On countable cofinality of definable chains in Borel partial orders *

Vladimir Kanovei[†]

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Abstract

We prove that in some cases definable chains of Borel partial orderings are necessarily countably cofinal. This includes the following cases: analytic chains, **ROD** chains in the Solovay model, and $\mathbf{\Sigma}^1_2$ chains in the assumption that $\omega_1^{\mathbf{L}[x]} < \omega_1$ for all reals x.

Introduction

Studies of maximal chains in partially ordered sets go back to as early as Hausdorff [7, 8], where this issue appeared in connection with Du Bois Reymond's investigations [1, 2] of orders of infinity. Using axiom of choice, Hausdorff proved the existence of maximal chains (called *pantachies*) in any partial ordering. On the other hand, Hausdorff clearly understood the difference between such a pure existence proof and an actual construction of a maximal chain — see *e.g.* [7, p. 110] or comments in [5] — which we would call now the existence of *definable* maximal chains.

The following theorem is the main content of this note. It shows that in some notable cases definable chains are necessarily countably cofinal.

Theorem 1. *If* $\le$ *is a Borel PQO on a (Borel) set* $D = \mathtt{dom}\,(\le)$, $X \subseteq D$, *and* $\le \restriction X$ *is a linear quasi-order* (= chain), *then* $\langle X; \le\rangle$ *is countably cofinal in each of the following three cases:*

(i) X *is a* $\mathbf{\Sigma}^1_1$ *set — and in this case, moreover, there is no strictly increasing* ω_1*-sequences in* X,

(ii) X *is a* **ROD** *set in the Solovay model,*

(iii) X *is a* $\mathbf{\Sigma}^1_2$ *set, and* $\omega_1^{\mathbf{L}[r]} < \omega_1$ *for every real* r.

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†IITP RAS and MIIT, Moscow, Russia, `kanovei@googlemail.com`

Therefore, if, in addition, it is known that $\langle D\,;\leq\rangle$ does not have maximal chains of countable cofinality, then in all three cases X is not a maximal chain.

Part (i) is proved by reduction to a result in [6]. Part (ii) is already known from [9], but we present here a simplified proof in order to make the exposition self-contained, since the result is used in the proof of (iii).

The additional condition in the theorem, of uncountable cofinality of all maximal chains, holds for many partial orders of interest, *e.g.*, the eventual domination order on sets like ω^ω or $\mathbb{R}^\omega$, or the *rate of growth order* defined on $\mathbb{R}^\omega$ by

$$x <_{\rm RG} y \quad \text{iff} \quad \lim_{n\to\infty} \frac{x(n)}{y(n)} = \infty\,.$$

Needless to say that chains, gaps, and similar structures related to these or similar orderings have been subject of extended studies, of which we mention [3, 4, 13, 10] among those in which the definability aspect is considered.

We end the introduction with a review of basic notation related to orderings.

PQO, *partial quasi-order*: $x \leq x$ and $x \leq y \wedge y \leq z \Longrightarrow x \leq z$ in the domain;

LQO, *linear quasi-order*: in addition, $x \leq y \vee y \leq x$ in the domain;

LO, *linear order*: in addition, $x \leq y \wedge y \leq x \Longrightarrow x = y$ in the domain;

sub-order: restriction of the given PQO to a subset of its domain.

$\leqslant_{\tt lex}$: the *lexicographical* order on sets of the form 2^ξ, $\xi \in \mathtt{Ord}$.

1 Analytic linear suborders of Borel PQOs

In this Section, we prove Theorem 1(i). Thus suppose that $\leq$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, $X \subseteq D$ is a $\mathbf{\Sigma}^1_1$ set, and $\leq\restriction X$ is a *linear* quasi-order. Prove that this restricted quasi-order $\langle X;\leq\rangle$ has no strictly increasing ω_1-chains.

The proof is based on the following well-known lemma.

Lemma 2. *Every Borel LQO $\leq$ is countably cofinal, and moreover, there is no strictly increasing ω_1-sequences.*

Proof (lemma)**.** By a result in Harrington – Marker – Shelah [6], there is an ordinal $\xi < \omega_1$ and a Borel map $f : X = \mathtt{dom}\,(\leq) \to 2^\xi$ such that

$$x \leq y \quad \text{iff} \quad f(x) \leqslant_{\tt lex} f(y)$$

for all $x, y \in X$. But the lemma easily holds for $\langle 2^\xi;\leqslant_{\tt lex}\rangle$. □

Coming back to the proof of Theorem 1(i), we describe the idea: find a Borel set $W \subseteq D$ such that $X \subseteq W$ and still $\leq\restriction W$ is linear, then use Lemma 2. We find such a set by means of the following two-step procedure.

