

CAUCHY'S INFINITESIMALS, HIS SUM THEOREM, AND FOUNDATIONAL PARADIGMS

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ABSTRACT. Cauchy's *sum theorem* is a prototype of what is today a basic result on the convergence of a series of functions in undergraduate analysis. We seek to interpret Cauchy's proof, and discuss the related epistemological questions involved in comparing distinct interpretive paradigms. Cauchy's proof is often interpreted in the modern framework of a Weierstrassian paradigm. We analyze Cauchy's proof closely and show that it finds closer proxies in a different modern framework.

Keywords: Cauchy's infinitesimal; sum theorem; quantifier alternation; uniform convergence; foundational paradigms.

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1. INTRODUCTION

Infinitesimals were to disappear from mathematical practice in the face of Weierstrass' ε and δ notation. [Bottazzini 1986, p. 208]

Nearly two centuries after Cauchy first published his controversial *sum theorem*, historians are still arguing over the nature of its hypotheses and Cauchy's modification thereof in a subsequent paper.

1.1. The sum theorem. In 1821 Cauchy presented the sum theorem as follows:

When the various terms of series $[u_0 + u_1 + u_2 + \dots + u_n + u_{n+1} \dots]$ are functions of the same variable x , continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value.

This definition is found in his textbook *Cours d'Analyse*, Theorem I in section 6.1; the translation above is from [Bradley & Sandifer 2009, p. 90]. Cauchy returned to the sum theorem in an article [Cauchy 1853], and the matter has been debated by mathematicians and historians alike ever since.

Cauchy is often claimed to have modified¹ the hypothesis of his sum theorem in 1853 to a stronger hypothesis of uniform convergence. What did that modification consist of precisely? Writes G. Arsac:

Assez curieusement, Robinson montre que si l'on interprète de façon non standard les hypothèses de l'énoncé de Cauchy de 1853 que nous venons d'étudier, alors cet énoncé implique la convergence uniforme. [Arsac 2013, p. 134]

Our text can be viewed as an extended commentary on Arsac's *rather curious* observation.

The received scholarship on this issue contains a tension between a pair of contradictory contentions. On the one hand, it holds that Cauchy worked with an Archimedean continuum (A-track for short) exclusively; on the other, it holds that Cauchy introduced the condition of uniform convergence in 1853. These contentions are contradictory because Cauchy's 1853 condition was not formulated in an A-track fashion and on the contrary involved infinitesimals.

In more detail, uniform convergence is traditionally formulated over an Archimedean continuum in terms of alternating quantifiers which are conspicuously absent from Cauchy's text, whereas infinitesimals are conspicuously present there. From the A-track viewpoint, the modification required for uniform convergence focuses on how the remainder term $r_n(x)$ behaves both (1) as n increases and (2) as a function of x . This is, however, not what Cauchy does in his 1853 paper. Namely,

¹A debate of long standing concerns the issue of whether Cauchy *modified* or *clarified* the hypothesis of the sum theorem of 1853 as compared to 1821. Our analysis of the 1853 text is independent of this debate and our main conclusions are compatible with either view.

Cauchy declares n to be infinite, and requires r_n to be infinitesimal for all x (*toujours*).

An infinitesimal-enriched continuum (B-track for short) enables a characterisation of uniform convergence by means of enlargement of the domain, along the lines of what Cauchy suggests by means of his term *toujours* (*always*) as widely discussed in the literature; see e.g., [Laugwitz 1987], [Cutland et al. 1988]. We argue that Cauchy's procedures as developed in his 1853 text admit better proxies in modern infinitesimal frameworks² than in modern Weierstrassian frameworks limited to an Archimedean continuum.

Thus, a B-framework allows us to follow Cauchy in presenting a statement of the sum theorem without quantifier alternations, and also to be faithful to Cauchy in retaining his infinitesimals rather than providing *paraphrases* for them in terms of either a variable subordinate to a universal quantifier, or a sequence tending to zero.

1.2. Wartofski's challenge. Wartofsky launched a challenge to the historians of science. [Wartofsky 1976] called for a philosophical analysis of the ontology of a field like mathematics (as practiced during a particular period) as a prior condition for a historiography that possesses vision. We propose a minor step in this direction by introducing a distinction between, on the one hand, the *procedures* exploited in the mathematical practice of that period, and, on the other, the *ontology* (in the narrow sense) of the mathematical entities like points, numbers, or functions, as used during that period. We use Cauchy's sum theorem as a case study.

We seek an approach to Cauchy that interprets his sum theorem in a way faithful to his own *procedures*. We argue that, leaving aside issues of the *ontology of mathematical entities* and their justification in terms of one or another *foundational* framework, a B-track (infinitesimal-enriched) framework provides better proxies for interpreting Cauchy's *procedures* and inferential moves than does an A-track (Archimedean) framework. More specifically, we will provide a proxy for Cauchy's proof of the sum theorem and show that our A-track opponents, who criticize Cauchy's proof, have to ignore some of Cauchy's procedures.

²Some historians are fond of recycling the claim that Abraham Robinson used *model theory* to develop his system with infinitesimals. What they tend to overlook is not merely the fact that an alternative construction of the hyperreals via an ultrapower requires nothing more than a serious undergraduate course in algebra (covering the existence of a maximal ideal), but more significantly the distinction between *procedures* and *foundations* (see Section 1.2) which highlights the point that whether one uses Weierstrass's foundations or Robinson's is of little import, procedurally speaking.

As I. Grattan-Guinness pointed out, Cauchy's procedures in 1853 are *difficult to interpret* in an A-framework (see Section 7.1 for more details). Indeed, nowhere does Cauchy rely on quantifier alternations; furthermore, he exploits the term *always* to refer to an extension of the possible inputs to the function, which must now include infinitesimals in addition to ordinary values. Meanwhile, an A-framework can't express uniform convergence without quantifier alternations, nor countenance any extension beyond the real numbers. We document strawmanship and crude mathematical errors by A-track scholars in response to B-track interpretations of Cauchy.

The goal of a Cauchy scholar is not to translate Cauchy's procedures into contemporary mathematics so as to obtain a correct result, but rather to understand them on his (Cauchy's) own terms, or as close as possible to his own terms. We argue that Cauchy's procedures are best understood in the sense of infinitesimal mathematics, rather than paraphrased to fit the *Epsilontik* mode. Recent articles dealing with Cauchy's foundational stance include [Sørensen 2005], [Ramis 2012], [Nakane 2014].

1.3. A re-evaluation. This article is part of a re-evaluation of the history of infinitesimal mathematics. Such studies have recently appeared in journals as varied as *Erkenntnis* [Katz & Sherry 2013], *Foundations of Science* [Bascelli et al. 2017], *HOPOS* [Bascelli et al. 2016], *JGPS* [Bair et al. 2017], *Notices of the American Mathematical Society* [Bair et al. 2013], *Perspectives on Science* [Katz, Schaps & Shnider 2013], *Studia Leibnitiana* [Sherry & Katz 2014].³ Abraham Robinson's framework has recently become more visible due to the activity of some high-profile advocates like Terry Tao; see e.g., Tao [Tao 2014], Tao–Vu [Tao & Vu 2016]. The field has also had its share of high-profile detractors like Errett Bishop [Bishop 1977] and Alain Connes [Connes et al. 2001]. Their critiques were analyzed in [Katz & Katz 2011a], [Katz & Leichtnam 2013], [Kanovei, Katz & Mormann 2013], and [Sanders 2017a], [Sanders 2017b]. Additional criticisms were voiced by J. Earman [Earman 1975], K. Easwaran [Easwaran 2014], H. M. Edwards [Edwards 2007], G. Ferraro [Ferraro 2004], J. Grabiner [Grabiner 1981], J. Gray [Gray 2015], P. Halmos [Halmos 1985], H. Ishiguro [Ishiguro 1990], K. Schubring [Schubring 2005], Y. Sergeyev [Sergeyev 2015], and D. Spalt [Spalt 2002]. These were dealt with respectively in [Katz & Sherry 2013], [Bascelli et al. 2014], [Kanovei et al. 2015], [Bair et al. 2017], [Borovik & Katz 2012], [Błaszczyk et al. 2017a], [Błaszczyk et al. 2016], [Bascelli et al. 2016], [Błaszczyk et al. 2017b], [Gutman et al. 2017], [Katz & Katz 2011b].

³See <http://u.cs.biu.ac.il/~katzmik/infinitesimals.html> for a more detailed list.

