1}\}$-superperfect by construction. Thus (II) holds.

Case 2: $\boldsymbol{U}$ has uncountably many F_1-classes. Then by the Silver dichotomy [24] there exists a perfect pairwise F_1-inequivalent set $X \subseteq \boldsymbol{U}$. If $x \in X$ then by definition the class $[x]_{\mathsf{F}_1}$ is not $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. Therefore by the inductive hypothesis there exists an $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect set $Y \subseteq [x]_{\mathsf{F}_1}$, above and hence a superperfect tree $T \subseteq \omega^{<\omega}$ such that $[T] = Y$. The next step is to get such a tree T by means of a Borel function defined on a smaller domain.

Lemma 11.4. *In our assumptions, there is a perfect set $X' \subseteq X$ and a Borel map $x \longmapsto T_x$ defined on X', such that if $x \in X'$ then T_x is a superperfect tree, $[T_x] \subseteq [x]_{\mathsf{F}_1}$, and $[T_x]$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect.*

Proof (Lemma)**.** Let $p \in \mathscr{N}$ be a parameter such that X is $\Pi^0_1(p)$.

Let $\mathbf{V}$ be the set universe considered, and let $\mathbf{V}^+$ be a generic extension of $\mathbf{V}$ such that $\omega_1^{\mathbf{L}[p]}$ is countable in $\mathbf{V}^+$. Let X^+ be the $\mathbf{V}^+$-extension of X, so that X^+ is $\Pi^0_1(p)$ in $\mathbf{V}^+$ and $X = X^+ \cap \mathbf{V}$. Let F_i^+ be a similar extension of F_i. It is true then in $\mathbf{V}^+$ by the Shoenfield absoluteness that each F_i^+ is a Δ^1_1 equivalence relation on $\mathscr{N}$, and X^+ is a perfect set in $\Pi^0_1(p)$. Moreover, it is true in $\mathbf{V}^+$ by the Shoenfield absoluteness that

$(*)$ if $x \in X^+$ then the F_1^+-class $[x]_{\mathsf{F}_1^+}$ is not $\{\mathsf{F}_2^+, \dots, \mathsf{F}_{n+1}^+\}$-$\sigma$-bounded

— simply because the formula

$$\forall\, x \in X\ ([x]_{\mathsf{F}_1} \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}\text{-}\sigma\text{-bounded})$$

is essentially Π^1_2 by Corollary 9.3, and is true in $\mathbf{V}$. It follows by the inductive hypothesis (applied in $\mathbf{V}^+$) that, in $\mathbf{V}^+$, the $\Pi^1_1(p)$ set W^+ of all pairs $\langle x, T\rangle$ such that $x \in X^+$, $T \subseteq \omega^{<\omega}$ is a superperfect tree, and

$$[T] \subseteq [x]_{\mathsf{F}_1^+} \ \wedge \ \text{the set } [T] \text{ is } \{\mathsf{F}_2^+, \dots, \mathsf{F}_{n+1}^+\}\text{-superperfect},$$

— satisfies $\mathtt{proj}\, W^+ = X^+$. Therefore by the Shoenfield absoluteness theorem the set $W = W^+ \cap \mathbf{V}$ is $\Pi^1_1(p)$ and satisfies $\mathtt{proj}\, W = X$ in $\mathbf{V}$.

Applying the Kondo — Addison uniformization in $\mathbf{V}^+$, we get a $\Pi^1_1(p)$ set $U^+ \subseteq W^+$ which uniformizes W^+, in particular, $\mathtt{proj}\, U^+ = \mathtt{proj}\, W^+ = X^+$. The corresponding set $U = U^+ \cap \mathbf{V}$ of type $\Pi^1_1(p)$ in $\mathbf{V}$ then uniformizes W and satisfies $\mathtt{proj}\, U = \mathtt{proj}\, W = X$ still by Shoenfield.

Now, by the choice of the universe $\mathbf{V}^+$, the uncountable $\Pi^1_1(p)$ set U^+ must contain a perfect subset $P^+ \subseteq U^+$ of class $\Pi^0_1(q)$ for a parameter $q \in \mathbf{L}[p]$, hence, $q \in \mathbf{V}$. The according set $P = P^+ \cap \mathbf{V}$ is then a perfect subset of U in $\mathbf{V}$, and hence $X' = \mathtt{proj}\, P \subseteq X$ is a perfect set.

Finally, if $x \in X'$ then let T_x be the only element such that $\langle x, T_x\rangle \in P$. The map $x \longmapsto T_x$ is Borel. On the other hand, still by the Shoenfield absoluteness, if $x \in X'$ then $[T_x] \subseteq [x]_{\mathsf{F}_1}$ and the set $[T_x]$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect. $\square$ (Lemma)

We continue the proof of Theorem 11.1.

Let $X' \subseteq X$ and a Borel map $x \longmapsto T_x$ be as in the lemma. If $x \in X'$ and $i = 2, \dots, n+1$, then every F_i-class $[y]_{\mathsf{F}_i}$ has at most one point common with the set $Y_x = [T_x]$. Thus if C is a $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded set then the intersection $C \cap Y_x$ is σ-bounded and hence $C \cap Y_x$ is meager in Y_x.

There is a Borel set $W \subseteq X' \times \mathscr{N}$ such that the collection of all cross-sections $(W)_x$, $x \in X'$, is equal to the family of all countable unions of

F_i-classes, $i = 2, \ldots, n+1$, plus a σ-bounded $\mathbf{F}_\sigma$ set. (Note that σ-bounded $\mathbf{F}_\sigma$ sets is the same as σ-compact sets, and that every σ-bounded set is a subset of a σ-bounded $\mathbf{F}_\sigma$ set.) Thus if $x \in X'$ then $(W)_x \cap Y_x$ is meager in Y_x by the above. Therefore, by a version of "comeager uniformization", there is a Borel map f defined on X' such that if $x \in X'$ then $f(x) \in Y_x \smallsetminus (W)_x$. Clearly f is $1-1$, hence the set $R = \{f(x) : x \in X'\}$ is Borel.

Moreover R is pairwise F_1-inequivalent by construction. We assert that R is non-$\{\mathsf{F}_2, \ldots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded, in particular, not σ-bounded!

Indeed suppose otherwise. Then there is $x \in X'$ such that $R \subseteq (W)_x$. But then $f(x) \in (W)_x$, which contradicts the choice of f.

Thus indeed R is non-$\{\mathsf{F}_2, \ldots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. It follows by the inductive hypothesis that there exists a $\{\mathsf{F}_2, \ldots, \mathsf{F}_{n+1}\}$-superperfect set $P \subseteq R$. And P is pairwise F_1-inequivalent since so is R. We conclude that P is even $\{\mathsf{F}_1, \ldots, \mathsf{F}_{n+1}\}$-superperfect, which leads to (II) of the theorem.

$\square$ (Theorem 11.1)

It is an interesting problem to figure out whether Theorem 11.1 is true for a countable infinite family of equivalence relations (as in Lemma 9.2). The inductive proof presented above is of little help, of course.

Another problem is to figure out whether the theorem still holds for $\mathbf{\Pi}^1_1$ equivalence relations, as the classical Silver dichotomy does. This is open even for the case of one $\mathbf{\Pi}^1_1$ equivalence relation, since the background result, Corollary 11.3, does not cover this case.

And finally we don't know whether Theorem 11.1 can be strengthened to yield the existence of sets free (as in Corollary 11.3) for a given (finite or countable) collection of Borel sets.

It remains to note that Theorem 11.1 (in its relativized form) implies the following theorem, perhaps, not known previously in such a generality.

Theorem 11.5. *Suppose that $\mathsf{F}_1, \ldots, \mathsf{F}_n$ are Borel equivalence relations on a Polish space $\mathbb{X}$, and $A \subseteq \mathbb{X}$ is a $\mathbf{\Sigma}^1_1$ set. Then either A is $\{\mathsf{F}_1, \ldots, \mathsf{F}_n\}$-$\sigma$-bounded, or there exists an $\{\mathsf{F}_1, \ldots, \mathsf{F}_n\}$-superperfect set $P \subseteq A$.* $\square$

Yet the case $n = 1$ is known in the form of the following (not yet published) superperfect dichotomy theorem of Zapletal:

Theorem 11.6. *If E be a Borel equivalence relation on $\mathscr{N}$ and $A \subseteq \mathscr{N}$ is a $\mathbf{\Sigma}^1_1$ set then either A is covered by countably many E-classes and a σ-bounded set or there is a superperfect pairwise E-inequivalent set $P \subseteq A$.* $\square$

Theorem 11.6 can be considered as a "superperfect" version of Silver's dichotomy (see [24] or [13, 10.1]), saying that if E is a Borel equivalence relation then either the domain of E is a union of countably many E-classes or there is a perfect pairwise E-inequivalent set $Y \subseteq D$.

12 The case of Σ^1_2 sets: preliminaries

In view of the counterexamples in Section 8, one can expect that positive results for Σ^1_2 sets similar to Theorems 4.1, 11.1, 5.1 should be expected in terms of ω_1-unions of compact sets. And indeed using a determinacy-style argument, Kechris proved in [15] that if $A \subseteq \mathscr{N}$ is a Σ^1_2 set then (in a somewhat abridged form) one of the following two claims holds:

(I) A is $\mathbf{L}$-*σ-bounded*, in the sense that it is covered by the union of all sets $[T]$, where $T \in \mathbf{L}$ is a compact tree[4] (hence not necessarily a countable union) — or equivalently, for each $x \in A$ there is $y \in \mathscr{N} \cap \mathbf{L}$ with $x \leq^* y$, where $\leq^*$ is the eventual domination order on $\mathscr{N}$,

(II) there is a superperfect set $P \subseteq A$.

