

# Definable minimal collapse functions at arbitrary projective levels

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## Abstract

Using a non-Laver modification of Uri Abraham's minimal  $\Delta_3^1$  collapse function, we define a generic extension  $L[a]$  by a real  $a$ , in which, for a given  $n \geq 3$ ,  $\{a\}$  is a lightface  $\Pi_n^1$  singleton,  $a$  effectively codes a cofinal map  $\omega \rightarrow \omega_1^L$  minimal over  $L$ , while every  $\Sigma_n^1$  set  $X \subseteq \omega$  is still constructible.

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## 1 Introduction

It is well-known that all sets  $x \subseteq \omega$  of the lightface class  $\Sigma_2^1$  or  $\Pi_2^1$  are Goedel-constructible. In fact this is an immediate corollary of the Shoenfield absoluteness theorem. But one gets models with nonconstructible sets which belong to the analytic hierarchy just above the mentioned threshold. In particular it is consistent with **ZFC** that there exists a  $\Delta_3^1$  real and a  $\Pi_2^1$  real singleton, see [10], and such a real can be of minimal **L**-degree, [11].

Many more results on definable sets of different kind have been obtained on the base of forcing methods invented in the abovementioned papers. Most of them employ versions of the *almost disjoint* coding method of [10]. A recent article [6] contains several powerful applications of almost disjoint coding, in particular, to the construction of models with  $\Delta_3^1$  well-orderings of the reals, in which the reals have some very special properties. The paper also contains further references.

Yet the almost disjoint coding technique is pretty useless in the case of models containing definable generic objects and *minimal* over the ground model with respect to this or another property. The first example of such a model was presented by Jensen [11]. Namely, Jensen's forcing notion  $\mathbf{J} \in \mathbf{L}$  consists of perfect trees in  $\omega^{<\omega}$  (a subset of the Sacks forcing), and if a real  $a \in \omega^\omega$  is  $\mathbf{J}$ -generic over  $\mathbf{L}$  then 1) it is true in  $\mathbf{L}[a]$  that  $\{a\}$  is a nonconstructible  $\Pi_2^1$  singleton, and 2)  $a$  is *minimal over  $\mathbf{L}$* , in the sense that if  $b \in \mathbf{L}[a] \cap \omega^\omega$  then either  $b \in \mathbf{L}$  or  $a \in \mathbf{L}[b]$ . (See also 28A in [9] on this forcing.)

Several variations of this forcing are known. In particular, a model in [13] in which, for a given  $n \geq 3$  there exists a minimal nonconstructible  $\Pi_n^1$  singleton but all  $\Sigma_n^1$  sets  $x \subseteq \omega$  are constructible, an  $\omega_2$ -long iteration of Jensen's forcing in [1], or a recent model in [12] in which there is an equivalence class of the equivalence relation  $E_0^1$  (a  $E_0$ -class, for brevity), which is a lightface  $\Pi_2^1$  set in  $\omega^\omega$ , not containing OD elements, and a related model in [7] containing a  $\Pi_2^1$  Groszek – Laver pair of  $E_0$ -classes.

The research of this paper was inspired by another minimal-style forcing construction, a generic extension  $\mathbf{L}[a]$  by Abraham [2] such that 1)  $\{a\}$  is a nonconstructible  $\Pi_2^1$  singleton in  $\mathbf{L}[a]$ , 2)  $\omega_1^{\mathbf{L}[a]} = \omega_2^{\mathbf{L}}$  (so  $a$  codes a collapse of  $\omega_1^{\mathbf{L}}$ ), and 3)  $a$  is a *minimal collapse over  $\mathbf{L}$* , in the sense that if  $b \in \mathbf{L}[a]$ ,  $b : \omega \rightarrow \omega_1^{\mathbf{L}}$ , and  $b$  is cofinal in  $\omega_1^{\mathbf{L}}$ , then  $a \in \mathbf{L}[b]$ . Abraham's forcing in [2] consists of Laver-style trees in  $\omega_1^{<\omega}$ , and its complicated construction in  $\mathbf{L}$ , while having a certain semblance of Jensen's method in [11], involves some crucial novel ideas.

Our main result extends this research line. The next theorem asserts the existence of a model of **ZFC**, in which, for a given  $n \geq 2$ , there is a  $\Pi_n^1$  real singleton which codes the collapse of  $\omega_1^{\mathbf{L}}$  in minimal way, and in the same time reals in  $\Sigma_n^1$  do not code the collapse. The abovementioned result of [2] corresponds to the case  $n = 2$  in this theorem, of course. We use the blackboard  $\mathfrak{n}$  to distinguish the fixed number  $n$  in the theorem from other numbers  $n$  in the text.

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<sup>1</sup>  $E_0$  is defined on the Baire space  $\omega^\omega$  so that  $x E_0 y$  iff the set  $\{n : x(n) \neq y(n)\}$  is finite.

**Theorem 1.1.** *Let  $\mathfrak{n} \geq 2$ . There is a generic extension  $\mathbf{L}[a]$  of  $\mathbf{L}$ , by a real  $a \in \omega^\omega$ , such that the following is true in  $\mathbf{L}[a]$ : (i)  $\omega_1^{\mathbf{L}[a]} = \omega_2^{\mathbf{L}}$ ;*

(ii) (minimality) *if  $b \in \mathbf{L}[a]$ ,  $b : \omega \rightarrow \omega_1^{\mathbf{L}}$ ,  $b$  is cofinal in  $\omega_1^{\mathbf{L}}$ , then  $a \in \mathbf{L}[b]$ ;*

(iii) *the singleton  $\{a\}$  is a (lightface)  $\Pi_{\mathfrak{n}}^1$  set;*

(iv) (vacuous for  $\mathfrak{n} = 2$ ) *every  $\Sigma_{\mathfrak{n}}^1$  set  $x \subseteq \omega$  belongs to  $\mathbf{L}$ .*

## 2 Structure of the proof

The proof of Theorem 1.1 is organized as follows. Basic notions, related to  $\omega_1$ -branching trees in  $\omega_1^{<\omega}$  (*wide trees*), are introduced in Section 3. Unlike [2], we'll not focus on Laver-style trees, which makes basic constructions somewhat simpler. Every set  $\mathbf{P}$  of wide trees  $T$ , closed under restrictions, is considered as a *forcing by wide trees*, a *WT-forcing* in brief, Section 4. Every WT-forcing adjoins a  $\mathbf{P}$ -generic “real”  $a \in \omega_1^\omega$ .

Section 6 presents a non-Laver modification of Abraham's method in [2, 2.14], designed to define uncountable decreasing sequences of wide trees. Basically, any collection  $F$  of wide trees, satisfying rather transparent conditions of Definition 6.1, yields a wide tree  $\mathbf{wr}(F)$ , so that if  $F \subseteq F'$  then  $\mathbf{wr}(F') \subseteq \mathbf{wr}(F)$ . We apply the method to prove Theorem 5.3 in sections 7, 8, which allows, given a wide tree  $S$  and a family of continuous functions  $f_\alpha : \omega_1^\omega \rightarrow \omega_1^\omega$ ,  $\alpha < \omega_1^{\mathbf{L}}$ , to define a smaller wide tree  $T \subseteq S$ , regular in some sense with respect to each  $f_\alpha$ .

Another technical device, also having its roots in [2], is introduced in Section 9. It allows to shrink a given wide tree  $S$  to a smaller wide tree  $T$  such that any pre-dense set  $U \subseteq S$  in a given family of  $\aleph_1$ -many such sets meets every infinite branch in  $T$  except for a bounded set of them (Corollary 9.5).

Then, arguing in the constructible universe  $\mathbf{L}$ , we define a forcing notion to prove Theorem 1.1 in Section 11 in the form  $\mathbb{P} = \bigcup_{\alpha < \omega_2} \mathbb{P}_\alpha$ . The summands  $\mathbb{P}_\alpha$  are  $\aleph_1$ -large WT-forcings defined by induction. Any  $\mathbb{P}$ -generic extension of  $\mathbf{L}$  happens to be a model for Theorem 1.1, which we prove in the remainder.

The inductive construction of  $\mathbb{P}_\alpha$  involves two key genericity ideas. The first idea, essentially by Jensen [11], is to make every level  $\mathbb{P}_\alpha$  of the construction *generic* in some sense over the union of lower levels  $\mathbb{P}_\xi$ ,  $\xi < \alpha$ . This is based on a construction developed in sections 10, 11, which includes the abovementioned modification of Abraham's method. The iterated genericity of the levels  $\mathbb{P}_\alpha$  implies that the two sets are equal in any  $\mathbb{P}$ -generic extension of  $\mathbf{L}$ :

- 1) the singleton  $\{\mathbf{a}[G]\}$  of the principal generic element  $\mathbf{a}[G] \in \omega_1^\omega$ ,
- 2) the intersection  $\bigcap_{\alpha < \omega_2} \bigcup_{T \in \mathbb{P}_\alpha}$ .

This equality, eventually leading to (ii) of Theorem 1.1, is established in sections 12, 13, on the base of studies of continuous functions in sections 7, 8.

The second idea goes back to old papers [8], [13]. In  $\mathbf{L}$ , let  $\overline{\text{WTF}}$  be the set of all countable sequences  $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi < \alpha}$  ( $\alpha < \omega_1$ ), compatible with the

first genericity idea at each step  $\xi < \alpha$ . Then a whole sequence  $\langle \mathbf{P}_\alpha \rangle_{\alpha < \omega_1}$  can be interpreted as a maximal chain in  $\overline{\mathbf{WTF}}$ . It happens that if such a chain is *generic*, in some sense precisely defined in Section 11, (ii) of Theorem 11.4, with respect to all  $\Sigma_{n-1}^1$  subsets of  $\overline{\mathbf{WTF}}$ , then the ensuing forcing notion  $\mathbb{P} = \bigcup_{\alpha < \omega_1} \mathbb{P}_\alpha$  inherits some basic forcing properties of the whole forcing by (all) wide trees, up to the  $n$ -th level of projective hierarchy. This includes, in particular, the invariance of the forcing relation with respect to some natural transformations of wide trees, leading eventually to the proof of (iv) of Theorem 1.1 in sections 15 – 18.

### 3 Wide trees

Let  $\omega_1^{<\omega}$  be the set of all *strings* (finite sequences) of ordinals  $\xi < \omega_1$  — including *the empty string*  $\Lambda$ . If  $s \in \omega_1^{<\omega}$  then  $\text{lh}(s) < \omega$  is the *length* of a string  $s$ , and  $\max s < \omega_1$  is the largest term in  $s$ . Let  $\omega_1^n = \{s \in \omega_1^{<\omega} : \text{lh}(s) = n\}$  (strings of length  $n$ ). If  $t \in \omega_1^{<\omega}$  and  $\xi < \omega_1$ , then  $t \hat{\ } \xi$  is the extension of  $t$  by  $\xi$  as the rightmost term. If  $s, t \in \omega_1^{<\omega}$  then  $s \subseteq t$  means that the string  $t$  extends  $s$ , while  $s \subset t$  means a proper extension. A set  $T \subseteq \omega_1^{<\omega}$  is a *tree* iff  $s \in T \implies t \in T$  whenever  $s, t \in \omega_1^{<\omega}$  and  $t \subset s$ . Then:

- if  $s \in T$  then  $\text{succ}_T(s) = \{t \in T : s \subset t \wedge \text{lh}(t) = \text{lh}(s) + 1\}$ , the set of all *successors* of  $s$  in  $T$ . If  $\text{succ}_T(s) = \emptyset$  then  $s$  is an *endnode* of  $T$ ;
- $\mathbf{BN}(T) = \{s \in T : \text{card}(\text{succ}_T(s)) \geq 2\}$ , all *branching nodes* of  $T$ , and  $\mathbf{BN}_n(T) = \{s \in \mathbf{BN}(T) : \text{card}(\{u \in \mathbf{BN}(T) : u \subset s\}) = n\}$ ;
- if  $u \in T$  then define  $T \upharpoonright_u = \{t \in T : u \subseteq t \vee t \subseteq u\}$ , a *restricted tree*;
- if  $T$  is not pairwise  $\subseteq$ -compatible then there is a largest string  $u \in T$  such that  $T \upharpoonright_u = T$ , denoted by  $u = \text{stem}(T)$ , then  $\{\text{stem}(T)\} = \mathbf{BN}_0(T)$ ;
- $[T] = \{x \in \omega_1^\omega : \forall m (x \upharpoonright m \in T)\}$ , a closed set in  $\omega_1^\omega$ .

**Definition 3.1.** A set  $U \subseteq T$  is *dense* in a tree  $T$  if  $\forall s \in T \exists u \in U (s \subseteq u)$ , *open dense*, if in addition  $s \in U$  holds whenever  $s \in T$ ,  $u \in U$ ,  $u \subseteq s$ , and *pre-dense*, if the set  $U' = \{s \in T : \exists u \in U (u \subseteq s)\}$  is dense.  $\square$

**Definition 3.2.** A tree  $\emptyset \neq T \subseteq \omega_1^{<\omega}$  is a *wide tree*, in symbol  $T \in \mathbf{WT}$ , if any  $s \in T$  can be extended to a branching node  $t \in \mathbf{BN}(T)$ ,  $s \subseteq t$ , and if  $t \in \mathbf{BN}(T)$  then  $\text{card}(\text{succ}_T(s)) = \aleph_1$  — i.e., all branching nodes are  $\omega_1$ -branching.  $\square$

A bigger set  $\mathbf{WT}'$  consists of all trees  $\emptyset \neq T \subseteq \omega_1^{<\omega}$  such that each subtree of the form  $T \upharpoonright_s$ ,  $s \in T$ , is uncountable. Clearly  $\mathbf{WT} \subsetneq \mathbf{WT}'$ , but  $\mathbf{WT}$  is still dense in  $\mathbf{WT}'$ , so that every tree  $T \in \mathbf{WT}'$  contains a subtree  $S \in \mathbf{WT}$ ,  $S \subseteq T$ .

Generally,  $\mathbf{WT}$  and  $\mathbf{WT}'$  belong to the category of *uncountably-splitting* versions of the perfect set forcing. Similar forcing notions of this kind, as well as their Laver-style versions (which require every node above the stem to be a wide-splitting node), have been thoroughly studied in set theoretic papers, see e.g. Namba [14], Bukovsky [4], Abraham [2], Jech [9, Chap. 28], to mention a few.

**Lemma 3.3.** *Suppose that  $T \in \mathbf{WT}$ . If  $s \in T$  then  $T \upharpoonright_s \in \mathbf{WT}$ . If  $x \in X \subseteq [T]$ ,  $X$  is open in  $[T]$ , then there is  $s \in T$  such that  $s \subset x$  and  $T \upharpoonright_s \subseteq X$ .  $\square$*

**Definition 3.4.** We introduce two notions of inclusion between trees which partially honor the branching structure. If  $S, T \subseteq \omega_1^{<\omega}$  are trees then define:

- $S \subseteq_n T$  iff  $\mathbf{BN}_n(T) \subseteq S \subseteq T$ ;
- $S \subseteq'_n T$  iff  $S \subseteq T$  and  $\mathbf{BN}_{n-1}(S) = \mathbf{BN}_{n-1}(T)$ .  $\square$

**Lemma 3.5.** (i) *The relations  $\subseteq_0$  and  $\subseteq'_0$  coincide with just  $\subseteq$  ;*

(ii)  *$S \subseteq'_{n+1} T \implies S \subseteq_n T \implies S \subseteq'_n T$  ;*

(iii) *if  $S \subseteq_n T$  then  $\forall u \in \mathbf{BN}_n(T)$  (there is a unique  $v \in \mathbf{BN}_n(S)$  with  $u \subseteq v$ ) ;*

(iv) *if  $S \subseteq'_n T$  then  $S \subseteq_n T$  iff  $\forall u \in \mathbf{BN}_{n-1}(T)$  ( $\mathbf{succ}_T(u) = \mathbf{succ}_S(u)$ ).  $\square$*

**Lemma 3.6.** *Let  $T \in \mathbf{WT}$ ,  $n < \omega$ . Assume that if  $u \in \mathbf{BN}_n(T)$  then  $T_u \in \mathbf{WT}$ ,  $T_u \subseteq T \upharpoonright_u$ . Then the tree  $S = \bigcup_{u \in \mathbf{BN}_n(T)} T_u$  belongs to  $\mathbf{WT}$  and satisfies  $S \subseteq_n T$  and  $S \upharpoonright_u = T_u$  for all  $u \in \mathbf{BN}_n(T)$ .  $\square$*

Note that under the conditions of the lemma, if  $u \in \mathbf{BN}_n(T)$  then  $u \subseteq \mathbf{stem}(T_u)$ , and in addition  $\mathbf{BN}_n(S) = \{\mathbf{stem}(T_u) : u \in \mathbf{BN}_n(T)\}$ .

**Lemma 3.7.** *Assume that  $\dots \subseteq_4 T_4 \subseteq_3 T_3 \subseteq_2 T_2 \subseteq_1 T_1 \subseteq_0 T_0$  is an infinite decreasing sequence of trees in  $\mathbf{WT}$ . Then the tree  $T = \bigcap_n T_n$  belongs to  $\mathbf{WT}$ , and we have  $T \subseteq_n T_n$ , and hence  $\mathbf{BN}_n(T) = \mathbf{BN}_n(T_{n+1})$ , for all  $n$ .  $\square$*

## 4 Wide tree forcing notions and dense sets

A non-empty set  $\mathbb{P} \subseteq \mathbf{WT}$  is a *wide tree forcing*, WT-forcing in brief, if we have  $T \upharpoonright_u \in \mathbb{P}$  whenever  $u \in T \in \mathbb{P}$ . Thus  $\mathbf{WT}$  itself is a WT-forcing, and if  $S \in \mathbf{WT}$  then the set  $\{S \upharpoonright_t : t \in S\}$  is a WT-forcing.