2. THE REVISED SUM THEOREM

Cauchy denotes the n th partial sum of the series

$$s(x) = u_0(x) + u_1(x) + u_2(x) + \dots$$

by $s_n(x)$. He also denotes by $r_n(x) = s(x) - s_n(x)$ the n th remainder of the series. Cauchy's revised sum theorem from [Cauchy 1853, p. 456–457], as reprinted in [Cauchy 1900], states the following.⁴

Théorème 1. Si les différents termes de la série

$$u_0, u_1, u_2, \dots, u_n, u_{n+1}, \dots \quad (1)$$

sont des fonctions de la variable réelle x , continues, par rapport à cette variable, entre des limites données; si, d'ailleurs, la somme

$$u_n + u_{n+1} + \dots + u_{n'-1} \quad (3)$$

devient *toujours* infiniment petite pour des valeurs infiniment grandes des nombres entiers n et $n' > n$, la série (1) sera convergente et la somme s de la série sera, entre les limites données, fonction continue de la variable x . [Cauchy 1900, p. 33] (emphasis added)

We will highlight several significant aspects of the procedures exploited by Cauchy in connection with the sum theorem. Cauchy proceeds to present the following argument:

Si à la série (1) on substitue la série (2), l'expression (3), réduite à la somme

$$\frac{\sin(n+1)x}{n+1} + \frac{\sin(n+2)x}{n+2} + \dots + \frac{\sin n'x}{n'} \quad (4)$$

s'évanouira pour $x = 0$; mais pour des valeurs de x très-voisines de zéro, par exemple pour $x = \frac{1}{n}$, n étant un très-grand nombre, elle pourra différer notablement de zéro; et si, en attribuant à n une très-grande valeur, on pose non-seulement $x = \frac{1}{n}$, mais encore $n' = \infty$, la somme (4), ou, ce qui revient au même, le reste r_n de la série (2) se réduira sensiblement à l'intégrale

$$\int_1^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} - 1 + \frac{1}{1.2.3} \frac{1}{3} - \frac{1}{1.2.3.4.5} \frac{1}{5} + \dots = 0,6244\dots$$

[Cauchy 1853, p. 457]

⁴Here the equation numbers (1) and (3) are in Cauchy's text as reprinted in [Cauchy 1900].

Cauchy's use of the substitution " $x = \frac{1}{n}$ " is puzzling to a mathematician trained in the Weierstrassian framework. We will show that Robinson's framework provides a more successful interpretation.

2.1. Infinite numbers. Cauchy considers two instantiations (which he refers to as *valeurs*) of integer numbers, denoted n and n' . Note that he refers to them as infinitely large *numbers*, rather than either *quantities* or *sequences* increasing without bound.

2.2. Cauchy's ε . Cauchy points out that in order to prove the sum theorem, one needs to show that "le module de r_n [soit] inférieur à un nombre ε aussi petit que l'on voudra." This comment can be formalized in two different ways, [A] and [B]:

$$[A] (\forall \varepsilon \in \mathbb{R})(\exists m \in \mathbb{N})(\forall n \in \mathbb{N}) [(\varepsilon > 0) \wedge (n > m) \implies (|r_n| < \varepsilon)].$$

$$[B] (\forall \varepsilon \in \mathbb{R}) [(\varepsilon > 0) \rightarrow (|r_n| < \varepsilon)].$$

In neither interpretation does ε assume infinitesimal values.

Here n is a bound variable in formula [A] in the sense of being subordinate to the universal quantifier. Meanwhile, n is interpreted as a specific value in [B], i.e., it is a free variable. For formula [B] to be true, n must be infinite (and r_n infinitesimal). Interpretations [A] and [B] correspond to the two tracks compared in Section 3.4.

Clarifying the formulas completely would require adding a quantifier over the variable x of r_n , as well as specifying explicitly in [B] that n is infinite.

2.3. When is $n' = \infty$? Cauchy examines the series⁵

$$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \quad (2)$$

closely related to Abel's purported counterexample to the sum theorem, namely to the sum theorem as stated in 1821. The series sums to a (discontinuous) sawtooth waveform. Cauchy argues that the series does not in fact satisfy the hypotheses of his 1853 theorem, as follows. He considers the difference $s_{n'} - s_n = u_n + u_{n+1} + \dots + u_{n'-1}$ which translates into the following sum in the case of the series (2):⁶

$$\frac{\sin(n+1)x}{n+1} + \frac{\sin(n+2)x}{n+2} + \dots + \frac{\sin(n')x}{n'}. \quad (4)$$

Cauchy proceeds to assign the value ∞ to the index n' , by writing $n' = \infty$, and points out that the sum (4) is then precisely the remainder term r_n .

⁵The equation number (2) is in Cauchy's original text.

⁶The equation number (4) is in Cauchy's original text.

2.4. **Evaluating at $\frac{1}{n}$.** Cauchy goes on to evaluate the remainder $r_n(x)$ at $x = \frac{1}{n}$ and points out in [Cauchy 1853] that the result

se réduira sensiblement à l'intégrale

$$\int_1^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} - 1 + \frac{1}{1.2.3} \frac{1}{3} - \frac{1}{1.2.3.4.5} \frac{1}{5} + \dots = 0,6244\dots$$

Here one can evaluate the integral of $\frac{\sin x}{x}$ by viewing the series as an infinite Riemann sum; see [Laugwitz 1989, p. 212]. Cauchy concludes that the remainder is not infinitesimal and that therefore the purported counterexample (2) in fact does not satisfy the hypotheses of the sum theorem as formulated in 1853. For a discussion of Cauchy's proof of the theorem see Section 8.4.

3. METHODOLOGICAL ISSUES

3.1. **Issues raised by Cauchy's article.** Cauchy's 1853 article raises several thorny issues:

- (1) If one is to interpret Cauchy's new hypothesis as uniform convergence, how can one explain the absence of any quantifier alternations in the article (cf. Section 7.1, claim GG4)?
- (2) If the term *always* is supposed to signal a transition from the hypothesis of convergence in 1821 to a hypothesis of uniform convergence in 1853, what amplification does the term provide, given that the convergence was already required for every x in the domain $[x_0, X]$, and there are no further real inputs at which one could impose additional conditions?
- (3) The traditional reading of convergence, whether ordinary or uniform, leaves no room for indices given by *infinite* numbers n and n' referred to by Cauchy (see Section 2.1).
- (4) How can $|r_n|$ be smaller than every real $\varepsilon > 0$ if the option of interpreting this inequality in terms of alternating quantifiers is not available (see Section 2.2)?
- (5) How can $r_n(x)$ evaluated at $x = \frac{1}{n}$ produce the definite integral $\int_1^\infty \frac{\sin x}{x} dx$ when the answer should patently be dependent on n (see Section 2.4)?

Before seeking to answer the questions posed in Section 3.1 (see Section 5), we first address some methodological concerns and point out possible pitfalls.

3.2. **An ontological disclaimer.** Before answering such questions, one needs to clarify what kind of answer can be expected when dealing

with the work of a mathematician with no access to modern conceptualisations of mathematical entities like numbers and their set-theoretic ontology prevalent today.

Such a historiographic concern is ubiquitous. Thus, what could a modern historian mean, without falling into a trap of presentism, when he asserts that Lagrange was the first to define the notion of the derivative f' of a real function f ? Such an attribution is apparently problematic since Lagrange shared neither our set-theoretic ontology of a punctiform continuum⁷ given by the set of real numbers, nor our notion of a function as a relation of a certain type, again in a set-theoretic context.

Some helpful insights in this area were provided by Benacerraf and Quine, in terms of a dichotomy of *mathematical procedures and practice* versus *ontology of mathematical entities* like numbers. Thus, Quine wrote:

Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic. [Quine 1968, p. 198]

In a related development, [Benacerraf 1965] pointed out that if observer E learned that the natural numbers “are” the Zermelo ordinals $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots$, while observer J learned that they are the von Neumann ordinals $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$ then, strictly speaking, they are dealing with different things. Nevertheless, observer E’s actual mathematical practice (and the mathematical structures and procedures he may be interested in) is practically the same as observer J’s. Hence, different ontologies may underwrite one and the same practice.

The distinction between practice and ontology can be seen as a partial response to the challenge to the historians of science launched in [Wartofsky 1976], as discussed in Section 1.2.

When dealing with Lagrange, Cauchy, or for that matter any mathematician before 1872, we need to keep in mind that their ontology of mathematical entities like numbers is distinct from ours. However, the *procedures* Lagrange uses can often be interpreted and clarified in terms of their modern proxies, i.e., procedures used by modern mathematicians, as in the case of the concept of the derivative.

