A countable definable set of reals containing no definable elements

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Abstract

We make use of a finite support product of Jensen forcing to define a model in which there is a countable non-empty Π^1_2 set containing no ordinal-definable real.

1 Introduction

It is well-known that the existence of a non-empty OD (ordinal-definable) set of reals X with no OD element is consistent with **ZFC**; the set of all non-constructible reals gives an example in many generic models including *e.g.* the Solovay model or the extension of $\mathbf{L}$, the constructible universe, by a Cohen real. Can such a set X be countable?

This question was initiated and briefly discussed at the Mathoverflow exchange desk in 2010 [1] and at FOM [2]. In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support product $\mathbb{P}^{<\omega}$ of countably many copies of the Jensen "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [4] (see also Section 28A of [3]). Enayat proved that a symmetric part of the $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$ definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite OD set of reals with no OD elements. In fact both $\mathbb{P}^{<\omega}$-generic extensions and their symmetric submodels were considered in [1] (Theorem 3.3) with respect to some other questions.

Following the mentioned conjecture, we prove the next theorem in this paper:

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[1] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.

[2] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html

Theorem 1.1. *It is true in a $\mathbb{P}^{<\omega}$-generic extension of* $\mathbf{L}$, *the constructible universe, that the set of $\mathbb{P}$-generic reals is non-empty, countable, and Π^1_2, but it has no OD elements.*

The Π^1_2 definability is definitely the best one can get in this context since it easily follows from the Π^1_1 uniformisation theorem that any non-empty Σ^1_2 set of reals definitely contains a Δ^1_2 element.

Jindra Zapletal[3] informed us that there is a totally different model of **ZFC** with an OD E_0-class X containing no OD elements. The construction of such a model, not yet published, but described to us in a brief communication, looks quite complicated and involves a combination of several forcing notions and some modern ideas in descriptive set theory recently presented in [5]; it also does not look to be able to get X analytically definable, let alone Π^1_2.

It remains to note that a *finite* OD set of reals contains only OD reals by obvious reasons. On the other hand, by a result in [2] there can be two *sets* of reals X, Y such that the pair $\{X, Y\}$ is OD but neither X nor Y is OD.

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2 Trees and perfect-tree forcing

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\wedge k$ is the extension of t by k. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\mathtt{lh}\, s$ is the length of s, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* iff for any strings $s \subset t$ in $2^{<\omega}$, if $t \in T$ then $s \in T$. Thus every non-empty tree $T \subseteq 2^{<\omega}$ contains the empty string Λ. If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T\restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let **PT** be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a non-empty tree $T \subseteq 2^{<\omega}$ belongs to **PT** iff it has no endpoints and no isolated branches. Then there is a largest string $s \in T$ such that $T = T\restriction_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\wedge 1 \in T$ and $s^\wedge 0 \in T$ in this case.

Each perfect tree $T \in \mathbf{PT}$ defines $[T] = \{a \in 2^\omega : \forall n\, (a \restriction n \in T)\} \subseteq 2^\omega$, the perfect set of all *paths through* T.

By a **perfect-tree forcing** we understand any set $\mathbb{P} \subseteq \mathbf{PT}$ suct that

(1) $\mathbb{P}$ contains the full tree $2^{<\omega}$;

(2) if $u \in T \in \mathbb{P}$ then $T\restriction_u \in \mathbb{P}$.

[3] Personal communication, Jul 31/Aug 01, 2014.

Such a set $\mathbb{P}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition). The forcing $\mathbb{P}$ adds a real in 2^ω.

Let $\mathbb{P}^{<\omega}$ be the **product of ω-many copies of $\mathbb{P}$ with finite support.** Thus a typical element of $\mathbb{P}^{<\omega}$ is a sequence $\boldsymbol{\tau} = \{T_n\}_{n\in\omega}$, where each term $T_n = \boldsymbol{\tau}(n)$ belongs to $\mathbb{P}$ and the set $|\boldsymbol{\tau}| = \{n : T_n \neq 2^{<\omega}\}$ (the support of $\boldsymbol{\tau}$) is finite. We order $\mathbb{P}^{<\omega}$ componentwisely: $\boldsymbol{\sigma} \leqslant \boldsymbol{\tau}$ ($\boldsymbol{\sigma}$ is stronger) iff $\boldsymbol{\sigma}(n) \subseteq \boldsymbol{\tau}(n)$ in $\mathbb{P}$ for all n; $\mathbb{P}^{<\omega}$ adds an infinite sequence $\{x_n\}_{n<\omega}$ of $\mathbb{P}$-generic reals $x_n \in 2^\omega$.