**Remark 4.1.** Any WT-forcing  $\mathbf{P}$  can be considered as a forcing notion ordered so that if  $T \subseteq T'$ , then  $T$  is a stronger condition. The forcing  $\mathbf{P}$  adjoins a cofinal element  $x \in \omega_1^\omega$ . More exactly if a set  $G \subseteq \mathbf{P}$  is  $\mathbf{P}$ -generic over a given set universe  $\mathbf{V}$  (and  $\mathbf{P} \in \mathbf{V}$  is assumed) then the intersection  $\bigcap_{T \in G} [T]$  contains a unique element  $\mathbf{a}[G] \in (\omega_1^\mathbf{V})^\omega$ , and  $\mathbf{a}[G]$  satisfies  $G = \{T \in \mathbf{P} : \mathbf{a}[G] \in [T]\}$ ,  $\mathbf{V}[G] = \mathbf{V}[\mathbf{a}[G]]$ , and  $\sup \mathbf{a}[G] = \omega_1^\mathbf{V}$  (cardinality collapse).

Elements  $\mathbf{a}[G]$  of this kind are called  $\mathbf{P}$ -generic.  $\square$

To prove Theorem 1.1 we'll make use of a certain WT-forcing  $\mathbb{P} \subseteq \mathbf{WT}$ .

**Definition 4.2.** A set  $D \subseteq \mathbf{P}$  is *dense* in  $\mathbf{P}$  if for any  $S \in \mathbf{P}$  there is a tree  $T \in D$ ,  $T \subseteq S$ , *open dense*, if in addition  $S \in D$  holds whenever  $S \in \mathbf{P}$ ,  $T \in D$ ,  $S \subseteq T$ , and *pre-dense*, if the set  $D' = \{S \in \mathbf{P} : \exists T \in D (S \subseteq T)\}$  is dense.  $\square$

If  $T \in \mathbf{WT}$  and  $D \subseteq \mathbf{WT}$  then let  $D \upharpoonright T = \{s \in T : \exists S \in D (T \upharpoonright_s \subseteq S)\}$ .

**Lemma 4.3.** *Assume that  $\mathbf{P}$  is a WT-forcing, and  $D_n \subseteq \mathbf{P}$  is pre-dense in  $\mathbf{P}$  for all  $n$ . Let  $S_0 \in \mathbf{P}$ . Then there is a tree  $T \in \mathbf{WT}$  (not necessarily in  $\mathbf{P}$ !) such that  $T \subseteq S_0$  and if  $n < \omega$  then  $\mathbf{BN}_n(T) \subseteq D_n \upharpoonright T$ .*

**Proof.** We wlog assume that each  $D_n$  is open dense; otherwise replace it by  $D'_n = \{S' \in \mathbf{P} : \exists S \in D_n (S' \subseteq S)\}$ . Using Lemma 3.6 and the open density, define a sequence  $\dots \subseteq_4 T_4 \subseteq_3 T_3 \subseteq_2 T_2 \subseteq_1 T_1 \subseteq_0 T_0 \subseteq S_0$ , such that if  $n < \omega$  and  $s \in \mathbf{BN}_n(T_{n+1})$  then  $T_{n+1} \upharpoonright_s \in D_n$ . By Lemma 3.7, the tree  $T = \bigcap_n T_n$  is as required: if  $s \in \mathbf{BN}_n(T)$  then  $s \in \mathbf{BN}_n(T_{n+1})$ , so that  $T \upharpoonright_u \subseteq T_{n+1} \upharpoonright_u \in D_n$ .  $\square$

There is no way to directly extend Lemma 4.3 to the case of  $\omega_1$ -sequences of dense sets. But a somewhat weaker result of Lemma 9.4 will be possible.

## 5 Bounded sets and continuous maps

It is known from descriptive set theory that if a continuous map  $f : P \rightarrow \omega^\omega$  is defined on a perfect set  $P \subseteq \omega^\omega$  then  $f$  is a bijection or a constant on a suitable perfect subset  $P' \subseteq P$ . A similar but somewhat more complicated dichotomy holds for wide trees. Say that a set  $X \subseteq \omega_1^\omega$  is *bounded*, if there is an ordinal  $\beta < \omega_1$  such that  $X \subseteq \beta^\omega$ . Note that if  $T \in \mathbf{WT}$  then the set  $[T]$  is **unbounded**.

**Lemma 5.1.** *Let  $S \in \mathbf{WT}$  and  $f : [S] \rightarrow \omega_1^\omega$  be continuous. There is a tree  $T \subseteq S$ ,  $T \in \mathbf{WT}$ , such that either  $f''[T]$  is bounded or  $f \upharpoonright [T]$  is a bijection.*

**Proof.** Suppose that for no  $T \in \mathbf{WT}$ ,  $T \subseteq S$ ,  $f \upharpoonright [T]$  is bounded. Then, as the set  $\mathbf{BN}_1(S)$  is uncountable, by a simple cardinality argument there exist: an uncountable set  $U \subseteq \mathbf{BN}_1(S)$ , a number  $k$ , and for each  $t \in U$  — an ordinal  $\xi_t < \omega_1$  and a tree  $U_t \in \mathbf{WT}$  satisfying  $U_t \subseteq S \upharpoonright_t$ ,  $f(x)(k) = \xi_t$  for all  $x \in [U_t]$  (same  $k$  for all  $t \in U$ !), and if  $t \neq t'$  belong to  $U$  then  $\xi_t \neq \xi_{t'}$ .

Then the tree  $S_1 = \bigcup_{t \in U} U_t$  belongs to  $\mathbf{WT}$  and satisfies  $S_1 \subseteq'_1 S$ . In addition, there is a number  $k = k_1$  such that if  $u \neq u'$  belong to  $\mathbf{BN}_1(S_1)$  and  $x, x' \in [S_1]$ ,  $u \subset x$ ,  $u' \subset x'$ , then  $f(x)(k_1) \neq f(x')(k_1)$ .

Similarly, there is a tree  $S_2 \in \mathbf{WT}$ ,  $S_2 \subseteq'_2 S_1$ , and a number  $k_2$ , such that if  $u \neq u'$  belong to  $\mathbf{BN}_2(S_2)$  and  $x, x' \in [S_1]$ ,  $u \subset x$ ,  $u' \subset x'$ , then  $\langle f(x)(k_1), f(x)(k_2) \rangle \neq \langle f(x')(k_1), f(x')(k_2) \rangle$ .

Iterating this construction appropriately by induction, we get a required tree  $T = \bigcap_n S_n \in \mathbf{WT}$  by Lemma 3.6.  $\square$

The next theorem presents a dichotomy somewhat different than the one considered by Lemma 5.1, and related to the case of  $\aleph_1$ -many maps.

**Definition 5.2.** If  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is a continuous map, and  $U, V \in \mathbf{WT}$ , then  $\mathbb{H}(U, f, V)$  is the set of all strings  $s \in U$  such that (1)  $[V] \cap (f''[U \upharpoonright_s])$  is bounded or (2)  $f \upharpoonright [U \upharpoonright_s]$  is a *total identity*, that is,  $f(x) = x$  for all  $x \in [U \upharpoonright_s]$ .

Note that (1) and (2) are incompatible provided  $U \subseteq V$ .  $\square$

**Theorem 5.3** (the proof ends in Section 8). *Assume that  $S \in \mathbf{WT}$  and, for each  $\alpha < \omega_1$ ,  $f_\alpha : \omega_1^\omega \rightarrow \omega_1^\omega$  is a continuous function. Then there is a tree  $T \in \mathbf{WT}$ ,  $T \subseteq_1 S$ , such that, for any  $\alpha < \omega_1$ , the set  $\mathbb{H}(T, f_\alpha, T)$  is dense in  $T$ .*

## 6 Iteration of wide trees

Here we develop another method of construction of trees in  $\mathbf{WT}$ , similar to a construction introduced in [2, 2.14], and designed for the proof of Theorem 5.3.

**Definition 6.1.** A function  $J$  is an *iteration (of wide trees)*, in symbol  $J \in \mathbf{IWT}$ , if its *core*  $C = \text{dom } J$  is a subtree of  $\omega_1^{<\omega}$  (possibly with endnodes and/or isolated branches), all values  $J(u)$  are trees in  $\mathbf{WT}$ , and in addition

- (1) if  $u \subseteq v$  belong to  $C$  then  $v \in J(u)$  and  $J(v) \subseteq J(u) \upharpoonright_v$ ;
- (2) if  $u \subset v$  belong to  $C$ ,  $\text{lh}(v) = \text{lh}(u) + 1$ , and  $u \notin \mathbf{BN}(J(u))$  then  $J(v) = J(u)$ .

In this case we define the *wrap* of  $J$ ,

- (3)  $\mathbf{wr}(J) = \{s \in \omega_1^{<\omega} : \forall u \in \text{dom } J (u \subset s \implies s \in J(u))\}$ .

If  $\mathbf{P} \subseteq \mathbf{WT}$  then  $\mathbf{IWT}(\mathbf{P})$  consists of all iterations  $J \in \mathbf{IWT}$  with  $\text{ran } J \subseteq \mathbf{P}$ . An iteration  $J \in \mathbf{IWT}$  is *small* if the core  $C = \text{dom } J$  is at most countable.  $\square$

In particular  $\emptyset \in \mathbf{IWT}$  and  $\mathbf{wr}(\emptyset) = \omega_1^{<\omega}$ .

If  $C \subseteq \omega_1^{<\omega}$  is a tree and  $s \in \omega_1^{<\omega}$  then let  $\mathbf{proj}_C(s)$  (the *projection*) be the largest string in  $C$  with  $u \subseteq s$ ;  $\mathbf{proj}_C(s) = s$  provided  $s \in C$ .

**Lemma 6.2.** *If  $J \in \mathbf{IWT}$  then  $T = \mathbf{wr}(J) \in \mathbf{WT}$ ,  $C = \text{dom } J \subseteq T$ , and*

- (i) *if  $s \in C$  then  $s \subseteq \mathbf{stem}(J(s))$ ,  $T \upharpoonright_s \subseteq J(s)$ , and  $\mathbf{succ}_T(s) = \mathbf{succ}_{J(s)}(s)$ ;*
- (ii) *if  $s \in T \setminus C$  and  $u = \mathbf{proj}_C(s)$  then  $s \in J(u)$  and  $T \upharpoonright_s = J(u) \upharpoonright_s$ ;*
- (iii) *if  $s \in C$  is an endnode in  $C$  then we have  $T \upharpoonright_s = J(s)$ .*

**Proof.** If  $u \in C$  then  $u \in T$  by 6.1(1), so we have  $C \subseteq T$ .

(i) If  $s \in C$  then  $J(s) = J(s) \upharpoonright_s$  by 6.1(1) with  $u = v = s$ , so that obviously  $s \subseteq \mathbf{stem}(J(s))$ . If now  $t \in T$  and  $s \subseteq t$  then  $t \in J(s)$  by 6.1(3), therefore  $T \upharpoonright_s \subseteq J(s)$ . This implies  $\mathbf{succ}_T(s) \subseteq \mathbf{succ}_{J(s)}(s)$ . To get the equality, let  $t = s \hat{\ } \xi \in \mathbf{succ}_{J(s)}(s)$ . Then  $t \in T$  by 6.1(1),(3), so  $t \in \mathbf{succ}_T(s)$ , as required.

(ii) If  $s \notin C$  then by 6.1(1)(3) the criterion of  $s \in T = \mathbf{wr}(J)$  is just  $s \in J(u)$ , where  $u = \mathbf{proj}_C(s)$ . This easily implies the result. And (iii) is similar to (ii).

To prove  $T \in \mathbf{WT}$ , let  $s \in T$ . We have to prove that (a) if  $s \in \mathbf{BN}(T)$  then  $\mathbf{succ}_T(s)$  is uncountable, and (b) there is a string  $s' \in \mathbf{BN}(T)$  with  $s \subseteq s'$ . By (i), (ii) we have (a) immediately, so it remains to check (b).

*Case 1:*  $s \in T \setminus C$ . Then  $T \upharpoonright_s = J(u) \upharpoonright_s$  by (ii), where  $u = \mathbf{proj}_C(s)$ . But  $J(u) \upharpoonright_s \in \mathbf{WT}$  by Lemma 3.3, which easily implies (b).

Case 2:  $s$  is an endnode in  $C$ , so  $T \upharpoonright_s = J(s) \in \mathbf{WT}$  by (iii), follow Case 1.

Case 3: there is an endnode  $s'$  in  $C$  with  $s \subseteq s'$  — apply Case 2 for  $s'$ .

Case 4: if all the above fails then there is an infinite branch in  $C$  containing  $s$ , that is,  $b \in \omega_1^\omega$  such that  $b \upharpoonright n \in C$ ,  $\forall n$ , and  $s = b \upharpoonright n_0$ , where  $n_0 = \text{lh}(s)$ . Then  $b \upharpoonright n \in J(s)$  for all  $n$  by (i). Therefore, as  $J(s) \in \mathbf{WT}$ , there is a least number  $k \geq n_0$  with  $t = b \upharpoonright k \in \mathbf{BN}(J(s))$ . Then by the way  $J(t) = J(s)$  by 6.1(2), hence  $t \in \mathbf{BN}(J(t))$ , and finally  $t \in \mathbf{BN}(T)$  by (i), as required.  $\square$

The lemma allows to maintain infinite, even uncountable  $\subseteq$ -decreasing sequences of trees in  $\mathbf{WT}$ , with the help of the following two rather obvious results.

**Lemma 6.3.** *If  $J \subseteq J'$  are iterations in  $\mathbf{IWT}$  then  $\mathbf{wr}(J') \subseteq \mathbf{wr}(J)$ .*

*If  $\langle J_\xi \rangle_{\xi < \lambda}$  is a  $\subseteq$ -increasing sequence of iterations  $J_\xi \in \mathbf{IWT}$  then  $J = \bigcup_{\xi < \lambda} J_\xi \in \mathbf{IWT}$  and  $\mathbf{wr}(J) = \bigcap_{\xi} \mathbf{wr}(J_\xi)$ .*  $\square$

**Lemma 6.4.** *Let  $J \in \mathbf{IWT}$ ,  $\text{dom } J = C \subseteq C' \subseteq T = \mathbf{wr}(J)$ ,  $C'$  be a tree.*

- Define a **natural extension**  $J'$  of  $J$  to  $C'$  by  $\text{dom } J' = C'$ ,  $J'(s) = J(s)$  for  $s \in C$ , and if  $s \in C' \setminus C$  and  $u = \mathbf{proj}_C(s)$  then  $J'(s) = J(u) \upharpoonright_s$ .

*Then  $J' \in \mathbf{IWT}$ ,  $J \subseteq J'$ ,  $\mathbf{wr}(J') = \mathbf{wr}(J)$ .*  $\square$

Condition (2) of Definition 6.1 imposes important restrictions on the construction of iterations, basically justifying proper shrink only at successors of branching nodes. Nevertheless it leaves us enough freedom.

**Lemma 6.5.** *Assume that  $\mathbf{P}$  is a WT-forcing,  $J \in \mathbf{IWT}(\mathbf{P})$ ,  $C = \text{dom } J$ ,  $s \in T = \mathbf{wr}(J)$ , and  $s \notin C$  or  $s$  is an endnode in  $C$ . Let  $U \in \mathbf{P}$ ,  $U \subseteq T \upharpoonright_s$ . Then there exists an iteration  $J' \in \mathbf{IWT}(\mathbf{P})$  and a string  $s' \in \text{dom } J'$  such that  $J \subseteq J'$ ,  $s \subseteq s'$ , and  $J'(s') \subseteq U$ .*

**Proof.** Let  $t = \mathbf{stem}(U)$ , thus  $s \subseteq t \in \mathbf{BN}(U)$  and all shorter strings  $v \subset t$  do not belong to  $\mathbf{BN}(U)$ . Pick any  $s' \in U$  with  $\text{lh}(s') = \text{lh}(t) + 1$ ; then  $t \subset s' \notin C$ . Let  $u = \mathbf{proj}_C(s)$ . Let  $J' \in \mathbf{IWT}(\mathbf{P})$  be the extension of  $J$  to the domain  $C' = C \cup \{v : u \subset v \subseteq t\} \cup \{s'\}$  by  $J'(u) = J(s) = J(s) \upharpoonright_u$  whenever  $s \subset u \subseteq t$ , and finally  $J'(s') = U \upharpoonright_{s'}$ . To see that 6.1(2) is satisfied for  $J'$  at  $u = t$  and  $v = s'$ , recall that  $t \in \mathbf{BN}(U)$ , hence  $t \in \mathbf{BN}(J(s)) = \mathbf{BN}(J(t))$  as well.  $\square$

## 7 Key dichotomy lemma

**Lemma 7.1.** *Assume that  $\mathbf{P} \subseteq \mathbf{WT}$  is a WT-forcing,  $J \in \mathbf{IWT}(\mathbf{P})$  is a small iteration,  $S = \mathbf{wr}(J)$ ,  $g_0 \in C = \text{dom } J$ , and  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is continuous. There is a small iteration  $J' \in \mathbf{IWT}(\mathbf{P})$  and a string  $g \in C' = \text{dom } J'$ , such that  $g_0 \subseteq g$ ,  $J \subseteq J'$ , and  $g \in \mathbb{H}(T, f, T)$ , where  $T = \mathbf{wr}(J')$ , i. e.,*

- (1)  $[T] \cap (f''[T \upharpoonright_g])$  is bounded,    **or**    (2)  $f$  is a total identity on  $[T \upharpoonright_g]$ .