⁷Historians often use the term *punctiform continuum* to refer to a continuum made out of points (as for example the traditional set-theoretic \mathbb{R}). Earlier notions of continuum are generally thought to be non-punctiform (i.e., not made out of points). The term *punctiform* also has a technical meaning in topology unrelated to the above distinction.

The dichotomy of practice, procedures, and inferential moves, on the one hand, versus foundational issues of ontology of mathematical entities, on the other, is distinct from the familiar opposition between intuition and rigor. In a mathematical framework rigorous up to modern standards, the syntactic procedures and inferential moves are just as rigorous as the semantic foundational aspects, whether set-theoretic or category-theoretic.⁸

3.3. Felix Klein on two tracks. In addition to an ontological disclaimer as in Section 3.2, one needs to bear in mind a point made by Felix Klein in 1908, involving the non-uniqueness of a conceptual framework for interpreting pre-1872 procedures.

Felix Klein described a rivalry of dual approaches in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that

The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries [Klein 1908, p. 214].

Such a different conception, according to Klein,

harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of [...] infinitely small parts (ibid.).

Klein went on to formulate a criterion,⁹ in terms of the mean value theorem, for what would qualify as a successful theory of infinitesimals, and concluded:

I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive [Klein 1908, p. 219].

Thus, the approach based on notions of infinitesimals is not limited to “the work of Fermat, Newton, Leibniz and many others in the 17th and

⁸Nelson’s syntactic recasting of Robinson’s framework is a good illustration, in that the logical procedures in Nelson’s framework are certainly up to modern standards of rigor and are expressed in the classical Zermelo–Fraenkel set theory (ZFC). With respect to an enriched set-theoretic language, infinitesimals in Nelson’s Internal Set Theory (IST) can be found within the real number system \mathbb{R} itself. The semantic/ontological issues are handled in an appendix to [Nelson 1977], showing Nelson’s IST to be a conservative extension of ZFC.

⁹A similar criterion was formulated in [Fraenkel 1928, pp. 116–117]. For a discussion of the Klein–Fraenkel criterion see [Kanovei, Katz & Mormann 2013, Section 6.1].

18th centuries,” as implied by Victor J. Katz [KatzV 2014]. Rather, it was very much a current research topic in Felix Klein’s mind.

Of course, Klein in 1908 had no idea at all of modern infinitesimal frameworks, such as those of A. Robinson (see [Robinson 1961]), Lawvere–Moerdijk–Reyes–Kock (see [Kock 2006]), or J. Bell [Bell 2006]. What Klein was referring to is the *procedural* issue of how analysis is to be presented, rather than the *ontological* issue of a specific realisation of an infinitesimal-enriched number system in the context of a traditional set theory.

3.4. Track A and track B in Leibniz. The dichotomy mentioned by Klein (see Section 3.3) can be reformulated in the terminology of dual methodology as follows. One finds both A-track (i.e., Archimedean) and B-track (Bernoullian, i.e., involving infinitesimals) methodologies in historical authors like Leibniz, Euler, and Cauchy. Note that scholars attribute the first systematic use of infinitesimals as a foundational concept to Johann Bernoulli. While Leibniz exploited both infinitesimal methods and “exhaustion” methods usually interpreted in the context of an Archimedean continuum, Bernoulli never wavered from the infinitesimal methodology. To note the fact of such systematic use by Bernoulli is not to say that Bernoulli’s foundation is adequate, or that it could distinguish between manipulations with infinitesimals that produce only true results and those manipulations that can produce false results.

In addition to ratios of differentials like $\frac{dy}{dx}$, Leibniz also considered ratios of ordinary values which he denoted $(d)y$ and $(d)x$, so that the ratio $\frac{(d)y}{(d)x}$ would be what we call today the derivative. Here dx and $(d)x$ were distinct entities since Leibniz described them as respectively *inassignable* and *assignable* in his text *Cum Prodiisset* [Leibniz 1701c]:

... although we may be content with the assignable quantities $(d)y$, $(d)v$, $(d)z$, $(d)x$, etc., ... yet it is plain from what I have said that, at least in our minds, the unassignables [*inassignables* in the original Latin] dx and dy may be substituted for them by a method of supposition even in the case when they are evanescent; ... (as translated in [Child 1920, p. 153]).

Echoes of Leibnizian terminology are found in Cauchy when he speaks of *assignable numbers* while proving various convergence tests, see e.g., [Bradley & Sandifer 2009, pp. 37, 40, 87, 369]. An alternative term used by Cauchy and others is *a given quantity*. [Fisher 1978] shares the view that historical evaluations of mathematical analysis in general

and Cauchy’s work in particular can profitably be made in the light of both A-track and B-track approaches.

Leibniz repeatedly asserted that his infinitesimals, when compared to 1, violate what is known today as the Archimedean property, viz., Euclid’s *Elements*, Definition V.4; see e.g., [Leibniz 1695b, p. 322], as cited in [Bos 1974, p. 14].

4. FROM INDIVISIBLES TO INFINITESIMALS: RECEPTION

4.1. Indivisibles banned by the jesuits. Indivisibles and infinitesimals were perceived as a theological threat and opposed on doctrinal grounds in the 17th century. The opposition was spearheaded by clerics and more specifically by the jesuits. Tracts opposing indivisibles were composed by jesuits Paul Guldin, Mario Bettini, and André Tacquet [Redondi 1987, p. 291]. P. Mancosu writes:

Guldin is taking Cavalieri to be composing the continuum out of indivisibles, a position rejected by the Aristotelian orthodoxy as having atomistic implications. . . . Against Cavalieri’s proposition that “all the lines” and “all the planes” are magnitudes - they admit of ratios - Guldin argues that “all the lines . . . of both figures are infinite; but an infinite has no proportion or ratio to another infinite.” [Mancosu 1996, p. 54]

Tacquet for his part declared that the method of indivisibles “makes war upon geometry to such an extent, that if it is not to destroy it, it must itself be destroyed.” [Festa 1992, p. 205], [Alexander 2014, p. 119].

In 1632 (the year Galileo was summoned to stand trial over heliocentrism) the Society’s Revisors General led by Jacob Bidermann banned teaching indivisibles in their colleges [Festa 1990], [Festa 1992, p. 198]. Indivisibles were placed on the Society’s list of permanently banned doctrines in 1651 [Hellyer 1996]. The effects of anti-indivisible bans were still felt in the 18th century, when most jesuit mathematicians adhered to the methods of Euclidean geometry, to the exclusion of the new infinitesimal methods:

. . . le grand nombre des mathématiciens de [l’Ordre] resta jusqu’à la fin du XVIII^e siècle profondément attaché aux méthodes euclidiennes. [Bosmans 1927, p. 77]

Echoes of such bans were still heard in the 19th century when jesuit Moigno, who considered himself a student of Cauchy, wrote:

In effect, either these magnitudes, smaller than any *given* magnitude, still have substance and are divisible, or they are simple and indivisible: in the first case their existence is a chimera, since, necessarily greater than their half, their quarter, etc., they are not actually less than any *given* magnitude; in the second hypothesis, they are no longer mathematical magnitudes, but take on this quality, this would renounce the idea of the continuum divisible to infinity, a necessary and fundamental point of departure of all the mathematical sciences (ibid., xxiv f.) (as quoted in [Schubring 2005, p. 456]) (emphasis added)

Moigno saw a contradiction where there is none. Indeed, an infinitesimal is smaller than any *assignable*, or *given* magnitude; and has been so at least since Leibniz, as documented in Section 3.4. Thus, infinitesimals need not be less than “their half, their quarter, etc.,” because the latter are not assignable.

In fact, the dichotomy of Moigno’s critique is reminiscent of the dichotomy of a critique of Galileo’s indivisibles penned by Moigno’s fellow jesuit Orazio Grassi some three centuries earlier. P. Redondi summarizes it as follows:

As for light - composed according to Galileo of indivisible atoms, more mathematical than physical - in this case, logical contradictions arise. Such indivisible atoms must be finite or infinite. If they are finite, mathematical difficulties would arise. If they are infinite, one runs into all the paradoxes of the separation to infinity which had already caused Aristotle to discard the atomist theory ... [Redondi 1987, p. 196].

