In Cohen generic extension, every countable OD set of reals belongs to the ground model

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Abstract

It is true in the Cohen generic extension of $\mathbf{L}$, the constructible universe, that every countable ordinal-definable set of reals belongs to $\mathbf{L}$.

Theorem 1. *Let $a \in \omega^\omega$ be a Cohen-generic real over $\mathbf{L}$. Then it is true in $\mathbf{L}[a]$ that if $X \subseteq \omega^\omega$ is a countable OD set then $X \in \mathbf{L}$.*

One may expect such a result of any homogeneous forcing notion. For instance, Theorem 1 is true for the Solovay model (the extension of $\mathbf{L}$ by Levy cardinal collapse up to an inaccessible cardinal [4]) — but by a different argument. One hardly can doubt that any typical homogeneous extension (Solovay-random, Sacks, Hehler, and the like) also satisfies the same result, but it's not easy to manufacture a proof of sufficient generality.

On the contrary, non-homogeneous forcing notions may lead to models with countable OD non-empty sets of reals with no OD elements [2], and such a set can even have the form of a Π^1_2 E_0-equivalence class [3].

Proof. Let $C = \omega^{<\omega}$ be the Cohen forcing. First of all, it suffices to prove that (it is true in $\mathbf{L}[a]$ that) if $X \subseteq \omega^\omega$ is a countable OD set then $X \subseteq \mathbf{L}$. Indeed, as the Cohen forcing is homogeneous, any statement about sets in $\mathbf{L}$, the ground model, is decided by the weakest condition.

There is a formula $\varphi(x)$ with an unspecified ordinal α_0 as a parameter, such that $X = \{x \in \omega^\omega : \varphi(x)\}$ in $\mathbf{L}[a]$, and then there is a condition $p_0 \in C$ such that $p_0 \subset a$ and p_0 C-forces that $\{x \in \omega^\omega : \varphi(x)\}$ is a countable set. Suppose to the contrary that $X \not\subseteq \mathbf{L}$, so that p_0 also forces $\exists\, x\, (x \notin \mathbf{L} \wedge \varphi(x))$.

There is a sequence $\{t_n\}_{n<\omega} \in \mathbf{L}$ of C-names, such that if $b \in \omega^\omega$ is Cohen generic and $p_0 \subset b$ then it is true in $\mathbf{L}[b]$ that $\{x \in \omega^\omega : \varphi(x)\} = \{t_n[\![b]\!] : n < \omega\}$,

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where $t[\![x]\!]$ is the interpretation of a C-name t by a real $x \in \omega^\omega$. Let $T \in \mathbf{L}$ be the C-name for $\{t_n[\![\dot a]\!] : n < \omega\}$. Thus we assume that p_0 forces

$$T[\![\dot a]\!] = \{t_n[\![\dot a]\!] : n < \omega\} = \{x \in \omega^\omega : \varphi(x)\} \not\subseteq \mathbf{L}\,, \tag{1}$$

where $\dot a$ is the canonical name for the C-generic real.

Let $\dot a_{\mathtt{lef}}, \dot a_{\mathtt{rig}}$ be canonical $(C \times C)$-names for the left, resp., right of the terms of a $(C \times C)$-generic pair of reals $\langle a_{\mathtt{lef}}, a_{\mathtt{rig}}\rangle$.

Corollary 2. *The pair $\langle p_0, p_0\rangle$ $(C \times C)$-forces over $\mathbf{L}$ that $T[\![\dot a_{\mathtt{lef}}]\!] \neq T[\![\dot a_{\mathtt{rig}}]\!]$.*

Proof. $\mathbf{L}[a_{\mathtt{lef}}] \cap \mathbf{L}[a_{\mathtt{rig}}] \cap \omega^\omega \subseteq \mathbf{L}$ due to the mutual genericity of $a_{\mathtt{lef}}, a_{\mathtt{rig}}$. □

Now pick a regular cardinal $\kappa > \alpha_0$. Consider, in $\mathbf{L}$, a countable submodel $\mathfrak{M}$ of $\mathbf{L}_\kappa$ containing α_0 and all names t_n and T. Let $\pi : \mathfrak{M} \to \overline{\mathfrak{M}}$ be the Mostowski collapse onto a transitive set $\overline{\mathfrak{M}}$.

Corollary 3. *It is true in $\overline{\mathfrak{M}}$ that $\langle p_0, p_0\rangle$ $(C \times C)$-forces $T[\![\dot a_{\mathtt{lef}}]\!] \neq T[\![\dot a_{\mathtt{rig}}]\!]$.*

Proof. By the elementarity, this holds in $\mathfrak{M}$. Further we have $\pi(t_n) = t_n$ and $\pi(T) = T$ because the names t_n and T belong to the transitive part of $\mathfrak{M}$. □

Corollary 4. *If $\langle a_{\mathtt{lef}}, a_{\mathtt{rig}}\rangle$ is a $(C \times C)$-generic pair over $\overline{\mathfrak{M}}$ with $p_0 \subset a_{\mathtt{lef}}$, $p_0 \subset a_{\mathtt{rig}}$, then $T[\![a_{\mathtt{lef}}]\!] \neq T[\![a_{\mathtt{rig}}]\!]$.* □

By the countability, there is a real $z \in \omega^\omega \cap \mathbf{L}$ satisfying $z(j) = 0$ for all $j < \mathtt{dom}\, p_0$ and C-generic over $\overline{\mathfrak{M}}$, so that $\overline{\mathfrak{M}}[z]$ is a set in $\mathbf{L}$. Let $x \in \omega^\omega$ be C-generic over $\mathbf{L}$, with $p_0 \subset x$. Then, as $z \in \mathbf{L}$, the real y defined by $y(k) = z(k) + x(k)$, $\forall\, k$, is C-generic over $\mathbf{L}$ as well, and we have $\mathbf{L}[x] = \mathbf{L}[y]$ and still $p_0 \subset y$. It follows from (1) that $T[\![\dot a_{\mathtt{lef}}]\!] = T[\![\dot a_{\mathtt{rig}}]\!]$ (an OD set of reals in $\mathbf{L}[x] = \mathbf{L}[y]$).

But on the other hand by the product forcing theorem and the choice of z, the pair $\langle x, y\rangle$ is $(C \times C)$-generic over $\overline{\mathfrak{M}}$, and hence $T[\![\dot a_{\mathtt{lef}}]\!] \neq T[\![\dot a_{\mathtt{rig}}]\!]$ by Corollary 4, which is a contradiction. □ (Theorem 1)

Remark 5. The Solovay model [4] admits a somewhat stronger result established in [1], namely, any countable non-empty OD *set of sets of reals* consists of OD elements (sets of reals). We don't know whether this is true in the Cohen generic extension $\mathbf{L}[a]$. □

Remark 6. Is Theorem 1 true for other popular forcing notions like *e.g.* the random forcing? The proof above crucially employs the countability of the Cohen forcing. □

References

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