The lemma will be crucial in the proof of Theorem 5.3 in Section 8.

**Proof.** Pick any  $g_1 \in S \setminus C$  satisfying  $g_0 \subseteq g_1$ . Let  $u = \mathbf{proj}_C(g_1)$ . If  $f''[S \upharpoonright_{g_1}] \subseteq [C]$  (a bounded set) then let  $J_1 \in \mathbf{IWT}(\mathbf{P})$  be the natural extension of  $J$  to the domain  $C_1 = C \cup \{s : u \subset s \subseteq g_1\}$  by Lemma 6.4. Thus  $J \subseteq J_1$ ,  $\mathbf{dom} J_1 = C_1$ ,  $J_1(g_1) = J(u) \upharpoonright_{g_1}$ , and  $\mathbf{wr}(J_1) = S$ . Therefore  $J' = J_1$  and  $g = g_1$  satisfy (1).

Thus suppose that  $x_1 \in [S \upharpoonright_{g_1}]$ , and  $y_1 = f(x_1) \in [S] \setminus [C]$ . As  $f$  is continuous while  $\omega_1^\omega \setminus [C]$  open, there is a longer string  $g_2 \in S \setminus C$ ,  $g_1 \subset g_2$ , such that  $f(x) \notin [C]$  for all  $x \in S \upharpoonright_{g_2}$ . If  $f$  is a total identity on  $[T \upharpoonright_{g_2}]$  then let  $J_2 \in \mathbf{IWT}(\mathbf{P})$  be the natural extension of  $J$  to the domain  $C_2 = C \cup \{s : u \subset s \subseteq g_2\}$  by Lemma 6.4; now  $J' = J_2$  and  $g = g_2$  satisfy (2).

Thus suppose that  $x_2 \in [S \upharpoonright_{g_2}]$ , and  $y_2 = f(x_2) \neq x_2$ . There is a yet longer string  $g_3 \in S \setminus C$ ,  $g_2 \subset g_3$ , such that  $f(x) \neq x$  and  $f(x) \notin [C]$  for all  $x \in S \upharpoonright_{g_3}$ . If  $(f''[S \upharpoonright_{g_3}]) \cap [S] = \emptyset$  then let  $J_3 \in \mathbf{IWT}(\mathbf{P})$  be the natural extension of  $J$  to the domain  $C_3 = C \cup \{s : u \subset s \subseteq g_3\}$ ; now  $J' = J_3$ ,  $g = g_3$  satisfy (1).

Thus suppose that  $x_3 \in S \upharpoonright_{g_3}$  and  $y_3 = f(x_3) \in [S]$ . In addition,  $x_3 \neq y_3 \notin [C]$  holds as  $g_2 \subseteq g_3$ , hence there is  $m \geq \mathbf{lh}(g_3)$  such that  $t = y_3 \upharpoonright m \in \mathbf{BN}(S) \setminus C$  and  $t \neq s = x_3 \upharpoonright m$ . Let  $t' = y_3 \upharpoonright (m+1)$  (a successor of  $t$  in  $S$ ). There is a string  $h \in S$  such that  $t' \subset h$  but  $h \neq t'' = y_3 \upharpoonright \ell$ , where  $\ell = \mathbf{lh}(h)$ . As  $f$  is continuous, pick a number  $n \geq n_3 = \mathbf{lh}(g_3)$  such that  $t'' \subset f(x)$  holds for all  $x \in [S \upharpoonright_g]$ , where  $g = x_3 \upharpoonright n$ . Recall that  $u = \mathbf{proj}_C(g)$ . Let  $v = \mathbf{proj}_C(t)$ ,

$$C' = C \cup \{w \in \omega_1^{<\omega} : u \subset w \subseteq g\} \cup \{w \in \omega_1^{<\omega} : v \subset w \subseteq t'\},$$

and extend the iteration  $J$  to the domain  $C'$  by  $J'(w) = J(u) \upharpoonright_w$  whenever  $u \subset w \subseteq g$ ,  $J'(w) = J(v) \upharpoonright_w$  whenever  $v \subset w \subseteq t$ , and  $J'(t') = J(v) \upharpoonright_h$ .

Now it suffices to prove (1) in the form  $[T] \cap (f''[T \upharpoonright_g]) = \emptyset$ . Let  $g \subset x \in [S]$ . Then  $y = f(x)$  satisfies  $t'' \subset y$ , hence  $h \not\subset y$ . Let's show that  $y \notin [T]$ . It suffices to check  $t'' \notin T$ . Suppose otherwise. Then, as  $t' \in C'$ , we have  $t'' \in J'(t')$  by 6.1(3). However  $J'(t') = J(v) \upharpoonright_h$ , so it follows that  $t''$  and  $h$  are compatible, which contradicts to the construction, as required.  $\square$

## 8 The proof of the restriction theorem

Here we accomplish the proof of Theorem 5.3 on the base of the results above. We argue in the assumptions of Theorem 5.3.

The set  $\mathbf{P} = \{S \upharpoonright_t : t \in S\}$  is a WT-forcing and  $S \in \mathbf{P}$ .

By Lemmas 7.1 and 6.3, 6.4, there is a  $\subseteq$ -increasing sequence of small iterations  $J_\gamma \in \mathbf{IWT}(\mathbf{P})$ ,  $\gamma < \omega_1$ , with domains  $C_\gamma = \mathbf{dom} J_\gamma$  and trees  $S_\gamma = \mathbf{wr}(J_\gamma)$ , such that  $C_0 = \{u : u \subseteq \sigma\}$ , where  $\sigma = \mathbf{stem}(S)$ , and  $J_0(u) = S$  for all  $u \in C_0$ , the sets  $C = \bigcup_{\gamma < \omega_1} C_\gamma$  and  $T = \bigcap_{\gamma < \omega_1} T_\gamma$  coincide (Lemma 6.4 is responsible), and in addition (Lemma 7.1 is responsible), if  $s_0 \in C = T$  and  $\alpha < \omega_1$  then there is an index  $\gamma = \gamma(s_0, \alpha) < \omega_1$  and a string  $s \in C_\gamma$  such that  $s_0 \subseteq s$  and  $s \in \mathbb{H}(T_\gamma, f_\alpha, T_\gamma)$ . Then  $J = \bigcup_\alpha J_\alpha \in \mathbf{IWT}(\mathbf{P})$ ,  $C = \mathbf{dom} J$ , and  $T = \mathbf{wr}(J)$ , by

Lemma 6.3. Moreover, as  $T \subseteq T_\gamma$ , we have  $s \in \mathbb{H}(T, f_\alpha, T)$  as well. It follows that the set  $\mathbb{H}(T, f_\alpha, T)$  is dense in  $T$ , and obviously *open* dense. And finally we have  $T \subseteq_1 S$  by Lemma 6.2(i) with  $s = \sigma = \text{stem}(S)$ . (Recall that  $J_0(\sigma) = S$ .)

□ (Theorem 5.3)

## 9 Belts and covering

Here we introduce the last major tool employed in the definition of the forcing notion for Theorem 1.1. It is based on the following definition.

**Definition 9.1.** A set  $H \subseteq \omega_1^{<\omega}$ :

- *meets*  $x \in \omega_1^\omega$  iff  $\exists m (x \upharpoonright m \in H)$ ;
- is a *belt* for a tree  $T \in \mathbf{WT}$ , if it meets every  $x \in [T]$ ;
- *weakly covers*  $T$ , in symbol  $T \subseteq^w B$ , if there is an ordinal  $\beta < \omega_1$  such that  $H$  is a belt for each subtree  $T \upharpoonright_s$ , where  $s \in T$  and  $\max s \geq \beta$  — in other words, we require  $H$  to meet every  $x \in [T]$  with  $\sup x \geq \beta$ .

For instance, if  $n < \omega$  then  $\mathbf{BN}_n(T)$  is a belt for  $T \in \mathbf{WT}$ . □

**Lemma 9.2.** Let  $H \subseteq T$  weakly cover  $T \in \mathbf{WT}$  with a parameter  $\beta < \omega_1$ . Then

- (i)  $H$  is pre-dense in  $T$ ;
- (ii)  $H$  weakly covers any tree  $S \in \mathbf{WT}$ ,  $S \subseteq T$ , with the same  $\beta$ ;
- (iii) the set  $X = \{x \in [T] : H \text{ does not meet } x\}$  satisfies  $X \subseteq \beta^\omega$ .

**Proof.** (iii) Let  $x \in [T] \setminus \beta^\omega$ ,  $x(j) \geq \beta$  for some  $j$ . Let  $s = x \upharpoonright (j+1)$ . Then  $H$  is a belt for  $T \upharpoonright_s$ , hence  $H$  meets  $x$ . □

**Remark 9.3.** Being a belt is equivalent to the wellfoundedness of the subtree  $T' = \{s \in T : \neg \exists t \in H (t \subseteq s)\}$ , hence it is an *absolute* notion. It follows that to weakly cover with a parameter  $\beta$  is an *absolute* notion, too.

Now assume that  $H \subseteq T$  weakly covers  $T \in \mathbf{WT}$  with a parameter  $\beta < \omega_1$ . Let  $x \in [T]$  be an element cofinal in  $\omega_1$  ( $=\omega_1^{\mathbf{V}}$  of the given set universe  $\mathbf{V}$ ), which may exist in an extension of  $\mathbf{V}$ , Remark 4.1. **We claim that  $H$  meets  $x$ .** Indeed,  $x \notin \beta^\omega$  by the cofinality, and on the other hand, the absoluteness of the weak covering allows to apply Lemma 9.2(iii) in the extension containing  $x$ . □

**Lemma 9.4.** Assume that  $\mathbf{P}$  is a WT-forcing,  $T \in \mathbf{P}$ , and  $D_\xi \subseteq \mathbf{P}$  is open dense in  $\mathbf{P}$  for all  $\xi < \omega_1$ . Then there is a tree  $S \in \mathbf{WT}$  such that  $S \subseteq_1 T$ , and each set  $D_\xi \upharpoonright S = \{t \in S : \exists U \in D_\xi (S \upharpoonright_t \subseteq U)\}$  weakly covers  $S$ .

**Proof.** If  $\alpha < \omega_1$  then fix an enumeration of the countable set  $\{D_\xi : \xi \leq \alpha\} = \{D_k^\alpha : k < \omega\}$ . Using Lemma 3.6 and the open-density of each  $D_\xi$  in  $\mathbf{P}$ , define a sequence  $\dots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 = T$  of trees in  $\mathbf{WT}$ , such that if

$n \geq 1$  and  $u \in \mathbf{BN}_n(T_n)$  then  $T_n \upharpoonright_u \in \bigcap_{j,k \leq n} D_k^{u(j)}$ . The tree  $S = \bigcap_n T_n$  belongs to **WT** and satisfies  $S \subseteq_{n+1} T_n$  and  $\mathbf{BN}_n(S) = \mathbf{BN}_n(T_n)$  for all  $n$ , by Lemma 3.7. In particular  $S \subseteq_1 T$ . Now suppose that  $\xi < \omega_1$ .

We claim that  $\xi$  itself witnesses  $D_\xi \upharpoonright S$  to weakly cover  $S$ . Let  $x \in [S]$  and  $x(j) = \alpha \geq \xi$  for some  $j$ . Then  $D_\xi = D_k^\alpha$  for some  $k$ . Let  $n = 1 + \max\{j, k\}$ . There is a number  $m \geq n$  such that  $u = x \upharpoonright m$  belongs to  $\mathbf{BN}_n(S) = \mathbf{BN}_n(T_n)$ . Then  $S \upharpoonright_u \subseteq T_n \upharpoonright_u \in D_k^{u(j)} = D_k^\alpha = D_\xi$  by construction, and we are done.  $\square$

**Corollary 9.5.** *If  $T \in \mathbf{WT}$  and  $H_\xi \subseteq T$  is open dense in  $T$  for all  $\xi < \omega_1$  then there is  $S \in \mathbf{WT}$  such that  $S \subseteq_1 T$  and each  $H_\xi \cap S$  weakly covers  $S$ .*

**Proof.** Apply the lemma for  $\mathbf{P} = \{T \upharpoonright_s : s \in T\}$  and  $D_\xi = \{T \upharpoonright_s : s \in H_\xi\}$ .  $\square$

## 10 Extensions of wide tree forcing notions

The forcing notion to prove Theorem 1.1 will be defined in the form of an  $\omega_1$ -union of its parts — WT-forcings of cardinality  $\leq \aleph_1$ .

**Definition 10.1.** Let  $\mathfrak{M}$  be any set and  $\mathbf{P}$  be a WT-forcing. Another WT-forcing  $\mathbf{Q}$  is an  $\mathfrak{M}$ -extension of  $\mathbf{P}$ , in symbol  $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$ , if the following holds:

- (A)  $\mathbf{Q}$  is dense in  $\mathbf{Q} \cup \mathbf{P}$ ;
- (B)  $\mathbf{Q}$  refines  $\mathbf{P}$ : if  $Q \in \mathbf{Q}$  then there exists  $T \in \mathbf{P}$  satisfying  $Q \subseteq T$ ;
- (C) if a set  $D \in \mathfrak{M}$ ,  $D \subseteq \mathbf{P}$  is pre-dense in  $\mathbf{P}$  and  $U \in \mathbf{Q}$  then the set  $D \upharpoonright U = \{s \in U : \exists S \in D (U \upharpoonright_s \subseteq S)\}$  weakly covers  $U$ ;
- (D) if  $T_0 \in \mathbf{P}$  and  $\langle D_n \rangle_{n < \omega} \in \mathfrak{M}$  is a sequence of pre-dense sets  $D_n \subseteq \mathbf{P}$  then there is a tree  $T \in \mathbf{Q}$  such that  $T \subseteq T_0$ , and  $\mathbf{BN}_n(T) \subseteq D_n \upharpoonright T$  for all  $n$ ;
- (E) if  $T_0 \in \mathbf{P}$  and  $f : \omega_1^\omega \rightarrow \omega_1^\omega$ ,  $f \in \mathfrak{M}$ , is continuous, then there is  $T \in \mathbf{Q}$  such that  $T \subseteq T_0$ , and either  $f''[T]$  is bounded or  $f \upharpoonright [T]$  is a bijection;
- (F) if  $f \in \mathfrak{M}$ ,  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is a continuous map, and  $U, V \in \mathbf{Q}$ , then the set  $\mathbb{H}(U, f, V)$ , of all strings  $s \in U$  such that  $[V] \cap (f''[U \upharpoonright_s])$  is bounded or  $f \upharpoonright [U \upharpoonright_s]$  is a total identity, weakly covers  $U$ .  $\square$

If  $\mathfrak{M} = \emptyset$  then we write  $\mathbf{P} \sqsubset \mathbf{Q}$  instead of  $\mathbf{P} \sqsubset_{\emptyset} \mathbf{Q}$ ; in this case (C) – (F) are trivial. Generally, in the role of  $\mathfrak{M}$  we'll consider transitive models of the theory **ZFC'** which includes all **ZFC** axioms except for the Power Set axiom, but an axiom is adjoined, which claims the existence of  $\omega_1$  and  $\mathcal{P}(\omega_1)$ . (Then the existence of sets like  $\omega_1^{<\omega}$  and **WT** easily follows.)

**Lemma 10.2.** *Let  $\mathbf{P}, \mathbf{Q}, \mathbf{R}$  be WT-forcings satisfying  $\mathbf{P} \sqsubset \mathbf{Q} \wedge \mathbf{Q} \sqsubset \mathbf{R}$ . Then  $\mathbf{P} \sqsubset \mathbf{R}$ , and if (K) is one of (C), (D), (E), (F) and the pair  $\mathbf{P} \sqsubset \mathbf{Q}$  satisfies (K) with some  $\mathfrak{M}$ , then the pair  $\mathbf{P} \sqsubset \mathbf{R}$  satisfies (K) with the same  $\mathfrak{M}$ .*

**Proof.** (C) Let  $R \in \mathbf{R}$ . As  $\mathbf{Q} \sqsubset \mathbf{R}$ , there is a tree  $Q \in \mathbf{Q}$  with  $R \subseteq Q$ . Then  $D \uparrow Q$  weakly covers  $Q$  by (C) for  $\mathbf{P}, \mathbf{Q}$ . Then easily  $D \uparrow R$  weakly covers  $R$ .

(D) If  $T' \subseteq T$  and  $t \in \mathbf{BN}_n(T')$  then there is a string  $s \in \mathbf{BN}_n(T)$  with  $s \subseteq t$ .

(F) If  $U' \subseteq U$  and  $V' \subseteq V$  then  $\mathbb{H}(U, f, V) \cap U' \subseteq \mathbb{H}(U', f, V')$ .  $\square$

**Lemma 10.3.** *Assume that  $\mathfrak{M} \models \mathbf{ZFC}'$  is a transitive model, and  $\mathbf{P} \in \mathfrak{M}$  and  $\mathbf{Q}$  are WT-forcings satisfying  $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$ . Then*

- (i) if a set  $D \in \mathfrak{M}$ ,  $D \subseteq \mathbf{P}$  is pre-dense in  $\mathbf{P}$  then  $D$  is pre-dense in  $\mathbf{P} \cup \mathbf{Q}$ ;
- (ii) if  $T, T' \in \mathbf{P}$  are incompatible in  $\mathbf{P}$  then  $T, T'$  are incompatible in  $\mathbf{P} \cup \mathbf{Q}$ .

