On intermediate extensions of generic extensions by a random real

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Abstract

The paper is the second of our series of notes aimed to bring back in circulation some bright ideas of early modern set theory, mainly due to Harrington and Sami, which have never been adequately presented in set theoretic publications. We prove that if a real a is random over a model M and $x \in M[a]$ is another real then either (1) $x \in M$, or (2) $M[x] = M[a]$, or (3) $M[x]$ is a random extension of M and $M[a]$ is a random extension of $M[x]$. This is a less-known result of old set theoretic folklore, and, as far as we know, has never been published.

As a corollary, we prove that $\mathbf{\Sigma}^1_n$-Reduction holds for all $n \geq 3$, in a model extending $\mathbf{L}$ by $\aleph_1$-many random reals.

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1 Introduction

It is known from Solovay [17], and especially Grigorieff [2] in most general form, that any subextension $\mathbf{V}[x]$ of a generic extension $\mathbf{V}[G]$, generated by a set $x \in \mathbf{V}[G]$, is itself a generic extension $\mathbf{V}[x] = \mathbf{V}[G_0]$ of the same ground universe $\mathbf{V}$, and the whole extension $\mathbf{V}[G]$ is equal to a generic extension $\mathbf{V}[G_0][G_1]$ of the subextension $\mathbf{V}[x] = \mathbf{V}[G_0]$. See a more recent treatment of this question in [3, 18, 10, 6]. In particular, it is demonstrated in [6] that if $\mathbb{P} = \langle \mathbb{P}\,; \leqslant \rangle \in \mathbf{V}$ is a forcing notion, a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$, $t \in \mathbf{V}[G]$ is a $\mathbb{P}$-name, $x = t[G] \in \mathbf{V}[G]$ is the G-valuation of t, and $x \subseteq \mathbf{V}$, then

(I) there is a set $\Sigma \subseteq \mathbb{P}$ such that $\mathbf{V}[\Sigma] = \mathbf{V}[x]$ and G is Σ-generic over $\mathbf{V}[x]$;

(II) there exists a stronger order $\leqslant_t$ on $\mathbb{P}$ (so that $p \leqslant q$ implies $p \leqslant_t q$) in $\mathbf{V}$ such that Σ itself is $\langle \mathbb{P}\,; \leqslant_t \rangle$-generic over $\mathbf{V}[\Sigma] = \mathbf{V}[x]$.

However the nature and forcing properties of the derived forcing notions $\mathbb{P}_0 = \langle \mathbb{P}\,; \leqslant_t \rangle \in \mathbf{V}$ and $\mathbb{P}_1(x) = \langle \Sigma\,; \leqslant \rangle \in \mathbf{V}[x]$ is not immediately clear.

At the trivial side, we have the Cohen forcing $\mathbb{P} = \mathbb{C} = 2^{<\omega}$. In this case, $\mathbb{P}_0$ and $\mathbb{P}_1(x)$ are countable forcing notions, hence the corresponding extensions, $\mathbf{V} \to \mathbf{V}[x]$ and $\mathbf{V}[x] \to \mathbf{V}[G]$ in the above scheme, are Cohen generic or trivial. As observed in [6], this leads to the following result of set theoretic folklore, never explicitly appeared in set theoretic publications, except for [16, Lemma 1.9]. (It can also be derived from some results in [2], especially 4.7.1 and 2.14.1.)

Theorem 1.1 (folklore, Sami). *Let $a \in 2^\omega$ be Cohen-generic over the ground set universe $\mathbf{V}$. Let x be a real in $\mathbf{V}[a]$. Then we have exactly one of the following*:

(C1) $x \in \mathbf{V}$; (C2) $\mathbf{V}[x] = \mathbf{V}[a]$;

(C3) (a) $\mathbf{V}[x]$ *is a Cohen-generic extension of* $\mathbf{V}$, *and*
(b) $\mathbf{V}[a]$ *is a Cohen-generic extension of* $\mathbf{V}[x]$. [1] □

A much more complex case is the Levy – Solovay extension of $\mathbf{L}$, the constructible universe. As established in [17], such an extension is equal to a Levy – Solovay extension of $\mathbf{L}[x]$ for any real x it contains.

The following theorem, proved below, is a result of the same type.