Our next goal is to generalise this result in the directions of Theorem 11.1. The logic of such a generalization forces us to change superperfect sets in (II) by $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-superperfect sets, where $\mathsf{F}_1, \dots, \mathsf{F}_n$ is a given collection of Δ^1_1 equivalence relations. As for a corresponding change in (I), one would naturally look for a condition like:

> for each $x \in A$, either there is $y \in \mathscr{N} \cap \mathbf{L}$ with $x \leq^* y$, or there is $j = 1, \dots, n$ and an "$\mathbf{L}$-presented" F_j-equivalence class containing x,

whatever being "$\mathbf{L}$-presented" would mean. The following example shows that the most elementary definition of "$\mathbf{L}$-presented" as "containing a constructible element" fails.

Example 12.1. Let F be the equivalence relation of equality of countable sets of reals. That is, its domain is the set $\mathscr{N}^\omega$ of all infinite sequences of reals, and for $x, y \in \mathscr{N}^\omega$, $x \mathrel{\mathsf{F}} y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$. Let $f : \omega \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$ is a generic collapse map. In $\mathbf{L}[f]$, let A be the Σ^1_2 set of all $x \in \mathscr{N}^\omega$ such that $\mathtt{ran}\, x$ (a set of reals) belongs to $\mathbf{L}$ (but x itself does not necessarily belong to $\mathbf{L}$). Then, if $x \in A$ then the F-class $[x]_{\mathsf{F}}$ is not σ-bounded, and the quotient A/F (the set of all F-classes inside A) is uncountable in $\mathbf{L}[f]$.

We believe that there is no perfect (let alone superperfect) pairwise F-inequivalentset $P \subseteq A$ in $\mathbf{L}[f]$, which is quite a safe conjecture in view of

[4] $\mathbf{L}$ is the constructible universe.

the results in [4]. Yet to make the example self-contained let us add to $\mathbf{L}[f]$ a set C of $\aleph_3^{\mathbf{L}} = \aleph_2^{\mathbf{L}[f]}$ Cohen reals. By a simple cardinality argument, there are no perfect pairwise F-inequivalent sets $P \subseteq A$ in $\mathbf{L}[f, C]$.

However, in $\mathbf{L}[f, C]$, the quotient A/F has uncountably many particular F-classes which are non-σ-bounded and even non-$\mathbf{L}$-σ-bounded in the sense of (I) above, but contain no constructible elements. Thus A neither contains an F-superperfect subset nor satisfies the condition that *for each* $x \in A$, *either there is* $y \in \mathscr{N} \cap \mathbf{L}$ *with* $x \leq^* y$, *or there is an* F-*equivalence class containing* x *and containing a constructible element.* □

Our model for "$\mathbf{L}$-presented" will be somewhat more complex than just "containing a constructible element". In fact we'll consider two (connected) models, one being based on a certain uniform version of Δ^1_1, with ordinals as background parameters, the other one being based on Borel coding. They are introduced in the following definitions.

Definition 12.2 (coding ordinals)**.** Let $\mathbf{WO} \subseteq \mathscr{N}$ be the Π^1_1 set of all codes of countable (including finite) ordinals, and if $\xi < \omega_1$ then we define $\mathbf{WO}_\xi = \{w \in \mathbf{WO} : w \text{ codes } \xi\}$. If $w \in \mathbf{WO}_\xi$ then put $|w| = \xi$. □

Definition 12.3. A Σ^1_2 map $h : \mathscr{N} \to \mathscr{N}$ is *absolutely total* if it remains total in any set-generic extension of the universe. In other words, it is required that there is a Σ^1_2 formula $\sigma(\cdot,\cdot)$ such that $h = \{\langle x, y\rangle : \sigma(x, y)\}$ and the sentence $\forall x \exists y \sigma(x, y)$ is forced by any set forcing. (Note that a total but not absolutely total map can be defined in $\mathbf{L}$ by $h(x)=$ the Goedel-least $w \in \mathbf{WO}$ such that x appears at the ξ-th step of the Goedel construction, where $\xi = |w| < \omega_1$ is the ordinal coded by w.)

Suppose that $\xi < \omega_1$. A set $X \subseteq \mathscr{N}$ is *essential* $\Sigma^1_n(\xi)$ if there is a Σ^1_n formula $\varphi(x, w)$ such that $X = \{x \in \mathscr{N} : \varphi(x, w)\}$ for every $w \in \mathbf{WO}_\xi$. Essential $\Pi^1_n(\xi)$ sets are defined similarly, while an essential $\Delta^1_n(\xi)$ set is any set both essential $\Sigma^1_n(\xi)$ and essential $\Pi^1_n(\xi)$.

A set X is *essential* $(\Delta^1_1/\Delta^1_2)(\xi)$ if there is an absolutely total Σ^1_2 map h, a Σ^1_1 formula $\chi(\cdot,\cdot)$, and a Π^1_1 formula $\chi'(\cdot,\cdot)$, such that if $w \in \mathbf{WO}_\xi$ then $X = \{x \in \mathscr{N} : \chi(x, h(w))\} = \{x \in \mathscr{N} : \chi'(x, h(w))\}$. □

Thus essential $(\Delta^1_1/\Delta^1_2)(\xi)$ sets belong in between essential $\Delta^1_1(\xi)$ and essential $\Delta^1_2(\xi)$. Each essential $(\Delta^1_1/\Delta^1_2)(\xi)$ set X is Borel, hence, it admits a Borel code. Moreover, if X is essential $(\Delta^1_1/\Delta^1_2)(\xi)$ via an absolutely total Σ^1_2 map h, and $w \in \mathbf{WO}_\xi$, then X admits a Borel code in $\mathbf{L}[w]$. We'll show (see 14.2 and 14.4) that such a set X admits a Borel code, even in $\mathbf{L}$, in some generalized sense which allows uncountable Borel operations.

Definition 12.4. Let $\mathbf{Ord}^{<\omega}$ be the class of all strings (finite sequences) of ordinals. If $s \in \mathbf{Ord}^{<\omega}$ and $\xi \in \mathbf{Ord}$ then $s^\wedge\xi$ denotes the string s extended by ξ. If $s \in \mathbf{Ord}^{<\omega}$ then $\mathtt{lh}\, s$ is the length of s. Λ is the empty string.

A set $T \subseteq \mathbf{Ord}^{<\omega}$ is a *tree* if $T \neq \varnothing$, and for any $s \in T$ and $m < \mathtt{lh}\, s$ we have $s \restriction m \in T$. Then let $\sup T$ be the least ordinal λ such that $T \subseteq \lambda^{<\omega}$, and let $\mathtt{Max}\, T$ be the set of all $\subseteq$-maximal elements $s \in T$.

If a tree T is *well-founded* then a *rank function* $s \longmapsto |s|_T \in \mathbf{Ord}$ can be associated with T so that $|t|_T = \mathtt{sup}_{t^\wedge\xi \in T}(|t^\wedge\xi|_T + 1)$ (the least ordinal strictly bigger than all ordinals of the form $|t^\wedge\xi|_T$, where $\xi \in \mathbf{Ord}$ and $t^\wedge\xi \in T$) for each $t \in T$. In particular $|s|_T = 0$ for any $s \in \mathtt{Max}\, T$.

Let $|T| = |\Lambda|_T$ (the *rank* of T).

Let $\mathbb{K}$ be the class of all *generalized Borel codes* in $\mathbf{L}$, that is, all pairs $c = \langle T, d\rangle = \langle T_c, d_c\rangle \in \mathbf{L}$, where $T \subseteq \mathbf{Ord}^{<\omega}$ is a well-founded tree and $d \subseteq T \times \omega^{<\omega}$. In this case, a set $[T, d, s] \subseteq \mathscr{N}$ can be defined for each $s \in T$ by induction on $|s|_T$ so that

$$[T,d,s] = \begin{cases} \mathscr{N} \smallsetminus \bigcup_{\langle s,u\rangle \in d} \mathscr{N}_u & \text{, whenever} \quad s \in \mathtt{Max}\, T\,; \\ \mathscr{N} \smallsetminus \bigcup_{s^\wedge\xi \in T} [T,d,s^\wedge\xi] & \text{, whenever} \quad |s|_T > 0\,. \end{cases}$$

Recall that $\mathscr{N}_u = \{a \in \mathscr{N} : u \subset a\}$ is a Baire interval.

Finally we put $[T, d] = [T, d, \Lambda]$.

If $\rho < \omega_1$ then let $\mathbb{K}_\rho \in \mathbf{L}$ be the set of all codes $\langle T, d\rangle \in \mathbb{K}$ such that $|T| \le \rho$ and $\sup T \le \omega_\rho^{\mathbf{L}}$. (Not necessarily $\sup T < \omega_1$.)

