Canonization of smooth equivalence relations on infinite-dimensional perfect cubes*

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Abstract

A canonization scheme for smooth equivalence relations on $\mathbb{R}^\omega$ modulo restriction to infinite perfect products is proposed. It shows that given a pair of Borel smooth equivalence relations E, F on $\mathbb{R}^\omega$, there is an infinite perfect product $P \subseteq \mathbb{R}^\omega$ such that either $\mathsf{F} \subseteq \mathsf{E}$ on P, or, for some $j < \omega$, the following is true for all $x, y \in P$: $x \mathrel{\mathsf{E}} y$ implies $x(j) = y(j)$, and $x \restriction (\omega \smallsetminus \{j\}) = y \restriction (\omega \smallsetminus \{j\})$ implies $x \mathrel{\mathsf{F}} y$.

1 Introduction

The canonization problem can be broadly formulated as follows. Given a class $\mathscr{E}$ of mathematical structures E, and a collection $\mathscr{P}$ of sets P considered as *large*, or *essential*, find a smaller and better structured subcollection $\mathscr{E}' \subseteq \mathscr{E}$ such that for any structure $E \in \mathscr{E}$ with the domain P there is a smaller set $P' \in \mathscr{P}$, $P' \subseteq P$, such that the restricted substructure $E \restriction P'$ belongs to $\mathscr{E}'$. For instance, the theorem saying that every Borel real map is either a bijection or a constant on a perfect set, can be viewed as a canonization theorem, with $\mathscr{E} = \{\text{Borel maps}\}$, $\mathscr{E}' = \{\text{bijections and constants}\}$, $\mathscr{P} = \{\text{perfect sets}\}$.

We refer to [3] as the background of the general canonization problem for Borel and analytic equivalence relations in descriptive set theory.

Among other results, it is established in [3, Section 9.3] (theorems 9.26 and 9.27) that if E belongs to one of two large families of analytic equivalence relations [1] on $(2^\omega)^\omega$ then there is and infinite perfect product $P \subseteq (2^\omega)^\omega$ such that $\mathsf{E} \restriction P$ is *smooth*, that is, simply there exists a Borel map $f : P \to 2^\omega$ satisfying $x \mathrel{\mathsf{E}} y \iff f(x) = f(y)$ for all $x, y \in P$. The canonization problem for smooth equivalence

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[1] The first family consists of equivalence relations classifiable by countable structures, the second of those Borel reducible to an analytic P-ideal.

relations themselves was not considered in [3].[2] Theorem 2.1, the main result of this note, contributes to this problem.

2 Perfect products

We consider sets in $(2^\omega)^\omega$. Let a *perfect product* be any set $P \subseteq (2^\omega)^\omega$, such that $P = \prod_{k<\omega} P(k)$, where $P(k) = \{x(k) : x \in P\}$ is the *projection* on the k-th coordinate, ant it is required that each set $P(k)$ is a perfect subset of 2^ω. Let **PP** be the set of all perfect products. To set up a convenient notation, say that an equivalence relation E on $(2^\omega)^\omega$:

captures $j \in \omega$ **on** $P \in \mathbf{PP}$**:** if $x \mathrel{\mathsf{E}} y$ implies $x(j) = y(j)$ for all $x, y \in P$;

is reduced to $U \subseteq \omega$ **on** $P \in \mathbf{PP}$**:** if $x \restriction U = y \restriction U$ implies $x \mathrel{\mathsf{E}} y$ for all $x, y \in P$.

Theorem 2.1. *If* E, F *are smooth Borel equivalence relations on* $(2^\omega)^\omega$ *then there is a perfect product* P *such that either* $\mathsf{F} \subseteq \mathsf{E}$ *on* P*, or, for some* $j < \omega$*,* E *captures* j *on* P *and* F *is reduced to* $\omega \smallsetminus \{j\}$ *on* P*.*

The two options of the theorem are incompatible on perfect products.