This criticism appeared in the first edition of Grassi’s book *Ratio ponderum librae et simbellae*, published in Paris in 1626. In fact, this criticism of Grassi’s

exhumed a discounted argument, copied word-for-word from almost any scholastic philosophy textbook. ... The Jesuit mathematician [Paul] Guldin, great opponent of the geometry of indivisibles, and an excellent Roman friend of [Orazio] Grassi, must have dissuaded him from repeating such obvious objections. Thus the second edition of the *Ratio*, the Neapolitan edition of 1627, omitted as superfluous the whole section on indivisibles. [Redondi 1987, p. 197].

Alas, unlike father Grassi, father Moigno had no Paul Guldin to dissuade him.

4.2. Unguru’s admonition. There are several approaches to uniform convergence. These approaches are equivalent from the viewpoint of modern mathematics (based on classical logic):

- [A] the epsilon-delta approach in the A-track context;
- [B] the approach involving a Bernoullian continuum;
- [C] the sequential approach in the A-track context.

The approach usually found in textbooks is the approach [A] via the epsilon-delta formulation involving quantifier alternations as detailed in Section 6.5. The B-track approach is detailed in Section 5. An alternative approach [C] via sequences runs as follows. A sequence of functions r_n as n runs through \mathbb{N} converges uniformly to 0 on a domain D if for each sequence of inputs x_n in D , the sequence of outputs $r_n(x_n)$ tends to zero. Giusti pursues approaches [A] and [C] to Cauchy respectively at [Giusti 1984, p. 38] and [Giusti 1984, p. 50] (see Section 8). Meanwhile, S. Unguru admonished:

It is . . . a historically unforgiveable sin . . . to assume wrongly that mathematical equivalence is tantamount to historical equivalence. [Unguru 1976, p. 783]

When seeking a mathematical interpretation of a historical text we must consider, in addition to questions of mathematical coherence, also questions of coherence with the actual procedures adopted by the historical author.

4.3. Bottazzini: What kept Cauchy from seeing? The points made in Sections 3.2 through 3.4 about ontology vs procedures, as well as multiple possibilities for modern interpretive frameworks, seem straightforward enough. However, the consequences of paying insufficient attention to such distinctions can be grave. Thus, U. Bottazzini opines that

the techniques that Abel used were those of Cauchy; his definitions of continuity and of convergence are the same, as are his use of infinitesimals in demonstrations. It is precisely this latter fact that kept both men from finding the weak point in Cauchy’s demonstration and from seeing that there is another form of convergence, what is today called the *uniform convergence* of a series of functions. [Bottazzini 1986, pp. 115-116] (emphasis in the original)

Bottazzini's claim that 'it is precisely' 'the use of infinitesimals in demonstrations' that 'kept [Cauchy] from finding ... and from seeing, etc.' only makes sense in the context of an interpretive paradigm exclusively limited to track A (see Section 3.4). Such an approach views the evolution of 19th century analysis as inevitable progress toward the foundations of analysis purged of infinitesimals as established by "the great triumvirate" [Boyer 1949, p. 298] of Cantor, Dedekind, and Weierstrass. In a similar vein, J. Dauben notes:

There is nothing in the language or thought of Leibniz, Euler, or Cauchy ... that would make them early Robinsonians. [Dauben 1988, p. 180]

"Early Robinsonians" perhaps not if this means familiarity with ultrafilters,¹⁰ but the extent to which the procedures as found in Leibniz, Euler, and Cauchy admit straightforward proxies in Robinson's framework does not emerge clearly from the above comment.

It seems almost as if, in an effort to save Cauchy as a herald for Weierstrassian arithmetical analysis, some A-track enforcers need carefully to purge the possibility that Cauchy might also be something quite different, something generally seen *nowadays* as contrary to the rational structure of his iconage; e.g., their image seems to be that this *other* was surely a remnant of the irrational in his methodology.

4.4. Fraser vs Laugwitz. In the abstract of his 1987 article in *Historia Mathematica*, Laugwitz is careful to note that he interprets Cauchy's sum theorem "with his [i.e., Cauchy's] own concepts":

It is shown that the famous so-called errors of Cauchy are correct theorems when interpreted with his own concepts. [Laugwitz 1987, p. 258]

In the same abstract, Laugwitz goes on to emphasize: "No assumptions on uniformity or on nonstandard numbers are needed." Indeed, in section 7 on pages 264–266, Laugwitz gives a lucid discussion of the sum theorem in terms of Cauchy's infinitesimals, with not a whiff of modern number systems. In particular this section does not mention the article [Schmieden & Laugwitz 1958]. In a final section 15 entitled "Attempts toward theories of infinitesimals," Laugwitz presents a rather general discussion, with no specific reference to the sum theorem, of how one might formalize Cauchyan infinitesimals in modern set-theoretic terms. A reference to [Schmieden & Laugwitz 1958] appears in this final section only. Thus, Laugwitz carefully distinguishes between his analysis of Cauchy's procedures, on the one hand, and the ontological issues of

¹⁰For the role of these in a possible construction of the hyperreals see Section 5.2.

possible implementations of infinitesimals in a set-theoretic context, on the other.

Alas, all of Laugwitz's precautions went for naught. In 2008, he became a target of damaging innuendo in the updated version of *The Dictionary of Scientific Biography*. Here C. Fraser writes as follows in his article on Cauchy:

Laugwitz's thesis is that certain of Cauchy's results that were criticized by later mathematicians are in fact valid *if one is willing to accept certain assumptions about Cauchy's understanding and use of infinitesimals. These assumptions reflect a theory of analysis and infinitesimals that was worked out by Laugwitz and Curt Schmieden during the 1950s.* [Fraser 2008, p. 76] (emphasis added)

What is particularly striking about Fraser's misrepresentation of Laugwitz's scholarship is Fraser's failure to recognize the dichotomy of procedure *vs* ontology. Fraser repeats the performance in 2015 when he writes:

Laugwitz, ... some two decades following the publication by Schmieden and him of the Ω -calculus commenced to publish a series of articles arguing that their non-Archimedean formulation of analysis is well suited to interpret Cauchy's results on series and integrals. [Fraser 2015, p. 27]

What Fraser fails to mention is that Laugwitz specifically separated his analysis of Cauchy's *procedures* from attempts to account *ontologically* for Cauchy's infinitesimals in modern terms.

4.5. The rhetoric of reaction. G. Schubring's disagreement with Laugwitz's interpretation of Cauchy found expression in the following comment:

[Giusti's article] spurred Laugwitz to even more detailed attempts to banish the error and *confirm that Cauchy had used hyper-real numbers.* On this basis, he claims, the errors vanish and the theorems become correct, or, rather, they always were correct (see Laugwitz 1990, 21). [Schubring 2005, p. 432] (emphasis added)

Here Schubring is *referring to* the article [Laugwitz 1990] though he is most decidedly not *quoting* it. In fact, there is no mention of hyperreals on page 21 in [Laugwitz 1990], contrary to Schubring's claim. What one does find here is the following comment:

The “mistakes” show rather, as *experimenta crucis* that one must understand Cauchy’s terms/definitions [*Begriffe*], in the spirit of the motto,¹¹ in an infinitesimal-mathematical sense. [Laugwitz 1990, p. 21] (translation ours)

We fully endorse Laugwitz’s comment to the effect that Cauchy’s procedures must be understood in the sense of infinitesimal mathematics, rather than paraphrased to fit the *Epsilontik* mode. Note that we are dealing with an author, namely Laugwitz, who published Cauchy studies in the leading periodicals *Historia Mathematica* [Laugwitz 1987] and *Archive for History of Exact Sciences* [Laugwitz 1989].¹² The idea that Laugwitz would countenance the claim that Cauchy “had used hyper-real numbers” whereas both the term *hyper-real* and the relevant construction were not introduced until [Hewitt 1948, p. 74], strikes us as far-fetched. Meanwhile, in a colorful us-against-“them” passage, J. Grabiner opines that

[Schubring] effectively rebuts the partisans of nonstandard analysis who wish to make Cauchy one of *them*, using the work of Cauchy’s disciple the Abbé Moigno to argue for Cauchy’s own intentions. [Grabiner 2006, p. 415]. (emphasis added)

Schubring indeed uses the work of Moigno in an apparent attempt to refute infinitesimals. Jesuit Moigno’s confusion on the issue of infinitesimals was detailed in Section 4.1.

Contrary to Schubring’s claim, Laugwitz did not attribute 20th century number systems to Cauchy, but rather argued that modern (B-track in the terminology of Section 3.4) infinitesimal frameworks provide better proxies for Cauchy’s *procedures* than modern Weierstrassian (A-track in the terminology of Section 3.4) frameworks. Laugwitz also sought to understand Cauchy’s inferential moves in terms of their modern proxies.