Remark 2.1. Sometimes we'll use tuples like $\langle T_0, \dots, T_n\rangle$ of trees $T_i \in \mathbb{P}$ to denote the infinite sequence $\langle T_0, \dots, T_n, 2^{<\omega}, 2^{<\omega}, 2^{<\omega}, \dots\rangle \in \mathbb{P}^{<\omega}$. $\square$

3 Splitting construction over a perfect set forcing

Assume that $\mathbb{P} \subseteq \mathbf{PT}$ is a perfect-tree forcing notion. The *splitting construction* $\mathbf{SC}(\mathbb{P})$ over $\mathbb{P}$ consists of all finite systems of trees of the form $\varphi = \{T_s\}_{s\in 2^{<n}}$, where $n = \mathtt{hgt}(\varphi) < \omega$ (the height of φ) and

(3) each tree $T_s = \varphi(s)$ belongs to $\mathbb{P}$;

(4) if $s^\wedge i \in 2^{<n}$ $(i = 0, 1)$ then $T_{s^\wedge i} \subseteq T_s$ and $\mathtt{stem}(T_s)^\wedge i \subseteq \mathtt{stem}(T_{s^\wedge i})$ — it easily follows that $[T_{s^\wedge 0}] \cap [T_{s^\wedge 0}] = \varnothing$.

The empty system Λ is the only one in $\mathbf{SC}(\mathbb{P})$ satisfying $\mathtt{hgt}(\Lambda) = 0$.

Let φ, ψ be systems in $\mathbf{SC}(\mathbb{P})$. Say that

- φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $n = \mathtt{hgt}(\psi) \leq \mathtt{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{<n}$;
- *properly extends* ψ, symbolically $\psi \prec \varphi$, if in addition $\mathtt{hgt}(\psi) < \mathtt{hgt}(\varphi)$;
- *reduces* ψ, if $n = \mathtt{hgt}(\psi) = \mathtt{hgt}(\varphi)$, $\varphi(s) \subseteq \psi(s)$ for all $s \in 2^{\mathtt{hgt}(\varphi)-1}$, and $\varphi(s) = \psi(s)$ for all $s \in 2^{<\mathtt{hgt}(\varphi)-1}$.

In other words, reduction allows to shrink trees in the top layer of the system, but keeps intact those in the lower layers.

Under the above assumption (2), there is a strictly $\prec$-increasing sequence $\{\varphi_n\}_{n<\omega}$ in $\mathbf{SC}(\mathbb{P})$. The limit system $\varphi = \bigcup_n \varphi_n = \{T_s\}_{s\in 2^{<\omega}}$ then satisfies (3) and (4) on the whole domain $2^{<\omega}$, and in this case, $T = \bigcap_n \bigcup_{s\in 2^n} T_s$ is still a perfect tree in $\mathbf{PT}$ (not necessarily in $\mathbb{P}$), and $[T] = \bigcap_n \bigcup_{s\in 2^n} [T_s]$.

Say that a tree T *occurs in* $\varphi \in \mathbf{SC}(\mathbb{P})$ if $T = \varphi(s)$ for some $s \in 2^{<\mathtt{hgt}(\varphi)}$.

We define $\mathbf{SC}^{<\omega}(\mathbb{P})$, **the finite-support product of $\mathbf{SC}(\mathbb{P})$,** to consist of all infinite sequences $\Phi = \{\varphi_k\}_{k\in\omega}$, where each $\varphi_k = \Phi(k)$ belongs to $\mathbf{SC}(\mathbb{P})$ and the set $|\Phi| = \{k : \varphi_k \neq \Lambda\}$ (the support of Φ) is finite.

Say that a tree T *occurs in* $\Phi = \{\varphi_k\}$ if it occurs in some φ_k, $k \in |\Phi|$.

We define $\Psi \preccurlyeq \Phi$ iff $\Psi(k) \preccurlyeq \Phi(k)$ (in $\mathbf{SC}(\mathbb{P})$) for all k. Then $\Psi \prec \Phi$ means that $\Psi \preccurlyeq \Phi$ and $\Psi(k) \prec \Phi(k)$ for at least one k. In addition we define $\Psi \prec\!\!\prec \Phi$ iff $|\Psi| \subseteq |\Phi|$ and $\Psi(k) \prec \Phi(k)$ for all $k \in |\Phi|$.

4 Jensen's extension of a perfect tree forcing

Let $\mathbf{ZFC}'$ be the subtheory of $\mathbf{ZFC}$ including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1 and continual sets like $\mathbf{PT}$ exist as well.) Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$.

Suppose that $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{PT}$ is a perfect-tree forcing notion. Then the sets $\mathbb{P}^{<\omega}$, $\mathbf{SC}(\mathbb{P})$, and $\mathbf{SC}^{<\omega}(\mathbb{P})$ belong to $\mathfrak{M}$, too.

Definition 4.1. Consider any $\preccurlyeq$-increasing sequence $\Phi = \{\Phi^j\}_{j<\omega}$ of systems $\Phi^j = \{\varphi_k^j\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{SC}^{<\omega}(\mathbb{P})$, dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$ [4] .