The result can be compared to canonization results related to *finite* perfect products and equivalence relations defined on spaces of the form $(2^\omega)^m$, $m < \omega$. Theorem 9.3 in [3, Section 9.1] implies that every analytic equivalence relation on $(2^\omega)^m$ coincides with one of the multiequalities D_U, $U \subseteq \{0, 1, \dots, m-1\}$, on some perfect product $P \subseteq (2^\omega)^m$, where $x \mathrel{\mathsf{D}_U} y$ iff $x \restriction U = y \restriction U$. One may ask whether such a result holds for equivalence relations on $(2^\omega)^\omega$ and accordingly infinite perfect products. This answers in the negative, even for smooth equivalences.

Example 2.2. Let E be defined on $(2^\omega)^\omega$ so that $x \mathrel{\mathsf{E}} y$ iff $x(0) = y(0)$, and also $x(j+1) = y(j+1)$ for all numbers j such that $x(0)(j) = 0$. That E is smooth can be witnessed by the map sending each $x \in (2^\omega)^\omega$ to $a = f(x) \in (2^\omega)^\omega$ defined so that $a(k) = x(k)$ whenever $k = 0$ or $k = j+1$ and $x(0)(j) = 0$, and $a(k)(n) = 0$ for all other k and all $n < \omega$. That E is not equal (and even not Borel bi-reducible) to any D_U on any perfect product $P \subseteq (2^\omega)^\omega$ is easy. □

The proof of Theorem 2.1 is based on splitting/fusion technique known in the theory of iterations and products of the perfect-set forcing (see, *e.g.*, [1, 2]).

3 Splitting

The *simple splitting* of a perfect set $X \subseteq 2^\omega$ consists of subsets $X(\to i) = \{x \in X : x(n) = i\}$, $i = 0, 1$, where $n = \mathtt{lh}(s)$ (the lenght of a string $r \in 2^{<\omega}$), and $s = \mathtt{stem}(X)$ is the largest string in $2^{<\omega}$ satisfying $r \subset x$ for all $x \in X$. Then $X = X(\to 0) \cup X(\to 1)$ is a disjoint partition of a perfect set $X \subseteq 2^\omega$ onto two

[2] *We avoid any attempt at organizing the very complicated class of smooth equivalence relations*, [3, page 232].

perfect subsets. Splittings can be iterated. We let $X(\to\Lambda) = X$ for the empty string Λ, and if $s \in 2^n$, $s \neq \Lambda$ then we define

$$X(\to s) = X(\to s(0))(\to s(1))(\to s(2)) \ldots (\to s(n-1)) .$$

If $X, Y \subseteq 2^\omega$ are perfect sets and $n < \omega$ then define $X \subseteq_n Y$ (*refinement*), if $X(\to s) \subseteq Y(\to s)$ for all $s \in 2^n$; $X \subseteq_0 Y$ is equivalent to $X \subseteq Y$. Clearly $X \subseteq_{n+1} Y$ implies $X \subseteq_n Y$ (and $X \subseteq Y$).

Lemma 3.1. *If $X \subseteq 2^\omega$ is a perfect set, $s_0 \in 2^n$, and $A \subseteq X(\to s_0)$ is a perfect set, then $Y = A \cup \bigcup_{u \in 2^n, u \neq s} X(\to u)$ is perfect, $Y \subseteq_n X$, and $Y(\to s_0) = A$.* $\square$

Now we extend the splitting technique to perfect products.

Definition 3.2. Fix once and for all a function $\phi : \omega \xrightarrow{\text{onto}} \omega$ taking each value infinitely many times, so that if $j < \omega$ then the following set is infinite:

$$\phi^{-1}(j) = \{k : \phi(k) = j\} = \{\mathbf{k}_{0j} < \mathbf{k}_{1j} < \mathbf{k}_{2j} < \ldots < \mathbf{k}_{lj} < \ldots\}.$$

If $m < \omega$ then let $\boldsymbol{\nu}_{mj}$ be the number of indices $k < m$, $k \in \phi^{-1}(j)$. $\square$

Let $m < \omega$ and $\sigma \in 2^m$ (a stringh of length m). If $j \in \phi"m = \{\phi(k) : k < m\}$, then the set $\phi^{-1}(j)$ cuts in σ a substring $\sigma[j] \in 2^{\boldsymbol{\nu}_{mj}}$, of length $\mathtt{lh}(\sigma[j]) = \boldsymbol{\nu}_{mj}$, defined by $(\sigma[j])(\ell) = \sigma(\mathbf{k}_{\ell j})$ for all $\ell < \boldsymbol{\nu}_{mj}$. Thus the string $\sigma \in 2^m$ splits in an array of strings $\sigma[j] \in 2^{\boldsymbol{\nu}_{mj}}$ $(j \in \phi"m)$ of total length $\sum_{j \in \phi"m} \boldsymbol{\nu}_{mj} = m$.