Accordingly let $[\mathbb{K}_\rho] = \{[T, d] : \langle T, d\rangle \in \mathbb{K}_\rho\}$. □

If $\langle T, d\rangle \in \mathbb{K}$ and $\sup T < \omega_1$ then $[T, d]$ is a Borel set in $\mathbf{\Pi}^0_{1+|T|}$.

We underline that only *constructible* codes are considered.

13 The case of Σ^1_2 sets: the result

We'll prove the next theorem which generalizes the result of Kechris in [15] sited above. If F is an equivalence relation on $\mathscr{N}$ then let a *σ-F-class* be any finite or countable union of F-equivalence classes.

Theorem 13.1. *Let $n < \omega$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ be Δ^1_1 equivalence relations on $\mathscr{N}$, and $A \subseteq \mathscr{N}$ be a Σ^1_2 set. Then one of the following* (I), (II) *holds*:

(I) *A is $\mathbf{L}$-$\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, in the sense that for each $x \in A$:*

- *either there is $y \in \mathscr{N} \cap \mathbf{L}$ such that $x \le^* y$,*
- *or (non-exclusively) we have both* (I)a *and* (I)b*, where*

(a) *there is* $j = 1, \dots, n$ *and a* σ-F_j*-class* C *which contains* x *and is essential* $(\varDelta^1_1/\varDelta^1_2)(\xi)$ *for some* $\xi < \omega_1$,

(b) *if, in addition,* $\rho < \omega_1^{\mathbf{L}}$ *and all* F_j *are* $\mathbf{\Pi}^0_{1+\rho}$ *then there is* $j = 1, \dots, n$ *and an* F_j*-class* X *in* $[\mathbb{K}_\rho]$ *which contains* x ;

(II) *there exists an* $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$*-superperfect set* $P \subseteq A$.

A point of certain dissatisfaction is $\omega_\rho^{\mathbf{L}}$ as the measure of borelness in Definition 12.4 and subsequently in (I)b of Theorem 13.1. Can it be reduced, to present the borelness involved by considerably narrower trees (of the same height)? Examples given in [22] and more resently in [3] allow to conjecture that the value $\omega_\rho^{\mathbf{L}}$ cannot be reduced in any essential way. A similar question can be addressed to the inequality $\omega_{\rho+1}^{\mathbf{L}} < \omega_1$ in the next remark.

Suppose that $\omega_{\rho+1}^{\mathbf{L}} < \omega_1$. Then both $\mathscr{N} \cap \mathbf{L}$ and $\mathbb{K}_\rho$ are countable sets, and hence the number of points y and classes X involved in (I) of Theorem 13.1. Thus, assuming $\omega_{\rho+1}^{\mathbf{L}} < \omega_1$, condition (I) of Theorem 13.1 can be replaced by just the $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-boundedness of A.

The proof of Theorem 13.1 will consist of two major parts. First of all, we prove, in this section, the version sans condition (I)b in (I) of the theorem. Then we prove, in Section 14, that (I)a implies (I)b in (I).

Proof (Theorem 13.1 sans (I)b)**.** We'll make use of Theorems 10.1 and 11.1 in key arguments. To begin with, we reveal a certain uniformity in Theorem 10.1(iii), which was not convenient to deal with in Section 10.

Proposition 13.2. *Under the conditions of Theorem 10.1, if* $\xi < \omega_1$*, all relations* F_j *are essential* $\varDelta^1_1(\xi)$*, and* X *is essential* $\varSigma^1_1(\xi)$*, then the sets* Y_j *and* X_{n+1} *satisfying* (i), (ii) *can be chosen to be essential* $(\varDelta^1_1/\varDelta^1_2)(\xi)$.

Proof (Sketch)**.** We come back to the proof of Theorem 10.1. The map $h(p) = \bar{p}$ is defined in finitely many steps, such that each step is governed by a combination of $\varSigma^1_1$ and $\varPi^1_1$ formulas, so it is absolutely total $\varSigma^1_2$. $\square$

In continuation, note that, by Kondo's uniformization, A is the projection of a uniform $\varPi^1_1$ set $B \subseteq \mathscr{N} \times 2^\omega$. Let $B = \bigcup_{\xi<\omega_1} B_\xi$ be an ordinary decomposition of B into pairwise disjoint Borel sets B_ξ (called *constituents*). There is a $\varSigma^1_1$ formula $\beta(w, x, y)$ and a $\varPi^1_1$ formula $\beta'(w, x, y)$ such that

(A) if $\xi < \omega_1$ and $w \in \mathbf{WO}_\xi$ then
$$B_\xi = \{\langle x, y\rangle : \beta(w, x, y)\} = \{\langle x, y\rangle : \beta'(w, x, y)\}.$$

We put $A_\xi = \mathtt{proj}\, B_\xi$; then $A = \bigcup_{\xi<\omega_1} A_\xi$ and all sets A_ξ are Borel, and moreover A_ξ is $\varDelta^1_1(w)$ whenever $w \in \mathbf{WO}_\xi$, because

(B) if $\xi < \omega_1$ and $w \in \mathbf{WO}_\xi$ then

$$x \in A_\xi \iff \underbrace{\exists\, y\, \beta(w,x,y)}_{\alpha(w,x)} \iff \underbrace{\exists\, y \in \Delta^1_1(x,w)\, \beta'(w,x,y)}_{\alpha'(w,x)},$$

where $\alpha(\cdot,\cdot)$ is a Σ^1_1 formula while $\alpha'(\cdot,\cdot)$ is a Π^1_1 formula.

Case 1: There is an ordinal $\xi < \omega_1$ such that A_ξ is **not** $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded. Then we have (II) of the theorem by Theorem 11.1.

Case 2: All sets A_ξ are $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded. We claim that, *under this assumption, if $\xi < \omega_1$ then the set A_ξ is $\mathbf{L}$-$\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded in the sense of* (I) *of Theorem 13.1, hence* (I) *of Theorem 13.1 holds for A.*

To prove the claim, fix $\xi < \omega_1$.

The set $A_\xi = \{x : \alpha(w,x)\} = \{x : \alpha'(w,x)\}$ (for any $w \in \mathbf{WO}_\xi$) is essential $\Delta^1_1(\xi)$. The relations F_j are just Δ^1_1. By Theorem 10.1 and Proposition 13.2 there exist Borel sets $Y_1,\dots,Y_n,X_{n+1} \subseteq \mathscr{N}$ satisfying (i), (ii) of Theorem 10.1 for $X = A_\xi$, and essential $(\Delta^1_1/\Delta^1_2)(\xi)$. Thus X_{n+1} is σ-bounded, each set Y_j is a σ-F_j-class, and $A_\xi \subseteq Y_1 \cup \dots \cup Y_n \cup X_{n+1}$.

Now the claim and Theorem 13.1 immediately follow from:

Lemma 13.3. *If $y \in X_{n+1}$ then there is a real $a \in \mathbf{L}$ such that $y \le^* a$.*

Proof (Lemma)**.** Let $\mathbf{V}$ be the whole set universe in which we prove the lemma. Thus $\xi < \omega_1^{\mathbf{V}}$ but not necessarily $\xi < \omega_1^{\mathbf{L}}$.

Case L1: $\xi < \omega_1^{\mathbf{L}}$. We assert that the set $(X_{n+1})^{\mathbf{L}} = X_{n+1} \cap \mathbf{L}$ is σ-bounded in $\mathbf{L}$. Indeed otherwise by Theorem 4.1 (relativized) there is a superperfect tree $T \in \mathbf{L}$ such that $[T] \subseteq (X_{n+1})^{\mathbf{L}}$ in $\mathbf{L}$. Then by Shoenfield $[T] \subseteq X_{n+1}$ in the universe, contrary to the σ-boundedness of X_{n+1}.

Thus $(X_{n+1})^{\mathbf{L}}$ is σ-bounded in $\mathbf{L}$, so there is a real $a \in \mathbf{L}$ such that if $y \in (X_{n+1})^{\mathbf{L}}$ then $y \le^* a$. Then, again by Shoenfield, it is true in the universe that if $y \in X_{n+1}$ then $y \le^* a$, as required.

Case L2: $\xi \ge \omega_1^{\mathbf{L}}$. Recall that the set X_{n+1} is essential $(\Delta^1_1/\Delta^1_2)(\xi)$, via a certain absolutely total Σ^1_2 map $h(w) = \bar{w}$ and formulas $\chi(\cdot,\cdot)$ of type Σ^1_1 and $\chi'(\cdot,\cdot)$ of type Π^1_1, as in Definition 12.4, so that

(C) if $w \in \mathbf{WO}_\xi$ then $X_{n+1} = \{x : \chi(\bar{w},x)\} = \{x : \chi'(\bar{w},x)\}$.

Let $f,g \in \xi^\omega$ be collapse functions generic over $\mathbf{V}$, such that the pair $\langle f,g\rangle$ is generic over $\mathbf{V}$ as well. Then $\xi < \omega_1^{\mathbf{L}[f]}$ and $\xi < \omega_1^{\mathbf{L}[g]}$. For X_{n+1} being σ-bounded is a Σ^1_2 formula (make use of (C) and Corollary 9.3), hence by Shoenfield it is true in $\mathbf{L}[f]$ that the set

$$(X_{n+1})^{\mathbf{L}[f]} = \{x \in \mathbf{L}[f] : \chi(\bar{w},x)\} = \{x \in \mathbf{L}[f] : \chi'(\bar{w},x)\}$$

($w \in \mathbf{WO}_\xi \cap \mathbf{L}[f]$ is arbitrary) is σ-bounded. Thus there is a real $x \in \mathscr{N} \cap \mathbf{L}[f]$ such that $y \leq^* x$ for all $y \in (X_{n+1})^{\mathbf{L}[f]}$. Then again by the Shoenfield absoluteness $y \leq^* x$ holds even for all $y \in (X_{n+1})^{\mathbf{L}[f,g]}$.