**Proof.** (i) Let  $U \in \mathbf{Q}$ . Then  $D \uparrow U$  weakly covers  $U$  by 10.1(C). Let  $s \in D \uparrow U$ . Then  $U' = U \upharpoonright_s \in \mathbf{Q}$ ,  $U' \subseteq U$ , and  $U' \subseteq S$  for some  $S \in D$ .

(ii) The sets  $D(T) = \{S \in \mathbf{P} : S \subseteq T \vee [S] \cap [T] = \emptyset\}$  and  $D(T')$  belong to  $\mathfrak{M}$  and are open dense in  $\mathbf{P}$  by Lemma 3.3. Therefore  $D = D(T) \cap D(T')$  is open dense either, and in fact  $S \in D \implies [S] \cap [T] = \emptyset \vee [S] \cap [T'] = \emptyset$  by the incompatibility. It follows that if  $U \in \mathbf{Q}$  and, by (i),  $S \in D$  and  $U' \in \mathbf{Q}$ ,  $U' \subseteq U$ ,  $U' \subseteq S \cap U$ , then  $[U'] \cap [T] = \emptyset$  or  $[U'] \cap [T'] = \emptyset$ , hence  $U$  cannot witness the compatibility of  $T, T'$ .  $\square$

We now establish the existence of extensions.

**Theorem 10.4.** *Assume that  $\mathfrak{M} \models \mathbf{ZFC}'$  is a transitive model of cardinality  $\leq \aleph_1$ , and  $\mathbf{P} \in \mathfrak{M}$  is a WT-forcing,  $\text{card } \mathbf{P} \leq \aleph_1$  in  $\mathfrak{M}$ . Then there exists a WT-forcing  $\mathbf{Q}$  of cardinality  $\aleph_1$ , satisfying  $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$ .*

**Proof.** *Step 1.* If  $P \in \mathbf{P}$  then by Lemma 9.4 there is a tree  $T(P) \in \mathbf{WT}$ ,  $T(P) \subseteq P$ , such that  $D \uparrow T(P)$  weakly covers  $T(P)$  for each  $D \in \mathfrak{M}$ ,  $D \subseteq \mathbf{P}$ , predense in  $\mathbf{P}$ . The set  $\mathbf{P}' = \{T(P) \upharpoonright_s : P \in \mathbf{P} \wedge s \in T(P)\}$  is a WT-forcing of cardinality  $\aleph_1$  and 10.1(A),(B),(C) hold for  $\mathbf{Q} = \mathbf{P}'$ .

*Step 2.* To fulfill 10.1(E), if  $P' \in \mathbf{P}'$  and  $f : \omega_1^\omega \rightarrow \omega_1^\omega$ ,  $f \in \mathfrak{M}$  is continuous, then by Lemma 5.1 there is a tree  $T(P', f) \in \mathbf{WT}$ , such that  $T(P', f) \subseteq T$ , and either  $f''[T(P', f)]$  is bounded or  $f \upharpoonright [T(P', f)]$  is a bijection. We let  $\mathbf{P}'' = \{T(P', f) \upharpoonright_s : P' \in \mathbf{P}' \wedge s \in T(P', f)\}$ . Now 10.1(A),(B),(C),(E) hold for  $\mathbf{Q} = \mathbf{P}''$ .

*Step 3.* To fulfill 10.1(D), note first of all that each set  $D \in \mathfrak{M}$ ,  $D \subseteq \mathbf{P}$ , pre-dense in  $\mathbf{P}$ , remains pre-dense in  $\mathbf{P} \cup \mathbf{P}''$  by Lemma 10.3(i). If  $P'' \in \mathbf{P}''$  and  $\mathbf{d} = \langle D_n \rangle_{n < \omega} \in \mathfrak{M}$  is a sequence of pre-dense sets  $D_n \subseteq \mathbf{P}$ , then by Lemma 4.3 there is a tree  $T(P'', \mathbf{d}) \in \mathbf{WT}$  such that  $T(P'', \mathbf{d}) \subseteq P''$ , and if  $n < \omega$  and  $s \in \mathbf{BN}_n(T(P'', \mathbf{d}))$  then  $\exists S \in D_n (T(P'', \mathbf{d}) \upharpoonright_s \subseteq S)$ . We let

$$\mathbf{P}''' = \{T(P'', \mathbf{d}) \upharpoonright_s : P'' \in \mathbf{P}'' \wedge \mathbf{d} \in \mathfrak{M}\}.$$

Now 10.1(A),(B),(C),(D),(E) hold for  $\mathbf{Q} = \mathbf{P}'''$ .

To fulfill 10.1(F), we begin with some notation. If  $S \in \mathbf{WT}$  and  $\alpha < \omega_1$  then let  $\alpha \wedge S = \{\alpha \wedge s : s \in S\}$ ; then  $\alpha \wedge S \in \mathbf{WT}$  and  $\langle \alpha \rangle \subseteq \text{stem}(\alpha \wedge S)$ . Conversely,

if  $W \in \mathbf{WT}$  and  $\langle \alpha \rangle \subseteq \mathbf{stem}(W)$  then let  $W \downarrow = \{s \in \omega_1^{<\omega} : \alpha \wedge s \in W\}$ ; then  $W \downarrow \in \mathbf{WT}$  and  $W = \alpha \wedge (W \downarrow)$ . We have  $(\alpha \wedge S) \downarrow = S$ , of course.

*Step 4.* Let  $\mathbf{P}''' = \{R_\alpha : \alpha < \omega_1\}$ . We convert  $\mathbf{P}'''$  into a single tree

$$R = \{\Lambda\} \cup \bigcup_{\alpha < \omega_1} (\alpha \wedge R_\alpha) \in \mathbf{WT}; \quad \text{then } R_\alpha = (R \upharpoonright_{\langle \alpha \rangle}) \downarrow, \forall \alpha.$$

If  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is continuous and  $\alpha, \beta < \omega_1$  then define  $f_{\alpha\beta} : \omega_1^\omega \rightarrow \omega_1^\omega$  so that  $f_{\alpha\beta}(\alpha \wedge x) = \beta \wedge f(x)$ ,  $f_{\alpha\beta}(\beta \wedge x) = \alpha \wedge f(x)$ , and  $f_{\alpha\beta}(y) = y$  whenever  $y(0) \neq \alpha, \beta$ . The set of continuous functions  $F = \{f_{\alpha\beta} : f \in \mathfrak{M} \wedge \alpha, \beta < \omega_1\}$  is still of cardinality  $\aleph_1$ . By Theorem 5.3 there exists a tree  $T \in \mathbf{WT}$ ,  $T \subseteq_1 R$ , such that if  $h \in F$  then the set  $\mathbb{H}(T, h, T)$  is open dense in  $T$ . Therefore by Corollary 9.5 there is a tree  $Q \in \mathbf{WT}$  such that  $Q \subseteq_1 T$  (hence  $Q \subseteq_1 R$  as well) and if  $h \in F$  then  $\mathbb{H}(T, h, T)$  weakly covers  $Q$ . Then  $\mathbb{H}(Q, h, Q)$  weakly covers  $Q$  as well by Lemma 9.2(ii) since  $\mathbb{H}(T, h, T) \cap Q \subseteq \mathbb{H}(Q, h, Q)$ .

*Step 5.* Note that if  $\alpha < \omega_1$  then the one-term string  $\langle \alpha \rangle$  belongs to  $Q$  since  $Q \subseteq_1 R$ . Now let  $Q_\alpha = (Q \upharpoonright_{\langle \alpha \rangle}) \downarrow = \{q \in \omega_1^{<\omega} : \alpha \wedge q \in Q\}$ . We claim that *the WT-forcing*  $\mathbf{Q} = \{Q_\alpha \upharpoonright_q : \alpha < \omega_1 \wedge q \in Q_\alpha\}$  *satisfies*  $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$ .

First of all,  $\mathbf{P} \sqsubset \mathbf{P}' \sqsubset \mathbf{P}'' \sqsubset \mathbf{P}''' \sqsubset \mathbf{Q}$  by construction, and hence  $\mathbf{P} \sqsubset \mathbf{Q}$  holds, and we have 10.1(C),(D),(E) for the pair  $\mathbf{P} \sqsubset \mathbf{Q}$  by Lemma 10.2.

To check 10.1(F), let  $f \in \mathfrak{M}$ ,  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  be continuous, and  $U = Q_\alpha$ ,  $V = Q_\beta$  be trees in  $\mathbf{Q}$ . To prove that  $\mathbb{H}(U, f, V)$  weakly covers  $U$ , let  $h = f_{\alpha\beta}$ . Then  $\mathbb{H}(Q, h, Q)$  weakly covers  $Q$  by Step 4. Thus there is an ordinal  $\xi < \omega_1$  such that if  $x \in [Q]$  and  $\sup x \geq \xi$  then  $\mathbb{H}(Q, h, Q)$  meets  $x$ , so  $x \upharpoonright m \in \mathbb{H}(Q, h, Q)$  for some  $m$ . We claim that  $\xi$  witnesses that  $\mathbb{H}(U, f, V)$  weakly covers  $U$ .

Assume that  $y \in [U] = [Q_\alpha]$ ,  $\max y \geq \xi$ . Then  $x = \alpha \wedge y \in [Q]$ , so  $s = x \upharpoonright (m+1) \in \mathbb{H}(Q, h, Q)$  for some  $m$ , by the above. Then  $s = \alpha \wedge t$ , where  $t = y \upharpoonright m$ . It remains to prove that  $t \in \mathbb{H}(U, f, V)$ .

*Case 1:*  $[Q] \cap (h''[Q \upharpoonright_s])$  is bounded. However  $h = f_{\alpha\beta}$  and  $U = Q_\alpha$ ,  $V = Q_\beta$ , hence  $[Q] \cap (h''[Q \upharpoonright_s]) = \beta \wedge ([V] \cap (f''[U \upharpoonright_t]))$ . Thus the set  $[V] \cap (f''[U \upharpoonright_t])$  is bounded, therefore  $t \in \mathbb{H}(U, f, V)$ .

*Case 2:*  $h \upharpoonright [Q \upharpoonright_s]$  is a total identity,  $h(x) = x$  whenever  $x \in Q \upharpoonright_s$ . Then  $\beta = \alpha$ ,  $U = V$ , and  $f \upharpoonright [U \upharpoonright_t]$  is a total identity, thus still  $t \in \mathbb{H}(U, f, V)$ .  $\square$

## 11 Blocking sequences and the forcing

**We argue in the constructible universe  $\mathbf{L}$  in this section.**

The forcing to prove Theorem 1.1 will be defined as the union of a  $\omega_1$ -sequence of WT-forcings of size  $\aleph_1$ , increasing in the sense of a relation  $\sqsubset$  (Definition 10.1). We here introduce the notational system to be used in this construction.

**Definition 11.1.** Let  $\mathbb{WTF}$  be the set of all WT-forcings of cardinality  $\leq \aleph_1$ .

If  $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \lambda}$  is a sequence of forcings  $\mathbf{P}_\alpha \in \mathbb{WTF}$ , then let  $\bigcup \overline{\mathbf{P}} = \bigcup_{\alpha < \lambda} \mathbf{P}_\alpha$ , and let  $\mathfrak{M}(\overline{\mathbf{P}})$  be the least transitive model of  $\mathbf{ZFC}^-$  of the form  $\mathbf{L}_\theta$ , containing  $\overline{\mathbf{P}}$ , in which both  $\lambda$  and the set  $\bigcup \overline{\mathbf{P}}$  are of cardinality  $\leq \aleph_1$ .

If  $\lambda \leq \omega_2$  then let  $\overline{\text{WTF}}_\lambda$  be the set of all  $\lambda$ -sequences  $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \lambda}$  of forcings  $\mathbf{P}_\alpha \in \text{WTF}$ , satisfying the following:

- (\*) if  $\gamma < \lambda$  then  $\bigcup(\overline{\mathbf{P}} \upharpoonright \gamma) \sqsubset_{\text{m}(\overline{\mathbf{P}} \upharpoonright \gamma)} \mathbf{P}_\gamma$ .

Let  $\overline{\text{WTF}} = \bigcup_{\lambda < \omega_2} \overline{\text{WTF}}_\lambda$ . □

The set  $\overline{\text{WTF}} \cup \overline{\text{WTF}}_{\omega_2}$  is ordered by the extension relations  $\subset$  and  $\subseteq$ .

**Lemma 11.2.** *Assume that  $\kappa < \lambda < \omega_2$ , and  $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \kappa} \in \overline{\text{WTF}}$ . Then:*

- (i) *the union  $\mathbf{P} = \bigcup \overline{\mathbf{P}}$  belongs to  $\text{WTF}$ ;*
- (ii) *there is a sequence  $\overline{\mathbf{Q}} \in \overline{\text{WTF}}$  such that  $\text{dom}(\overline{\mathbf{Q}}) = \lambda$  and  $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$ .*

**Proof.** To prove (ii) apply Theorem 10.4 by induction on  $\lambda$ . □

**Definition 11.3** (key definition). A sequence  $\overline{\mathbf{P}} \in \overline{\text{WTF}}$  *blocks* a set  $W \subseteq \overline{\text{WTF}}$  if either  $\overline{\mathbf{P}} \in W$  or there is no sequence  $\overline{\mathbf{Q}} \in W$  satisfying  $\overline{\mathbf{P}} \subseteq \overline{\mathbf{Q}}$ . □

**Sets  $\mathbf{H}_\kappa$  and definability classes.** Recall that  $\mathbf{H}_\kappa$  is the set of all sets *hereditarily of cardinality*  $< \kappa$ . Thus  $x \in \mathbf{H}_\kappa$  if the *transitive closure*  $\text{TC}(x)$  is a set of cardinality  $< \kappa$ . In particular  $\text{HC} = \mathbf{H}_{\omega_1}$  is the set of all *hereditarily (at most) countable* sets, while  $\mathbf{H}_{\omega_2}$  is the set of all sets *hereditarily of cardinality*  $\leq \aleph_1$ ;  $\text{HC} = \mathbf{L}_{\omega_1}$  and  $\mathbf{H}_{\omega_2} = \mathbf{L}_{\omega_2}$  in the constructible universe  $\mathbf{L}$ .

$\Sigma_n(\mathbf{H}_\kappa)$ , resp.,  $\Sigma_n^{\mathbf{H}_\kappa}$  is the class of all sets  $X \subseteq \mathbf{H}_\kappa$ , definable in  $\mathbf{H}_\kappa$  by a  $\Sigma_n$  formula with parameters in  $\mathbf{H}_\kappa$ , resp., with no parameters. The classes  $\Pi_n(\mathbf{H}_\kappa)$ ,  $\Pi_n^{\mathbf{H}_\kappa}$  have the same meaning (with  $\Pi_n$  formulas), and  $\Delta_n(\mathbf{H}_\kappa) = \Sigma_n(\mathbf{H}_\kappa) \cap \Pi_n(\mathbf{H}_\kappa)$ ,  $\Delta_n^{\mathbf{H}_\kappa} = \Sigma_n^{\mathbf{H}_\kappa} \cap \Pi_n^{\mathbf{H}_\kappa}$ , as usual. In particular,  $\Delta_0(\mathbf{H}_\kappa) = \Sigma_0(\mathbf{H}_\kappa) = \Pi_0(\mathbf{H}_\kappa)$  and  $\Delta_0^{\mathbf{H}_\kappa} = \Sigma_0^{\mathbf{H}_\kappa} = \Pi_0^{\mathbf{H}_\kappa}$  (definability by bounded formulas, with/without parameters). See more on  $\in$ -definability in [3, Part B, Chap. 5, Sect. 4] or elsewhere.

In particular, we consider the classes  $\Sigma_n^{\mathbf{H}_{\omega_2}}$ ,  $\Pi_n^{\mathbf{H}_{\omega_2}}$ ,  $\Delta_n^{\mathbf{H}_{\omega_2}}$  of definability in  $\mathbf{H}_{\omega_2}$  (parameters not allowed) and  $\Sigma_n(\mathbf{H}_{\omega_2})$ ,  $\Pi_n(\mathbf{H}_{\omega_2})$ ,  $\Delta_n(\mathbf{H}_{\omega_2})$  (all parameters in  $\mathbf{H}_{\omega_2}$  allowed) — this is the case  $\kappa = \aleph_2$  in the above definitions.

**Theorem 11.4** (the blocking sequence theorem, in  $\mathbf{L}$ ). *Let  $n \geq 2$ . There exists a sequence  $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \omega_2} \in \overline{\text{WTF}}_{\omega_2}$  satisfying the following two conditions:*

- (i)  $\overline{\mathbf{P}}$ , as the set of pairs  $\langle \alpha, \mathbf{P}_\alpha \rangle$ , belongs to the definability class  $\Delta_{n-1}^{\mathbf{H}_{\omega_2}}$ ;
- (ii) if  $n \geq 3$  and  $W \subseteq \overline{\text{WTF}}$  is a  $\Sigma_{n-2}(\mathbf{H}_{\omega_2})$  set then there is an ordinal  $\gamma < \omega_2$  such that the restricted sequence  $\overline{\mathbf{P}} \upharpoonright \gamma = \langle \mathbf{P}_\alpha \rangle_{\alpha < \gamma} \in \overline{\text{WTF}}$  blocks  $W$ .