Let P is a perfect product. If $j < \omega$, $i = 0, 1$ then define a perfect product $Q = P(\xrightarrow{j} i)$ so that $Q(k) = P(k)$ for all $k \neq j$, but $q(j) = P(j)(\to i)$. If $\sigma \in 2^m$ then define a perfect product $P(\Rightarrow\sigma)$ by induction so that $P(\Rightarrow\Lambda) = P$ and $P(\Rightarrow\sigma^\frown i) = P(\Rightarrow\sigma)(\xrightarrow{j_0} i)$, where $j_0 = \phi(m)$, $m = \mathtt{lh}(\sigma)$. Note that $P(\Rightarrow\sigma)(j) = P(j)(\to\sigma[j])$ for all j. In particular, if $j \notin \phi"m$ then $P(\Rightarrow\sigma)(j) = P(j)$, because $\mathtt{lh}(\sigma[j]) = \boldsymbol{\nu}_{mj} = 0$ holds provided $j \notin \phi"m$.

Let P, Q be perfect products. Define $P \subseteq_m Q$, if $P(j) \subseteq_{\boldsymbol{\nu}_{mj}} Q(j)$ for all j. This is equivalent to $P(\Rightarrow\sigma) \subseteq Q(\Rightarrow\sigma)$ for all $\sigma \in 2^m$.

If $\sigma, \tau \in 2^m$ then let $\boldsymbol{\Delta}[\sigma, \tau] = \omega \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\}$.

Lemma 3.3. *Let $P \subseteq (2^\omega)^\omega$ is a perfect product and $m < \omega$. Then:*

(i) *if $\sigma, \tau \in 2^m$, then $P(\Rightarrow\sigma) \upharpoonright \boldsymbol{\Delta}[\sigma, \tau] = P(\Rightarrow\tau) \upharpoonright \boldsymbol{\Delta}[\sigma, \tau]$;*

(ii) *if $\sigma_0 \in 2^m$ and $B \subseteq P(\Rightarrow\sigma_0)$ is a perfect product, then there is a perfect product $Q \subseteq_m P$ satisfying $Q(\Rightarrow\sigma_0) = B$;*

(iii) *if B is clopen in P in* (ii) *then such a Q can be chosen to be clopen in P;*

(iv) *if $\sigma_0, \tau_0 \in 2^m$, $B \subseteq P(\Rightarrow\sigma_0)$ and $B' \subseteq P(\Rightarrow\tau_0)$ are perfect products, and $B \upharpoonright \boldsymbol{\Delta}[\sigma, \tau] = B' \upharpoonright \boldsymbol{\Delta}[\sigma, \tau]$, then there is a perfect product $R \subseteq_m P$ satisfying $R(\Rightarrow\sigma_0) = B$ and $R(\Rightarrow\tau_0) = B'$.*

Proof. (ii) Apply Lemma 3.1 componentwise with $A = B(j)$ for each $j < \omega$. Namely if $j = \phi(k)$, $k < m$, $\nu = \boldsymbol{\nu}_{mj}$, $s_0 = \sigma_0[j] \in 2^\nu$, then we put $Q(j) = B(j) \cup \bigcup_{s \in 2^\nu, s \neq s_0} P(j)(\to s)$, while if $j \notin \phi"m$ then simply $Q(j) = B(j)$.