In particular if $y \in (X_{n+1})^{\mathbf{L}[g]}$ then $y \leq^* x$.

On the other hand, as f, g are mutually generic, one can show that if $x \in \mathscr{N} \cap \mathbf{L}[f]$, $y \in \mathscr{N} \cap \mathbf{L}[g]$, and $y \leq^* x$ then there is a real $a \in \mathscr{N} \cap \mathbf{L}$ with $y \leq^* a \leq^* x$. We conclude that if $y \in (X_{n+1})^{\mathbf{L}[g]}$ then there is a real $a \in \mathbf{L}$ such that $y \leq^* a$. Now, as $\omega_1^{\mathbf{L}} \leq \xi < \omega_1^{\mathbf{L}[g]}$, there exists a sequence $\{a_n\}_{n<\omega} \in \mathbf{L}[g]$ such that $\mathscr{N} \cap \mathbf{L} = \{a_n : n < \omega\}$. Therefore we have

$$\forall\, y \in (X_{n+1})^{\mathbf{L}[g]}\, \exists\, n\, (y \leq^* a_n)$$

in $\mathbf{L}[g]$, and hence by Shoenfield $\forall\, y \in (X_{n+1})^{\mathbf{V}[g]}\, \exists\, n\, (y \leq^* a_n)$ in $\mathbf{V}[g]$. But as $\xi < \omega_1^{\mathbf{V}}$, there is a sequence $\{b_n\}_{n<\omega} \in \mathbf{V}$ such that $\mathscr{N} \cap \mathbf{L} = \{b_n : n < \omega\}$. Then we have $\forall\, y \in (X_{n+1})^{\mathbf{V}[g]}\, \exists\, n\, (y \leq^* b_n)$ in $\mathbf{V}[g]$, and by Shoenfield $\forall\, y \in X_{n+1}\, \exists\, n\, (y \leq^* b_n)$ in $\mathbf{V}$, and so on. $\square$ (Lemma)

$\square$ (Theorem 13.1 sans (I)b)

14 The case of Σ^1_2 sets: proof of the second part

To demonstrate that the abridged version of Theorem 13.1 implies the full version, it suffices to prove the following theorem.

Theorem 14.1. *Assume that, in the ground set universe* $\mathbf{V}$*,*

($*$) $\rho < \omega_1^{\mathbf{L}}$*,* $\xi < \omega_1$*,* E *is an equivalence relation on* $\mathscr{N}$ *in* $\Delta^1_1 \cap \mathbf{\Pi}^0_{1+\rho}$*,* $\varnothing \neq C \subseteq \mathscr{N}$ *is a* σ-E*-class and a set essential* $(\Delta^1_1/\Delta^1_2)(\xi)$*.*

Then each E*-class* $X \subseteq C$ *is a set in* $[\mathbb{K}_\rho]$*.*

Any essential $(\Delta^1_1/\Delta^1_2)(\xi)$ set is essential $\Delta^1_2(\xi)$, and hence $\Delta^{\mathrm{HC}}_1(\xi)$. (Recall that HC is the set of all hereditarily countable sets.) This simple fact will allow us to make use of the following result, explicitly proved in [8] (Lemma 4) on the base of ideas and technique developed in [29, 30].

Proposition 14.2. *Let* $X, Y \subseteq \mathscr{N}$ *are two disjoint sets in* $\Sigma^{\mathrm{HC}}_1(\omega_1)$ (that is, Σ^{HC}_1 with any finite number of parameters in ω_1)*. Suppose that* $\rho < \omega_1^{\mathbf{L}}$ *and* X *is* $\mathbf{\Pi}^0_{1+\rho}$*-separable from* Y*. Then there is a separating set in* $[\mathbb{K}_\rho]$*.*

In particular if $Z \subseteq \mathscr{N}$ *is a set in* $\Delta^{\mathrm{HC}}_1 \cap \mathbf{\Pi}^0_{1+\rho}$ *then* $Z \in [\mathbb{K}_\rho]$*.* $\square$

For instance, if $\rho = 0$, so that $\mathbf{\Pi}^0_{1+\rho}$ = closed sets, then the result takes the form: any closed Δ^{HC}_1 set $Z \subseteq \mathscr{N}$ has a code in the set

$$\mathbb{K}_0 = \{\langle T, d\rangle \in \mathbb{K} : |T| = 0 \text{ (hence just } T = \{\Lambda\}) \ \wedge\ \sup T \le \omega\},$$

but this can be easily established directly.

Thus sets essential $(\varDelta^1_1/\varDelta^1_2)(\xi)$, $\xi < \omega_1$, even those essential $\varDelta^1_2(\xi)$, admit a straight Borel coding by (not necessarily *countable*) codes in $\mathbf{L}$.

Proof (Theorem 14.1)**.** Assume that ρ, ξ, E, C are as in $(*)$ above. Then C is $\mathbf{\Sigma}^0_{1+\rho+1}$, therefore by Lemma 14.4 C belongs to $[\mathbb{K}_{\rho+2}]$ in $\mathbf{V}$. We'll show now that an appropriate coding can be chosen in **absolute** manner.

Remark 14.3. Suppose that our set C is essential $(\varDelta^1_1/\varDelta^1_2)(\xi)$, via an absolutely total $\varSigma^1_2$ map h and formulas χ, χ' as in Definition 12.3. Then the following is true in the ground universe $\mathbf{V}$:

($\dagger$) if $v, w \in \mathbf{WO}_\xi$ and $x \in \mathscr{N}$ then
$$\chi(x, h(v)) \iff \chi(x, h(w)) \iff \chi'(x, h(v)) \iff \chi'(x, h(w)).$$

If we eliminate h by a formula σ as in Definition 12.3 then ($\dagger$) becomes a $\varPi^1_2$ sentence. Therefore ($\dagger$) is true in any extension $\mathbf{V}[G]$ of $\mathbf{V}$ by Shoenfield, and moreover, in any generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ such that $\xi < \omega_1^{\mathbf{L}[G]}$. This allows us to unambiguously define extensions $h^{\mathbf{V}[G]}$ of h (a total map) and $C^{\mathbf{V}[G]}$ of C to $\mathbf{V}[G]$, using the same formulas, so that $C^{\mathbf{V}[G]}$ is an essential $(\varDelta^1_1/\varDelta^1_2)(\xi)$ set in $\mathbf{V}[G]$ still via $h^{\mathbf{V}[G]}, \chi, \chi'$. Then, assuming $\xi < \omega_1^{\mathbf{L}[G]}$, we define associated restrictions $h^{\mathbf{L}[G]} = h^{\mathbf{V}[G]} \cap \mathbf{L}[G]$ and $C^{\mathbf{L}[G]} = C^{\mathbf{V}[G]} \cap \mathbf{L}[G]$ to $\mathbf{L}[G]$, so that $C^{\mathbf{L}[G]}$ is essential $(\varDelta^1_1/\varDelta^1_2)(\xi)$ in $\mathbf{L}[G]$ via $h^{\mathbf{L}[G]}, \chi, \chi'$, too.

And as E is a $\varDelta^1_1$ equivalence relation in $\mathbf{V}$, then, even easier, we define an extension $\mathsf{E}^{\mathbf{V}[G]}$ of E to $\mathbf{V}[G]$, using the same formulas which define E, so that $\mathsf{E}^{\mathbf{V}[G]}$ is a $\varDelta^1_1$ equivalence relation in $\mathbf{V}[G]$ by Shoenfield, and then define $\mathsf{E}^{\mathbf{L}[G]} = \mathsf{E}^{\mathbf{V}[G]} \cap \mathbf{L}[G]$ (a $\varDelta^1_1$ equivalence relation in $\mathbf{L}[G]$). □

Lemma 14.4. *C **absolutely** belongs to $[\mathbb{K}_{\rho+2}]$, in the sense that there is a code $\langle T, d\rangle \in \mathbb{K}_{\rho+2}$ such that we have $C^{\mathbf{V}[G]} = [T, d]$ in any set generic extension $\mathbf{V}[G]$ of the universe $\mathbf{V}$.*

Note that then by Shoenfield the equality $C^{\mathbf{L}[G]} = [T, d]$ also holds in any generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ such that $\xi < \omega_1^{\mathbf{L}[G]}$.

Proof (Lemma)**.** Let a map $f : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+1}$ be collapse generic over $\mathbf{V}$. Let $C^{\mathbf{V}[f]} \in \mathbf{V}[f]$ be the extension of C to $\mathbf{V}[f]$, as above. Then $C^{\mathbf{V}[f]}$ is essential $(\varDelta^1_1/\varDelta^1_2)(\xi)$ in $\mathbf{V}[f]$, and hence by Proposition 14.2 there is a code $\langle T, d\rangle \in \mathbb{K}_\rho$ such that $C^{\mathbf{V}[f]} = [T, d]$ in $\mathbf{V}[f]$. To prove, that this code witnesses that C absolutely belongs to $[\mathbb{K}_\rho]$, consider any generic extension $\mathbf{V}[G]$. It can be assumed that G is generic even over $\mathbf{V}[f]$.