Then in particular it intersects every set of the form

$$D_k = \{\Phi \in \mathbf{SC}^{<\omega}(\mathbb{P}) : \forall\, k' \le k\ (k \le \mathtt{hgt}(\Phi(k')))\}\,.$$

Hence if $k < \omega$ then the sequence $\{\varphi_k^j\}_{j<\omega}$ of systems $\varphi_k^j \in \mathbf{SC}(\mathbb{P})$ is *eventually strictly increasing*, so that $\varphi_k^j \prec \varphi_k^{j+1}$ for infinitely many indices j (and $\varphi_k^j = \varphi_k^{j+1}$ for other j). Therefore there is a system of trees $\{\boldsymbol{T}_k(s)\}_{k<\omega\wedge s\in 2^{<\omega}}$ in $\mathbb{P}$ such that $\varphi_k^j = \{\boldsymbol{T}_k(s)\}_{s\in 2^{<h(j,k)}}$, where $h(j,k) = \mathtt{hgt}(\varphi_k^j)$. Then

$$\boldsymbol{U}_k = \textstyle\bigcap_n \bigcup_{s\in 2^n} \boldsymbol{T}_k(s) \quad \text{and} \quad \boldsymbol{U}_k(s) \textstyle\bigcap_{n\ge \mathtt{lh}\, s} \bigcup_{t\in 2^n,\, s\subseteq t} \boldsymbol{T}_k(t)$$

are trees in $\mathbf{PT}$ (not necessarily in $\mathbb{P}$) for each k and $s \in 2^{<\omega}$; thus $\boldsymbol{U}_k = \boldsymbol{U}_k(\Lambda)$. In fact $\boldsymbol{U}_k(s) = \boldsymbol{U}_k \cap \boldsymbol{T}_k(s)$ by (4). □

Lemma 4.2. *The set of trees* $\mathbb{U} = \{\boldsymbol{U}_k(s) : k < \omega \wedge s \in 2^{<\omega}\}$ *satisfies* (2) *while the union* $\mathbb{P} \cup \mathbb{U}$ *is a perfect-tree forcing.* □

Lemma 4.3. *The set* $\mathbb{U}$ *is dense in* $\mathbb{U} \cup \mathbb{P}$.

Proof. Suppose that $T \in \mathbb{P}$. The set $D(T)$ of all systems $\Phi = \{\varphi_k\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$, such that $\varphi_k(\Lambda) = T$ for some k, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$. It follows that $\Phi^j \in D(T)$ for some j, by the choice of Φ. Then $\boldsymbol{T}_k(\Lambda) = T$ for some k. However $\boldsymbol{U}_k(\Lambda) \subseteq \boldsymbol{T}_k(\Lambda)$. □

[4] Meaning that for any $\Psi \in \mathbf{SC}^{<\omega}(\mathbb{P})$ there is $\Phi \in D$ with $\Psi \preccurlyeq \Phi$.

Lemma 4.4. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$, and $U \in \mathbb{U}$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, that is, there is a finite $D' \subseteq D$ with $U \subseteq \bigcup D'$.*

Proof. Suppose that $U = \boldsymbol{U}_K(s)$, $K < \omega$ and $s \in 2^{<\omega}$. Consider the set $\Delta \in \mathfrak{M}$ of all systems $\Phi = \{\varphi_k\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$ such that $K \in |\Phi|$, $\mathtt{lh}\, s < h = \mathtt{hgt}(\varphi_K)$, and for each $t \in 2^{h-1}$ there is a tree $S_t \in D$ with $\varphi_K(t) \subseteq S$. The set Δ is dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$ by the pre-density of D. Therefore there is an index j such that Φ^j belongs to Δ. Let this be witnessed by trees $S_t \in D$, $t \in 2^{h-1}$, where $\mathtt{lh}\, s < h = \mathtt{hgt}(\varphi_K^J)$, so that $\varphi_K^J(t) \subseteq S_t$. Then

$$U = \boldsymbol{U}_K(s) \subseteq \boldsymbol{U}_K(\Lambda) \subseteq \bigcup_{t\in 2^{h-1}} \varphi_K^J(t) \subseteq \bigcup_{t\in 2^{h-1}} S_t \subseteq \bigcup D'$$

by construction, where $D' = \{S_t : t \in 2^{h-1}\} \subseteq D$ is finite. □

Lemma 4.5. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}^{<\omega}$ is pre-dense in $\mathbb{P}^{<\omega}$ then it remains pre-dense in $(\mathbb{P} \cup \mathbb{U})^{<\omega}$.*

Proof. Given a condition $\boldsymbol{\tau} \in (\mathbb{P}\cup\mathbb{U})^{<\omega}$, we have to prove that $\boldsymbol{\tau}$ is compatible in $(\mathbb{P} \cup \mathbb{U})^{<\omega}$ with a condition $\boldsymbol{\sigma} \in D$. For the sake of brevity, assume that $\boldsymbol{\tau} = \langle U, V\rangle$, where $U = \boldsymbol{U}_k(s)$ and $V = \boldsymbol{U}_\ell(t)$ belong to $\mathbb{U}$.

Consider the set $\Delta \in \mathfrak{M}$ of all systems $\Phi = \{\varphi_k\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$ such that there are strings $s', t' \in 2^{<\omega}$ with $s \subseteq s'$, $t \subseteq t'$, $\mathtt{lh}\, s' < \mathtt{hgt}(\varphi_k)$, $\mathtt{lh}\, t' < \mathtt{hgt}(\varphi_\ell)$, and trees $S, T \in \mathbb{P}$ such that $\langle S, T\rangle \in D$ and $\varphi_k(s') \subseteq U \cap S$, $\varphi_\ell(t') \subseteq V \cap T$. The set Δ is dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$ by the pre-density of D. Therefore there is an index j such that Φ^j belongs to Δ.