(iv) We first apply (ii) for $B \subseteq P(\Rightarrow \sigma_0)$, getting a perfect product $Q \subseteq_m P$ such that $Q(\Rightarrow \sigma_0) = B$. We claim that $B' \subseteq Q(\Rightarrow \tau_0)$. Indeed if $j \in \boldsymbol{\Delta}[\sigma_0, \tau_0]$ then still $\tau_0[j] = s_0$, hence $Q(\Rightarrow \tau_0)(j) = Q(j)(\to s_0) = B(j)$ by construction, therefore $B'(j) = B(j) = Q(\Rightarrow \tau_0)(j)$. If $j \notin \boldsymbol{\Delta}[\sigma_0, \tau_0]$ then the string $t_0 = \tau_0[j] \in 2^\nu$ differs from s_0, hence $Q(\Rightarrow \tau_0)(j) = Q(j)(\to t_0) = P(j)(\to t_0)$ by construction, therefore $B'(j) \subseteq P(j)(\to t_0) = Q(\Rightarrow \tau_0)(j)$ anyway. Thus indeed $B' \subseteq Q(\Rightarrow \tau_0)$.

Now we apply (ii) for $B' \subseteq Q(\Rightarrow \tau_0)$, getting a perfect product $R \subseteq_m Q$ such that $R(\Rightarrow \tau_0) = B'$. And $R(\Rightarrow \sigma_0) = B$ holds by the same reasons as above. □

4 Fusion

We begin with a basic fusion lemma, rather elementary.

Lemma 4.1 (fusion)**.** *Let* $\dots \subseteq_4 X_3 \subseteq_3 X_2 \subseteq_2 X_1 \subseteq_1 X_0$ *be an infinite sequence of perfect sets* $X_n \subseteq 2^\omega$*. Then* $X = \bigcap_n X_n$ *is perfect and* $X \subseteq_{n+1} X_n$, $\forall n$. □

A version for perfect products follows:

Lemma 4.2 (applying Lemma 4.1 componentwise)**.** *Let* $\dots \subseteq_5 P_4 \subseteq_4 P_3 \subseteq_3 P_2 \subseteq_2 P_1 \subseteq_1 P_0$ *be a sequence of perfect products. Then* $Q = \bigcap_n P_n$ *is a perfect product,* $Q(j) = \bigcap_m P_m(j)$ *for all* $j < \omega$*, and* $Q \subseteq_{m+1} P_m$ *for all* m. □

Corollary 4.3 (see Proposition 9.31 in [3, Section 9.3])**.** *If* $P \subseteq (2^\omega)^\omega$ *is a perfect product and* $B \subseteq P$ *a Borel set then there is a perfect product* $Q \subseteq P$ *such that* $Q \subseteq B$ *or* $Q \cap B = \varnothing$. □

Corollary 4.4. *If* $P \subseteq (2^\omega)^\omega$ *is a perfect product and* $f : P \to 2^\omega$ *a Borel map then there is a perfect product* $Q \subseteq P$ *such that* $f \restriction Q$ *is continuous.*

Proof. If $n < \omega$ and $i = 0, 1$ then let $B_{ni} = \{x \in P : f(x)(n) = i\}$. Using Corollary 4.3 and Lemma 3.3, we get a sequence $\dots \subseteq_3 P_2 \subseteq_2 P_1 \subseteq_1 P_0 \subseteq P$ of perfect products as in Lemma 4.2, such that if $m < \omega$ and $\sigma \in 2^m$ then $P_m(\Rightarrow \sigma) \subseteq B_{m0}$ or $P_m(\Rightarrow \sigma) \subseteq B_{m1}$. Then $Q = \bigcap_m P_m$ is as required. □

5 Proof of the main theorem

Beginning the proof of Theorem 2.1, we let Borel maps $\boldsymbol{e}, \boldsymbol{f} : 2^\omega \to 2^\omega$ witness the smoothness of the equivalence relations resp. E, F, so that

$$x \mathrel{\mathsf{E}} y \iff \boldsymbol{e}(x) = \boldsymbol{e}(y) \quad \text{and} \quad x \mathrel{\mathsf{F}} y \iff \boldsymbol{f}(x) = \boldsymbol{f}(y).$$

By Corollary 4.4, we can assume that in fact $\boldsymbol{e}, \boldsymbol{f}$ <u>are continuous</u>.