Theorem 1.2. *Let $a \in 2^\omega$ be Solovay-random over the ground set universe* $\mathbf{V}$, *Let x be a real in $\mathbf{V}[a]$. Then we have exactly one of the following*:

(R1) $x \in \mathbf{V}$; (R2) $\mathbf{V}[x] = \mathbf{V}[a]$;

(R3) (a) $\mathbf{V}[x]$ *is a Solovay-random extension of* $\mathbf{V}$, *and*
(b) $\mathbf{V}[a]$ *is a Solovay-random extension of* $\mathbf{V}[x]$.

[1] Theorem 1.1 dramatically fails for intermediate extensions not generated by sets, [11].

It is ***not*** asserted though that the real x itself is random over $\mathbf{V}$ in (a) and/or the real a itself is random over $\mathbf{V}[x]$ in (b).

Note that Theorem 1.2 contains two separate dichotomies: (R1) vs. (R3)(a) and (R2) vs. (R3)(b). In spite of obvious semblance of Theorem 1.1, this theorem takes more effort. Its proof (it begins in Section 4) involves some results related rather to real analysis and measure theory.

Now we proceed with an application of Theorem 1.2.

2 A corollary: Reduction in extensions by random reals

The reduction property for a pointclass K, or simply K-*Reduction*, is the assertion that for any two sets X, Y in K (in the same Polish space) there exist *disjoint* sets $X' \subseteq X$, $Y' \subseteq Y$ in the same class K, such that $X' \cup Y' = X \cup Y$.

It is known classically from studies of Kuratowski [13] that Reduction holds for $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$, but fails for $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_2$. As for the higher projective classes, Addison [1] proved that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies that Reduction holds for $\mathbf{\Sigma}^1_n$, $n \geq 3$, but fails for $\mathbf{\Pi}^1_n$, $n \geq 3$. On the other hand, by Martin [14], the axiom of projective determinacy **PD** implies that, similarly to projective level 1, $\mathbf{\Pi}^1_n$-Reduction holds for all odd numbers $n \geq 3$, and, similarly to projective level 2, $\mathbf{\Sigma}^1_n$-Reduction holds for all even $n \geq 4$.

Apparently not much is known on Reduction for higher projective classes in generic models. One can expect that rather homogeneous, well-behaved forcing notions produce generic extensions of $\mathbf{L}$, in which Reduction keeps to be true for projective classes $\mathbf{\Sigma}^1_n$ and accordingly fails for $\mathbf{\Pi}^1_n$, $n \geq 3$, while in specially designed non-homogeneous extensions this pattern can be violated. This idea is supported by a few known results. Ramez Sami [16] proved

Theorem 2.1 (Sami)**.** *It is true in any extension of* $\mathbf{L}$ *by* $\aleph_1$ *Cohen reals that if* $n \geq 3$ *then* Σ^1_n*-Reduction holds, and hence* $\mathbf{\Sigma}^1_n$*-Reduction holds, too.* [2] □

On the other hand, we proved in [5] that Reduction fails for $\mathbf{\Sigma}^1_3$ (and in fact Separation fails for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$) in a rather complicated model related to an $\aleph_1$-product of forcings similar to Jensen's minimal forcing [4]. See also [7, 9] on similar models in which the Uniformization principle fails for $\mathbf{\Pi}^1_2$ (or $\mathbf{\Pi}^1_n$ for a given $n \geq 3$) sets with countable sections, and [8] on some related (and very complex) models of Harrington. Here we prove the following theorem.

Theorem 2.2. *It is true in any extension of* $\mathbf{L}$ *by* $\aleph_1$ *Solovay-random reals that if* $n \geq 3$ *then* Σ^1_n*-Reduction holds, and hence* $\mathbf{\Sigma}^1_n$*-Reduction holds, too.*

Note that the theorem also holds in models obtained by adding any uncountable (not necessarily $\aleph_1$) number κ of random reals. (Because such models are elementarily equivalent to the extension by $\aleph_1$ random reals.)

[2] To prove that Σ^1_n-Reduction implies the boldface $\mathbf{\Sigma}^1_n$-Reduction, it suffices to use a double-universal pair of Σ^1_n sets, as those used in a typical proof that $\mathbf{\Sigma}^1_n$-Reduction and $\mathbf{\Sigma}^1_n$-Separation contadict each other. This argument does not work for Separation though.

Sami's proof of Theorem 2.1 involves Theorem 1.1. Accordingly, we'll use Theorem 1.2 rather similar way. The following lemma is the key ingredient.