But Schubring’s insights don’t even reach a quarter of his student K. Viertel’s, who writes:

¹¹Here Laugwitz is referring to Cauchy’s motto to the effect that “Mon but principal a été de concilier la rigueur, dont je m’étais fait une loi dans mon *Cours d’analyse* avec la simplicité que produit la consideration directe des quantités infiniment petites.”

¹²The fact that Laugwitz had published articles in leading periodicals does not mean that he couldn’t have said something wrong. However, it does suggest the existence of a strawman aspect of Schubring’s sweeping claims against him.

...one can find assertions of Cauchy that contradict a consequent NSA-interpretation. With respect to the function $\frac{-m}{x}$ and x^{-m} and their continuity he remarks: Both become infinite and, as a consequence, discontinuous when $x = 0$. (Bradley, Sandifer 2009, 28). ... In non-standard analysis, however, one has $x^{-1} \simeq \infty$ for all $x \simeq 0$ which entails the continuity of the function $\frac{1}{x}$ for the point zero. Thus Cauchy's concept of continuity is not always consistent (or is not always compatible) with the concept of continuity of nonstandard analysis. [Viertel 2014, Section 4.3.1.1, pp. 56-57] (translation ours)

Viertel's apparent intuition that all infinite numbers x^{-1} are infinitely close to each other makes one wonder how such intuitions could be squared with Cauchy's apparently distinct infinite integers n and n' (see Sections 2.3 and 8.3). Viertel's claims concerning the *reciprocal function* in Robinson's framework are of course incorrect.

Such crude mathematical errors are symptomatic of questionable priority scales. Viertel's attempt to "prove" that the *reciprocal function* $\frac{1}{x}$ is continuous at the origin in nonstandard analysis is doomed from the start since the extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ preserves all the first-order properties under transfer.¹³ Anti-infinitesimal rhetoric by A-track scholars appears to be *reciprocal* to mathematical competence in matters infinitesimal.

¹³The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system (or more general mathematical structure), still apply (i.e., are *transferred*) to an extended number system (or more general mathematical structure). Thus, the familiar extension $\mathbb{Q} \hookrightarrow \mathbb{R}$ preserves the properties of an ordered field. To give a negative example, the extension $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the properties of an ordered field. The hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ preserves *all* first-order properties. For example, the identity $\sin^2 x + \cos^2 x = 1$ remains valid for all hyperreal x , including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$. In particular, the properties of the reciprocal function remain the same after it is extended to the hyperreal domain.

Arsac tries to explain nonstandard analysis but he seems to be as unaware of the transfer principle as Viertel: "Une fonction continue au sens habituel est une fonction continue aux points standards, mais elle ne l'est pas obligatoirement aux points non standard." [Arsac 2013, p. 133] Contrary to his claim, the natural extension of a continuous function f will be continuous at *all* hyperreal points c in the sense of the standard definition $\forall \epsilon > 0 \exists \delta > 0: |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$, by the transfer principle. Indeed, Arsac confused S-continuity and continuity...

4.6. Gestalt switch. The debate over Cauchy's understanding of infinitesimals bears some of the marks of a paradigm shift in the sense of Thomas Kuhn. A change is taking place in the historiography of mathematics, involving the entry of professional mathematicians into the field. Mathematicians bring with them the understanding that the same mathematical object or structure can be defined and described in a variety of mathematical languages. Similarly, the same theorem can be proved (or the same theory developed) in a multitude of languages and conceptual frameworks.

When a mathematician of the past used (and in some cases, invented) several distinct mathematical languages (as was the case for Cauchy), it is natural for a modern mathematician to exploit modern formalisations of these languages in the analysis of his predecessor's work. This adds depth and added dimension to our vision of the past, amounting to a *gestalt switch* as formulated in [Kuhn 1996, pp. 150–151]:

[T]he transition between competing paradigms cannot be made a step at a time, forced by logic and neutral experience. Like a gestalt switch, it must occur all at once (though not necessarily in an instant) or not at all. How, then, are scientists brought to make this transition? Part of the answer is that they are very often not. Copernicanism made few converts for almost a century after Copernicus' death. Newton's work was not generally accepted, particularly on the Continent, for more than half a century after the *Principia* appeared. Priestley never accepted the oxygen theory, nor Lord Kelvin the electromagnetic theory, and so on. The difficulties of conversion have often been noted by scientists themselves. . . . Max Planck, surveying his own career in his *Scientific Autobiography*, sadly remarked that "a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."

Here Kuhn is quoting [Planck 1950, pp. 33–34]. In short, science (including its historiography) makes progress one funeral at a time.

5. INTERPRETING CAUCHY'S INFINITESIMAL MATHEMATICS

Cutland et al. agree with Grattan-Guinness (see Section 7.1) concerning the *difficulty* of interpreting Cauchy's procedures in an Archimedean framework:

[Cauchy’s] modification of his theorem is anything but clear if we interpret his conception of the continuum as identical with the ‘Weierstrassian’ concept. Abraham Robinson [1966] first discusses Cauchy’s original ‘theorem’ . . . [Cutland et al. 1988, p. 376]

5.1. Robinson’s reading. The first B-track interpretation of Cauchy’s sum theorem was developed in [Robinson 1966, pp. 271–273].

Modern versions of the sum theorem require an additional hypothesis to obtain a mathematically correct result, either (i) in terms of uniform convergence, or (ii) in terms of equicontinuity of the family of functions.

Robinson considers both possibilities in the context of Cauchy’s 1821 formulation of the sum theorem, which is sufficiently ambiguous to accommodate various readings. The last mention of the term *equicontinuous* occurs in [Robinson 1966] on page 272, line 10 from bottom.

Meanwhile, starting on page 272, line 2 from bottom, Robinson turns to an analysis of Cauchy’s 1853 formulation of the sum theorem and its proof. Robinson writes: “Over thirty years after the publication of [the sum theorem], Cauchy returned to the problem [in] (Cauchy [1853]).” (ibid.) Robinson concludes:

If we interpret this theorem in the sense of Non-standard Analysis, so that ‘infiniment petite’ is taken to mean ‘infinitesimal’ and translate ‘toujours’ by ‘for all x ’ (and not only ‘for all standard x ’), then the condition introduced by Cauchy . . . amounts precisely to uniform convergence in accordance with (i) above.” [Robinson 1966, p. 273]

Thus, in his analysis of Cauchy’s 1853 article, Robinson only envisions the possibility (i) of uniform convergence. The reason is that Cauchy’s proof contains hints pointing toward such an interpretation, while it contains no hints in the direction of (ii) equicontinuity.

Robinson’s interpretation is based on a characterisation of uniform convergence exploiting a hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$; cf. [Davis 1977, p. 62, Theorem 6.3]. Down-to-earth treatments of the hypereals can be found e.g., in [Lindstrøm 1988] and [Prestel 1990]; see Section 5.2.

Robinson wrote:

Theorem 4.6.1 The sequence of standard functions $\{f_n(x)\}$ converges to the standard function $f(x)$ uniformly on $B \subset T$ [where T is a metric space] if and only if $f(p) \approx f_n(p)$ for all $p \in {}^*B$ and for all infinite n . [Robinson 1966, p. 116]

Here the relation of infinite proximity $f_n(x) \approx f(x)$ signifies that the difference $f_n(x) - f(x)$ is infinitesimal.

5.2. Constructing a hyperreal field. We outline a construction of a hyperreal field ${}^*\mathbb{R}$. Let $\mathbb{R}^{\mathbb{N}}$ denote the ring of sequences of real numbers, with arithmetic operations defined termwise. Then we have

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \text{MAX}$$

where MAX is the maximal ideal consisting of all “negligible” sequences (u_n) , i.e., sequences which vanish for a set of indices of full measure ξ , namely, $\xi(\{n \in \mathbb{N} : u_n = 0\}) = 1$.

Here $\xi: \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}$ (thus ξ takes only two values, 0 and 1) is a finitely additive measure taking the value 1 on each cofinite set, where $\mathcal{P}(\mathbb{N})$ is the set of subsets of \mathbb{N} .

The subset $\mathcal{F}_\xi \subseteq \mathcal{P}(\mathbb{N})$ consisting of sets of full measure is called a *free ultrafilter*. These originate with [Tarski 1930]. The construction of a Bernoullian continuum outlined here was therefore not available prior to that date. Further details on the hyperreals can be found in [Fletcher et al. 2017]. The educational advantages of the infinitesimal approach are discussed in [Katz & Polev 2017].