Lemma 5.1. *If* P *is a perfect product,* $U_0, U_1, \dots \subseteq \omega$*, and* E *is reduced to each* U_k *on* P*, then* E *is reduced to* $U = \bigcap_k U_k$ *on* P*. The same for* F.

Proof. For just two sets, if $U = U_0 \cap U_1$ and $x, y \in P$, $x \restriction U = y \restriction U$, then, using the product structure, find a point $z \in P$ with $z \restriction U_0 = x \restriction U_0$ and $z \restriction U_1 = y \restriction U_1$. Then $\boldsymbol{e}(x) = \boldsymbol{e}(z) = \boldsymbol{e}(y)$, hence $x \mathsf{E} y$. The case of finitely many sets follows by induction. Therefore we can assume that $U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$ in the general case. Let $x, y \in P$, $x \restriction U = y \restriction U$. There exist points $x_k \in P$ satisfying $x_k \restriction U_k = x \restriction U_k$ and $x_k \restriction (B \smallsetminus U_k) = y \restriction (B \smallsetminus U_k)$. Then immediately $\boldsymbol{e}(x_k) = \boldsymbol{e}(x)$, $\forall k$. On the other hand, clearly $x_k \to y$, hence, $\boldsymbol{e}(x_k) \to \boldsymbol{e}(y)$ as $\boldsymbol{e}$ is continuous. Thus $\boldsymbol{e}(x) = \boldsymbol{e}(y)$, hence $x \mathsf{E} y$. □

Proof (Theorem 2.1)**.** We argue in terms of Definition 3.2. The plan is to define a sequence of perfect products as in Lemma 4.2, with some extra properties. Let $m < \omega$. A perfect product R is *m-good*, if (see definitions in Section 2):

(1)E: if $\sigma \in 2^m$ and $j = \phi(m)$ then either E is reduced to $\omega \smallsetminus \{j\}$ on $R(\Rightarrow\sigma)$, or there is no perfect product $R' \subseteq R(\Rightarrow\sigma)$ on which E is reduced to $\omega \smallsetminus \{j\}$;

(1)F: the same for F;

(2)E: if $\sigma, \tau \in 2^m$, then either (i) E is reduced on $R(\Rightarrow\sigma) \cup R(\Rightarrow\tau)$ to

$$\boldsymbol{\Delta}[\sigma, \tau] = \omega \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\},$$

or (ii) $\boldsymbol{e}[R(\Rightarrow\sigma)] \cap \boldsymbol{e}[R(\Rightarrow\tau)] = \varnothing$;[3]

(2)F: the same for F.

Lemma 5.2. *If $m < \omega$ and a perfect product R is m-good, then there is an $m+1$-good perfect product $Q \subseteq_{m+1} R$.*

Proof (Lemma)**.** Consider a string $\sigma' \in 2^{m+1}$, and first define a perfect product $Q \in \mathbf{MT}_B$, $Q \subseteq_{m+1} R$, satisfying (1)E relatively to this string only. Let $j = \phi(m+1)$. If there exists a perfect product $R' \subseteq R(\Rightarrow\sigma')$ on which E is reduced to $\omega \smallsetminus \{j\}$, then let U be such R'. If there is no such R' then put $U = R(\Rightarrow\sigma')$. By Lemma 3.3, there is a perfect product $Q \subseteq_{m+1} R$ such that $Q(\Rightarrow\sigma') = U$. Thus the perfect product Q satisfies (1)E with respect to σ'. Now take Q as the "new" perfect product R, consider another string $\sigma' \in 2^{m+1}$, and do the same as above. Consider all strings in 2^{m+1} consecutively, with the same procedure. This ends with a perfect product $Q \subseteq_{m+1} R$, satisfying (1)E for all strings in 2^{m+1}.