Let $C^{\mathbf{V}[G]}, C^{\mathbf{V}[f,G]}$ be the extensions of C (a set in $\mathbf{V}$) to resp. $\mathbf{V}[G]$, $\mathbf{V}[f,G]$ (see Remark 14.3). The code $\langle T,d\rangle$ is countable in $\mathbf{V}[f]$ and in $\mathbf{V}[f,G]$ by the choice of f. Therefore the equality $C^{\mathbf{V}[f]} = [T,d]$ can be expressed by a Shoenfield-absolute formula. We conclude that $C^{\mathbf{V}[f,G]} = [T,d]$ holds in $\mathbf{V}[f,G]$, hence $C^{\mathbf{V}[G]} = [T,d]$ is true in $\mathbf{V}[G]$ as well as easily $C^{\mathbf{V}[G]} = C^{\mathbf{V}[f,G]} \cap \mathbf{V}[G]$ and $[T,d]^{\mathbf{V}[G]} = [T,d]^{\mathbf{V}[f,G]} \cap \mathbf{V}[G]$. □ (Lemma)

It follows that there is a code $\langle T_0,d_0\rangle \in \mathbb{K}_{\rho+2}$ such that $[T_0,d_0] = C^{\mathbf{V}[G]}$ in any extension $\mathbf{V}[G]$ of $\mathbf{V}$, and hence we obtain by Shoenfield:

Corollary 14.5. *In any set-generic extension $\mathbf{V}[G]$ of $\mathbf{V}$, $[T_0,d_0]$ is a σ-$\mathsf{E}^{\mathbf{V}[G]}$-class containing only those $\mathsf{E}^{\mathbf{V}[G]}$-classes presented in $[T_0,d_0]\cap\mathbf{V}$.* □

We continue with a few definitions.

If $\langle T,d\rangle$, $\langle T',d'\rangle \in \mathbb{K}$ then let $\langle T,d\rangle \preccurlyeq \langle T',d'\rangle$ mean that $[T,d] \subseteq [T',d']$ in any set generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. Then, using appropriate collapse extensions, we conclude by Shoenfield, that $[T,d] \subseteq [T',d']$ also holds in any set generic extension $\mathbf{V}[G]$ of the ground universe $\mathbf{V}$, including $\mathbf{V}$ itself.

Say that a code $\langle T,d\rangle \in \mathbb{K}$ is *"essentially non-empty"* if $[T,d] \neq \varnothing$ in at least one set-generic extension of $\mathbf{L}$. By Shoenfield, this is equivalent to $[T,d] \neq \varnothing$ in some/any extension $\mathbf{L}[G]$ with $\sup T < \omega_1^{\mathbf{L}[G]}$.

Let $\mathbb{P} \in \mathbf{L}$ be the forcing notion which consists of all "essentially non-empty" codes $\langle T,d\rangle \in \mathbb{K}$ such that $\langle T,d\rangle \preccurlyeq \langle T_0,d_0\rangle$ and $\sup T \le \omega_{\rho+2}^{\mathbf{L}}$. We order $\mathbb{P}$ by $\preccurlyeq$, and $\langle T,d\rangle \preccurlyeq \langle T',d'\rangle$ is understood as $\langle T,d\rangle$ being a stronger forcing condition.

In particular condition $\langle T_0,d_0\rangle$ itself, as in Corollary 14.5, belongs to $\mathbb{P}$.

Lemma 14.6. *$\mathbb{P}$ forces a real over $\mathbf{L}$, so that if a set $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$ then the intersection $\bigcap_{\langle T,d\rangle\in G}[T,d]$ contains a single real in $\mathbf{L}[G]$.*

Proof. If $u \in \omega^{<\omega}$ is a string of length $n = \mathtt{lh}\,u$ then let $T^u = \{\Lambda\}$ and let d^u consist of all pairs $\langle\Lambda,v\rangle$ such that $v \in \omega^{<\omega}$, $v \neq u$, $\mathtt{lh}\,v = n$. Then $\langle T^u,d^u\rangle \in \mathbb{P}$ and $[T^u,d^u] = \mathscr{N}_u = \{a \in \mathscr{N} : u \subset a\}$. By the genericity, for any n there is a inuque $u = u[n] \in \omega^{<\omega}$ such that $\mathtt{lh}\,u[n] = n$ and $\langle T^{u[n]},d^{u[n]}\rangle \in G$, and in addition $u[n] \subset u[m]$ whenever $n < m$. It follows that there is a real $x_G = \bigcup_n u[n] \in \mathbf{L}[G]$ such that $x_G \restriction n = u[n]$, and hence $x_G \in [T^{u[n]},d^{u[n]}]$, $\forall n$. We claim that *if $\langle T,d\rangle \in \mathbb{P}$ then $\langle T,d\rangle \in G$ iff $x_G \in [T,d]$ in* $\mathbf{L}[G]$; this obviously proves the lemma.

We prove the claim by induction on the rank $|T|$.

Suppose that $|T| = 0$, so that $T = \{\Lambda\}$, $d \subseteq \{\Lambda\}\times\omega^{<\omega}$, and $[T,d] = \mathscr{N} \smallsetminus \bigcup_{v\in U}\mathscr{N}_v$, where $U = \{v \in \omega^{<\omega} : \langle\Lambda,v\rangle \in d\}$. We assert that

(1) any $\langle T', d' \rangle \in \mathbb{P}$ is compatible, in $\mathbb{P}$, either with $\langle T, d \rangle$ or with one of the codes $\langle T^v, d^v \rangle$, where $v \in U$ — therefore either $\langle T, d \rangle$ or one of the codes $\langle T^v, d^v \rangle$, $v \in U$, belongs to G.

Indeed we have $[T, d] = \mathscr{N} \setminus \bigcup_{v \in U} [T^v, d^v]$ in any universe.

With (1) in hands, if $v \in U$ and $\langle T^v, d^v \rangle \in G$ then on the one hand $\langle T, d \rangle \notin G$ by (1), and on the other hand, obviously $v = u[n]$, where $n = \mathtt{lh}\, v$, so that $x_G \in [T^v, d^v]$ and $x_G \notin [T, d]$. Conversely, if there is no $v \in U$ with $\langle T^v, d^v \rangle \in G$ then on the one hand $\langle T, d \rangle \in G$ by (1), and on the other hand, $x_G \notin \bigcup_{v \in U} [T^v, d^v]$, so that $x_G \in [T, d]$.

To carry out the step, suppose that $|T| > 0$. Let $\Xi = \{\xi : \langle \xi \rangle \in T\}$ (where $\langle \xi \rangle$ is a one-term string). If $\xi \in \Xi$ then let

$$T^\xi = \{s \in \mathbf{Ord}^{<\omega} : \xi^\wedge s \in T\} \quad \text{and} \quad d^\xi = \{\langle s, v \rangle : \langle \xi^\wedge s, v \rangle \in d\}.$$

Thus each $\langle T^\xi, d^\xi \rangle$ is a code in $\mathbb{P}$, $|T^\xi| < |T|$, and $[T, d] = \mathscr{N} \setminus \bigcup_{\xi \in \Xi} [T^\xi, d^\xi]$ in any universe containing $\langle T, d \rangle$. Similarly to (1) above, we have

(2) any $\langle T', d' \rangle \in \mathbb{P}$ is compatible, in $\mathbb{P}$, either with $\langle T, d \rangle$ or with one of the codes $\langle T^\xi, d^\xi \rangle$, where $\xi \in \Xi$ — therefore either $\langle T, d \rangle$ or one of the codes $\langle T^\xi, d^\xi \rangle$, $\xi \in \Xi$, belongs to G.

Now, if $\xi \in \Xi$ and $\langle T^\xi, d^\xi \rangle \in G$ then on the one hand $\langle T, d \rangle \notin G$ by (2), and on the other hand, $x_G \in [T^\xi, d^\xi]$ by the inductive hypothesis, and hence $x_G \notin [T, d]$. Conversely, if there is no $\xi \in \Xi$ with $\langle T^\xi, d^\xi \rangle \in G$ then on the one hand $\langle T, d \rangle \in G$ by (2), and on the other hand, $x_G \notin \bigcup_{\xi \in \Xi} [T^\xi, d^\xi]$, by the inductive hypothesis, so that $x_G \in [T, d]$. □

Reals of the form $x_G =$ the only element of $\bigcap_{\langle T, d \rangle \in G} [T, d]$ in $\mathbf{L}[G]$, where $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic, *e.g.*, over $\mathbf{V}$, will be called $\mathbb{P}$-*generic over* $\mathbf{V}$, too. Let $\mathbf{x}$ be a canonical $\mathbb{P}$-name for x_G. Let $\mathbf{x}_{\mathtt{left}}$, $\mathbf{x}_{\mathtt{right}}$ be canonical $(\mathbb{P} \times \mathbb{P})$-names for the left and the right copies of x_G.

Let $\underline{\mathsf{E}}$ be a canonical $\mathbb{P}$-name for the extension $\mathsf{E}^{\mathbf{V}[G]}$ or $\mathsf{E}^{\mathbf{L}[G]}$ of E to any class like $\mathbf{L}[G]$ or $\mathbf{V}[G]$, G being generic.