Let this be witnessed by $s', t' \in 2^{<\omega}$ and $\langle S, T\rangle \in D$. In other words, $\varphi_k^j(s') \subseteq U \cap S$ and $\varphi_\ell^j(t') \subseteq V \cap T$. However $U' = \boldsymbol{U}_k(s') \subseteq \varphi_k^j(s')$ and $V' = \boldsymbol{U}_\ell(t') \subseteq \varphi_\ell^j(t')$ by construction. It follows that condition $\langle U', V'\rangle \in \mathbb{U}^{<\omega}$ is stronger than both $\langle U, V\rangle$ and $\langle S, T\rangle$, as required. □

5 Forcing a real away of a pre-dense set

Let $\mathfrak{M}$ be still a countable transitive model of $\mathbf{ZFC}'$ and $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{PT}$ be a perfect-tree forcing notion. The goal of the following Theorem 5.3 is to prove that, in the conditions of Definition 4.1, for any $\mathbb{P}^{<\omega}$-name c of a real in 2^ω, it is forced by the extended forcing $(\mathbb{P} \cup \mathbb{U})^{<\omega}$ that c does not belong to sets $[U]$ where u is a tree in $\mathbb{U}$ — unless c is a name of one of generic reals x_k themselves. We begin with a suitable notation.

Definition 5.1. A $\mathbb{P}^{<\omega}$*-real name* is a system $\mathbf{c} = \{C_{ni}\}_{n<\omega, i<2}$ of sets $C_{ni} \subseteq \mathbb{P}^{<\omega}$ such that each set $C_n = C_{n0} \cup C_{n1}$ is dense or at least pre-dense in $\mathbb{P}^{<\omega}$ and if $\boldsymbol{\sigma} \in C_{n0}$ and $\boldsymbol{\tau} \in C_{n1}$ then $\boldsymbol{\sigma}, \boldsymbol{\tau}$ are incompatible in $\mathbb{P}^{<\omega}$.

If a set $G \subseteq \mathbb{P}^{<\omega}$ is $\mathbb{P}^{<\omega}$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_{ni} \neq \varnothing$. □

Thus any $\mathbb{P}^{<\omega}$-real name $\mathbf{c} = \{C_{ni}\}$ is a $\mathbb{P}^{<\omega}$-name for a real in 2^ω.
Recall that $\mathbb{P}^{<\omega}$ adds a generic sequence $\{x_k\}_{k<\omega}$ of reals $x_k \in 2^\omega$.

Example 5.2. Let $k < \omega$. Define a $\mathbb{P}^{<\omega}$-real name $\dot{\boldsymbol{x}}_k = \{C^k_{ni}\}_{n<\omega, i<2}$ such that each set C^k_{ni} contains a single condition $\boldsymbol{\rho}^k_{ni} \in \mathbb{P}^{<\omega}$, and $|\boldsymbol{\rho}^k_{ni}| = \{k\}$, $\boldsymbol{\rho}^k_{ni}(k) = R_{ni}$, where $R_{ni} = \{s \in 2^{<\omega} : \mathtt{lh}\, s > n \Longrightarrow s(n) = i\}$. Then $\dot{\boldsymbol{x}}_k$ is a $\mathbb{P}^{<\omega}$-name of a real x_k, the kth term of a $\mathbb{P}^{<\omega}$-generic sequence $\{x_k\}_{k<\omega}$. □

Let $\mathbf{c} = \{C_{ni}\}$ and $\mathbf{d} = \{C_{ni}\}$ be a $\mathbb{P}^{<\omega}$-real names. Say that $\boldsymbol{\tau} \in \mathbf{PT}^{<\omega}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff $\boldsymbol{\tau} \leqslant \boldsymbol{\rho}^k_{ni}$ (that is, the tree $T = \boldsymbol{\tau}(k) \in \mathbf{PT}$ satisfies $x(n) = i$ for all $x \in [T]$);
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}\, s$, $\boldsymbol{\tau}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \neq \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\boldsymbol{\tau}$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus T$ such that $\boldsymbol{\tau}$ directly forces $s \subset \mathbf{c}$;

Theorem 5.3. *In the assumptions of Definition 4.1, suppose that* $\mathbf{c} = \{C^i_m\}_{m<\omega, i<2} \in \mathfrak{M}$ *is a* $\mathbb{P}^{<\omega}$*-real name, and for every* k *the set*

$$D(k) = \{\boldsymbol{\tau} \in \mathbb{P}^{<\omega} : \boldsymbol{\tau} \text{ directly forces } \mathbf{c} \neq \dot{\boldsymbol{x}}_k\}$$

is dense in $\mathbb{P}^{<\omega}$*. Let* $\boldsymbol{u} \in (\mathbb{P} \cup \mathbb{U})^{<\omega}$ *and* $U \in \mathbb{U}$*. Then there is a stronger condition* $\boldsymbol{v} \in \mathbb{U}^{<\omega}$*,* $\boldsymbol{v} \leqslant \boldsymbol{u}$*, which directly forces* $\mathbf{c} \notin [U]$*.*

Proof. By construction $U \subseteq U_k$ for some k; thus we can assume that simply $U = U_k$. Let, say, $U = U_1$. Assume for the sake of brevity that $K = 1$, $|\boldsymbol{\tau}| = \{0, 1, 2, 3\}$, and $\boldsymbol{u} = \langle U_0, U_1, U_2, U_3 \rangle \in \mathbb{U}^{<\omega}$ (see Remark 2.1), where

$$U_0 = \boldsymbol{U}_0(t_0), \quad U_1 = \boldsymbol{U}_0(t_1), \quad U_2 = \boldsymbol{U}_1(t_2), \quad U_3 = \boldsymbol{U}_1(t_3),$$

and t_0, t_1, t_2, t_3 are strings in $2^{<\omega}$.