Now take care of (2)E. Let $\sigma', \tau' \in 2^{m+1}$. Note that if $\sigma'(m) = \tau'(m)$ then $\boldsymbol{\Delta}[\sigma', \tau'] = \boldsymbol{\Delta}[\sigma' \restriction m, \tau' \restriction m]$, so that (2)$\mathsf{E}$ relatively to σ', τ' follows from (2)E relatively to $\sigma' \restriction m, \tau' \restriction m$. Thus it suffices to consider only pairs in 2^{m+1} of the form $\sigma^\frown 0, \tau^\frown 1$, where $\sigma, \tau \in 2^m$. Consider one such a pair $\sigma' = \sigma^\frown 0$, $\tau' = \tau^\frown 1$, and define a perfect product $P \subseteq_{m+1} Q$, satisfying (2)E relatively to this pair.

The sets $U' = \boldsymbol{\Delta}[\sigma', \tau']$ and $U = \boldsymbol{\Delta}[\sigma, \tau]$ satisfy $U' = U \smallsetminus \{j_0\}$, where $j_0 = \phi(m)$, while the sets $Q(\Rightarrow\sigma')$, $Q(\Rightarrow\tau')$ satisfy $Q(\Rightarrow\sigma') \restriction U' = Q(\Rightarrow\tau') \restriction U'$.

[3] Given a function h and $X \subseteq \operatorname{dom} h$, the set $h[X] = \{h(x) : x \in X\}$ is the h-image of X.

If E is reduced to U' on $Z' = Q(\Rightarrow\sigma')\cup Q(\Rightarrow\tau')$ then (2)E(i) holds for $P = Q$ and the pair σ', τ'. Now suppose that E is **not** reduced to U' on Z', so that there are points $x_0, y_0 \in Z'$ with $x_0 \restriction U' = y_0 \restriction U'$ and $\boldsymbol{e}(x_0) \neq \boldsymbol{e}(y_0)$, *i.e.*, $\boldsymbol{e}(x_0)(k) = p \neq q = \boldsymbol{e}(y_0)(k)$ for some k and $\{p, q\} = \{0, 1\}$. As $Q(\Rightarrow\sigma')\restriction U' = Q(\Rightarrow\tau')\restriction U'$, we can w.l.o.g. assume that $x_0 \in Q(\Rightarrow\sigma')$ but $y_0 \in Q(\Rightarrow\tau')$.

As $\boldsymbol{e}$ is continuous, there exist relatively clopen perfect products $X \subseteq Q(\Rightarrow\sigma')$, $Y \subseteq Q(\Rightarrow\tau')$, such that $x_0 \in X$, $y_0 \in Y$, $\boldsymbol{e}(x)(k) = p$ and $\boldsymbol{e}(y)(k) = q$ for all $x \in X$, $y \in Y$. Define smaller perfect products $X' \subseteq X$ and $Y' \subseteq Y$ so that $X'(j) = Y'(j) = X(j) \cap Y(j)$ for all $j \in U'$ but $X'(j) = X(j)$, $Y'(j) = Y(j)$ for all $j \in \omega \smallsetminus U'$. Note that still $x_0 \in X'$, $y_0 \in Y'$, and now $X'\restriction U' = Y'\restriction U'$.

This allows to apply Lemma 3.3(iv), getting a perfect product $P \subseteq_{m+1} Q$ such that $P(\Rightarrow\sigma') = X'$ and $P(\Rightarrow\tau') = Y'$. Then $\boldsymbol{e}[P(\Rightarrow\sigma')] \cap \boldsymbol{e}[P(\Rightarrow\tau')] = \varnothing$ by construction, therefore (2)E(ii) holds for P and the pair of σ', τ'.

To conclude, we get a perfect product $P \subseteq_{m+1} Q$ such that (2)E holds for P and the pair of σ', τ' in both cases.

Consider all pairs of strings in 2^{m+1} consecutively. This yields a perfect product $R \subseteq_{m+1} Q$, satisfying (2)E for all $\sigma', \tau' \in 2^{m+1}$ (and still satisfying (1)E).

Then repeat the same procedure for F. $\square$ (Lemma)

Come back to the proof of the theorem. Lemma 5.2 yields an infinite sequence $\ldots \leqslant_3 Q_2 \leqslant_2 Q_1 \leqslant_1 Q_0$ of perfect products Q_m, such that each Q_m is a m-good. The limit perfect product $P = \bigcup_m Q_m \in \mathbf{MT}_B$ satisfies $P \subseteq_{m+1} Q_m$ for all m by Lemma 4.2. Therefore P is m-good for every m, hence we can freely use (1)E, F and (2)E, F for P in the following final argument.