Definition 14.7. A code $\langle T, d \rangle \in \mathbb{P}$ is *stable* if condition $(\langle T, d \rangle ; \langle T, d \rangle)$ $(\mathbb{P} \times \mathbb{P})$-forces, over $\mathbf{L}$, that $\mathbf{x}_{\mathtt{left}} \mathrel{\underline{\mathsf{E}}} \mathbf{x}_{\mathtt{right}}$. □

Lemma 14.8. *If* $\langle T, d \rangle \in \mathbb{P}$ *is stable then there is an element* $y \in C = [T_0, d_0] \cap \mathbf{V}$ *such that* $\langle T, d \rangle$ $\mathbb{P}$-*forces, over* $\mathbf{V}$*, that* $\mathbf{x} \mathrel{\underline{\mathsf{E}}} y$.

Proof. By Corollary 14.5 the contrary assumption leads to a pair of conditions $\langle T', d' \rangle \preccurlyeq \langle T, d \rangle$ and $\langle T'', d'' \rangle \preccurlyeq \langle T, d \rangle$ in $\mathbb{P}$ and elements $y', y'' \in [T_0, d_0] \cap \mathbf{V}$ such that

$$\langle T', d'\rangle \ \mathbb{P}\text{-forces } \mathbf{x} \,\underline{\mathsf{E}}\, y', \ \text{ and } \ \langle T'', d''\rangle \ \mathbb{P}\text{-forces } \mathbf{x} \,\underline{\mathsf{E}}\, y'' \ \text{ — over } \mathbf{V},$$

and $y' \not\mathsf{E}\, y''$. To get a contradiction consider a set $G' \times G''$, $(\mathbb{P} \times \mathbb{P})$-generic over $\mathbf{V}$, and containing condition $(\langle T', d'\rangle ; \langle T'', d''\rangle)$. Then, on the one hand, the generic reals $x_{G'}$ and $x_{G''}$ satisfy $x_{G'} \ \mathsf{E}^{\mathbf{V}[G']} \ y'$ and $x_{G''} \ \mathsf{E}^{\mathbf{V}[G']} \ y''$, but on the other hand, $x_{G'} \ \mathsf{E}^{\mathbf{V}[G', G'']} \ x_{G''}$ holds by stability. Therefore $y' \ \mathsf{E} \ y''$, which contradicts to the choice of y', y''. □

Lemma 14.9. *The set of all stable conditions $\langle T, d\rangle \in \mathbb{P}$ is dense in $\mathbb{P}$.*

Proof. By definition $\mathtt{card}\,\mathbb{P} = \omega^{\mathbf{L}}_{\rho+3}$ and $\mathtt{card}\,\mathscr{P}(\mathbb{P}) = \omega^{\mathbf{L}}_{\rho+4}$ in $\mathbf{L}$. Consider an extension $\mathbf{V}[g]$ by a collapse-generic map $g : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+4}$. Then, in $\mathbf{V}[g]$, there is an enumeration $\{D_n\}_{n<\omega}$ of all dense sets $D \subseteq \mathbb{P} \times \mathbb{P}$, $D \in \mathbf{L}$.

Now suppose towards the contrary that $\langle T^*, d^*\rangle \in \mathbb{P}$ and there is no stable $\langle T, d\rangle \preccurlyeq \langle T^*, d^*\rangle$ in $\mathbb{P}$. Then for any condition $\langle T, d\rangle \preccurlyeq \langle T^*, d^*\rangle$ there are stronger conditions $\langle T', d'\rangle \preccurlyeq \langle T, d\rangle$ and $\langle T'', d''\rangle \preccurlyeq \langle T, d\rangle$ such that $(\langle T', d'\rangle ; \langle T'', d''\rangle)$ $(\mathbb{P} \times \mathbb{P})$-forces $\neg\, \mathbf{x}_{\mathtt{left}} \,\underline{\mathsf{E}}\, \mathbf{x}_{\mathtt{right}}$ over $\mathbf{L}$. This allows to define, in $\mathbf{V}[g]$, a family $\{\langle T(u), d(u)\rangle\}_{u \in 2^{<\omega}}$ of conditions in $\mathbb{P}$ satisfying $\langle T(\Lambda), d(\Lambda)\rangle = \langle T^*, d^*\rangle$, and in addition

(i) $\langle T(u^\wedge i), d(u^\wedge i)\rangle \preccurlyeq \langle T(u), d(u)\rangle$ for each $i = 0, 1$ and $u \in \omega^{<\omega}$,

(ii) if $u \neq v$ are of length $n+1$ then $(\langle T(u), d(u)\rangle ; \langle T(v), d(v)\rangle) \in D_n$,

(iii) if $u \in 2^{<\omega}$ then the condition $(\langle T(u^\wedge 0), d(u^\wedge 0)\rangle ; T(u^\wedge 1), d(u^\wedge 1))$ $(\mathbb{P} \times \mathbb{P})$-forces $\neg\, \mathbf{x}_{\mathtt{left}} \,\underline{\mathsf{E}}\, \mathbf{x}_{\mathtt{right}}$ over $\mathbf{L}$.

Then, in $\mathbf{V}[g]$, if $a \in 2^\omega$ then the intersection $\bigcap_n [T(a \restriction n), d(a \restriction n)]$ contains a single point $x_a \in [T^*, d^*]$ by Lemma 14.6, and if $a \neq b$ then $\neg\, (x_a \ \mathsf{E}^{\mathbf{V}[g]} \ x_b)$. But by construction $[T^*, d^*] \subseteq [T_0, d_0]$ in $\mathbf{V}[g]$, so that $[T_0, d_0]$ contains uncountably many $\mathsf{E}^{\mathbf{V}[g]}$-classes in $\mathbf{V}[g]$ — a contradiction to Corollary 14.5.
□

Let H be the set of all codes $\langle T, d\rangle \in \mathbb{K}_\rho$ such that the $\omega^{\mathbf{L}}_{\rho+4}$-collapse forcing notion $\mathrm{Coll}(\omega^{\mathbf{L}}_{\rho+4}) = (\omega^{\mathbf{L}}_{\rho+4})^{<\omega}$ forces, over $\mathbf{L}$, that

$$[T, d] \subseteq [T_0, d_0] \text{ and } [T, d] \text{ is an } \underline{\mathsf{E}}\text{-equivalence class.}$$

Lemma 14.10. *If $\langle T, d\rangle \in H$ then it is true in in the ground set universe $\mathbf{V}$ that $[T, d] \subseteq [T_0, d_0]$ and $[T, d]$ is a E-class.*

Proof. By definition this is true for $\mathrm{Coll}(\omega^{\mathbf{L}}_{\rho+4})$-generic extensions of $\mathbf{L}$ — hence by Shoenfield also for all generic extensions $\mathbf{V}[G]$ in which $\omega^{\mathbf{L}}_{\rho+4}$ is countable, and then, by quite obvious downward absoluteness, for $\mathbf{V}$. □

Lemma 14.11. $H \neq \varnothing$.

Proof. By Lemma 14.11 there is a stable condition $\langle T', d' \rangle \in \mathbb{P}$. Using an $\omega^{\mathbf{L}}_{\rho+4}$-enumeration of all dense sets $D \subseteq \mathbb{P}$ in $\mathbf{L}$, we easily get a code $\langle T^*, d^* \rangle \in \mathbb{K}$ such that $\sup T^* \le \omega^{\mathbf{L}}_{\rho+4}$ and

$$[T^*, d^*] = \{x \in [T', d'] : x \text{ is } \mathbb{P}\text{-generic over } \mathbf{L}\}$$

in any class $\mathbf{V}[G]$. Lemma 14.8 implies that all elements $x \in [T^*, d^*]$ in $\mathbf{V}[G]$ are $\mathsf{E}^{\mathbf{V}[G]}$-equivalent to each other and to some $y^* \in [T_0, d_0] \cap \mathbf{V}$.

Let $g : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+4}$ be a collapse-generic map. *We argue in* $\mathbf{V}[g]$. By a simple cardinality argument, $[T^*, d^*] \neq \varnothing$ in $\mathbf{V}[g]$, and $[T^*, d^*]$ consists of pairwise $\mathsf{E}^{\mathbf{V}[g]}$-equivalent elements by the above. This allows us to define

$$Z = \{z : \exists\, x \in [T^*, d^*]\ (x \mathrel{\mathsf{E}^{\mathbf{V}[g]}} z)\} = \{z : \forall\, x \in [T^*, d^*]\ (x \mathrel{\mathsf{E}^{\mathbf{V}[g]}} z)\}.$$

in $\mathbf{V}[g]$, so that it is true in $\mathbf{V}[g]$ that Z is an entire $\mathsf{E}^{\mathbf{V}[g]}$-equivalence class, which includes $[T^*, d^*]$, hence, has a non-empty intersection with $[T', d'] \subseteq [T_0, d_0]$, therefore $Z \subseteq [T_0, d_0]$ as $[T_0, d_0]$ is an σ-$\mathsf{E}^{\mathbf{V}[g]}$-class in $\mathbf{V}[g]$ by $(*)$.