There is an index J such that the system $\Phi^J = \{\varphi^J_k\}_{k\in\omega}$ satisfies $\mathtt{hgt}(\varphi^J_0) > \mathtt{max}\{\mathtt{lh}\, t_0, \mathtt{lh}\, t_1\}$ and $\mathtt{hgt}(\varphi^J_1) > \mathtt{max}\{\mathtt{lh}\, t_2, \mathtt{lh}\, t_2\}$, so that the trees

$$T_0 = \varphi^J_0(t_0) = \boldsymbol{T}_0(t_0),\ T_1 = \varphi^J_0(t_1) = \boldsymbol{T}_0(t_1),\ T_2 = \varphi^J_1(t_2) = \boldsymbol{T}_1(t_2),$$

and $T_3 = \varphi_1^J(t_3) = \boldsymbol{T}_1(t_3)$ in $\mathbb{P}$ are defined and condition $\boldsymbol{\tau} = \langle T_0, T_1, T_2, T_3 \rangle$ belongs to $\mathbb{P}^{<\omega}$. Note that $\boldsymbol{u} \leqslant \boldsymbol{\tau}$.

Consider the set $\mathscr{D}$ of all systems $\Phi = \{\varphi_k\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$ such that $\Phi^J \preccurlyeq \Phi$ and there is a condition $\boldsymbol{\sigma} = \langle S_0, \dots, S_n \rangle \in \mathbb{P}^{<\omega}$, $\boldsymbol{\sigma} \leqslant \boldsymbol{\tau}$ (*i.e.*, stronger that $\boldsymbol{\tau}$), such that

(5) $\boldsymbol{\sigma}$ directly forces $\mathbf{c} \notin [T]$, where $T = \bigcup_{s\in 2^{h_1-1}} \varphi_1(s)$ and $h_k = \mathtt{hgt}(\varphi_k)$;

(6) each tree S_i occurs in Φ (see Section 3);

(7) more specifically, $S_0 = \varphi_0(s_0)$, $S_1 = \varphi_0(s_1)$, $S_2 = \varphi_1(s_2)$, $S_3 = \varphi_1(s_3)$, where $s_0, s_1 \in 2^{h_0-1}$, $s_2, s_3 \in 2^{h_1-1}$, and $t_i \subseteq s_i$, $i = 0, 1, 2, 3$.

Lemma 5.4. *$\mathscr{D}$ is dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$ above Φ^J.*

Proof. Consider any system $\Phi = \{\varphi_k\}_{k\in\omega} \in \mathbf{SC}^{<\omega}(\mathbb{P})$ with $\Phi^J \preccurlyeq \Phi$; the goal is to define a system $\Phi' \in \mathscr{D}$ such that $\Phi \preccurlyeq \Phi'$. We can assume that in fact $\Phi^J \prec\!\!\prec \Phi$; then any system $\Phi' \in \mathbf{SC}^{<\omega}(\mathbb{P})$ which is a reduction of Φ still satisfies $\Phi^J \prec\!\!\prec \Phi'$ and $\Phi^J \preccurlyeq \Phi'$. Let $h_0 = \mathtt{hgt}(\varphi_0)$ and $h_1 = \mathtt{hgt}(\varphi_1)$. Then by the assumption $\mathtt{hgt}(\varphi_0^J) < h_0$ and $\mathtt{hgt}(\varphi_1^J) < h_1$ strictly.

Pick strings $s_0, s_1 \in 2^{h_0-1}$ and $s_2, s_3 \in 2^{h_1-1}$ satisfying $t_i \subset s_i$, $i = 0, 1, 2, 3$. Consider the condition $\boldsymbol{\rho} = \langle R_0, R_1, R_2, R_3, R_4, \dots, R_N \rangle \in \mathbb{P}^{<\omega}$, where $N = 1 + 2^{n_1}$, $R_0 = \varphi_0(s_0)$, $R_1 = \varphi_0(s_1)$, $R_2 = \varphi_1(s_2)$, $R_3 = \varphi_1(s_3)$, and $\{R_4, \dots, R_N\}$ is an arbitrary enumeration of $\{\varphi_1(s) : s \in 2^{n_1-1}, s \neq s_2, s_3\}$.

It follows from the density of sets $D(k)$ that there is a stronger condition $\boldsymbol{\sigma} = \langle S_0, S_1, S_2, S_3, \dots, S_N, \dots, S_M \rangle \in \mathbb{P}^{<\omega}$, where $M \geq N$ and $S_i \subseteq R_i$ for all $i \leq N$, which directly forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_k$ for all $k = 2, \dots, N$. Then there exist strings $u, v_2, \dots, v_N \in 2^{<\omega}$ such that $\boldsymbol{\sigma}$ directly forces each of the formulas

$$u \subset \mathbf{c}, \text{ and also } v_2 \subseteq \dot{\boldsymbol{x}}_2, \ v_3 \subseteq \dot{\boldsymbol{x}}_3 \ , \ \dots \ , \ v_N \subseteq \dot{\boldsymbol{x}}_N,$$

and u is incompatible in $2^{<\omega}$ with each v_k.