**Proof.** Let  $\leq_{\mathbf{L}}$  be the canonical  $\Delta_1$  wellordering of  $\mathbf{L}$ ; thus its restriction to  $\mathbf{H}_{\omega_2} = \mathbf{L}_{\omega_2}$  is  $\Delta_1^{\mathbf{H}_{\omega_2}}$ . As  $n \geq 3$ , there exists a universal  $\Sigma_{n-2}^{\mathbf{H}_{\omega_2}}$  set  $\mathfrak{U}^n \subseteq \omega_2 \times \mathbf{H}_{\omega_2}$ . That is,  $\mathfrak{U}^n$  is  $\Sigma_{n-2}^{\mathbf{H}_{\omega_2}}$  (parameter-free  $\Sigma_{n-2}$  definable in  $\mathbf{H}_{\omega_2}$ ), and for every set  $X \subseteq \mathbf{H}_{\omega_2}$  of type  $\Sigma_{n-2}(\mathbf{H}_{\omega_2})$  ( $\Sigma_{n-2}$  definable in  $\mathbf{H}_{\omega_2}$  with arbitrary parameters) there is an ordinal  $\delta < \omega_1$  such that  $X = \mathfrak{U}_\delta^n = \{x : \langle \delta, x \rangle \in \mathfrak{U}^n\}$ . The choice of

$\omega_2$  as the domain of parameters is validated by the assumption  $\mathbf{V} = \mathbf{L}$ , which implies the existence of a  $\Delta_1^{\mathbf{H}\omega_2}$  surjection  $\omega_2 \xrightarrow{\text{onto}} \mathbf{H}\omega_2$ .

Coming back to Definition 11.3, note that for any sequence  $\overline{\mathbf{P}} \in \overline{\mathbf{WTF}}$  and any set  $W \subseteq \overline{\mathbf{WTF}}$  there is a sequence  $\overline{\mathbf{Q}} \in \overline{\mathbf{WTF}}$  which satisfies  $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$  and blocks  $W$ . This allows us to define  $\overline{\mathbf{Q}}_\alpha \in \overline{\mathbf{WTF}}$  by induction on  $\alpha < \omega_1$  so that  $\overline{\mathbf{Q}}_0 = \emptyset$ ,  $\overline{\mathbf{Q}}_\lambda = \bigcup_{\alpha < \lambda} \overline{\mathbf{Q}}_\alpha$ , and each  $\overline{\mathbf{Q}}_{\alpha+1}$  is equal to the  $\leq_{\mathbf{L}}$ -least sequence  $\overline{\mathbf{Q}} \in \overline{\mathbf{WTF}}$  which satisfies 1)  $\overline{\mathbf{Q}}_\alpha \subset \overline{\mathbf{Q}}$  and 2) if  $n \geq 3$  then  $\overline{\mathbf{Q}}$  blocks  $\mathfrak{U}_\alpha^n$ .

Then  $\overline{\mathbf{P}} = \bigcup_{\alpha < \omega_2} \overline{\mathbf{Q}}_\alpha \in \overline{\mathbf{WTF}}_{\omega_2}$ . Condition (ii) holds by construction, while (i) follows by a routine verification, based on the fact that  $\overline{\mathbf{WTF}} \in \Delta_1^{\mathbf{H}\omega_2}$  and  $\mathfrak{U}^n \in \Sigma_{n-2}^{\mathbf{H}\omega_2}$  (provided  $n \geq 3$ ).  $\square$

**Definition 11.5** (in  $\mathbf{L}$ ). We fix a sequence  $\overline{\mathbf{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_2} \in \overline{\mathbf{WTF}}_{\omega_2}$ , given by Theorem 11.4 for a number  $n \geq 2$ , for which Theorem 1.1 is to be established.

In particular  $\overline{\mathbf{P}}$  satisfies (i) and (ii) of Theorem 11.4.

If  $\gamma < \omega_2$  then let  $\mathfrak{M}_\gamma = \mathfrak{M}(\overline{\mathbf{P}} \upharpoonright \gamma)$ , and  $\mathbb{P}_{< \gamma} = \bigcup_{\alpha < \gamma} \mathbb{P}_\alpha$ ,  $\mathbb{P} = \bigcup_{\alpha < \omega_2} \mathbb{P}_\alpha$ .  $\square$

## 12 Some forcing properties

The WT-forcing  $\mathbb{P} \in \mathbf{L}$  defined by 11.5 will be the forcing notion for the proof of Theorem 1.1. The next lemma establishes some properties of  $\mathbb{P}$ .

**We continue to argue in  $\mathbf{L}$  in the conditions and notation of Definition 11.5.**

**Lemma 12.1.**  $\mathbb{P}$  is a WT-forcing, all sets  $\mathbb{P}_\alpha$ ,  $\mathbb{P}_{< \gamma}$  belong to  $\mathbf{WTF}$ . In addition:

- (i) if  $\alpha < \omega_2$  then  $\mathbb{P}_{< \gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$ ;
- (ii) if  $\alpha < \omega_2$  and the set  $D \in \mathfrak{M}_\alpha$ ,  $D \subseteq \mathbb{P}_{< \alpha}$  is pre-dense in  $\mathbb{P}_{< \alpha}$  then it is pre-dense in  $\mathbb{P}$ , too;
- (iii) every set  $\mathbb{P}_\alpha$  is pre-dense in  $\mathbb{P}$ ;
- (iv) if  $\alpha < \omega_2$  and trees  $T, T' \in \mathbb{P}_{< \alpha}$  are incompatible in  $\mathbb{P}_{< \alpha}$  then  $T, T'$  are incompatible in  $\mathbb{P}$ , too;
- (v) if  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is continuous then the set of all trees  $T \in \mathbb{P}$  such that  $f''[T]$  is bounded or  $f \upharpoonright [T]$  is a bijection, is dense in  $\mathbb{P}$ ;
- (vi) if  $f : \omega_1^\omega \rightarrow \omega_1^\omega$  is continuous then the set of all trees  $T \in \mathbb{P}$  such that (1)  $f \upharpoonright [T]$  is a total identity, **or**, for some  $\gamma < \omega_2$ , (2)  $f \upharpoonright [T]$  **avoids**  $\mathbb{P}_\gamma$  in the sense that if  $V \in \mathbb{P}_\gamma$  then the subset  $\{s \in T : [V] \cap (f''[T \upharpoonright_s]) \text{ is bounded}\}$  weakly covers  $T$ , is dense in  $\mathbb{P}$ ;
- (vii) if  $n \geq 3$  and a set  $Q \subseteq \mathbf{WT}$  belongs to  $\Sigma_{n-2}(\mathbf{H}\omega_2)$ , then  $\mathbb{P} \cap (Q \cup Q^-)$  is dense in  $\mathbb{P}$ , where  $Q^- = \{T \in \mathbf{WT} : \neg \exists S \in Q (S \subseteq T)\}$ .

**Proof.** (i) holds by (\*) of Definition 11.1.

(ii) We use induction on  $\gamma$ ,  $\alpha \leq \gamma < \omega_2$ , to check that if  $D$  is pre-dense in  $\mathbb{P}_{<\gamma}$  then it remains pre-dense in  $\mathbb{P}_{<\gamma} \cup \mathbb{P}_\gamma = \mathbb{P}_{<\gamma+1}$  by (i) and Lemma 10.3(i). Limit steps, including the final step to  $\mathbb{P}$  ( $\gamma = \omega_2$ ) are routine.

(iii)  $\mathbb{P}_\alpha$  is dense in  $\mathbb{P}_{<\alpha+1} = \mathbb{P}_{<\alpha} \cup \mathbb{P}_\alpha$  by 10.1(A). It remains to refer to (ii).

(iv) Prove by induction on  $\gamma$  that if  $\alpha < \gamma \leq \omega_1$  then  $T, T'$  are incompatible in  $\mathbb{P}_{<\gamma}$ , using (i) and Lemma 10.3(ii).

To prove (v) and (vi) let  $T_0 \in \mathbb{P}$ . There is an ordinal  $\gamma < \omega_2$  such that  $T_0 \in \mathbb{P}_{<\gamma}$  and  $f \in \mathfrak{M}_\gamma$ . We have  $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$  by (i). Therefore by (E) of Definition 10.1 there is a tree  $T \in \mathbb{P}_\gamma$  such that  $T \subseteq T_0$  and  $f''[T]$  is bounded or  $f \upharpoonright [T]$  is a bijection, so we get (v). Further by (F) of Definition 10.1 if  $V \in \mathbb{P}_\gamma$  then the set  $H(T, f, V)$ , of all strings  $s \in T$  such that  $[V] \cap (f''[T \upharpoonright_s])$  is bounded or  $f \upharpoonright [T \upharpoonright_s]$  is a total identity, weakly covers  $T$ . We have two cases.

**Case 1:**  $f \upharpoonright [T \upharpoonright_s]$  is a total identity for at least one  $s \in T$ . Then the corresponding subtree  $T' = T \upharpoonright_s$  satisfies (1) of (vi).

**Case 2:** for each  $V \in \mathbb{P}_\gamma$ , the set  $H(V)$  of all strings  $s \in T$  such that  $[V] \cap (f''[T \upharpoonright_s])$  is bounded, weakly covers  $T$ , thus  $T$  itself satisfies (2) of (vi).

(vii) Suppose that  $n \geq 3$ . Let  $T_0 \in \mathbb{P}$ , that is,  $T_0 \in \mathbb{P}_{<\alpha_0}$ ,  $\alpha_0 < \omega_2$ . The set  $W$  of all sequences  $\overline{\mathbf{P}} \in \overline{\text{WTF}}$ , such that  $\overline{\mathbf{P}} \upharpoonright \alpha_0 \subseteq \overline{\mathbf{P}}$  and  $\exists T \in Q \cap (\bigcup \overline{\mathbf{P}}) (T \subseteq T_0)$ , belongs to  $\Sigma_{n-2}(\mathbf{H}\omega_2)$  along with  $Q$ . Therefore there is an ordinal  $\alpha < \omega_2$  such that  $\overline{\mathbf{P}} \upharpoonright \alpha$  blocks  $W$ . We have two cases.

**Case 1:**  $\overline{\mathbf{P}} \upharpoonright \alpha \in W$ . Then the related tree  $T \subseteq T_0$  belongs to  $Q \cap \mathbb{P}$ .

**Case 2:** there is no sequence in  $W$  which extends  $\overline{\mathbf{P}} \upharpoonright \alpha$ . Let  $\gamma = \max\{\alpha, \alpha_0\}$ . Then  $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$  by (i). As  $\alpha_0 \leq \gamma$ , there is a tree  $T \in \mathbb{P}_\gamma$ ,  $T \subseteq T_0$ . We claim that  $T \in Q^-$ , which completes the proof in Case 2.

Suppose to the contrary that  $T \notin Q^-$ , thus there is a tree  $S \in Q$ ,  $S \subseteq T$ . The set  $\mathbf{R} = \mathbb{P}_\gamma \cup \{S \upharpoonright_t : t \in S\}$  is a WT-forcing and obviously  $\mathbb{P}_\gamma \sqsubset \mathbf{R}$ , hence still  $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbf{R}$  holds by Lemma 10.2. It follows that the sequence  $\overline{\mathbf{R}}$  defined by  $\text{dom } \overline{\mathbf{R}} = \gamma + 1$ ,  $\overline{\mathbf{R}} \upharpoonright \gamma = \overline{\mathbf{P}} \upharpoonright \gamma$ , and  $\overline{\mathbf{R}}(\gamma) = \mathbf{R}$ , belongs to  $\overline{\text{WTF}}$ , and even  $\overline{\mathbf{R}} \in W$  since  $S \in Q \cap \mathbf{R}$ . Yet  $\overline{\mathbf{P}} \upharpoonright \alpha \subset \overline{\mathbf{R}}$ , which contradicts to the Case 2 hypothesis.  $\square$

To prove a chain condition for  $\mathbb{P}$ , we'll need the following general lemma. See Definition 11.5 on models  $\mathfrak{M}_\alpha$ .

**Lemma 12.2** (in  $\mathbf{L}$ ). *If  $X \subseteq \mathbf{H}\omega_2 = \mathbf{L}_{\omega_2}$  then the set  $\mathcal{O}_X$  of all ordinals  $\alpha < \omega_2$  such that the model  $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$  is an elementary submodel of  $\langle \mathbf{L}_{\omega_2}; X \rangle$  and  $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ , is unbounded in  $\omega_2$ .*

**Proof.** Let  $\alpha_0 < \omega_2$ . There is an elementary submodel  $M$  of  $\langle \mathbf{L}_{\omega_3}; \in \rangle$ , of cardinality  $\text{card } M = \aleph_1$ , which contains  $\alpha_0, \omega_2, X$  and is such that the set  $M \cap \mathbf{L}_{\omega_2}$  is transitive. Consider the Mostowski collapse  $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ . Let  $\alpha = \phi(\omega_2)$ . Then  $\alpha_0 < \alpha < \lambda < \omega_2$  and  $\phi(X) = X \cap \mathbf{L}_\alpha$  by the choice of  $M$ . We conclude that  $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$  is an elementary submodel of  $\langle \mathbf{L}_{\omega_2}; X \rangle$ . And  $\text{card } \alpha > \aleph_1$  in  $\mathbf{L}_\lambda$ , hence  $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$ . Then  $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ , as  $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$  by construction.  $\square$

**Corollary 12.3** (in  $\mathbf{L}$ ). (i) *If  $A \subseteq \mathbb{P}$  is an antichain then  $\text{card } A \leq \aleph_1$ .*

- (ii) Let  $D_n \subseteq \mathbb{P}$  be pre-dense in  $\mathbb{P}$ , for each  $n$ . Then the set of all trees  $T \in \mathbb{P}$ , satisfying  $\forall n (\mathbf{BN}_n(T) \subseteq D_n \uparrow T)$ , is dense in  $\mathbb{P}$ .

**Proof.** (i) Let  $A \subseteq \mathbb{P}$  be a maximal antichain. By Lemma 12.2 there is an ordinal  $\alpha$  such that  $\langle \mathbf{L}_\alpha; \mathbb{P}', A' \rangle$  is an elementary submodel of  $\langle \mathbf{L}_{\omega_2}; \mathbb{P}, A \rangle$ , where  $\mathbb{P}' = \mathbb{P} \cap \mathbf{L}_\alpha$  and  $A' = A \cap \mathbb{P}_{<\alpha}$ , and in addition  $\mathbb{P}', A' \in \mathfrak{M}_\alpha$ . By the elementarity, we have  $\mathbb{P}' = \mathbb{P}_{<\alpha}$  and  $A' = A \cap \mathbb{P}_{<\alpha} \in \mathfrak{M}_\alpha$ , and  $A'$  is a maximal antichain, hence a pre-dense set, in  $\mathbb{P}_{<\alpha}$ . But then  $A'$  is a pre-dense set, hence, a maximal antichain, in the whole set  $\mathbb{P}$  by Lemma 12.1(ii). Thus  $A = A'$ , and  $\text{card } A = \text{card } A' \leq \aleph_1$ .

(ii) We wlog assume that all  $D_n$  are open dense, for if not then replace  $D_n$  by the set  $\{T \in \mathbb{P} : \exists S \in D_n (T \subseteq S)\}$ . Let  $T_0 \in \mathbb{P}$ . Pick a maximal antichain  $A_n \subseteq D_n$  in each  $D_n$ . Then all sets  $A_n$  are maximal antichains in  $\mathbb{P}$  by the open density, and  $\text{card } A_n \leq \aleph_1$  by (i). Therefore there is an ordinal  $\alpha < \omega_2$  such that the set  $A = \bigcup_n A_n$  satisfies  $A \subseteq \mathbb{P}_{<\alpha}$  and  $A, T_0$ , and the sequence  $\langle A_n \rangle_{n < \omega}$  belong to  $\mathfrak{M}_\alpha$ . By the maximality of  $D_n$  and Lemma 12.1(iv), each  $D'_n = D_n \cap \mathbb{P}_{<\alpha}$  is dense in  $\mathbb{P}_{<\alpha}$ . It follows by Lemma 12.1(i) and (D) of Definition 10.1 that there is a tree  $T \in \mathbb{P}_\alpha$  such that  $T \subseteq T_0$  and  $\mathbf{BN}_n(T) \subseteq D_n \uparrow T$  for all  $n$ .  $\square$

### 13 The model

This section presents some key properties of  $\mathbb{P}$ -generic extensions  $\mathbf{L}[G]$  of  $\mathbf{L}$  obtained by adjoining a  $\mathbb{P}$ -generic set  $G \subseteq \mathbb{P}$  to  $\mathbf{L}$ . Recall that the forcing notion  $\mathbb{P} \in \mathbf{L}$  was introduced by Definition 11.5, along with some related notation.

**Corollary 13.1.** *If a set  $G \subseteq \mathbb{P}$  is  $\mathbb{P}$ -generic over  $\mathbf{L}$  then  $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$ .*

**Proof.** That  $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]}$  follows from the fact that  $\mathbf{a}[G]$  is a cofinal map  $\omega \rightarrow \omega_1^{\mathbf{L}}$ . To prove  $\omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$  use Corollary 12.3.  $\square$

**Blanket agreement 13.2.** Arguing in generic extensions of  $\mathbf{L}$ , we'll use standard notation like  $\omega_\xi^{\mathbf{L}}$  to denote  $\mathbf{L}$ -cardinals. We also use  $(\mathbf{WT})^{\mathbf{L}}$  to denote “the set  $\mathbf{WT}$  defined in  $\mathbf{L}$ ”. Thus for instance  $\mathbb{P} \subseteq (\mathbf{WT})^{\mathbf{L}}$ .  $\square$

We'll make use of a **coding system for continuous maps**, helpful whenever “the same” continuous  $f : \omega_1^\omega \rightarrow \mathbf{Ord}^\omega$  is considered in different models.