5.3. An example. To illustrate the mathematical issue involved, we let $B = [0, 1]$ and consider the sequence of real functions $f_n(x) = (1 - x)^n$ on the interval B . Then the limiting function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

is discontinuous. The condition that $f(x) \approx f_n(x)$ for an infinite n is satisfied at all real inputs x , but fails at the infinitesimal input $x = \frac{1}{n}$ since $f_n(\frac{1}{n}) = (1 - \frac{1}{n})^n \approx \frac{1}{e}$ and $\frac{1}{e}$ is appreciable: $\frac{1}{e} \not\approx 0$. The limit of the sequence can of course be re-written as the sum of a series, producing a series of continuous real functions with discontinuous sum.

A similar phenomenon can be illustrated by means of the family $f_n(x) = \arctan nx$.

5.4. Cauchy's B-track procedures. We will now interpret Cauchy's *procedures* (as opposed to *ontology*; see Section 3.2) in such a framework. The functions f_n in Robinson's Theorem 4.6.1 correspond to Cauchy's partial sums s_n . Our main thesis, in line with the comment by Grattan-Guinness cited in Section 6.2, is the following.

Main Thesis. Cauchy's term *toujours* suggests *extending* the possible inputs to the function.

We interpret this *extension* procedure in terms of the following hyperreal proxy. We extend the domain B of a real function $f: B \rightarrow \mathbb{R}$ to its hyperreal extension *B , obtaining a hyperreal function $f: {}^*B \rightarrow {}^*\mathbb{R}$. Thus, if B is Cauchy's interval $[x_0, X]$ with, say, $x_0 = 0$ (as in the example of the series (4) in Section 2.3), then the hyperreal extension ${}^*B = {}^*[0, X]$ will contain in particular positive infinitesimals of the form $\alpha = \frac{1}{n}$ where n is an infinite hyperinteger.

The condition $f_n(x) \approx f(x)$ is equivalent to the remainder term $r_n = f(x) - f_n(x)$ being infinitesimal. This is required to hold *at all hyperreal inputs* x , serving as a proxy for Cauchy's 1853 modified hypothesis with its implied extension of the domain.

Note that Robinson's Theorem 4.6.1 cited in Section 5.1 contains no quantifier alternations. This characterisation of uniform convergence closely parallels the condition found in Cauchy's text.

The condition $|r_n| < \varepsilon$ for all real positive ε amounts to requiring r_n (more precisely, $r_n(x)$) to be infinitesimal, as per item [B] in Section 2.2. As Cauchy pointed out, the condition is not satisfied by the series $\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$

In the hyperreal framework, the quantity $r_n(\frac{1}{n})$ for all infinite n will possess nonzero standard part (shadow) given precisely by the integral $\int_1^\infty \frac{\sin x}{x} dx$, in line with Cauchy's comment cited in Section 2.4 (a more detailed discussion can be found in [Cleave 1971]). This answers all the queries formulated in Section 3.1.

5.5. An issue of adequacy and refinement.

Bottazzini opines that the language of infinites and infinitesimals that Cauchy used here seemed ever more *inadequate* to treat the sophisticated and complex questions then being posed by analysis. . . . The problems posed by the study of nature, such as those Fourier had faced, now reappeared everywhere in the most delicate questions of "pure" analysis and necessarily led to the elaboration of techniques of inquiry considerably more *refined* than those that had served French mathematicians at the beginning of the century. Infinitesimals were to disappear from mathematical practice in the face of Weierstrass' ε and δ notation, . . . [Bottazzini 1986, p. 208]. (emphasis added)

Are infinitesimal techniques "inadequate," or less "refined" than epsilon-delta ones, as Bottazzini appears to suggest? We have argued that, *procedurally* speaking, the infinitesimal techniques of the classical masters ranging from Leibniz to Euler to Cauchy are more robust than is

generally recognized by historians whose training is limited to a Weierstrassian framework.

6. GRABINER'S READING

In this section we will analyze the comments by J. Grabiner related to Cauchy's alleged confusion in the context of the sum theorem. Two subtle mathematical distinctions emerged in the middle of the 19th century: (i) continuity *vs* uniform continuity, and (ii) convergence *vs* uniform convergence. We will show that Grabiner herself is confused about whether Cauchy's purported error consisted in his failure to distinguish continuity from uniform continuity, or his failure to distinguish convergence from uniform convergence.

6.1. Whose confusion? Grabiner alleges that Cauchy's 1821 theorem failed to distinguish pointwise from uniform convergence, and as a result he left his sum theorem open to some obvious counterexamples. Grabiner presents her case for Cauchy's alleged "confusion" in the following terms:

... in treating series of functions, Cauchy did not distinguish between pointwise and uniform convergence. The verbal formulations like "for all" that are involved in choosing deltas did not distinguish between "for any epsilon and for all x " and "for any x , given any epsilon" [19]. Nor was it at all clear in the 1820's how much depended on this distinction, since proofs about continuity and convergence were in themselves so novel. We shall see the same *confusion* between uniform and point-wise convergence as we turn now to Cauchy's theory of the derivative. [Grabiner 1983a, p. 191] (emphasis added)

Here the bracketed number [19] refers to her footnote 19 that reads as follows:

19. I. Grattan-Guinness, *Development of the Foundations of Mathematical Analysis from Euler to Riemann*, M. I. T. Press, Cambridge and London, 1970, p. 123, puts it well: "Uniform convergence was tucked away in the word 'always,' with no reference to the variable at all." [Grabiner 1983a, p. 194]

The views of Grattan-Guinness will be examined in Section 7. We now summarize Grabiner's position as follows:

- G1. Cauchy did not distinguish between pointwise convergence and uniform convergence.

- G2. Cauchy did not distinguish between “for any epsilon and for all x ” and “for any x , given any epsilon.”
- G3. The failure to distinguish between the quantified clauses in item (2) is indicative of a “confusion between uniform and pointwise convergence” in Cauchy’s procedures.
- G4. Cauchy’s term *always* is somehow related to uniform convergence.
- G5. Cauchy’s use of the term *always* made no reference to the variable of the function.
- G6. No mention of infinitesimals whatsoever is made by Grabiner in connection with Cauchy’s treatment of the sum theorem.

6.2. **Quantifier order.** What does Cauchy’s mysterious term *always* signify? A reader sufficiently curious about it to trace the sources will discover that Grattan-Guinness (as cited by Grabiner) is not referring to the 1821 *Cours d’Analyse*, but rather to the article [Cauchy 1853], where Cauchy

stated a new theorem which embodied the *extension* of the necessary and sufficient conditions for convergence, and a revised theorem 4.4 to cover uniform convergence ... [Grattan-Guinness 1970, p. 122]. (emphasis added)

We will get back to Cauchy’s *always* in due time. We can now analyze Grabiner’s claims G1, G2, and G3 as stated in Section 6.1. Her claim G1 is directly contradicted by Grattan-Guinness in the passage just cited. Here it turns out that in 1853 Cauchy *did modify* the original 1821 condition by *extending* it.

Grabiner’s claim G2 involves terms “for any”, “for all”, and “given any”, each of which expresses the universal quantifier. Thus claim G2 postulates a distinction between a pair of quantifier clauses, namely, $(\forall x)(\forall \epsilon)$ and $(\forall \epsilon)(\forall x)$. However, these two clauses are logically equivalent. Indeed, the order of the quantifiers is significant only when one has alternating existential and universal quantifiers, contrary to her claim G2. Quantifier alternation is irrelevant when only universal quantifiers are involved.

Grabiner’s claim G3 postulates a “confusion” on Cauchy’s part but is based on incorrect premises, and is therefore similarly incorrect. Grabiner’s indictment of Cauchy in [Grabiner 1983a, p. 191] does not stand up to scrutiny, at least if we are to believe her source, namely Grattan-Guinness. Since her claims G5 and G6 are based on Grattan-Guinness, we will postpone their analysis until Section 7.