However $\boldsymbol{\sigma}$ directly forces $v_k \subseteq \dot{\boldsymbol{x}}_k$ iff $v_k \subseteq \mathtt{stem}(S_k)$. We conclude that $\boldsymbol{\sigma}$ directly forces $\mathbf{c} \notin [S]$, where $S = \bigcup_{2\leq k\leq M} S_k$.

Now let $\Phi' \in \mathbf{SC}^{<\omega}(\mathbb{P})$ be defined as follows. We begin with Φ.

Step 1. Recall that $R_0 = \varphi_0(s_0)$, $R_1 = \varphi_0(s_1)$, $R_2 = \varphi_1(s_2)$, $R_3 = \varphi_1(s_3)$ in Φ. Now let $\varphi_0'(s_0) = S_0$, $\varphi_0'(s_1) = S_1$, $\varphi_1'(s_2) = S_2$, $\varphi_1'(s_3) = S_3$.

Step 2. By construction each R_k, $4 \leq k \leq M$, was equal to some $\varphi_1(s_k)$, $s_k \in 2^{n_1-1}$, $s_k \neq s_2, s_3$; we let $\varphi_1'(t) = S_k$.

Step 3. Each S_k, $N+1 \leq k < M$, is a tree in $\mathbb{P}$. Let $\mu = \mathtt{max}\,|\Phi|$ and define a system $\varphi'_{\mu+k} \in \mathbf{SC}(\mathbb{P})$ so that $\mathtt{hgt}(\varphi'_{\mu+k}) = 1$ and $\varphi'_{\mu+k}(\Lambda) = S'_k$.

After all these changes in Φ, we obtain another system $\Phi' = \{\varphi'_k : k \in \omega\}$ in $\mathbf{SC}^{<\omega}(\mathbb{P})$ which is a reduction of Φ, hence, satisfies $\Phi^J \preccurlyeq \Phi'$, and every tree S_k in the condition $\boldsymbol{\sigma} = \langle S_0, S_1, S_2, S_3, \dots, S_N, \dots, S_M \rangle$ occurs in Φ'. Moreover $\boldsymbol{\sigma}$ witnesses that $\Phi' \in \mathscr{D}$, as required. □ (Lemma)

Come back to the proof of the theorem. It follows from the lemma that there is an index $j \geq J$ such that the system $\Phi^j = \{\varphi^j_k\}_{k\in\omega}$ belongs to $\mathscr{D}$, and let this be witnessed by a condition $\boldsymbol{\sigma} = \langle S_0, S_1, S_2, S_3, \dots, S_n \rangle \in \mathbb{P}^{<\omega}$ satisfying (5), (6), (7). In particular $\boldsymbol{\sigma} \leqslant \boldsymbol{\tau}$ by (7).

Finally consider a condition $\boldsymbol{v} = \langle V_0, V_1, V_2, V_3, \dots, V_n \rangle \in \mathbb{U}^{<\omega}$ defined so that $V_0 = \boldsymbol{U}_0(s_0)$, $V_1 = \boldsymbol{U}_0(s_1)$, $V_2 = \boldsymbol{U}_1(s_2)$, $V_3 = \boldsymbol{U}_1(s_3)$, and if $4 \leq k \leq n$ then let V_k be any tree in $\mathbb{U}$ satisfying $V_k \subseteq S_k$ (Lemma 4.3). Recall that $t_i \subseteq s_i$ for $i = 0, 1, 2, 3$ by construction, therefore $\boldsymbol{v} \leqslant \boldsymbol{u}$. On the other hand, $\boldsymbol{v} \leqslant \boldsymbol{\sigma}$, therefore $\boldsymbol{v}$ directly forces $\mathbf{c} \notin [T]$ by (5), where $T = \bigcup_{s\in 2^{h-1}} \varphi^j_1(s) = \bigcup_{s\in 2^{h-1}} \boldsymbol{T}_1(s)$ and $h = \mathtt{hgt}(\varphi_1)$. And finally by definition $\boldsymbol{U}_1 \subseteq \bigcup_{s\in 2^{h-1}} \varphi^j_1(s)$, so $\boldsymbol{v}$ directly forces $\mathbf{c} \notin [U_1]$, as required. □

6 Jensen's forcing

In this section, **we argue in L, the constructible universe.** Let $\leqslant_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 6.1 (in $\mathbf{L}$)**.** Following [4, Section 3], define, by induction on $\xi < \omega_1$, a countable set of trees $\mathbb{U}_\xi \subseteq \mathbf{PT}$ satisfying (2) of Section 2, as follows.

Let $\mathbb{U}_0$ consist of all clopen trees $\varnothing \neq S \subseteq 2^{<\omega}$, including $2^{<\omega}$ itself.

Suppose that $0 < \lambda < \omega_1$, and countable sets $\mathbb{U}_\xi \subseteq \mathbf{PT}$ are already defined. Let $\mathfrak{M}_\xi$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\{\mathbb{U}_\xi\}_{\xi<\lambda}$ and such that $\alpha < \omega_1^{\mathfrak{M}}$ and all sets $\mathbb{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$.