Case 1: if $m < \omega$, $\sigma, \tau \in 2^m$, and $\boldsymbol{e}[P(\Rightarrow\sigma)]\cap\boldsymbol{e}[P(\Rightarrow\tau)] = \varnothing$, then $\boldsymbol{f}[P(\Rightarrow\sigma)]\cap \boldsymbol{f}[P(\Rightarrow\tau)] = \varnothing$. Prove that $\mathsf{F} \subseteq \mathsf{E}$ on P in this case, as required by the "either" option of Theorem 2.1. Assume that $x, y \in P$ and $x \mathrel{\mathsf{E}} y$ **fails**, that is, $\boldsymbol{e}(x) \neq \boldsymbol{e}(y)$; show that $\boldsymbol{f}(x) \neq \boldsymbol{f}(y)$. Pick $a, b \in 2^\omega$ satisfying $\{x\} = \bigcap_m P(\Rightarrow a\restriction m)$ and $\{y\} = \bigcap_m P(\Rightarrow b\restriction m)$. As $x \neq y$, we have $\boldsymbol{e}[Q(\Rightarrow a\restriction m)] \cap \boldsymbol{e}[Q(\Rightarrow b\restriction m)] = \varnothing$ for some m by the continuity and compactness. Then by the Case 1 assumption, $\boldsymbol{f}[P(\Rightarrow a\restriction m)] \cap \boldsymbol{f}[P(\Rightarrow b\restriction m)] = \varnothing$ holds, hence $\boldsymbol{f}(x) \neq \boldsymbol{f}(y)$, and $x \mathrel{\mathsf{F}} y$ **fails**.

Case 2 = not Case 1. Then, by (2)F, there is a pair of strings $\sigma' = \sigma^\frown i$, $\tau' = \tau^\frown k \in 2^{m+1}$, $m < \omega$, such that $\boldsymbol{e}[P(\Rightarrow\sigma')] \cap \boldsymbol{e}[P(\Rightarrow\tau')] = \varnothing$, but F is reduced to $U' = \boldsymbol{\Delta}[\sigma', \tau']$ on $Z' = P(\Rightarrow\sigma') \cup P(\Rightarrow\tau')$. Assume that m is the least possible witness of this case. We are going to prove that the perfect product $P(\Rightarrow\sigma)$ satisfies the "or" option of Theorem 2.1, with the number $j_0 = \phi(m)$, that is, (*) F is reduced to $\omega \smallsetminus \{j_0\}$ on $P(\Rightarrow\sigma)$, and (**) E captures j_0 on $P(\Rightarrow\sigma)$.

Lemma 5.3. *The relation* E *is*:

(A) *reduced to* $U = \boldsymbol{\Delta}[\sigma, \tau]$ *on the set* $Z = P(\Rightarrow\sigma) \cup P(\Rightarrow\tau)$,

(B) **not** *reduced to* $U' = \boldsymbol{\Delta}[\sigma', \tau']$ *on* $Z' = P(\Rightarrow\sigma') \cup P(\Rightarrow\tau')$,

(C) **not** *reduced to* $\omega \smallsetminus \{j_0\}$ *on any perfect product* $U \subseteq P(\Rightarrow\sigma)$.

In addition, (D) $U \neq U'$, *hence* $j_0 \in U$ *and* $U' = U \smallsetminus \{j_0\}$.

Proof. (A) Otherwise $\boldsymbol{e}[P(\Rightarrow\sigma)]\cap\boldsymbol{e}[P(\Rightarrow\tau)]=\varnothing$ by (2)E, hence F is **not** reduced to U on $P(\Rightarrow\sigma)\cup P(\Rightarrow\tau)$ by the choice of m, thus $\boldsymbol{f}[P(\Rightarrow\sigma)]\cap\boldsymbol{f}[P(\Rightarrow\tau)]=\varnothing$ by (2)F, then $\boldsymbol{f}[P(\Rightarrow\sigma')]\cap\boldsymbol{f}[P(\Rightarrow\tau')]=\varnothing$, which contradicts to the fact that F is reduced to U' on $P(\Rightarrow\sigma')\cup P(\Rightarrow\tau')$, as $P(\Rightarrow\sigma')\restriction U'=P(\Rightarrow\tau')\restriction U'$ by Lemma 3.3.