Lemma 2.3 (proof see Section 7)**.** *If $n \geq 2$ and $\varphi(x)$ is a parameter-free Σ^1_n formula then there is a parameter-free Σ^1_n formula $\varphi^*(x)$ such that if x is a real in an $\aleph_1$-random extension N of $\mathbf{L}$ then $\varphi(x)$ holds in N iff $\mathbf{L}[x] \models \varphi^*(x)$.*

A similar result was obtained by Solovay [17] (w.r.t. Levy – Solovay extensions) and by Sami [16] (w.r.t. extensions by $\aleph_1$ Cohen reals).

Proof (Theorem 2.2, sketch)**.** The idea, due to Sami [16, Lemma 1.11], is to closely emulate Addison's proof of Σ^1_n-Reduction in $\mathbf{L}$.

Arguing in an $\aleph_1$-random extension N of $\mathbf{L}$, we suppose that $n \geq 3$, and $X = \{x : \varphi(x)\}$ and $Y = \{x : \psi(x)\}$ are sets of reals, φ and ψ being Σ^1_n formulas. Then, by Lemma 2.3, we have $X = \{x : \mathbf{L}[x] \models \varphi^*(x)\}$ and $Y = \{x : \mathbf{L}[x] \models \psi^*(x)\}$, where φ^* and ψ^* are still Σ^1_n-formulas. Thus $\varphi^*(x)$ is $\exists\, y\, \Phi(x, y)$ and $\psi^*(x)$ is $\exists\, y\, \Psi(x, y)$, Φ and Ψ being Π^1_{n-1}.

Still arguing in N, if $x \in 2^\omega$ then let $<_{\mathbf{L}[x]}$ be the canonical Gödel wellordering of the reals in $\mathbf{L}[x]$, of order type ω_1. The crucial property of this system of order relations says that the *bounded quantifiers* $\forall\, y' <_{\mathbf{L}[x]} y$ and $\forall\, y' \leqslant_{\mathbf{L}[x]} y$, applied to a Σ^1_n formula, yield a Σ^1_n formula. It follows that the sets

$$
\begin{aligned}
X' &= \{x : \mathbf{L}[x] \models \exists\, y \big(\Phi(x, y) \wedge \forall\, y' <_{\mathbf{L}[x]} y \neg \Psi(x, y')\big)\}\\
Y' &= \{x : \mathbf{L}[x] \models \exists\, y \big(\Psi(x, y) \wedge \forall\, y' \leqslant_{\mathbf{L}[x]} y \neg \Phi(x, y')\big)\}
\end{aligned}
$$

are Σ^1_n, because the relativization to $\mathbf{L}[x]$ does not violate being Σ^1_n $(n \geq 2)$.

□ (Theorem 2.2, modulo Lemma 2.3 and Theorem 1.2)

3 Randomness is measure-independent

Random (or *Solovay-random*) reals, over a set universe $\mathbf{V}$, are usually defined as those reals in 2^ω, or true reals in the unit interval $[0, 1] = \mathbb{I}$, which avoid Borel sets, coded in $\mathbf{V}$ and null with respect to, resp., the usual product probability measure μ on 2^ω, or the Lebesgue measure λ on $\mathbb{I}$.

That the μ-random reals in 2^ω and λ-random reals in $\mathbb{I}$ produce the same generic extensions and thereby both notions can be identified, is witnessed by the Borel map $f(a) = \sum_{a(n)=1} 2^{-n-1} : 2^\omega \xrightarrow{\text{onto}} \mathbb{I}$. It satisfies $\lambda(f[X]) = \mu(X)$ for any Borel $X \subseteq 2^\omega$, therefore if $a \in 2^\omega$ and $x = f(a) \in \mathbb{I}$ then a is μ-random iff x is λ-random, and $\mathbf{V}[a] = \mathbf{V}[x]$, of course. There is a general version of such a correspondence, which will be used in the proof of Theorem 1.2 below.

Lemma 3.1. *Assume that ν is a continuous* (that is, all singletons are null sets) *Borel probability measure defined on 2^ω in a set universe $\mathbf{V}$. Then there is a Borel map $g : 2^\omega \xrightarrow{\text{onto}} \mathbb{I}$, coded in $\mathbf{V}$, and such that if $a \in 2^\omega$ and $x = g(a) \in \mathbb{I}$ then a is ν-random over $\mathbf{V}$ iff x is λ-random over $\mathbf{V}$, and $\mathbf{V}[a] = \mathbf{V}[x]$.*