Then $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{U}_\xi$ is countable in $\mathfrak{M}$, too. Let $\{\Phi^j\}_{j<\omega}$ be the $\leq_{\mathbf{L}}$-least sequence of systems $\Phi^j \in \mathbf{SC}^{<\omega}(\mathbb{P}_\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$, and let $\mathbb{U}_\lambda = \mathbb{U}$ be defined, on the base of this sequence, as in Definition 4.1.

Modulo technical details, $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$ is the Jensen forcing of [4], and the finite-support product $\mathbb{P}^{<\omega}$ is the forcing we'll use to prove Theorem 1.1. □

Proposition 6.2 (in $\mathbf{L}$)**.** *The sequence* $\{\mathbb{U}_\xi\}_{\xi<\omega_1}$ *belongs to* Δ^{HC}_1*.* □

Lemma 6.3 (in $\mathbf{L}$)**.** *If a set* $D \in \mathfrak{M}_\xi$*,* $D \subseteq {\mathbb{P}_\xi}^{<\omega}$ *is pre-dense in* ${\mathbb{P}_\xi}^{<\omega}$ *then it remains pre-dense in* $\mathbb{P}^{<\omega}$*. Hence if* $\xi < \omega_1$ *then* ${\mathbb{U}_\xi}^{<\omega}$ *is pre-dense in* $\mathbb{P}^{<\omega}$*.*

Proof. By induction on λ, $\xi \leq \lambda < \omega_1$, if D is pre-dense in ${\mathbb{P}_\lambda}^{<\omega}$ then it remains pre-dense in ${\mathbb{P}_{\lambda+1}}^{<\omega} = (\mathbb{P}_\lambda \cup \mathbb{U}_\lambda)^{<\omega}$ by Lemma 4.5. Limit steps are obvious. To prove the second part, note that ${\mathbb{U}_\xi}^{<\omega}$ is dense in ${\mathbb{P}_{\xi+1}}^{<\omega}$ by Lemma 4.3, and $\mathbb{U}_\xi$ belongs to $\mathfrak{M}_{\xi+1}$. □

Lemma 6.4 (in $\mathbf{L}$)**.** *If* $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ *then the set* W_X *of all ordinals* $\xi < \omega_1$ *such that* $\langle \mathbf{L}_\xi\,; X \cap \mathbf{L}_\xi \rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}\,; X \rangle$ *and* $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ *is unbounded in* ω_1. *More generally, if* $X_n \subseteq \mathrm{HC}$ *for all* n *then the set* W *of all ordinals* $\xi < \omega_1$, *such that* $\langle \mathbf{L}_\xi\,; \{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}\,; \{X_n\}_{n<\omega} \rangle$ *and* $\{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \in \mathfrak{M}_\xi$, *is unbounded in* ω_1.

Proof. Let $\xi_0 < \omega_1$. By standard arguments, there are ordinals $\xi < \lambda < \omega_1$, $\xi > \xi_0$, such that $\langle \mathbf{L}_\lambda\,; \mathbf{L}_\xi, X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_2}\,; \mathbf{L}_{\omega_1}, X \rangle$. Then $\langle \mathbf{L}_\xi\,; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,; X \rangle$, of course. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. It follows that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ since $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction. The second claim does not differ much. □

Corollary 6.5 (in $\mathbf{L}$, = Lemma 6 in [4])**.** *The forcing* $\mathbb{P}^{<\omega}$ *satisfies CCC.*

Proof. Suppose that $A \subseteq \mathbb{P}^{<\omega}$ is a maximal antichain. By Lemma 6.4, there is an ordinal ξ such that $A' = A \cap {\mathbb{P}_\xi}^{<\omega}$ is a maximal antichain in ${\mathbb{P}_\xi}^{<\omega}$ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, maximal, in the whole set $\mathbb{P}$ by Lemma 6.3. It follows that $A = A'$ is countable. □

7 The model

We consider the sets $\mathbb{P}, \mathbb{P}^{<\omega} \in \mathbf{L}$ (Definition 6.1) as forcing notions over $\mathbf{L}$.

Lemma 7.1 (= Lemma 7 in [4])**.** *A real* $x \in 2^\omega$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$ *iff* $x \in Z = \bigcap_{\xi < \omega_1^{\mathbf{L}}} \bigcup_{U \in \mathbb{U}_\xi} [U]$.

Proof. All sets $\mathbb{U}_\xi$ are pre-dense in $\mathbb{P}$ by Lemma 6.3. On the other hand, if $A \subseteq \mathbb{P}$, $A \in \mathbf{L}$ is a maximal altichain in $\mathbb{P}$, then easily $A \subseteq \mathbb{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 6.5. But then every tree $U \in \mathbb{U}_\xi$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 4.4, so that $\bigcup_{U \in \mathbb{U}_\xi} [U] \subseteq \bigcup_{T \in A} [T]$. □

Corollary 7.2 (= Corollary 9 in [4])**.** *In any generic extension of* $\mathbf{L}$, *the set of all reals in* 2^ω $\mathbb{P}$*-generic over* $\mathbf{L}$ *is* $\mathit{\Pi}_1^{\mathrm{HC}}$ *and* $\mathit{\Pi}_2^1$.