Note that the set Y of all elements in D, $\le$-comparable with every element $x \in X$, is $\mathbf{\Pi}^1_1$, and $X \subseteq Y$ (as $\le$ is linear on X). By the Luzin Separation theorem, there is a Borel set Z such that $X \subseteq Z \subseteq Y$. This ends step 1.

Now, at the 2nd step, the set U of all elements in Z, comparable with every element in Z, is $\mathbf{\Pi}^1_1$, and we have $X \subseteq U$. Once again, by Separation, there is a Borel set W such that $X \subseteq W \subseteq U$.

By construction, $\le$ is linear on U, and hence on W. Therefore, there is no increasing ω_1-sequence in W by Lemma 2. But $X \subseteq W$.

□ (Theorem 1(i))

The next immediate corollary says that maximal chains cannot be analytic provided they cannot be countably cofinal.

Corollary 3. *If $\le$ is a Borel PQO, and every countable set $D' \subseteq \mathtt{dom} \le$ has a strict upper bound, then there is no maximal $\mathbf{\Sigma}^1_1$ chains in $\langle D; \le \rangle$.* □

2 Near-counterexamples

The following examples show that Theorem 1(i) is not true any more for different extensions of the domain of $\mathbf{\Sigma}^1_1$ suborders of a Borel partial quasi-orders, such as $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ linear quasi-orders — not necessarily suborders of Borel orderings, as well as $\mathbf{\Delta}^1_2$ and $\mathbf{\Pi}^1_1$ suborders of Borel orderings. In each of these classes, a counterexample of cofinality ω_1 will be defined.

Example 1 ($\mathbf{\Sigma}^1_1$ LQO)**.** Consider a recursive coding of sets of rationals by reals. Let Q_x be the set coded by a real x. Let X_α be the set of all reals x such that the maximal well-ordered initial segment of Q_x has the order type α. We define

$$x \le y \quad \text{iff} \quad \exists \alpha \, \exists \beta \, (x \in X_\alpha \wedge y \in X_\beta \wedge \alpha \le \beta).$$

Then $\le$ is a Σ^1_1 LQO of cofinality ω_1. □

Example 2 ($\mathbf{\Pi}^1_1$ LQO)**.** Let $D \subseteq \omega^\omega$ be the Π^1_1 set of codes of (countable) ordinals. Then

$$x \le y \quad \text{iff} \quad x, y \in D \wedge |x| \le |y|$$

is a Π^1_1 LQO of cofinality ω_1. □

Example 3 ($\mathbf{\Pi}^1_1$ LO)**.** To sharpen Example 2, define

$$x \le y \quad \text{iff} \quad x, y \in D \;\wedge\; \big(|x| < |y| \;\vee\; (|x| = |y| \wedge x <_{\mathtt{lex}} y)\big);$$

this is a Π^1_1 LO of cofinality ω_1. □

Example 4 ($\mathbf{\Delta}^1_2$ suborders)**.** Let $\le$ be the eventual domination order on ω^ω. Assuming the axiom of constructibility $\mathbf{V} = \mathbf{L}$, one can define a strictly $\le$-increasing Δ^1_2 ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω. □

Example 5 ($\mathbf{\Pi}^1_1$ suborders)**.** Define a PQO $\le$ on $(\omega \smallsetminus \{0\})^\omega$ so that

$$x \le y \quad \text{iff} \quad \lim_{n\to\infty} y(n) \,/\, x(n) = \infty\,.$$

Assuming the axiom of constructibility $\mathbf{V} = \mathbf{L}$, define a strictly $\le$-increasing Δ^1_2 ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω. By the Novikov – Kondo – Addison Π^1_1 Uniformization theorem, there is a Π^1_1 set $\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha<\omega_1}$, such that $\alpha \ne \beta \Longrightarrow y_\alpha \ne y_\beta$, and we may assume that each y_α belongs to 2^ω.

Let $z_\alpha(n) = 2^{x_\alpha(n)} \cdot 3^{y_\alpha(n)}$, $\forall n$. Then the ω_1-sequence $\{z_\alpha\}_{\alpha<\omega_1}$ is Π^1_1 and strictly $\le$-increasing. □

3 Definable linear suborders in the Solovay model

Here we prove Theorem 1(ii). Arguing in the Solovay model (a model of **ZFC** defined in [11], in which all **ROD** sets of reals are Lebesgue measurable), we assume that $\le$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, $X \subseteq D$ is a **ROD** (real-ordinal definable) set, and $\le\restriction X$ is a *linear* quasi-order.