Proof. Let $<_{\mathtt{lex}}$ be the lexicographical order on 2^ω, and let $(a,b)_{\mathtt{lex}} = \{a' : a <_{\mathtt{lex}} a' <_{\mathtt{lex}} b\}$ denote $<_{\mathtt{lex}}$-intervals. Let $g(a) = \mu((0_{\mathtt{lex}}, a)_{\mathtt{lex}})$, where $0_{\mathtt{lex}} \in 2^\omega$ is the $<_{\mathtt{lex}}$-least element, $0_{\mathtt{lex}}(k) = 0, \forall\, k$. Easily g is measure-presirving: if $X \subseteq 2^\omega$ is Borel then $\nu(X) = \lambda(g[X])$. (See *e.g.* the proof of Theorem 17.41 in Kechris [12].) It follows that a is ν-random iff x is λ-random, whenever $a \in 2^\omega$ and $x = g(a)$. To see that $a \in \mathbf{V}[x]$, note that $J = g^{-1}[x]$ is a closed $\leqslant_{\mathtt{lex}}$-interval in 2^ω, the interior of which (if non-empty) is a ν-null set, hence a is equal to oneof the two endpoints of J. □

4 Intermediate submodels of random extensions: kase split

We begin here a proof of Theorem 1.2. It will use only basic forcing ideas and some classical theorems related to real analysis.

Thus let $\boldsymbol{a}_0 \in 2^\omega$ be Solovay-random over the background set universe $\mathbf{V}$. We shall assume that $\boldsymbol{x}_0 \in \mathbf{V}[\boldsymbol{a}_0]$ is a real in the unit segment $[0,1]$ of the true real line $\mathbb{R}$. As the Solovay-random forcing admits continuous reading of names, there is a continuous map $f : 2^\omega \to \mathbb{I}$, coded in $\mathbf{V}$, such that $\boldsymbol{x}_0 = f(\boldsymbol{a}_0)$. Let μ_0 be the usual product probability measure on 2^ω, and λ be the Lebesgue measure on the segment $\mathbb{I} = [0,1]$.

We have to prove the trichotomy (R1) **vs.** (R2) **vs.** (R3) **of Theorem 1.2**.

First split. *Arguing in* $\mathbf{V}$, consider the set $C = \{x \in \mathbb{I} : \mu_0(f^{-1}[x]) > 0\}$. It is at most countable. Consider the complementary sets $D = f^{-1}[C]$ and $A_1 = 2^\omega \smallsetminus D$. These are resp. $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$ sets coded in $\mathbf{V}$, we identify them with "the same" (*i.e.*, coded by the same codes) sets in the extensions $\mathbf{V}[\boldsymbol{a}_0]$, $\mathbf{V}[\boldsymbol{x}_0]$.

Case 1: $\boldsymbol{a}_0 \in D$. Then there is a real $\bar{y} \in \mathbb{I} \cap \mathbf{V}$ such that $\boldsymbol{a}_0 \in f^{-1}[\bar{y}]$, hence $\boldsymbol{x}_0 = \bar{y} \in \mathbf{V}$, and (R1) holds.

Case 2: $\boldsymbol{a}_0 \in A_1$. Then $\mu_0(A_1) > 0$ by the randomness. In $\mathbf{V}$, there is an $\mathbf{F}_\sigma$ set $A'_1 \subseteq A_1$ of the same measure, so the Borel set $A_1 \smallsetminus A'_1$, coded in $\mathbf{V}$, is null, and hence $\boldsymbol{a}_0 \in A'_1$. Therefore there is, in $\mathbf{V}$, a perfect set $A_2 \subseteq A'_1$, satisfying $\boldsymbol{a}_0 \in A_2$ and $\mu_0(A_2) > 0$. We let $\mu(A) = \mu_0(A)/\mu_0(A_2)$, for any measurable $A \subseteq A_2$, so μ is a continuous probability measure on P, and the real $\boldsymbol{a}_0 \in P$ is μ-random over $\mathbf{V}$. The set $Y_2 = f[A_2]$ is closed, and by construction we have

($*$) if $x \in Y_2$ then $\mu(f^{-1}[x]) = 0$ (*i.e.*, f-preimages of singletons are μ-null).

The set R of all rational intervals $J \subseteq \mathbb{I}$, such that $\mu(f^{-1}[J \cap Y_2]) = 0$, is at most countable. Therefore $\mathbb{A}_0 = A_2 \smallsetminus \bigcup_{J \in R} f^{-1}[J \cap Y_2]$ is a closed subset of A_2, of the same measure, f maps $\mathbb{A}_0$ onto the closed set $\mathbb{Y}_0 = Y_2 \smallsetminus \bigcup R$, and we have

(†) if J is an open interval in $\mathbb{I}$ and $\mathbb{Y}_0 \cap J \neq \varnothing$ then $\mu(f^{-1}[\mathbb{Y}_0 \cap J]) > 0$.