It follows that Z is $\mathbf{\Pi}^0_{1+\rho}$ in $\mathbf{V}[g]$. Moreover, by the choice of g it is true in $\mathbf{V}[g]$ that $\langle T^*, d^* \rangle \in \mathbf{L} \cap \mathrm{HC}$, and hence $\langle T^*, d^* \rangle$ is $\varDelta^{\mathrm{HC}}_1(\eta)$ in $\mathbf{V}[g]$ for an ordinal $\eta < \omega_1^{\mathbf{V}[g]}$. (Indeed let η be the first ordinal such that $\langle T^*, d^* \rangle$ is the η-th set in the Gödel construction of $\mathbf{L}$.) Then Z is $\varDelta^{\mathrm{HC}}_1(\eta)$ in $\mathbf{V}[g]$. Therefore by Proposition 14.2 that there is a code $\langle T, d \rangle \in \mathbb{K}_\rho$ such that $Z = [T, d]$ in $\mathbf{V}[g]$. Let us demonstrate that $\langle T, d \rangle \in H$.

Consider a collapse-generic map $g' : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+4}$; we can assume that g' is $\mathrm{Coll}(\omega^{\mathbf{L}}_{\rho+4})$-generic even over $\mathbf{V}[g]$. We have to prove that

(‡) in $\mathbf{L}[g']$: $[T, d] \subseteq [T_0, d_0]$ and $[T, d]$ is an $\mathsf{E}^{\mathbf{L}[g']}$-equivalence class.

Recall that by construction $Z = [T, d] \subseteq [T_0, d_0]$ and $[T, d]$ is an $\mathsf{E}^{\mathbf{V}[g]}$-class in $\mathbf{V}[g]$. But the Borel codes involved are countable in both classes $\mathbf{V}[g]$ and $\mathbf{L}[g']$. This implies (‡) by Shoenfield. $\square$

Now we have gathered everything necessary to end the proof of the theorem in a few lines. It suffices to prove that $C = [T_0, d_0] \subseteq \bigcup_{\langle T, d \rangle \in H} [T, d]$ in $\mathbf{V}$. Suppose tovards the contrary that this is not the case.

The set $H \subseteq \mathbb{K}_\rho$ belongs to $\mathbf{L}$ and $\mathtt{card}\, H \le \omega^{\mathbf{L}}_{\rho+1}$ in $\mathbf{L}$, of course. As $\langle T_0, d_0 \rangle \in \mathbb{K}_{\rho+2}$, we can easily define a code $\langle T_1, d_1 \rangle \in \mathbb{K}_{\rho+2}$ such that $[T_1, d_1] = [T_0, d_0] \smallsetminus \bigcup_{\langle T, d \rangle \in H} [T, d]$ in any universe, and hence $[T_1, d_1] \neq \varnothing$ in $\mathbf{V}$ by the contrary assumption, and still $[T_1, d_1]$ is a σ-E-class in $\mathbf{V}$ since so is $C = [T_0, d_0]$ while each $[T, d]$, $\langle T, d \rangle \in H$, is a E-class by Lemma 14.10.

In other words, the code $\langle T_1, d_1\rangle \preccurlyeq \langle T_0, d_0\rangle$ has the same properties (see Corollary 14.5) as $\langle T_0, d_0\rangle$ does. In fact by exactly the same arguments as above, but with the forcing notion $\mathbb{P}_1 = \{\langle T, d\rangle \in \mathbb{P} : \langle T, d\rangle \preccurlyeq \langle T_1, d_1\rangle\}$, the set H_1 of all codes $\langle T, d\rangle \in H$ such that the $\mathrm{Coll}(\omega^{\mathbf{L}}_{\rho+4})$ forces "$[T, d] \subseteq [T_0, d_0]$" over $\mathbf{L}$, is non-empty. (Compare Lemma 14.11.) Let $\langle T, d\rangle \in H_1$. Then $\varnothing \neq [T, d] \subseteq [T_1, d_1]$ in $\mathbf{V}$ (similarly to Lemma 14.10), which contradicts to the definition of $\langle T_1, d_1\rangle$.

$\square$ (Theorem 14.1)

$\square$ (Theorem 13.1, full version)

15 OD sets in Solovay's model: generalizations

It is a rather common practice that results like Theorems 4.1 and 5.1 generalize this or another way in the Solovay model.[5] We'll prove below the following such generalizations of our main results. Recall that OD means ordinal-definable, and also denotes the class of all ordinal-definable sets.

Theorem 15.1. *The following is true in the Solovay model. If $A \subseteq \mathscr{N}$ is an* OD *set then one and only one of the next two claims* (I), (II) *holds*:

(I) *A is* OD*-effectively σ-bounded, so that there exists an* OD *sequence $\{T_\xi\}_{\xi<\omega_1^{\mathbf{L}}}$ of compact trees $T_\xi \subseteq \omega^{<\omega}$ such that $A \subseteq \bigcup_\xi [T_\xi]$;*

(II) *there is a superperfect set $Y \subseteq A$.*

Here conditions (I) and (II) are incompatible. Indeed, the union U in (I) is countable, hence the set U is σ-compact. Therefore if Y is a set as is (II) then Y cannot be covered by U.

Note that condition (I) cannot be strengthened to the form that *there is an* OD *sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ such that $A \subseteq \bigcup_n [T_n]$*. For a counterexample take $A = \mathscr{N} \cap \mathbf{L}$ (all constructible reals). This is a countable set in the Solovay model, hence (I) of Theorem 15.1 holds and (II) fails, but the existence of an OD (hence, constructible) sequence of trees as indicated is clearly impossible.

Theorem 15.2. *The following is true in the Solovay model. If $A \subseteq \mathscr{N}$ is an* OD *set then one and only one of the next two claims* (I), (II) *holds*:

[5] By *the Solovay model* we'll always mean a model of **ZFC**, a generic extension of $\mathbf{L}$ introduced in [27], in which all projective sets are Lebesgue measurable, rather than the other model of [27], in which only $\mathbf{ZF} + \mathrm{DC}$ holds but all sets of reals are measurable.

(I) *A is* OD*-effectively σ-compact, so that there exists an* OD *sequence* $\{T_\xi\}_{\xi<\omega_1^{\mathbf{L}}}$ *of compact trees $T_\xi \subseteq \omega^{<\omega}$ such that $A = \bigcup_\xi [T_\xi]$;*

(II) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and relatively closed in A.*

The proof of both theorems follows in the next section.

Let's start with some definitions and a couple of special results related to the Solovay model. If Ω is an ordinal then by Ω-SM we denote the following sentence: "$\Omega = \omega_1$, Ω is strongly inaccessible in $\mathbf{L}$, the constructible universe, and the whole universe $\mathbf{V}$ is a generic extension of $\mathbf{L}$ via a known collapse forcing $\mathrm{Coll}(\omega, <\Omega)$, as in [27]". Thus Ω-SM says that the universe is a Solovay-type extension of $\mathbf{L}$.

Lemma 15.3 (assuming Ω-SM)**.** *If X is a countable* OD *set then there exist an ordinal $\lambda < \Omega$ and an* OD *$1-1$ map $f : \lambda \xrightarrow{\text{onto}} X$.*

Proof. Let $F : \mathbf{Ord} \xrightarrow{\text{onto}} \mathrm{OD}$ be a canonical OD map. Recall that under Ω-SM the universe is a homogeneous generic extension of $\mathbf{L}$. Therefore the relations $F(\xi) \in X$ and $F(\xi) = F(\eta)$ (with arguments ξ, η) are OD. $\square$

Definition 15.4 (assuming Ω-SM)**.** Let $\mathbf{P}$ be the collection of all **non-empty** OD sets $Y \subseteq \mathscr{N}$. We consider $\mathbf{P}$ as a forcing notion (smaller sets are stronges conditions). A set $G \subseteq \mathbf{P}$ is $\mathbf{P}$*-generic over* OD if it non-emptily intersects every OD dense set $D \subseteq \mathbf{P}$. $\square$

Proposition 15.5 (see, *e.g.*, [11])**.** *Assuming Ω-SM, if a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic then the intersection $\bigcap G = \{a_G\}$ consists of a single real.* $\square$

As the set $\mathbf{P}$ is definitely uncountable, the existence of $\mathbf{P}$-generic sets does not immediately follow from Ω-SM by a cardinality argument. Yet fortunately $\mathbf{P}$ is *locally countable*, in a sense.

Definition 15.6 (assuming Ω-SM)**.** A set $X \in \mathrm{OD}$ is OD-*1st-countable* if the OD power set $\mathscr{P}^{\mathrm{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD}$ is at most countable.[6]

Let $\mathbf{P}^*$ be the set of all OD-1st-countable sets $X \in \mathbf{P}$. $\square$

For instance, assuming Ω-SM, the set $X = \mathscr{N} \cap \mathrm{OD} = \mathscr{N} \cap \mathbf{L}$ of all OD reals belongs to $\mathbf{P}^*$. Indeed $\mathscr{P}^{\mathrm{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD} = \mathscr{P}(X) \cap \mathbf{L}$, and hence $\mathscr{P}^{\mathrm{OD}}(X)$ admits an OD bijection onto the ordinal $\omega_2^{\mathbf{L}} < \omega_1$.