Prove that the restricted quasi-order $\langle X;\le\rangle$ is countably cofinal.

It is known that in the Solovay model any **ROD** set in ω^ω is a union of a **ROD** ω_1-sequence of analytic sets. Thus there is a $\subseteq$-increasing **ROD** sequence $\{X_\alpha\}_{\alpha<\omega_1}$ of $\mathbf{\Sigma}^1_1$ sets X_α, such that $X = \bigcup_{\alpha<\omega_1} X_\alpha$.

As the sets X_α are ctbly $\le$-cofinal by claim (i) of Theorem 1, it suffices to prove that one of X_α is cofinal in X.

Suppose otherwise. Then the sets $D_\alpha = \{z \in D : \exists\, x \in X_\alpha\,(z \le x)\}$ contain $\aleph_1$ different sets and form a ROD sequence.

We claim that *all sets D_α belong to the same class $\mathbf{\Sigma}^0_\rho$ as the given Borel order $\le$*. This will contradict to the following lemma by Stern [12], and therefore complete the proof of item (ii) of Theorem 1.

Lemma 4 (in the Solovay model)**.** *If $\rho < \omega_1$ then there is no ROD ω_1-sequence of pairwise different sets $X \subseteq \omega^\omega$ in the class $\mathbf{\Sigma}^0_\rho$.* □

To prove the claim, let $x_0 \le_{\rm lex} x_1 \le_{\rm lex} x_2 \le_{\rm lex} \dots$ be an arbitrary cofinal sequence in X_α, countable by the above. Then $D_\alpha = \{z \in D : \exists\, n\,(z \le x_n)\}$ is $\mathbf{\Sigma}^0_\rho$ by obvious reasons.

□ (Theorem 1(ii))

4 $\mathbf{\Sigma}^1_2$ linear suborders of Borel PQOs

Here we prove Theorem 1(iii). Assume that $\le$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, $X \subseteq D$ is a $\mathbf{\Sigma}^1_2$ set, and $\le\restriction X$ is a *linear* quasi-order. We also assume that $\omega_1^{\mathbf{L}[r]} < \omega_1$ for every real r.

Prove that the ordering $\langle X;\le\rangle$ is countably cofinal.

Pick a real r such that X is $\Sigma^1_2(r)$ and $\le$ is $\Delta^1_1(r)$. To prepare for an absluteness argument, fix canonical formulas,

$$\begin{array}{rll} \varphi(\cdot,\cdot) & \text{of type} & \Sigma^1_2\,, \\ \sigma(\cdot,\cdot,\cdot) & \text{of type} & \Sigma^1_1\,, \\ \pi(\cdot,\cdot,\cdot) & \text{of type} & \Pi^1_1\,, \end{array}$$

which define $\le$ and X in the set universe $\mathbf{V}$, so that it is true in $\mathbf{V}$ that

$$x \le y \iff \sigma(r,x,y) \iff \pi(r,x,y) \quad \text{and} \quad x \in X \iff \varphi(r,x)\,.$$

for all $x, y \in \omega^\omega$. We let $X_\varphi = \{x \in \omega^\omega : \varphi(r,x)\}$ and

$$x \le_{\sigma\pi} y \quad \iff \quad \sigma(r,x,y) \quad \iff \quad \pi(r,x,y)$$

so that $X_\varphi = X$ and $\le_{\sigma\pi}$ is $\le$ in $\mathbf{V}$, but X_φ and $\le_{\sigma\pi}$ can be defined in any transitive universe containing all ordinals (to preserve the equivalence of formulas σ and π). In particular, $X_\varphi = X$ and $\le_{\sigma\pi}$ is $\le$ in the background universe $\mathbf{V}$.

Let $\mathbf{WO}$ be the canonical Π^1_1 set of codes of (countable) ordinals, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w.

Let $X_\varphi = \bigcup_{\alpha<\omega_1} X_\varphi(\alpha)$ be a canonical representation of X_φ as an increasing union of $\boldsymbol{\Sigma}^1_1$ sets. Thus to define $X_\varphi(\alpha)$ we fix a $\Pi^1_1(r)$ set $P \subseteq (\omega^\omega)^2$ such that $X = \{x : \exists\, y\, P(x,y)\}$, fix a canonical $\Pi^1_1(r)$ norm $f : P \to \omega_1$, and let

$$P_\alpha = \{\langle x,y\rangle : f(x,y) < \alpha\} \quad \text{and} \quad X_\varphi(\alpha) = \{x : \exists\, y\, (\langle x,y\rangle \in P_\alpha)\}\,.$$