6.3. What does it take to elucidate a distinction? Turning now to Grabiner's 1981 book, we find the following:

The elucidation of the difference between convergence and uniform convergence by men like Stokes, Weierstrass, and Cauchy himself was still more than a decade away. [Grabiner 1981, p. 12]

Apparently Cauchy did provide an "elucidation of the difference between convergence and uniform convergence," after all. The passage immediately following is particularly revealing:

The verbal formulations of limits and *continuity* used by Cauchy and Bolzano obscured the distinction between "for any epsilon, there is a delta that works for all x " and "for any epsilon and for all x , there is a delta." The only tools for handling such distinctions were words, and the usual formulation with the word "always" suggested "for all x " as well as "for any epsilon." (ibid., pp. 12–13) (emphasis added)

This passage immediately follows the passage on p. 12 on uniform *convergence*, but it appears to deal instead with uniform *continuity*. Indeed, the quantifier clause $\forall\epsilon\exists\delta\forall x$ that Grabiner refers to customarily appears in the definition of uniform continuity of a function rather than uniform convergence of a series of functions.¹⁴ Grabiner's position as expressed in the 1981 book involves the following claims (cf. those of her 1983 article enumerated in Section 6.1):

- G7. More than a decade later, Cauchy elucidated the difference between convergence and uniform convergence;
- G8. To handle the distinction between continuity and uniform continuity, Cauchy needed to handle the quantifier-order distinction between $\forall\epsilon\exists\delta\forall x$, on the one hand, and $\forall\epsilon\forall x\exists\delta$, on the other.
- G9. Cauchy only had the term *always* at his disposal.

We will analyze these in Section 6.5.

6.4. Cauchy's definition of continuity. Having mentioned both continuity and convergence in Section 6.3, we should point out that, while Cauchy's definition of continuity in 1821 was ambiguous to the extent that it is not clear whether he was defining ordinary or uniform continuity, what is clear is that he exploited the term *toujours* to refer to the possible inputs to the function. He formulated the definition as follows in 1853:

¹⁴The symbol δ is not used in reference to an integer tending to infinity.

... une fonction u de la variable réelle x sera *continue*, entre deux limites données de x , si, cette fonction admettant pour chaque valeur intermédiaire de x une valeur unique et finie, un accroissement infiniment petit attribué à la variable produit *toujours*, entre les limites dont il s'agit, un accroissement infiniment petit de la fonction elle-même. [Cauchy 1853, pp. 455–456] (emphasis on *continue* in the original; emphasis on *toujours* ours)

The phrase “*toujours*, entre les limites dont il s'agit” suggests that the term *toujours* refers to the possible inputs to the function. While Cauchy apparently did not distinguish between distinct notions of continuity (such as modern notions of pointwise continuity as opposed to uniform continuity), he did use the term *toujours* in reference to the possible inputs.

One can surmise that when he uses the term *toujours* in the definition of convergence in the same article, he is also referring to the possible inputs.

Cauchy's definition of convergence in 1821 cited in Section 1.1 did not use the term *toujours*. Thus Cauchy was possibly referring to ordinary convergence. By contrast, in 1853 the definition did use the term *toujours* in defining convergence, which arguably corresponds to uniform convergence (see Section 5 for a more detailed discussion).

6.5. Uniform convergence today. To fix notation, recall that today a series $\sum_{n=0}^{\infty} u_n(x)$ is said to converge uniformly to a function $s(x)$ on a domain I if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in I)(\forall m \in \mathbb{N}) \left[(m > N) \Rightarrow \left| s(x) - \sum_{n=0}^m u_n \right| < \epsilon \right]$$

where all variables and functions are real. Grabiner's claim G8 to the effect that Cauchy failed to distinguish $\forall \epsilon \exists \delta \forall x$ from $\forall \epsilon \forall x \exists \delta$ (see Section 6.3 above) is puzzling coming as it does in the midst of a discussion of the dichotomy of convergence *vs* uniform convergence of series of functions, rather than continuity *vs* uniform continuity. Furthermore, we are not familiar with any text where Cauchy dealt explicitly with a distinction related to continuity *vs* uniform continuity. In fact, Grabiner herself writes:

Weierstrass and his school distinguished—as Cauchy had not—between pointwise and uniform continuity.
[Grabiner 1981, p. 97]

We will therefore assume that what Grabiner meant on page 13 of her book was actually the following modification of her claim G8:

G8'. To handle the distinction between *convergence* and *uniform convergence*, Cauchy needed to handle the quantifier-order distinction between $\forall\epsilon\exists N\forall x$, on the one hand, and $\forall\epsilon\forall x\exists N$, on the other.

By page 140 Grabiner seems to have changed her mind once again about the status of convergence in Cauchy, as she writes:

Actually, [Cauchy's] proof implicitly assumed the function to be uniformly continuous, though he did not distinguish between continuity and uniform continuity, *just as he had not distinguished between convergence and uniform convergence*. [Grabiner 1981, p. 140] (emphasis added)

We will assume that she is referring to [Cauchy 1821] here rather than [Cauchy 1853]. Grabiner does not explain Cauchy's use of the term *always*, but she assumes in claims G7–G9 that the use of this term somehow corresponds to requiring the stronger *uniform convergence*. She also assumes in G8' that successfully dealing with the dichotomy of convergence *vs* uniform convergence necessarily requires a quantifier-order distinction, indicative of an A-track *parti pris*.

6.6. A-tracking Cauchy's infinitesimals. Grabiner wrote in 1983:

Now we come to the last stage in our chronological list: definition. In 1823, Cauchy defined the derivative of $f(x)$ as the limit, when it exists, of the quotient of differences $\frac{f(x+h)-f(x)}{h}$ as h goes to zero [4, pp. 22–23]. [Grabiner 1983b, p. 204]

Her reference “4, pp. 22–23” is to [Cauchy 1823]. Here is Cauchy's definition that Grabiner claims to report on:

Lorsque la fonction $y = f(x)$ reste continue entre deux limites données de la variable x , et que l'on assigne à cette variable une valeur comprise entre les deux limites dont il s'agit, un accroissement *infinitement petit*, attribué à la variable, produit un accroissement *infinitement petit* de la fonction elle-même. Par conséquent, si l'on pose alors $\Delta x = i$, les deux termes du rapport aux différences

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$$

seront des quantités *infinitement petites*. [Cauchy 1823]
(emphasis added)

Notice that the infinitely small have been mentioned *three times*. Cauchy continues:

Mais, tandis que ces deux termes s’approcheront indéfiniment et simultanément de la limite zéro, le rapport lui-même pourra converger vers une autre limite, soit positive, soit négative. Cette limite, lorsqu’elle existe, a une valeur déterminée, pour chaque valeur particulière de x ... on donne à la nouvelle fonction le nom de fonction dérivée, ... (ibid)

Grabiner’s paraphrase of Cauchy’s definition is a blatant example of failing to see what is right before one’s eyes; for Cauchy’s *infinitely small* i have been systematically purged by Grabiner, only to be replaced by a post-Weierstrassian imposter “as h goes to zero.” The difference between the Cauchyan and Weierstrassian notions of limit was recently analyzed in [Nakane 2014].

7. INTERPRETATION BY GRATTAN-GUINNESS

As we saw in Section 6, Grabiner restricts her discussion of the sum theorem to mentioning that in 1853, Cauchy introduced uniform convergence (which is a sufficient condition for the validity of the sum theorem as stated today). Grattan-Guinness treats the sum theorem in more detail. He notes that

Cauchy lived through the appearance of modes of convergence and returned to his [sum theorem] in a short paper of 1853... [Cauchy] admitted that

$$\frac{\pi - x}{2} = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$$

... was a counterexample with its discontinuity of magnitude π when x equals a multiple of π .

[Grattan-Guinness 1970, p. 122].

7.1. Difficult to interpret. Recall that Cauchy denotes by $s_n(x)$ the n th partial sum of the series $s(x) = u_0(x) + u_1(x) + \dots$. He also denotes by $r_n(x) = s(x) - s_n(x)$ the n th remainder of the series. Grattan-Guinness writes:

So a repair was needed, precisely at the point in his original proof of [the sum theorem] where he had said that $[r_n(x_0 + \alpha) - r_n(x_0)]$ “becomes insensible at the same time” as $r_n(x_0)$. (ibid.)