**Definition 13.3.** Let  $\vartheta \in \mathbf{Ord}$ . A *code of continuous function* from  $(\omega_1^{\mathbf{L}})^\omega$  to  $\vartheta^\omega$  is any map  $\mathbf{c} : \text{dom } \mathbf{c} \rightarrow \vartheta$  with  $\text{dom } \mathbf{c} \subseteq (\omega_1^{\mathbf{L}})^{<\omega} \times \omega$ , such that the sets  $S_{n\xi}^{\mathbf{c}} = \{s \in (\omega_1^{\mathbf{L}})^{<\omega} : \langle s, n \rangle \in \text{dom } \mathbf{c} \wedge \mathbf{c}(s, n) = \xi\}$  satisfy the following for any  $n$ :

- (1) if  $\xi \neq \eta$ ,  $u \in S_{n\xi}^{\mathbf{c}}$ ,  $v \in S_{n\eta}^{\mathbf{c}}$ , then  $u, v$  are incompatible, and
- (2)  $S_n^{\mathbf{c}} = \bigcup_\xi S_{n\xi}^{\mathbf{c}}$  is a belt for  $(\omega_1^{\mathbf{L}})^{<\omega}$ , i. e.,  $\forall x \in (\omega_1^{\mathbf{L}})^\omega \exists m (x \upharpoonright m \in S_n^{\mathbf{c}})$ .

Let  $\mathbf{CCF}_\vartheta$  be the set of all such codes. If  $\mathbf{c} \in \mathbf{CCF}_\vartheta$  then a continuous  $f_{\mathbf{c}} : (\omega_1^{\mathbf{L}})^\omega \rightarrow \vartheta^\omega$  is defined as follows. If  $x \in (\omega_1^{\mathbf{L}})^\omega$  and  $n < \omega$ , then by definition there is a unique  $\xi < \vartheta$  such that  $x \upharpoonright k \in S_{n\xi}^{\mathbf{c}}$  for some  $k$ . Let  $f_{\mathbf{c}}(x)(n) = \xi$ .  $\square$

If  $f : (\omega_1^{\mathbf{L}})^\omega \rightarrow \vartheta^\omega$  is continuous then its code  $\mathbf{c} = \mathbf{code}(f) \in \mathbf{CCF}_\vartheta$  is defined by  $S_{n\xi}^{\mathbf{c}} = \{s \in (\omega_1^{\mathbf{L}})^{<\omega} : \forall x \in (\omega_1^{\mathbf{L}})^\omega (s \subset x \implies f(x)(n) = \xi)\}$ ; then  $f_{\mathbf{c}} = f$ .

**Remark 13.4** (absoluteness). Being a code in  $\mathbf{CCF}_\vartheta$  is absolute since so is the condition of being a belt, see Remark 9.3.  $\square$

**Lemma 13.5.** *If  $G \subseteq \mathbb{P}$  is generic over  $\mathbf{L}$ ,  $\vartheta \in \mathbf{Ord}$ ,  $y \in \vartheta^\omega \cap \mathbf{L}[G]$ , then*

- (i) *there is a code  $\mathbf{c} \in \mathbf{CCF}_\vartheta \cap \mathbf{L}$  such that  $y = f_{\mathbf{c}}(\mathbf{a}[G])$ ;*
- (ii) *if  $\vartheta = \omega_1^{\mathbf{L}}$  then  $y$  is bounded in  $\omega_1^{\mathbf{L}}$  or  $G \in \mathbf{L}[y]$ ;*
- (iii) *if  $\vartheta = \omega_1^{\mathbf{L}}$  and  $y$  is unbounded in  $\omega_1^{\mathbf{L}}$  then  $y = \mathbf{a}[G]$  or there is an ordinal  $\gamma < \omega_2^{\mathbf{L}}$  such that  $y \notin \bigcup_{V \in \mathbb{P}_\gamma} [V]$ .*

**Proof.** (i) There is a  $\mathbb{P}$ -name  $t \in \mathbf{L}$  satisfying  $y = t[G]$  (the  $G$ -valuation of  $t$ ). It can be assumed that  $\mathbb{P}$  forces that  $t$  is valuated as an element of  $\vartheta^\omega$ .

**Arguing in  $\mathbf{L}$ ,** let  $\tau_{n\xi} = \{T \in \mathbb{P} : T \text{ forces } t(n) = \xi\}$  ( $n < \omega$  and  $\xi < \vartheta$ ). The sets  $\tau_n = \bigcup_\xi \tau_{n\xi}$  are open dense in  $\mathbb{P}$ . It follows by Corollary 12.3(ii) that there is a tree  $T \in G$  such that  $T \upharpoonright_s \in \tau_n$  whenever  $n < \omega$  and  $s \in \mathbf{BN}_n(T)$ . This allows us to define, **still in  $\mathbf{L}$ ,** a continuous  $f' : [T] \rightarrow \vartheta^\omega$  by  $f'(x)(n) = \xi$  iff the only string  $s \in \mathbf{BN}_n(T)$  with  $s \subset x$  belongs to  $\tau_{n\xi}$ . Let  $f : \omega_1^\omega \rightarrow \vartheta^\omega$  be a continuous extension of  $f'$ . Then  $\mathbf{c} = \mathbf{code}(f) \in \mathbf{CCF}_\vartheta \cap \mathbf{L}$ , and easily  $y = f_{\mathbf{c}}(\mathbf{a}[G])$ .

(ii) Let, by (i),  $\mathbf{c} \in \mathbf{CCF}_{\omega_1^{\mathbf{L}}} \cap \mathbf{L}$  and  $y = f_{\mathbf{c}}(\mathbf{a}[G])$ . By Lemma 12.1(v), there is a tree  $T \in G$  such that, in  $\mathbf{L}$ ,  $f_{\mathbf{c}}''[T]$  is bounded or  $f_{\mathbf{c}} \upharpoonright [T]$  is a bijection.

*Case 1:* in  $\mathbf{L}$ ,  $f_{\mathbf{c}}''[T]$  is bounded, that is, there is an ordinal  $\beta < \omega_1^{\mathbf{L}}$  satisfying  $f_{\mathbf{c}}(x) \in \beta^\omega$  for all  $x \in [T] \cap \mathbf{L}$ . But  $f_{\mathbf{c}}$  is continuous while  $[T] \cap \mathbf{L}$  is dense in  $[T]$  in  $\mathbf{L}[G]$ . It follows that  $f_{\mathbf{c}}(x) \in \beta^\omega$  for all  $x \in [T] \cap \mathbf{L}[G]$ . In particular  $y = f_{\mathbf{c}}(\mathbf{a}[G]) \in \beta^\omega$  since  $\mathbf{a}[G] \in [T]$  (because  $T \in G$ ), so  $y$  is bounded.

*Case 2:* in  $\mathbf{L}$ ,  $f_{\mathbf{c}} \upharpoonright [T]$  is a bijection. The bijectivity is equivalent to the wellfoundedness of the tree  $W_{\mathbf{c}}$  of all pairs  $\langle s, t \rangle$  of strings  $s, t \in T$  such that  $\mathbf{lh}(s) = \mathbf{lh}(t)$  and there exist no strings  $u, v$  satisfying:  $u \subseteq s$ ,  $v \subseteq t$ , and  $u \in S_{n\xi}^{\mathbf{c}}$ ,  $v \in S_{n\eta}^{\mathbf{c}}$  for some  $n$  and  $\xi \neq \eta$ . Therefore the bijectivity of  $f_{\mathbf{c}} \upharpoonright [T]$  is an absolute property of  $\mathbf{c}, T$ . Thus  $f_{\mathbf{c}} \upharpoonright [T]$  is a bijection in  $\mathbf{L}[G]$ , and we have  $\mathbf{a}[G] = f_{\mathbf{c}}^{-1}(y) \in \mathbf{L}[y]$ , as required.

(iii) We still assume that, by (i),  $y = f_{\mathbf{c}}(\mathbf{a}[G])$ , where  $\mathbf{c} \in \mathbf{CCF}_{\omega_1^{\mathbf{L}}} \cap \mathbf{L}$ . By Lemma 12.1(vi), there is a tree  $T \in G$  such that, in  $\mathbf{L}$ ,  $f_{\mathbf{c}} \upharpoonright [T]$  is a total identity or, for some  $\gamma < \omega_2^{\mathbf{L}}$ ,  $f_{\mathbf{c}} \upharpoonright [T]$  avoids  $\mathbb{P}_\gamma$  in the sense of 12.1(vi).

*Case 1:* in  $\mathbf{L}$ ,  $f_{\mathbf{c}} \upharpoonright [T]$  is a total identity, that is,  $f_{\mathbf{c}}(x) = x$  for all  $x \in [T] \cap \mathbf{L}$ . By the same simple continuity/density argument, we have  $f_{\mathbf{c}}(x) = x$  for all  $x \in [T] \cap \mathbf{L}[G]$ , in particular  $y = f_{\mathbf{c}}(\mathbf{a}[G]) = \mathbf{a}[G]$ .

*Case 2:*  $\gamma < \omega_2^{\mathbf{L}}$  and, in  $\mathbf{L}$ :  $f_{\mathbf{c}} \upharpoonright [T]$  avoids  $\mathbb{P}_\gamma$ , that is, if  $V \in \mathbb{P}_\gamma$  then the subset  $T(V) = \{s \in T : [V] \cap (f''[T \upharpoonright_s]) \text{ is bounded}\}$  (defined in  $\mathbf{L}$ ) weakly covers  $T$ . Now let  $V \in \mathbb{P}_\gamma$  and check that  $y \notin [V]$ . By the Case 2 assumption,  $T(V)$  weakly covers  $T$ . Therefore, as  $\mathbf{a}[G] \in [T]$  is definitely unbounded, there is a string  $s \in T(V)$  satisfying  $s \subseteq \mathbf{a}[G]$ . Then  $S = T \upharpoonright_s \in G$  and  $[V] \cap (f''[S])$

is bounded, so that there is an ordinal  $\beta < \omega_1^{\mathbf{L}}$  satisfying: if  $x \in [S] \cap \mathbf{L}$  and  $f_{\mathbf{c}}(x) \in [V]$  then  $f_{\mathbf{c}}(x) \in \beta^\omega$ . We claim that *the implication  $f_{\mathbf{c}}(x) \in [V] \implies f_{\mathbf{c}}(x) \in \beta^\omega$  also holds for all  $x \in [S] \cap \mathbf{L}[G]$* . Assume that this is established. As  $x = \mathbf{a}[G] \in [S]$  (because  $S \in G$ ), we then have  $y \in [V] \implies y \in \beta^\omega$ . (Recall that  $y = f_{\mathbf{c}}(\mathbf{a}[G])$ .) But  $y$  is unbounded, hence  $y \notin [V]$ , as required.

To prove the claim, let  $x_0 \in [S] \cap \mathbf{L}[G]$  be a counterexample, so  $y_0 = f_{\mathbf{c}}(x_0) \in [V]$  but  $y_0(n_0) = \xi$  for some  $n_0$  and  $\xi \geq \beta$ . The existence of such  $x_0$  is equivalent to the non-wellfoundedness of the tree  $W$  of all strings  $s \in S$  such that  $s \in S_{n_0\eta}^{\mathbf{c}}$  for all  $\eta \neq \xi$ , and there is no string  $u \notin V$  satisfying:  $\forall j < \mathbf{lh}(u) (s \in S_{ju(j)}^{\mathbf{c}})$ . Therefore the existence of  $x_0$  is an absolute property of  $\mathbf{c}, S, V$ . Thus such an  $x_0 \in [S]$  exists already in  $\mathbf{L}$ , contrary to the Case 2 assumption.  $\square$

**Corollary 13.6.** *Let  $G \subseteq \mathbb{P}$  be generic over  $\mathbf{L}$ . Then it is true in  $\mathbf{L}[G]$  that*

- (i)  $\mathbf{a}[G]$  is the only member of the intersection  $\bigcap_{\gamma < \omega_1} \bigcup_{T \in \mathbb{P}_\gamma} [T]$ ;
- (ii)  $\{\mathbf{a}[G]\}$  is a  $\Pi_{n-1}^{\text{HC}}$  singleton;
- (iii) there is a  $\Pi_n^1$  real singleton  $\{r\}$ ,  $r \in \omega^\omega$ , such that  $\mathbf{L}[r] = \mathbf{L}[\mathbf{a}[G]]$ .

**Proof.** (i) Each  $\mathbb{P}_\gamma$  is pre-dense in  $\mathbb{P}$  by Lemma 12.1(iii). It follows that  $\mathbf{a}[G] \in \bigcup_{T \in \mathbb{P}_\gamma} [T]$ , by the genericity. The uniqueness follows from Lemma 13.5(iii).

(ii) The sequence  $\overline{\mathbb{P}} = \{\langle \gamma, \mathbb{P}_\gamma \rangle : \gamma < \omega_2^{\mathbf{L}}\}$  is of type  $\Delta_{n-1}^{\mathbf{HC}\omega_2}$  in  $\mathbf{L}$  by Definition 11.5. However  $\mathbf{H}\omega_2$  in the sense of  $\mathbf{L}$  coincides with the constructible part of HC (= hereditarily countable sets) in the sense of  $\mathbf{L}[G]$ , because  $\omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$  by Corollary 13.1. It easily follows that  $\overline{\mathbb{P}}$  is  $\Delta_{n-1}^{\text{HC}}$  in  $\mathbf{L}[G]$ . On the other hand,

$$\{\mathbf{a}[G]\} = \{x : \forall \gamma \forall \mathbb{Q} (\langle \gamma, \mathbb{Q} \rangle \in \overline{\mathbb{P}} \implies \exists T \in \mathbb{Q} (x \in [T]))\}$$

by (i). This yields the result since  $\exists T \in \mathbb{Q}$  is a bounded quantifier.

(iii) If  $r \in \omega^\omega$  then let  $(r)_n(k) = r(2^n(2k+1) - 1)$ , thus  $(r)_n \in \omega^\omega$ . Let  $W$  be the  $\Pi_1^1$  set of all reals which code an ordinal, and let  $|w| < \omega_1$  be the ordinal coded by  $w \in W$ . Let  $r \in \omega^\omega$  be defined so that each  $(r)_n$  belongs to  $W \cap \mathbf{L}$  and is  $\leq_{\mathbf{L}}$ -minimal of all  $w \in W \cap \mathbf{L}$  satisfying  $|w| = \mathbf{a}[G](n)$ . Thus  $r$  is a real in  $\mathbf{L}[G]$ . The singleton  $\{r\}$  is defined in HC of  $\mathbf{L}[G]$  by the following formula:

$$\forall n, (r)_n \in W \cap \mathbf{L} \text{ and } (r)_n \text{ is } \leq_{\mathbf{L}}\text{-minimal of all } w \in W \cap \mathbf{L} \text{ with } |w| = |(r)_n|, \text{ and } \forall x \in \mathbf{Ord}^\omega (\forall n (x(n) = |(r)_n|) \implies x = \mathbf{a}[G]).$$

It easily follows by the result of (ii) that  $\{r\}$  is a  $\Pi_{n-1}^{\text{HC}}$  singleton as well, hence a  $\Pi_n^1$  singleton.  $\square$

Corollary 13.1 and Corollary 13.6(ii),(iii) account for items (i), (ii), (iii) of Theorem 1.1. Item (iv) of the theorem is based on different ideas related to claim (vii) of Lemma 12.1. From now on we work towards this goal.

## 14 Shoenfield's transformation of $\Sigma_2^1$ formulas

The following useful transformation of  $\Sigma_2^1$  formulas involves an idea in the proof of the Shoenfield absoluteness theorem.

**Blanket agreement 14.1.** From now on  $p, q, r$  denote reals in  $\omega^\omega$ . □

**Theorem 14.2.** Let  $\varphi(p_1, \dots, p_n)$  be a  $\Sigma_2^1$  formula of the form

$$\left. \begin{aligned} \varphi(p_1, \dots, p_n) &:= \exists q \forall r \exists m R(q \upharpoonright m, r \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m), \\ \text{where } R &\subseteq (\omega^{<\omega})^{n+2}, R \in \mathbf{L}, \text{ and } q, r, p_i \text{ are variables over } \omega^\omega \end{aligned} \right\}, \quad (*)$$

and  $\vartheta \geq \aleph_1^{\mathbf{L}}$  a cardinal in  $\mathbf{L}$ . Then there is a relation  $Q = Q_\vartheta(R) \subseteq \vartheta^{<\omega} \times (\omega^{<\omega})^{n+1}$ ,  $Q \in \mathbf{L}$ , such that  $Q$  is  $\Delta_0^{\mathbf{H}_\vartheta}(R)$  as a subset of  $\mathbf{H}_\vartheta$  in  $\mathbf{L}^2$ , and it holds in any generic extension  $M$  of  $\mathbf{L}$  with  $\vartheta \geq \omega_1^M$  that if  $p_1, \dots, p_n \in \omega^\omega$  then

$$\varphi(p_1, \dots, p_n) \iff \exists \chi \in \vartheta^\omega \exists q \in \omega^\omega \forall m Q(\chi \upharpoonright m, q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m).$$

**Proof.**  $\varphi(p_1, \dots, p_n)$  is equivalent to  $\exists q (W_{q, p_1, \dots, p_n} \text{ is wellfounded})$ , where

$$W_{q, p_1, \dots, p_n} = \{u \in \omega^{<\omega} : \forall j \leq \text{lh}(u) \neg R(q \upharpoonright j, u \upharpoonright j, p_1 \upharpoonright j, \dots, p_n \upharpoonright j)\},$$

hence — in any universe  $M$  as in the theorem — to the formula:

$$\exists q \in \omega^\omega \exists f : W_{q, p_1, \dots, p_n} \rightarrow \vartheta \text{ (} f \text{ is order-preserving)}.$$

By “order-preserving” we mean: if  $u, v \in W_{q, p_1, \dots, p_n}$  then  $u \leq_{\text{LS}} v \iff f(u) \leq f(v)$ , where  $\leq_{\text{LS}}$  is the Lusin – Sierpinski (= Kleene – Brouwer) order on strings.