In our assumptions, the ordinal $\Omega = \omega_1$ is inaccessible in $\mathbf{L}[r]$. Let $\mathscr{P} = \text{Coll}({<}\,\Omega,\omega) \in \mathbf{L}[r]$ be the corresponding Levy collapse forcing. Consider a $\mathscr{P}$-generic extension $\mathbf{V}[G]$ of the universe. Then $\mathbf{L}[r][G]$ is a Solovay-model generic extension of $\mathbf{L}[r]$. The plan is to compare the models $\mathbf{V}$ and $\mathbf{L}[r][G]$. Note that $\mathbf{L}[r]$ is their common part, $\mathbf{V}[G]$ is their common extension, and the three models have the same cardinal $\omega_1^{\mathbf{V}} = \omega_1^{\mathbf{L}[r][G]} = \omega_1^{\mathbf{V}[G]} = \Omega > \omega_1^{\mathbf{L}[r]}$.

By Theorem 1(ii), it holds in $\mathbf{L}[r][G]$ that the ordering $\langle X_\varphi\,; \le_{\sigma\pi}\rangle$ is countably cofinal, hence there is an ordinal $\alpha < \Omega = \omega_1^{\mathbf{L}[r][G]}$ such that the sentence

($*$) the subset $X_\varphi(\alpha)$ is $\le_{\sigma\pi}$-cofinal in the whole set X_φ

is true in $\mathbf{L}[r][G]$. However ($*$) can be expressed by a Π^1_2 formula with r and an arbitrary code $w \in \mathbf{WO} \cap \mathbf{L}[r][G]$ such that $|w| = \alpha$ — as the only parameters. It follows, by the Shoenfield absoluteness, that ($*$) is true in $\mathbf{V}[G]$ as well.

And then, by exactly the same absoluteness argument, ($*$) is true in the set universe $\mathbf{V}$, too. In other words, it is true in $\mathbf{V}$ that $X_\varphi(\alpha)$, a $\boldsymbol{\Sigma}^1_1$ set, is cofinal in the whole set $X = X_\varphi$. But $X_\varphi(\alpha)$ is countably cofinal by Theorem 1(i).

$\square$ (Theorem 1(iii))

References

[1] P. Du Bois Reymond, Sur la grandeur relative des infinis des fonctions. *Ann. di Mat. (2)*, 1870, 4, pp. 338–353.

[2] P. Du Bois Reymond, *Die allgemeine Funktionentheorie.* Tübingen, 1882.
[French translation: *Théorie générale des fonctions*, 1887, reprinted in 1995 by Éditions Jacques Gabay, Sceaux.]

[3] I. Farah, Analytic quotients. Theory of liftings for quotients over analytic ideals on the integers. *Mem. Am. Math. Soc.*, 2000, 702, 171 pp.

[4] I. Farah, Analytic Hausdorff gaps. II: The density zero ideal. *Isr. J. Math.*, 2006, 154, pp. 235–246.

[5] G. Fisher, The infinite and infinitesimal quantities of du Bois-Reymond and their reception, *Arch. Hist. Exact Sci.*, 1981, 24, pp. 101–163.

[6] L.A. Harrington, D. Marker, S. Shelah, Borel orderings, *Trans. Amer. Math. Soc.* 1988, 310, pp. 293302.

[7] F. Hausdorff, Untersuchungen über Ordnungstypen IV, V. *Ber. über die Verhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Kl.*, 1907, 59, pp. 84–159.

[8] F. Hausdorff, Die Graduierung nach dem Endverlauf. *Abhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Kl.*, 1909, 31, pp. 295–334. [1]

[9] V. Kanovei, V. Lyubetsky, An infinity which depends on the axiom of choice, *Applied Mathematics and Computation*, 2012, 218, 16, pp. 8196–8202.

[10] Yu. Khomskii, Projective Hausdorff gaps, *Arch. Math. Logic*, 2013, Online September 2013.

[11] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 1970, 92, pp. 1–56.

[12] J. Stern, On Lusin's restricted continuum problem, *Ann. Math.*, 1984, 120 (2), pp. 7–37.

[13] S. Todorcevic, Gaps in analytic quotients, *Fundam. Math.*, 1998, 156, 1, pp. 85–97.

[1] Hausdorff's early papers on ordered sets, including our references [7, 8], were reprinted in F. Hausdorff, *Gesammelte Werke, Band IA: Allgemeine Mengenlehre*, Berlin: Springer, 2013, and translated in F. Hausdorff, *Hausdorff on ordered sets*, Translated, edited, and commented by J. M. Plotkin, AMS and LMS, 2005.