Here an understanding of the term $r_n(x_0 + \alpha)$ is crucial to interpreting Cauchy's proof, and we will return to it in Section 7.2. Grattan-Guinness proceeds to make the following key comment:

This remark [of Cauchy's] is *difficult to interpret* against [i.e., in the context of] the classification of modes of uniform convergence given here . . . since α is an infinitesimally small increment of x . (ibid.) (emphasis added)

The “modes of convergence” he is referring to here appear in his book [Grattan-Guinness 1970, pp. 119–120] (rather than in Cauchy's article). They are all formulated in terms of quantifier alternations, while the variable x ranges through the real domain of the function. Grattan-Guinness then paraphrases the revised version of Cauchy's theorem from [Cauchy 1853, p. 33], asserts that the proof is based on “the combined smallness of $r_n(x_0)$ and $r_n(x_0 + \alpha)$ ” [Grattan-Guinness 1970, p. 123] among other ingredients, and concludes that “uniform convergence itself was tucked away in the word ‘always’ with no reference to the variable at all.” (ibid.) We summarize his position as follows:

- GG1. Cauchy exploits infinitesimals in his attempted proof of the sum theorem.
- GG2. More specifically, Cauchy evaluates the remainder term r_n at an input $x_0 + \alpha$ where α is an infinitesimal.
- GG3. Cauchy's proof is based on the smallness of $r_n(x_0 + \alpha)$.
- GG4. The procedures in [Cauchy 1821] and [Cauchy 1853] involve no quantifier alternations.¹⁵
- GG5. Cauchy's arguments relying on infinitesimals are difficult to interpret in the context of the traditional characterisations of uniform convergence which rely on quantifier alternations.
- GG6. Cauchy's term *always* alludes to uniform convergence.
- GG7. Cauchy's term *always* makes no reference to the variable x .

7.2. Shortcomings. The analysis of Cauchy's procedures by Grattan-Guinness contains three shortcomings. First, claim GG3 concerning the expression $r_n(x_0 + \alpha)$ does not mention the crucial relation

$$\alpha = \frac{1}{n}$$

¹⁵Grattan-Guinness acknowledges this point GG4 implicitly in his summary of Cauchy's proof in [Grattan-Guinness 1970, p. 123]. Meanwhile, an ε does occur in Cauchy on p. 32; see Section 2.2.

between n and the infinitesimal. This relation is crucial to Cauchy’s analysis of series (2) namely $\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$ already discussed in Section 2.3. In fact in 1853 Cauchy did not use the symbol α at all but worked directly with the infinitesimal $\frac{1}{n}$.

As far as claim GG6 is concerned, Grattan-Guinness makes no attempt to explain the relation between uniform convergence and the term *always* exploited in the formulation of Cauchy’s theorem.

Finally, his claim GG7 is misleading. The term *always* is indeed meant to refer to the variable x (as it did in Cauchy’s definition of continuity; see Section 6.4). Neither Grabiner nor Grattan-Guinness properly address this point. But in order to explain precisely in what way the term refers to x , one needs to abandon the limitations of the particular interpretive paradigm they are both laboring under, and avail oneself of a modern framework where Cauchy’s procedures are *not* “difficult to interpret”; see Section 3.

8. GIUSTI ON THE MEANING OF *toujours*

E. Giusti proposed two different interpretations of Cauchy’s sum theorem, which we explore in Sections 8.1 and 8.2.

8.1. Giusti’s epsilon track. In a 1984 study of Cauchy’s sum theorem, E. Giusti wrote:

The comparison between the statement and the proof *clarifies immediately the meaning* of the word “*toujours*” in this context. The sentence: “the sum (3)¹⁶ always becomes infinitesimal for infinite n and $n' > n$ ” turns out to mean nothing but that “it is possible to assign to n a value sufficiently large so that it follows that $|s_{n'} - s_n| < \varepsilon$ for every $n' > n$ and every x ”; [Giusti 1984, p. 38] (emphasis added)

Giusti’s comment indeed “clarifies immediately the meaning” of the sum theorem but only in the sense of providing a mathematically correct result loosely corresponding to Cauchy’s theorem. Giusti continues:

in other words, it indicates the *uniform* convergence of the series. As support for this, Cauchy shows how in the case of Abel’s series the condition is not satisfied, and the sum (3) does *not* become infinitely small for

¹⁶This is a reference to Cauchy’s sum (3) namely $u_n + u_{n+1} \dots u_{n'-1}$ discussed in Section 2.

infinitely large values of n and n' . (ibid.) (emphasis in the original; translation ours)

To summarize, Giusti's formulation relies on thinly veiled quantifier alternations, requires interpreting what Cauchy calls an "infinite number n " as a variable subordinate to a universal quantifier, and sheds little light on Cauchy's term *always*. Cauchy's own *procedures* are therefore not fully clarified.

8.2. Giusti's sequential track. A few pages later, Giusti offers an alternative interpretation in the following terms:

[In Cauchy's sum theorem] the magnitude which plays a crucial role is the remainder term $r_n(x)$, depending on two variables. The convergence of that series implies that $r_n(x)$ always tends to 0 for infinite values of x [sic], which means, from the standpoint of our interpretation, that for each sequence x_n the variable $r_n(x_n)$ is infinitesimal. Once again, this [procedure] is equivalent to [proving] uniform convergence. . . . [In Cauchy's analysis of the series of Abel], as we have already explained, Cauchy proves that if the sum $u_n + u_{n+1} + \dots + u_{n'-1}$ becomes always infinitely small for some infinitely big values of n and n' , then the series $u_1 + u_2 + u_3 + \dots$ converges and its sum is a continuous function. [Giusti 1984, p. 50] [translation ours]

Thus, Giusti provides a sequential interpretation of Cauchy's argument, still in an Archimedean context (see Section 4.2, item [C]). However, as Laugwitz pointed out,

Giusti gives a correct translation of the example into the language of sequences, . . . [Giusti 1984, 50]. But he fails to translate the general theorem and its proof. Actually both x (or $x + \alpha$) and n will have to be replaced by sequences which becomes troublesome as soon as they are not connected as in the example where $x = 1/n$. Moreover, the theorem shows the power of Cauchy's "direct consideration of infinitesimals." [Laugwitz 1987, p. 266]

A paraphrase of Cauchy's proof along the lines of Giusti's sequential reading may be harder to implement than the analysis of Abel's example, and may depart significantly from Cauchy's own proof as summarized in [Grattan-Guinness 1970, formula (6.30), p. 123], which we present in Section 8.4. An additional difficulty (with Giusti's [C]-track

reading; see Section 4.2) not mentioned by Laugwitz is detailed in Section 8.3.

8.3. Order $n' > n$. If n and n' are (infinite) *numbers* then one naturally expects as Cauchy does that they are comparable; as Cauchy writes, we have $n' > n$. However if they are *sequences* $n = (n_k)$ and $n' = (n'_k)$ tending to infinity, then there is no reason to assume that they are comparable. For example, if $n_k = k$ whereas $n'_k = k + (-1)^k$ then who is to say which sequence is bigger, n' or n ? Meanwhile in any ordered Bernoullian continuum (see Section 3.4) comparability of numbers is automatic. This is not to say that Cauchy *constructed* an ordered Bernoullian continuum, but rather that his *procedures* presume that the infinite numbers are ordered, and this idea finds more faithful proxies in modern infinitesimal theories than in Giusti's [C]-track reading.

8.4. Cauchy's track. To show that $s(x_0 + \alpha) - s(x_0)$ is infinitesimal, Cauchy writes $r_n = s - s_n$, so that

$$s(x_0 + \alpha) - s(x_0) = [s_n(x_0 + \alpha) - s_n(x_0)] + [r_n(x_0 + \alpha) - r_n(x_0)]$$

(see Section 5.4). The terms $r_n(x_0 + \alpha)$ and $r_n(x_0)$ are infinitesimal for *each* infinite n by Cauchy's *always* hypothesis. To account for the smallness of the first summand, Grattan-Guinness appeals to the "continuity of $s_n(x)$." However, this is immediate only for finite n , whereas Cauchy's proof of continuity requires $\Delta s_n = s_n(x_0 + \alpha) - s_n(x_0)$ to be infinitesimal for *some* infinite n .

Can one guarantee that if Δs_n is infinitesimal for each finite n , this property will persist for some infinite n ? In the context of hyperreal proxies for Cauchy's procedures, the answer is affirmative (every internal sequence which is infinitesimal for all finite n will remain infinitesimal for *some* infinite n). What is perhaps even more remarkable than the validity of this type of *permanence principle* is the fact that a related principle can indeed be found in [Cauchy 1829], as reported in [Robinson 1966, p. 274]. The relevant technical result in a hyperreal framework is [Robinson 1966, Theorem 3.3.20, p. 65]. [McKinzie & Tuckey 1997, p. 49] follow [Laugwitz 1990] in interpreting Euler in terms of *hidden lemmas* using a related result called the *overspill theorem*; see also the *sequential theorem* in [McKinzie & Tuckey 2001, p. 361].

9. CONCLUSION

Interpretation of texts written in the nineteenth century, and the meaning we give to technical terms, procedures, theories, and the like

are closely related to what we already know as well as our expectations and assumptions. This paper provides evidence that a change in the cultural-technical framework of a historian provides new explanations, which are arguably more natural, and new insights into Cauchy's work.

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