Fix a recursive bijection  $k \mapsto s_k : \omega \xrightarrow{\text{onto}} \omega^{<\omega}$ , with the inverse bijection  $\text{num} : \omega^{<\omega} \rightarrow \omega$ , so that  $s = s_{\text{num}(s)}$ . We assume that  $\text{lh}(s) \leq \text{num}(s)$ ,  $\forall s$ . Let

$$W_{q, p_1, \dots, p_n}^m = \{s \in W_{q, p_1, \dots, p_n} : \text{num}(s) < m\},$$

a finite set. Then  $\varphi(p_1, \dots, p_n)$  is equivalent to the formula

$$\exists q \in \omega^\omega \exists \chi \in \vartheta^\omega \forall m (\chi \circ \text{num} \text{ is order-preserving on } W_{q, p_1, \dots, p_n}^m).$$

( $\chi \circ \text{num}$  is the superposition.) The subformula in brackets depends on  $\chi \upharpoonright m$  and  $q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m$  only. In other words, we have a relation  $Q = Q_\vartheta(R) \subseteq \vartheta^{<\omega} \times (\omega^{<\omega})^{n+1}$ , still  $Q \in \mathbf{L}$ , such that  $\varphi(p_1, \dots, p_n)$  is equivalent to the formula

$$\exists \chi \in \vartheta^\omega \exists q \in \omega^\omega \forall m Q(\chi \upharpoonright m, q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m). \quad (\dagger)$$

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<sup>2</sup> Meaning that the equality  $Q = \{w \in \mathbf{H}_\vartheta : \psi(w)\}$  holds in  $\mathbf{L}$ , where  $\psi$  is a bounded formula with  $R$  as the only parameter.

Namely  $Q$  contains all tuples  $\langle \sigma, v, u_1, \dots, u_n \rangle$  of strings  $\sigma \in \vartheta^{<\omega}$  and  $v, u_i \in \omega^{<\omega}$  of same length  $\text{lh}(\sigma) = \text{lh}(v) = \text{lh}(u_i) = \text{some } m$ , such that the superposition  $\sigma \circ \text{num}$  (defined on the set  $S_m = \{s_j : j < m\}$ ) is order-preserving on the set

$$W_{v, u_1, \dots, u_n}^m = \{u \in S_m : \forall j \leq \text{lh}(u) \neg R(v \upharpoonright j, u \upharpoonright j, u_1 \upharpoonright j, \dots, u_n \upharpoonright j)\}.$$

To see that  $Q$  is a  $\Delta_0^{\mathbf{H}\vartheta}(R)$  subset of  $\mathbf{H}\vartheta$ , note first of all that  $\vartheta = \mathbf{Ord} \cap \mathbf{H}\vartheta$ , which eliminates  $\vartheta$  and  $\vartheta^{<\omega}$  from the list of parameters. In the rest, we skip a routine verification of all elements of the definition of  $Q$  being expressible by bounded formulas.  $\square$

## 15 Auxiliary forcing relation

Here we introduce a key tool for the proof of claim (iv) of Theorem 1.1. This is a forcing-like relation **forc**. It is not explicitly connected with the forcing notion  $\mathbb{P}$  (but rather connected with the full wide tree forcing **WT**), however it will be compatible with  $\mathbb{P}$  for formulas of certain quantifier complexity (Theorem 17.1). The crucial advantage of **forc** will be its invariance a certain group of transformations (Lemma 16.3), a property that cannot be expected for  $\mathbb{P}$ . This will be the key argument in the proof of Theorem 1.1 below in Section 18.

**Blanket agreement 15.1.** From now on, we let  $\Theta = \omega_2^{\mathbf{L}}$ , so  $\Theta = \omega_2$  in  $\mathbf{L}$  but  $\Theta = \omega_1$  in  $\mathbb{P}$ -generic extensions of  $\mathbf{L}$ .  $\square$

**We argue in  $\mathbf{L}$ .** We consider a language  $\mathcal{L}$  whose elementary formulas, called  $\mathcal{L}\Sigma_2^1$  (in spite that they are looking more like  $\Sigma_1^1$ ), are those of the form

$$\left. \begin{aligned} \varphi(p_1, \dots, p_n) &:= \exists \chi \in \Theta^\omega \exists q \in \omega^\omega \forall m Q(\chi \upharpoonright m, q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m), \\ \text{where } Q \in \mathbf{L}, Q &\subseteq \Theta^{<\omega} \times (\omega^{<\omega})^{n+1}, Q \text{ is a } \Delta_0(\mathbf{H}\omega_2) \text{ set,} \\ \text{and } q, p_i &\text{ are variables over } \omega^\omega. \end{aligned} \right\} (1)$$

The dual class  $\mathcal{L}\Pi_2^1$  consists of formulas

$$\left. \begin{aligned} \varphi(p_1, \dots, p_n) &:= \forall \chi \in \Theta^\omega \forall q \in \omega^\omega \exists m Q(\chi \upharpoonright m, q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m), \\ \text{with the same specifications.} \end{aligned} \right\} (2)$$

Higher classes  $\mathcal{L}\Sigma_k^1$  and  $\mathcal{L}\Pi_k^1$  are defined naturally, e.g.  $\mathcal{L}\Sigma_5^1$  contains formulas of the form  $\exists q_1 \forall q_2 \exists q_3 \Phi(q_1, q_2, q_3)$ , where  $\Phi$  is  $\mathcal{L}\Pi_2^1$  and  $q_i$  vary over  $\omega^\omega$ .

We allow codes  $\mathbf{c} \in \mathbf{CCF}_\omega$  to substitute free variables over  $\omega^\omega$ . If  $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$  is an  $\mathcal{L}$ -formula, and  $x \in \omega_1^\omega$ , then  $\varphi[x]$  denotes the formula  $\varphi(f_{\mathbf{c}_1}(x), \dots, f_{\mathbf{c}_n}(x))$ , where all  $f_{\mathbf{c}_i}(x)$  are reals in  $\omega^\omega$ , of course.

**Definition 15.2** (in  $\mathbf{L}$ ). We define a relation  $T \text{ forc } \varphi$  between trees  $T \in \mathbf{WT}$  and closed  $\mathcal{L}$ -formulas in  $\bigcup_{k \geq 2} (\mathcal{L}\Sigma_k^1 \cup \mathcal{L}\Pi_k^1)$ . Recall that  $\Theta = \omega_2$  (in  $\mathbf{L}$ ).

- (A) Let  $\varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$  be a  $\mathcal{L}\Sigma_2^1$  formula as in (1), and  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbf{CCF}_\omega$ . Let finally  $T \in \mathbf{WT}$ . We define  $T \text{ forc } \varphi$  iff there exist codes  $\mathbf{c} \in \mathbf{CCF}_\omega$  and  $\mathbf{d} \in \mathbf{CCF}_\emptyset$  such that the following holds for all  $x \in [T]$ :

$$\forall m Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m).$$

- (B) If  $\varphi$  is a closed  $\mathcal{L}\Pi_k^1$  formula,  $k \geq 2$ , then  $T \text{ forc } \varphi$  iff there is no tree  $S \in \mathbf{WT}$  such that  $S \subseteq T$  and  $S \text{ forc } \varphi^-$ , where  $\varphi^-$  is the result of canonical transformation of  $\neg \varphi$  to  $\mathcal{L}\Sigma_k^1$  form.
- (C) If  $\varphi := \exists x \psi(x)$  is a closed  $\mathcal{L}\Sigma_{k+1}^1$  formula,  $k \geq 2$  ( $\psi$  being of type  $\mathcal{L}\Pi_k^1$ ), then  $T \text{ forc } \varphi$  iff there is a code  $\mathbf{c} \in \mathbf{CCF}_\omega$  such that  $T \text{ forc } \psi(\mathbf{c})$ .

If  $\varphi(p_1, \dots, p_n)$  is an  $\mathcal{L}$ -formula then let

$$\mathbf{Forc}(\varphi) = \{\langle T, \mathbf{c}_1, \dots, \mathbf{c}_n \rangle : T \in \mathbf{WT} \wedge \mathbf{c}_i \in \mathbf{CCF}_\omega \wedge T \text{ forc } \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)\}.$$

In particular if  $\varphi$  is closed then  $\mathbf{Forc}(\varphi) = \{T \in \mathbf{WT} : T \text{ forc } \varphi\}$ . We also define  $\mathbf{Des}(\varphi) = \mathbf{Forc}(\varphi) \cup \mathbf{Forc}(\varphi^-)$  in this case.  $\square$

**Theorem 15.3** (in  $\mathbf{L}$ ). *If  $k \geq 2$  and  $\varphi$  is a formula in  $\mathcal{L}\Sigma_k^1$ , resp.,  $\mathcal{L}\Pi_k^1$ , then the set  $\mathbf{Forc}(\varphi)$  belongs to  $\Sigma_{k-1}(\mathbf{H}\omega_2)$ , resp.,  $\Pi_{k-1}(\mathbf{H}\omega_2)$ .*

**Proof.** The proof goes on by induction on  $k$ . We begin with  $\mathcal{L}\Sigma_2^1$  formulas. We argue in the assumptions and notation of (1) above. According to definition 15.2(A), the existence quantifiers over  $\mathbf{c}$  and  $\mathbf{d}$  are in line with the  $\Sigma_1$  definability, but we have to prove that the set

$$W = \{\langle \mathbf{d}, \mathbf{c}, \mathbf{c}_1, \dots, \mathbf{c}_n, T, m \rangle \in \mathbf{CCF}_\emptyset \times (\mathbf{CCF}_\omega)^{n+1} \times \mathbf{WT} \times \omega : \\ \forall x \in [T] Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m)\}$$

belongs to  $\Sigma_1(\mathbf{H}\omega_2)$ . Recall that  $Q$  is  $\Delta_0(\mathbf{H}\omega_2)$  by (1). It can also be mentioned that  $\mathbf{CCF}_\emptyset \cup \mathbf{CCF}_\omega \cup \mathbf{WT} \subseteq \mathbf{H}\omega_2$ , so that  $W \subseteq \mathbf{H}\omega_2$  anyway.

The hostile elements in the definition of  $W$ , which do not allow it to be  $\Sigma_1(\mathbf{H}\omega_2)$  straightaway, are the quantifier  $\forall x \in [T]$  in the second line, and the quantifier  $\forall x \in \omega_1^\omega$  in (2) of Definition 13.3. (As we argue in  $\mathbf{L}$ , the upper index  $\mathbf{L}$  as in 13.3 is removed.) But,  $\omega_1^\omega \in \mathbf{H}\omega_2$  (under  $\mathbf{V} = \mathbf{L}$ ), hence, as we don't care here about the choice of parameters in  $\mathbf{H}\omega_2$ ,<sup>3</sup> we can pick up  $\omega_1^\omega$  as the extra parameter. The quantifier  $\forall x \in \omega_1^\omega$  in (2) of 13.3 then immediately becomes bounded, while the quantifier  $\forall x \in [T] (\dots x \dots)$  in the definition of  $W$  changes to  $\forall x \in \omega_1^\omega (x \in [T] \implies \dots x \dots)$ , hence becomes bounded as well, and overall we get even  $W \in \Delta_0(\mathbf{H}\omega_2)$ , as required.

The induction steps are easy applications of 15.2(B),(C).  $\square$

Recall that a number  $\mathfrak{n} \geq 2$  is fixed by Definition 11.5.

<sup>3</sup> If we do care then the result holds too but by means of more thoroughful arguments.

**Lemma 15.4** (in **L**). *Let  $\varphi$  be a closed formula in  $\mathcal{L}\Sigma_k^1 \cup \mathcal{L}\Pi_k^1$ ,  $k \geq 2$ . Then the set  $\mathbf{Des}(\varphi)$  is dense in **WT**. If  $k < \aleph$ , then  $\mathbf{Des}(\varphi) \cap \mathbb{P}$  is dense in  $\mathbb{P}$ .*

**Proof.** The first claim is a simple application of Definition 15.2(B). The second claim follows from the first one by lemmas 15.3 and 12.1(vii).  $\square$

## 16 Invariance

It happens that the relation **forc** is invariant under some natural transformations of wide trees. Here we prove the invariance. **We still argue in L.**

Let  $S \in \mathbf{WT}$ . To define a canonical homeomorphism  $h_S : [S] \xrightarrow{\text{ont}\varrho} \omega_1^\omega$ , assume that  $x \in [S]$ . Let  $k < \omega$ . Then  $x \upharpoonright m_k \in \mathbf{BN}_k(T)$  for some (unique) number  $m_k$ . The set  $\Xi(x, k) = \{\xi < \omega_1 : (x \upharpoonright m_k) \wedge \xi \in S\}$  has cardinality  $\text{card}(\Xi(x, k)) = \aleph_1$ ; let  $\Xi(x, k) = \{\xi_\gamma : \gamma < \omega_1\}$  be the enumeration in the increasing order. In particular,  $x(m_k) = \xi_\gamma$  for some (unique)  $\gamma = \gamma(x, k)$ . Define  $y = h_S(x) \in \omega_1^\omega$  by  $y(k) = \gamma(x, k)$ ,  $\forall k$ . The map  $h_S$  is a required homeomorphism.

It follows that if  $T \in \mathbf{WT}$  is another tree then  $h_{ST} = h_T^{-1} \circ h_S$  (the superposition) is a homeomorphism of  $[S]$  onto  $[T]$ . Moreover, in this case, if  $U \subseteq S$  is a subtree then the according subtree  $h_{ST} \cdot U = \{h_{ST}(x) \upharpoonright m : x \in [U] \wedge m < \omega\} \subseteq [T]$  satisfies  $U \in \mathbf{WT}$  iff  $h_{ST} \cdot U \in \mathbf{WT}$ , and  $[h_{ST} \cdot U] = \{h_{ST}(x) : x \in [U]\}$ .

**Lemma 16.1** (in **L**). *If  $S, T \in \mathbf{WT}$  and  $U \in \mathbf{WT}$ ,  $U \subseteq S$ , then  $V = h_{ST} \cdot U \in \mathbf{WT}$ ,  $V \subseteq T$ , and  $h_{UV} = h_{ST} \upharpoonright [U]$ .*  $\square$

If  $\lambda \in \mathbf{Ord}$  and  $f : [S] \rightarrow \lambda^\omega$  then a function  $h_{ST} \cdot f = f \circ h_{ST}^{-1} : [T] \rightarrow \lambda^\omega$  is defined by  $(h_{ST} \cdot f)(x) = f(h_{ST}(x))$ , equivalently,  $(h_{ST} \cdot f)(h_{ST}(x)) = f(x)$ . If  $\mathbf{c}, \mathbf{c}' \in \mathbf{CCF}_\lambda$  then we symbolically write  $\mathbf{c}' \upharpoonright T = h_{ST} \cdot (\mathbf{c} \upharpoonright S)$ , in case the associated functions  $f_{\mathbf{c}}$  and  $f_{\mathbf{c}'}$  satisfy:  $f_{\mathbf{c}'} \upharpoonright [T] = f_{ST} \cdot (f_{\mathbf{c}} \upharpoonright [S])$ .

**Lemma 16.2** (in **L**). *If  $S, T \in \mathbf{WT}$ ,  $\lambda \in \mathbf{Ord}$ , and  $\mathbf{c} \in \mathbf{CCF}_\lambda$  then there is a code  $\mathbf{c}' \in \mathbf{CCF}_\lambda$  satisfying  $\mathbf{c}' \upharpoonright T = f_{ST} \cdot (\mathbf{c} \upharpoonright S)$*

**Proof.** The map  $f = f_{\mathbf{c}} \upharpoonright [S] : [S] \rightarrow \lambda^\omega$  is continuous, hence so is the transformed map  $f' = h_{ST} \cdot f : [T] \rightarrow \lambda^\omega$ . Let  $g : \omega_1^\omega \rightarrow \lambda^\omega$  be any continuous extension of  $f'$ , and let  $\mathbf{c}' = \mathbf{code}(g)$ .  $\square$

Finally if  $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$  is a  $\mathcal{L}$ -formula, and  $\varphi' := \varphi(\mathbf{c}'_1, \dots, \mathbf{c}'_n)$ , where  $\mathbf{c}'_1, \dots, \mathbf{c}'_n$  is another set of codes  $\mathbf{c}'_i \in \mathbf{CCF}_\omega$ , then we symbolically write  $\varphi' \upharpoonright T = h_{ST} \cdot (\varphi \upharpoonright S)$ , in case  $\mathbf{c}'_i \upharpoonright T = h_{ST} \cdot (\mathbf{c}_i \upharpoonright S)$  holds for each  $i = 1, \dots, n$ .