The set $\mathbf{Coh}$ of all reals $x \in \mathscr{N}$ Cohen generic over $\mathbf{L}$ belongs to $\mathbf{P}^*$ as well. Indeed if $Y \subseteq \mathbf{Coh}$ is OD and $x \in Y$ then "$x \in Y$" is Cohen-forced over $\mathbf{L}$. It follows that there is a set $S \subseteq \omega^{<\omega}$, $S \in \mathbf{L}$, such that

[6] Then the set $\mathscr{P}^{\mathrm{OD}}(X)$ is not necessarily OD-countable. Take for instance $X = \omega$.

$Y = X \cap \bigcup_{t \in S} \mathscr{N}_t$. But the collection of all such sets S belongs to $\mathbf{L}$ and has cardinality ω_1 in $\mathbf{L}$, hence, is countable under Ω-SM.

Proposition 15.7. *Assuming Ω-SM, $\mathbf{P}^*$ is dense in $\mathbf{P}$, that is, if $X \in \mathbf{P}$ then there is a set $Y \in \mathbf{P}^*$ such that $Y \subseteq X$.* $\square$

Proof (sketch, see details in [11])**.** For any ordinal λ, let $\mathbf{Coh}_\lambda$ be the set of all elements $f \in \lambda^\omega$, $(\lambda^{<\omega})$-generic over $\mathbf{L}$. Suppose that $X \in \mathbf{P}$. Then by definition $X \neq \varnothing$, hence, there is a real $x \in X$. Then it follows from Ω-SM that there exist: an ordinal $\lambda < \omega_1 = \Omega$, an element $f \in \mathbf{Coh}_\lambda$, and an OD map $H : \lambda^\omega \to \mathscr{N}$, such that $x = H(f)$. The set $P = \{f' \in \mathbf{Coh}_\lambda : H(f') \in X\}$ is then OD and non-empty (contains f), and hence so is its image $Y = \{H(f') : f' \in P\} \subseteq X$ (contains x).

It remains to prove that $Y \in \mathbf{P}^*$. As H is an OD map, it is sufficient to show that $\mathbf{Coh}_\lambda$ is $\mathbf{P}^*$. But this is true by the same reasons as for the set $\mathbf{Coh}$ (see just before Proposition 15.7). $\square$

Remark 15.8. One may want to know whether Theorem 11.1 also admits a version similar to Theorem 15.1 — that is, for a finite sequence of OD equivalence relations F_j and an OD set A in the Solovay model.

But here we have a grave obstacle just from the beginning. Indeed, coming back to the derivation of 11.3 from Theorem 11.2, we'll have to prove that, in the Solovay model, any OD set $E \subseteq \mathscr{N} \times \mathscr{N}$ with σ-bounded sections splits into a countable union of OD sets with bounded sections. But this claim fails even for sets with *countable* sections: consider *e.g.* the $\varSigma^1_1$ set $E = \{\langle x, y\rangle : y \in \mathbf{L}[x]\}$.

Whether a more modest version holds in the Solovay model, with still $\varDelta^1_1$ relations F_j and an OD set A, remains to be seen. $\square$

16 OD sets in Solovay's model: proofs

Here we prove Theorems 15.1 and 15.2. The proofs strongly resemble those in Section 4 and Section 5, hence we skip some details. There are two notable differences. First, the Gandy – Harrington type of arguments is replaced by the OD forcing, and second, various niceties related to classes $\varSigma^1_1$ and $\varDelta^1_1$ become obsolete as OD is a more robust definability class.

Proof (Theorem 15.1)**.** *We argue in the Solovay model*, that is, we assume Ω-SM. Consider an arbitrary OD set $A \subseteq \mathscr{N}$. Let U be the union of all sets of the form $[T]$, where $T \subseteq \omega^{<\omega}$ is a compact OD tree. Clearly the set U and the difference $A' = A \smallsetminus U$ are OD.

Lemma 16.1. *Under the conditions of Theorem 15.1, if $Y \subseteq A'$ is a non-empty* OD *set then its topological closure $\overline{Y}$ in $\mathscr{N}$ is not compact.*

Proof. If $\overline{Y}$ is compact then $T = \mathtt{tree}(Y)$ is a compact OD tree, hence $Y \subseteq \overline{Y} = [T] \subseteq U$, a contradiction to the assumption $Y \subseteq A'$. $\square$ (Lemma)

Case 1: $A' = \varnothing$, that is, $A \subseteq U$. To check (I) of Theorem 15.1, note that under Ω-SM OD reals are the same as constructible reals, and hence there is an OD enumeration of all OD trees by ordinals $\xi < \omega_1^{\mathbf{L}}$.

Case 2: the set $A' = A \smallsetminus U$ is non-empty. By Proposition 15.7, there is a set $A'' \subseteq A'$, $A'' \in \mathbf{P}^*$. Then the power set $P = \mathscr{P}^{\mathrm{OD}}(A'') = \mathscr{P}(A'') \cap \mathrm{OD}$ is at most countable. By Lemma 15.3, there exist an ordinal $\lambda < \Omega$ and an OD map $f : \lambda \xrightarrow{\text{onto}} P$. But the power set $\mathscr{P}^{\mathrm{OD}}(\lambda)$ is obviously countable, therefore so is $\mathscr{P}^{\mathrm{OD}}(P)$. Fix an arbitrary enumeration $\{\mathscr{D}_n^{\mathrm{OD}}\}_{n \in \omega}$ of all OD sets $\mathscr{D} \subseteq P = \mathscr{P}^{\mathrm{OD}}(A'')$, dense in $\mathbf{P}^*$ below A''. We assert that then there is a system of non-empty OD sets $Y_s \subseteq A'$ satisfying conditions (1), (2), (3), (5) in Section 4, along with the following condition instead of (4):

(4^{OD}) if $s \in \omega^{<\omega}$ then $Y_s \in \mathscr{D}_{\mathtt{lh}\, s}^{\mathrm{OD}}$.

If such a construction is accomplished then $\bigcap_m Y_{a \restriction m} = \{f(a)\}$ for each $a \in \mathscr{N}$ by Proposition 15.5, and $f : \mathscr{N} \xrightarrow{\text{onto}} Y = \{f(a) : a \in \mathscr{N}\}$ is a homeomorphism. Moreover the set Y is closed in $\mathscr{N}$ by exactly the same reasons as in Section 4, and hence we have (II) of Theorem 15.1.

The construction of sets Y_s goes on exactly as in Section 4, with the only difference that Σ_1^1 and Lemma 4.3 are replaced by OD and Lemma 16.1.

$\square$ (Theorem 15.1)

Proof (Theorem 15.2)**.** Assuming Ω-SM, consider any OD set $A \subseteq \mathscr{N}$. Let U be the union of all sets $[T]$, where $T \subseteq \omega^{<\omega}$ is a compact OD tree and $[T] \subseteq A$. The set U and the difference $A' = A \smallsetminus U$ are OD.

By Theorem 15.1, we can w.l.o.g. assume that A is σ-bounded, and hence if $F \subseteq A$ is a closed set then F is σ-compact.

Lemma 16.2. *If $F \subseteq A'$ is a non-empty* OD *set then $\overline{F} \not\subseteq A$.*

Recall that $\overline{F}$ is the closure of a set $F \subseteq \mathscr{N}$.

Proof. Suppose towards the contrary that $\varnothing \neq F \subseteq A'$ is an OD set but $\overline{F} \subseteq A$. By the w.l.o.g. assumption above, $\overline{F} = \bigcup_n F_n$ is σ-compact, where all F_n are compact. There is a Baire interval $\mathscr{N}_s$ such that the set $X = \mathscr{N}_s \cap \overline{F}$ is non-empty and $X \subseteq F_n$ for some n. Thus $X \subseteq A$ is a non-empty compact OD set, hence by definition $X \subseteq U$ and $A' \cap X = \varnothing$. In other words, $\mathscr{N}_s \cap \overline{F} \cap A' = \varnothing$. It follows that $\mathscr{N}_s \cap F = \varnothing$ (because $F \subseteq A'$), which contradicts to $X = \mathscr{N}_s \cap \overline{F} \neq \varnothing$. □ (Lemma)

We come back to the proof of Theorem 15.2.

Case 1: $A' = \varnothing$, that is, $A = U$. This implies (I) of the theorem.

Case 2: $A' \neq \varnothing$. As in the proof of Theorem 15.1, choose a set $A'' \subseteq A'$, $A'' \in \mathbf{P}^*$, and fix an arbitrary enumeration $\{\mathscr{D}_n^{\text{OD}}\}_{n\in\omega}$ of all OD sets $\mathscr{D} \subseteq P = \mathscr{P}^{\text{OD}}(A'')$, dense in $\mathbf{P}^*$ below A''. To get a set $Y \subseteq A''$, *relatively* closed in A and homeomorphic to $\mathscr{N}$, we make use of a system of non-empty OD sets $Y_s \subseteq A''$ satisfying conditions (1), (2), (3) in Section 4, (4^{OD}) as in the proof of Theorem 15.1, and ($5'$) in Section 5.

If such a system of sets is defined , then the associated map $f : \mathscr{N} \to A''$ is $1-1$ and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \mathtt{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A''$. In addition, the set Y is relatively closed in A by the same arguments (based on condition ($5'$)) as in Section 5, and hence we have (II) of Theorem 15.2. The construction of sets Y_s also goes on as in Section 5, but we have to apply Lemma 16.2 instead of Lemma 5.3.

□ (Theorem 15.2)

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