(B) The otherwise assumption contradicts to $\boldsymbol{e}[P(\Rightarrow\sigma')]\cap\boldsymbol{e}[P(\Rightarrow\tau')]=\varnothing$.

(D) follows from (A) and (B).

(C) Otherwise E is reduced to $\omega\smallsetminus\{j_0\}$ on $P(\Rightarrow\sigma)$ by (1)E. Then E is reduced to U' on $P(\Rightarrow\sigma)$ by Lemma 5.1 since $U'=U\smallsetminus\{j_0\}$ by (D). It follows that E is reduced to U' on Z,[4] hence on $Z'\subseteq Z$ as well. But this contradicts to (B). □

Now, as $U'=U\smallsetminus\{j_0\}\subseteq\omega\smallsetminus\{j_0\}$, the perfect product $P(\Rightarrow\sigma')$ witnesses that F is reduced to $\omega\smallsetminus\{j_0\}$ on $P(\Rightarrow\sigma)$ by (1)F. Thus we have (*).

To prove (**), let $x,y\in P(\Rightarrow\sigma)$ and $x\;\mathsf{E}\;y$; prove that $x(j_0)=y(j_0)$. Indeed we have $\{x\}=\bigcap_n P(\Rightarrow a\restriction n)$ and $\{y\}=\bigcap_n P(\Rightarrow b\restriction n)$, where $a,b\in 2^\omega$, $\sigma\subset a$, $\sigma\subset b$. Let $\boldsymbol{\Delta}[a,b]=\bigcap_n\boldsymbol{\Delta}[a\restriction n,b\restriction n]$. Then $x\restriction\boldsymbol{\Delta}[a,b]=y\restriction\boldsymbol{\Delta}[a,b]$, since $P(\Rightarrow a\restriction n)\restriction\boldsymbol{\Delta}[a\restriction n,b\restriction n]=P(\Rightarrow b\restriction n)\restriction\boldsymbol{\Delta}[a\restriction n,b\restriction n]$ for all n. Thus it suffices to check that $j_0\in\boldsymbol{\Delta}[a\restriction n,b\restriction n]$ for all n.

Suppose towards the contrary that $j_0=\phi(m)\notin\boldsymbol{\Delta}[a\restriction n,b\restriction n]$ for some n. Note that $n>m$ because $a\restriction m=b\restriction m=\sigma$. However E is reduced to $\boldsymbol{\Delta}[a\restriction n,b\restriction n]$ on $P(\Rightarrow a\restriction n)$ by (2)E, since $x\;\mathsf{E}\;y$. Yet we have $j_0\notin\boldsymbol{\Delta}[a\restriction n,b\restriction n]$, therefore, $\boldsymbol{\Delta}[a\restriction n,b\restriction n]\subseteq\omega\smallsetminus\{j_0\}$. It follows that E is reduced to $\omega\smallsetminus\{j_0\}$ on $P(\Rightarrow a\restriction n)$. But this contradicts to Lemma 5.3(C) with $U=P(\Rightarrow a\restriction n)$.

To conclude Case 2, we have checked (*) and (**). □ (Theorem 2.1)

References

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[4] Let $x,y\in Z=P(\Rightarrow\sigma)\cup P(\Rightarrow\tau)$ and $x\restriction U'=y\restriction U'$. As $P(\Rightarrow\sigma)\restriction U=P(\Rightarrow\tau)\restriction U$ by Lemma 3.3, there are $x',y'\in P(\Rightarrow\sigma)$ with $x\restriction U=x'\restriction U$ and $y\restriction U=y'\restriction U$. We have $x\;\mathsf{E}\;x'$ and $y\;\mathsf{E}\;y'$ by (A), and $x'\;\mathsf{E}\;y'$ since E is reduced to U' on $P(\Rightarrow\sigma)$. We conclude that $x\;\mathsf{E}\;y$.