Proof. Use Lemma 7.1 and Proposition 6.2. □

Definition 7.3. From now on, let $G \subseteq \mathbb{P}^{<\omega}$ be a set $\mathbb{P}^{<\omega}$-generic over $\mathbf{L}$. If $k < \omega$ then let $G_k = \{\boldsymbol{\tau}(k) : \boldsymbol{\tau} \in G\}$, so that each G_k is $\mathbb{P}$-generic over $\mathbf{L}$ and $X_k = \bigcap_{T \in G_k} [T]$ is a singleton $X_k = \{x_k\}$ whose only element $x_k \in 2^\omega$ is a real $\mathbb{P}$-generic over $\mathbf{L}$. □

The whole extension $\mathbf{L}[G]$ is then equal to $\mathbf{L}[\{x_k\}_{k<\omega}]$, and our goal is now to prove that it contains no other $\mathbb{P}$-generic reals.

Lemma 7.4 (in the assumptions of Definition 7.3)**.** *If* $x \in \mathbf{L}[G] \cap 2^\omega$ *and* $x \notin \{x_k : k < \omega\}$ *then* x *is not a* $\mathbb{P}$*-generic real over* $\mathbf{L}$*.*

Proof. Otherwise there is a condition $\boldsymbol{\tau} \in \mathbb{P}^{<\omega}$ and a $\mathbb{P}^{<\omega}$-real name $\mathbf{c} = \{C_{ni}\}_{n<\omega,\, i=0,1} \in \mathbf{L}$ such that $\boldsymbol{\tau}$ $\mathbb{P}^{<\omega}$-forces that $\mathbf{c}$ is $\mathbb{P}$-generic while $\mathbb{P}^{<\omega}$ forces that $\mathbf{c} \neq \dot{\boldsymbol{x}}_k$ for all k. (Recall that $\dot{\boldsymbol{x}}_k$ is a $\mathbb{P}^{<\omega}$-real name for x_k.)

Let $C_n = C_{n0} \cup C_{n1}$; this is a pre-dense set in $\mathbb{P}^{<\omega}$. It follows from Lemma 6.4 that there is an ordinal $\lambda < \omega_1$ such that each set $C'_n = C_n \cap {\mathbb{P}_\lambda}^{<\omega}$ is pre-dense in ${\mathbb{P}_\lambda}^{<\omega}$, and the sequence $\{C'_{ni}\}_{n<\omega,\, i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni} = C'_n \cap C_{ni}$ — then C'_n is pre-dense in $\mathbb{P}^{<\omega}$, too, by Lemma 6.3. Thus we can assume that in fact $C_n = C'_n$, that is, $\mathbf{c} \in \mathfrak{M}_\lambda$ and $\mathbf{c}$ is a ${\mathbb{P}_\lambda}^{<\omega}$-real name.

Further, as $\mathbb{P}^{<\omega}$ forces that $\mathbf{c} \neq \dot{\boldsymbol{x}}_k$, the set D_k of all conditions $\boldsymbol{\sigma} \in \mathbb{P}^{<\omega}$ which directly force $\mathbf{c} \neq \dot{\boldsymbol{x}}_k$, is dense in $\mathbb{P}^{<\omega}$ — for every k. Therefore, still by Lemmas 6.4, we may assume that the same ordinal λ as above satisfies the following: each set $D'_k = D_k \cap {\mathbb{P}_\lambda}^{<\omega}$ is dense in ${\mathbb{P}_\lambda}^{<\omega}$.

Applying Theorem 5.3 with $\mathbb{P} = \mathbb{P}_\lambda$, $\mathbb{U} = \mathbb{U}_\lambda$, and $\mathbb{P} \cup \mathbb{U} = \mathbb{P}_{\lambda+1}$, we conclude that for each $U \in \mathbb{U}_\lambda$ the set Q_U of all conditions $\boldsymbol{v} \in {\mathbb{P}_{\lambda+1}}^{<\omega}$ which directly force $\mathbf{c} \notin [U]$, is dense in ${\mathbb{P}_{\lambda+1}}^{<\omega}$. As obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in the whole forcing $\mathbb{P}^{<\omega}$ by Lemma 6.3. This implies that $\mathbb{P}^{<\omega}$ forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}_\lambda} [U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}^{<\omega}$-generic, by Lemma 7.1. But this contradicts to the choice of $\boldsymbol{\tau}$. □

Finally the next lemma is a usual property of finite-support product forcing.

Lemma 7.5 (in the assumptions of Definition 7.3)**.** *If* $k < \omega$ *then* x_k *is not OD in* $\mathbf{L}[G]$*.* □

Now, arguing in the $\mathbb{P}^{<\omega}$-generic model $\mathbf{L}[G] = \mathbf{L}[\{x_k\}_{k<\omega}]$, we observe the countable set $X = \{x_k : k < \omega\}$ is exactly the set of all $\mathbb{P}$-generic reals by Lemma 7.4, hence it belongs to Π^1_2 by Corollary 7.2, and finally it contains no OD elements by Lemma 7.5.

□ (Theorem 1.1)

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