**Lemma 16.3** (in **L**). *Let  $S, T \in \mathbf{WT}$  and let  $\varphi, \varphi'$  be closed formulas in  $\mathcal{L}\Sigma_k^1 \cup \mathcal{L}\Pi_k^1$ ,  $k \geq 2$ , and finally  $\varphi' \upharpoonright T = h_{ST} \cdot (\varphi \upharpoonright S)$ . Then  $S \text{ forc } \varphi$  iff  $T \text{ forc } \varphi'$ .*

**Proof.** We argue by induction. Let  $\varphi, \varphi'$  be  $\mathcal{L}\Sigma_2^1$ , so that  $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$  and  $\varphi' := \varphi(\mathbf{c}'_1, \dots, \mathbf{c}'_n)$ , where  $\mathbf{c}_1, \dots, \mathbf{c}'_n, \mathbf{c}'_1, \dots, \mathbf{c}'_n$  are codes in  $\mathbf{CCF}_\omega$ , and

$$\varphi(p_1, \dots, p_n) := \exists \chi \in \mathbb{Q}^\omega \exists q \in \omega^\omega \forall m Q(\chi \upharpoonright m, q \upharpoonright m, p_1 \upharpoonright m, \dots, p_n \upharpoonright m)$$

is a formula as in (1) of Section 15, and  $\mathbf{c}'_i \upharpoonright T = h_{ST} \cdot (\mathbf{c}_i \upharpoonright S)$  holds for each  $i$ .

Assume that  $S \text{ forc } \varphi$ . Then by definition (Definition 15.2(A)) there are codes  $\mathbf{c} \in \mathbf{CCF}_\omega$  and  $\mathbf{d} \in \mathbf{CCF}_\emptyset$  such that

$$\forall x \in [S] \forall m Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m).$$

Pick, by Lemma 16.2, codes  $\mathbf{c}' \in \mathbf{CCF}_\omega$  and  $\mathbf{d}' \in \mathbf{CCF}_\emptyset$  with  $\mathbf{c}' \upharpoonright T = h_{ST} \cdot (\mathbf{c} \upharpoonright S)$  and  $\mathbf{d}' \upharpoonright T = h_{ST} \cdot (\mathbf{d} \upharpoonright S)$ . Then we obtain

$$\forall y \in [T] \forall m Q(f_{\mathbf{d}'}(y) \upharpoonright m, f_{\mathbf{c}'}(y) \upharpoonright m, f_{\mathbf{c}'_1}(y) \upharpoonright m, \dots, f_{\mathbf{c}'_n}(y) \upharpoonright m),$$

and hence the codes  $\mathbf{c}'$  and  $\mathbf{d}'$  witness  $T \text{ forc } \varphi'$ .

*Step  $\mathcal{L}\Sigma_k^1 \rightarrow \mathcal{L}\Pi_k^1$ .* Let  $\varphi$  be a closed formula in  $\mathcal{L}\Pi_k^1$ , so that  $\varphi$  is  $\psi^-$ , where  $\psi$  is  $\mathcal{L}\Sigma_k^1$ , and accordingly  $\varphi'$  is  $(\psi')^-$ ,  $\psi' \upharpoonright T = h_{ST} \cdot (\psi \upharpoonright S)$ . Assuming that  $S \text{ forc } \varphi$ , prove that  $T \text{ forc } \varphi'$ . Suppose to the contrary that  $T \text{ forc } \varphi'$  fails. Then, by Definition 15.2(B), there is a tree  $V \in \mathbf{WT}$ ,  $V \subseteq T$ ,  $V \text{ forc } \psi'$ . We let  $U = h_{TS} \cdot V$ , so that  $U \in \mathbf{WT}$ ,  $U \subseteq S$ ,  $V = h_{st} \cdot U$ . And, by the way,  $h_{UV} = h_{ST} \upharpoonright [U]$  by Lemma 16.1, thus still  $\psi' \upharpoonright V = h_{UV} \cdot (\psi \upharpoonright [U])$ . It follows that  $U \text{ forc } \psi$ , by the inductive hypothesis, which contradicts to  $S \text{ forc } \varphi$ .

*Step  $\mathcal{L}\Pi_k^1 \rightarrow \mathcal{L}\Sigma_{k+1}^1$ .* Let  $\varphi$  be a closed formula in  $\mathcal{L}\Sigma_{k+1}^1$ , so that  $\varphi$  is  $\exists q \psi(q)$ , where  $\psi(q)$  is  $\mathcal{L}\Pi_k^1$ , and accordingly  $\varphi'$  is  $\exists q \psi'(q)$ ,  $\psi' \upharpoonright T = h_{ST} \cdot (\psi \upharpoonright S)$ . Assuming that  $S \text{ forc } \varphi$ , prove that  $T \text{ forc } \varphi'$ . By Definition 15.2(C), there is a code  $\mathbf{c} \in \mathbf{CCF}_\omega$  satisfying  $S \text{ forc } \psi(\mathbf{c})$ . By Lemma 16.2, there exists a code  $\mathbf{c}' \in \mathbf{CCF}_\omega$  such that  $\mathbf{c}' \upharpoonright T = h_{ST} \cdot (\mathbf{c} \upharpoonright S)$ . Then  $\psi'(\mathbf{c}') \upharpoonright T = h_{ST} \cdot (\psi(\mathbf{c}) \upharpoonright S)$ . It follows that  $T \text{ forc } \psi'(\mathbf{c}')$ , by the inductive hypothesis, hence  $T \text{ forc } \varphi'$ .  $\square$

**Corollary 16.4.** *Let  $S, T \in \mathbf{WT}$  and let  $\varphi$  be a closed formula in  $\mathcal{L}\Sigma_k^1 \cup \mathcal{L}\Pi_k^1$ ,  $k \geq 2$ , with no codes in  $\mathbf{CCF}_\omega$  as parameters. Then  $S \text{ forc } \varphi$  iff  $T \text{ forc } \varphi$ .  $\square$*

## 17 Forcing and truth

Recall that  $\mathfrak{n} \geq 2$  is fixed by Definition 11.5.

Moreover we'll assume that  $\mathfrak{n} \geq 3$ , because we now focus on the proof of claim (iv) of Theorem 1.1, vacuous in the case  $\mathfrak{n} = 2$ .

The last part of the proof of Theorem 1.1 will be the next theorem which connects the forcing relation  $\text{forc}$  with the truth in  $\mathbb{P}$ -generic extensions. This will be the key ingredient of the proof of Theorem 1.1(iv): we use the invariant relation  $\text{forc}$  to surprisingly approximate the forcing  $\mathbb{P}$ , definitely non-invariant under the transformations considered in Section 16.

**Theorem 17.1.** *Assume that  $2 \leq k < \mathfrak{n}$ ,  $\varphi \in \mathbf{L}$  is a closed formula in  $\mathcal{L}\Pi_k^1 \cup \mathcal{L}\Sigma_{k+1}^1$ , and a set  $G \subseteq \mathbb{P}$  is generic over  $\mathbf{L}$ . Then the sentence  $\varphi[\mathbf{a}[G]]$  is true in  $\mathbf{L}[G]$  if and only if  $\exists T \in G (T \text{ forc } \varphi)$ .*

**Proof. We argue in  $\mathbf{L}[G]$ . Base of induction:**  $\varphi$  is a closed  $\mathcal{L}\Sigma_2^1$  formula,

$$\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n) := \exists \chi \in \mathbb{Q}^\omega \exists q \in \omega^\omega \forall m Q(\chi \upharpoonright m, q \upharpoonright m, \mathbf{c}_1 \upharpoonright m, \dots, \mathbf{c}_n \upharpoonright m),$$

as in 15.2(A) and (1) of Section 15. Assume that  $T \in G$  and  $T \text{ forc } \varphi$ . Then by Definition 15.2(A) there are codes  $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$  and  $\mathbf{d} \in \mathbf{CCF}_\emptyset \cap \mathbf{L}$  such that

$$\forall m \forall x \in [T] \cap \mathbf{L} Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m).$$

(Recall Remark 13.4 on the absoluteness of being a code in any  $\mathbf{CCF}_\lambda$ .) However all functions  $f_{\mathbf{d}}, f_{\mathbf{c}}, f_{\mathbf{c}_i}$  are continuous. It follows that the last displayed formula can be strengthened to

$$\forall x \in [T] \forall m Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m).$$

Therefore, as  $\mathbf{a}[G] \in [T]$  (because  $T \in G$ ), we obtain

$$\forall m Q(f_{\mathbf{d}}(\mathbf{a}[G]) \upharpoonright m, f_{\mathbf{c}}(\mathbf{a}[G]) \upharpoonright m, f_{\mathbf{c}_1}(\mathbf{a}[G]) \upharpoonright m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[G]) \upharpoonright m).$$

Thus elements  $\chi = f_{\mathbf{d}}(\mathbf{a}[G])$  and  $q = f_{\mathbf{c}}(\mathbf{a}[G])$  witness  $\varphi[\mathbf{a}[G]]$  to be true.

To establish the inverse, suppose that  $\varphi[\mathbf{a}[G]]$  is true in  $\mathbf{L}[G]$ , that is,

$$\forall m Q(\chi \upharpoonright m, q \upharpoonright m, f_{\mathbf{c}_1}(\mathbf{a}[G]) \upharpoonright m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[G]) \upharpoonright m)$$

true for some  $\chi \in \Theta^\omega$  and  $q \in \omega^\omega$  in  $\mathbf{L}[G]$ . By Lemma 13.5 there are codes  $\mathbf{d} \in \mathbf{CCF}_\emptyset \cap \mathbf{L}$  and  $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$  such that  $\chi = f_{\mathbf{d}}(\mathbf{a}[G])$  and  $q = f_{\mathbf{c}}(\mathbf{a}[G])$ . Thus there is a tree  $T \in G$  which  $\mathbb{P}$ -forces the formula

$$\forall m Q(f_{\mathbf{d}}(\mathbf{a}[G]) \upharpoonright m, f_{\mathbf{c}}(\mathbf{a}[G]) \upharpoonright m, f_{\mathbf{c}_1}(\mathbf{a}[G]) \upharpoonright m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[G]) \upharpoonright m) \quad (*)$$

over  $\mathbf{L}$ . We claim that the codes  $\mathbf{c}$  and  $\mathbf{d}$  witness  $T \text{ forc } \varphi$  as in 15.2(A). Indeed otherwise there are  $x \in [T]$  and  $m$  such that

$$\neg Q(f_{\mathbf{d}}(x) \upharpoonright m, f_{\mathbf{c}}(x) \upharpoonright m, f_{\mathbf{c}_1}(x) \upharpoonright m, \dots, f_{\mathbf{c}_n}(x) \upharpoonright m). \quad (\dagger)$$

But, the maps  $f_{\mathbf{d}}, f_{\mathbf{c}}, f_{\mathbf{c}_i}$  are continuous. It follows that there is a string  $u = x \upharpoonright j$  for some  $j$  such that  $(\dagger)$  holds for **all**  $x \in [S]$ , where  $S = T \upharpoonright_u \in \mathbb{P}$ . But then clearly  $T$  cannot  $\mathbb{P}$ -force  $(*)$  as  $S$  forces the opposite.

**Step**  $\mathcal{L}\Sigma_k^1 \implies \mathcal{L}\Pi_k^1$ ,  $k < \aleph_1$ . Let  $\varphi$  be a  $\mathcal{L}\Pi_k^1$  formula. By Lemma 15.4, there is a tree  $T \in G$  such that either  $T \text{ forc } \varphi$  or  $T \text{ forc } \varphi^-$ . Assume that  $T \text{ forc } \varphi$ ; we have to prove that  $\varphi[\mathbf{a}[G]]$  is true. Suppose otherwise. Then  $\varphi^-[\mathbf{a}[G]]$  is true. By the inductive hypothesis, there is a tree  $S \in G$  such that  $S \text{ forc } \varphi^-$ . But the trees  $S, T$  belong to the same generic set  $G$ , hence they are compatible, which leads to a contradiction with the assumption  $T \text{ forc } \varphi$ , according to Definition 15.2(B). Now assume that  $T \text{ forc } \varphi^-$ . Then  $\varphi^-[\mathbf{a}[G]]$  is true by the inductive hypothesis, hence  $\varphi[\mathbf{a}[G]]$  is false. On the other hand, there is no tree  $S \in G$  such that  $S \text{ forc } \varphi^-$ , just as above.

**Step**  $\mathcal{L}\Pi_k^1 \implies \mathcal{L}\Sigma_{k+1}^1$ ,  $k < \aleph_1$ . Let  $\varphi$  be  $\exists x \psi(x)$  where  $\psi$  is  $\mathcal{L}\Pi_k^1$ . Assume that  $T \in G$  and  $T \text{ forc } \varphi$ . Then by Definition 15.2(C) there is a code  $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$

such that  $T \text{ forc } \psi(\mathbf{c})$ . By the inductive hypothesis, the formula  $\psi(\mathbf{c})[\mathbf{a}[G]]$ , that is,  $\psi[\mathbf{a}[G]](f_{\mathbf{c}}(\mathbf{a}[G]))$ , is true in  $\mathbf{L}[G]$ . But then  $\varphi[\mathbf{a}[G]]$  is true as well.

Conversely assume that  $\varphi[\mathbf{a}[G]]$  is true. Then there is a real  $y \in \mathbf{L}[G] \cap \omega^\omega$  such that  $\psi[\mathbf{a}[G]](y)$  is true. By Lemma 13.5(i),  $y = f_{\mathbf{c}}(\mathbf{a}[G])$  for a code  $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$ . But then  $\psi(\mathbf{c})[\mathbf{a}[G]]$  is true in  $\mathbf{L}[G]$ . By the inductive hypothesis, there is a tree  $T \in G$  satisfying  $T \text{ forc } \psi(\mathbf{c})$ . Then  $T \text{ forc } \varphi$  as well.  $\square$

## 18 The final argument

**Proof (Theorem 1.1, the main theorem).** We assert that any  $\mathbb{P}$ -generic extension  $\mathbf{L}[G] = \mathbf{L}[\mathbf{a}[G]]$  satisfies conditions (i), (ii), (iii), (iv) of the theorem. Regarding (i), (ii), (iii) see a review in the very end of Section 13. Let's concentrate on (iv). Let  $\Phi(j)$  be a parameter-free  $\Sigma_n^1$  formula.<sup>4</sup> Thus

$$\Phi(j) := \exists r_1 \forall r_2 \dots \forall (\exists) r_n \exists (\forall) m R_j(r_1 \upharpoonright m, r_2 \upharpoonright m, \dots, r_n \upharpoonright m),$$

where  $r_i$  are variables over  $\omega^\omega$ ,  $R_j \subseteq (\omega^{<\omega})^n$ ,  $R_j \in \mathbf{L}$ , and the map  $j \mapsto R_j$  is arithmetically definable in  $\mathbf{L}$ . Applying Theorem 14.2 in  $\mathbf{L}$  with  $\Theta = \omega_2^{\mathbf{L}} = \omega_1^{\mathbf{L}[G]}$  and  $M = \mathbf{L}[G]$ , we get relations  $Q_j = Q_\Theta(R_j)$ , and closed  $\mathcal{L}\Sigma_n^1$  formulas

$$\begin{aligned} \varphi_j &:= \exists r_1 \forall r_2 \dots \exists (\forall) r_{n-2} \forall (\exists) \chi \in \Theta^\omega \forall (\exists) q \exists (\forall) m \\ &\quad Q_j(\chi \upharpoonright m, q \upharpoonright m, r_1 \upharpoonright m, r_2 \upharpoonright m, \dots, r_{n-2} \upharpoonright m), \end{aligned}$$

satisfying  $\Phi(j) \iff \varphi_j, \forall j$ , both in  $\mathbf{L}$  and in any  $\mathbb{P}$ -generic extension  $\mathbf{L}[G]$  of  $\mathbf{L}$ . It follows, by Theorem 17.1, that the set  $X = \{j : \Phi(j)^{\mathbf{L}[G]}\}$  (defined in  $\mathbf{L}[G]$ ) satisfies  $X = \{j : \exists T \in G (T \text{ forc } \varphi_j)\}$ . Furthermore, as the formulas  $\varphi_j$  do not contain codes in  $\mathbf{CCF}_\omega$ , it follows, by Corollary 16.4, that  $X = \{j : T \text{ forc } \varphi_j\}$ , where  $T$  is any particular tree in  $(\mathbf{WT})^{\mathbf{L}}$ , one and the same for all  $j$ . We conclude that  $X \in \mathbf{L}$ , as required.  $\square$

## 19 A problem

It is a challenge to figure out what kind of models the method of the proof of Theorem 1.1 gives for cardinals bigger than  $\aleph_1$ . For instance, let  $\mathbf{WT}_{\omega_2}$  be the set of all trees  $T \subseteq \omega_2^{<\omega}$  whose all branching nodes are  $\omega_2$ -branching nodes. This is a non-Laver version of the Namba forcing; the Namba forcing per se requires that in addition every node above the stem is a branching node. The forcing  $\mathbf{WT}_{\omega_2}$  (or an equivalent forcing) is considered e.g. in [4], [9, Section 28], and [5, 18.4].

Clearly  $\mathbf{WT}_{\omega_2}$  adds a cofinal infinite sequence, say  $\vec{a} = \langle \alpha_n \rangle_{n < \omega}$ , in  $\omega_2$ . On the other hand, if CH holds in the ground universe then, essentially by Namba,  $\mathbf{WT}_{\omega_2}$  does not add new reals, hence, does not collapse  $\omega_1$ . (See [9, Section 28] for a simple proof.) Thus  $\vec{a} \in \mathbf{H}_\lambda$  in the extension  $\mathbf{V}[\vec{a}]$ , where  $\lambda = \omega_2^{\mathbf{V}[\vec{a}]} > \omega_2^{\mathbf{V}}$ . (Where  $\mathbf{V}$  is the ground set universe, as usual.) It is then an interesting problem

<sup>4</sup> The case when  $\Phi$  has real parameters in  $\mathbf{L}$  can also be handled with some extra care.

to check whether there are results for the definability of  $\vec{a}$  in  $\mathbf{H}_\lambda$  similar to the results in [2] and those of this paper.

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