Definition 4.1. If $x \in \mathbb{I}$ then let $\hat{f}(x) = \mu(f^{-1}[\mathbb{Y}_0 \cap [0,x)])$, so $\hat{f} : \mathbb{I} \to \mathbb{I}$. □

Lemma 4.2. *The map* $\hat{f}$ *is continous,* $\mathtt{ran}\,\hat{f} = \mathbb{I}$*, and* $\hat{f}$ *is strictly increasing, except that* $\hat{f}(x) = \hat{f}(x')$ *in case when* $x < x'$ *belong to* $\mathbb{I}$ *and* $\mathbb{Y}_0 \cap (x, x') = \varnothing$.

Proof. Let $x < x'$ belong to $\mathbb{I}$. Then $\hat{f}(x) \leq \hat{f}(x')$ is clear. To prove the strict inequality, note that $\hat{f}(x') - \hat{f}(x) = \mu(f^{-1}[\mathbb{Y}_0 \cap [x, x')]) > 0$ provided $\mathbb{Y}_0 \cap (x, x') \neq \varnothing$, and apply $(*)$, $(\dagger)$. □

Lemma 4.3. *The superposition map* $F(a) = \hat{f}(f(a)) : \mathbb{A}_0 \xrightarrow{\text{onto}} \mathbb{I}$ *is continuous and measure-preserving in the sense that if* $X \subseteq \mathbb{I}$ *is Borel then* $\mu(F^{-1}[X]) = \lambda(X)$*, while if* $A \subseteq \mathbb{A}_0$ *is Borel then* $\lambda(F[A]) \geq \mu(A)$*.*

Proof. Consider any interval $X = [0, m)$ in $\mathbb{I}$; $0 \leq m \leq 1$. By definition, $\hat{f}(x) \in X$ iff $\mu(f^{-1}[\mathbb{Y}_0 \cap [0, x)]) < m$. Therefore the $\hat{f}$-preimage $\hat{f}^{-1}[X]$ is equal to $Z = [0, M)$, where M is the largest real in $\mathbb{I}$ satisfying the inequality $\mu(f^{-1}[\mathbb{Y}_0 \cap [0, M)]) \leq m$. Then clearly $\mu(f^{-1}[\mathbb{Y}_0 \cap Z]) = m$.

But $f^{-1}[\mathbb{Y}_0 \cap Z] = f^{-1}[\hat{f}^{-1}[X]] = F^{-1}[X]$. We conclude that $\mu(F^{-1}[X]) = \lambda(X) = m$ for any $X = [0, m)$, as above. By induction, this implies $\mu(F^{-1}[X]) = \lambda(X)$ for any Borel $X \subseteq \mathbb{I}$, the first claim. The second claim follows, since $A \subseteq F^{-1}[F[A]]$, and any analytic set has a Borel subset of the same measure. □

Corollary 4.4 (under Case 2)**.** *The real* $\boldsymbol{y}_0 = F(\boldsymbol{a}_0) = \hat{f}(\boldsymbol{x}_0) \in \mathbb{I}$ *is* λ*-random over* $\mathbf{V}$*. Therefore* $\mathbf{V}[\boldsymbol{x}_0] = \mathbf{V}[\boldsymbol{y}_0]$ *is a Solovay-random extension of* $\mathbf{V}$*.*

Proof. To prove the second claim, note that $\hat{f}$ is "almost" $1-1$ on $\mathbb{Y}_0$ by Lemma 4.2, and hence $\mathbf{V}[\boldsymbol{x}_0] = \mathbf{V}[\boldsymbol{y}_0]$. □

We have another split in cases. In $\mathbf{V}$, let $\mathscr{B}$ be the family of all Borel sets $B \subseteq \mathbb{A}_0$ such that $\mu(B) > 0$ and F is $1-1$ on B. The set $\mathscr{B}$ can be empty or not, but anyway there is a Borel set B_0, equal to a union of $\leq \aleph_0$ sets in $\mathscr{B}$, such that $\mu(B' \smallsetminus B_0) = 0$ for any $B' \in \mathscr{B}$. (If $\mathscr{B} = \varnothing$ then $B_0 = \varnothing$ either.) We let $\mathbb{A}_1 = \mathbb{A}_0 \smallsetminus B_0$ and $\mathbb{Y}_1 = F[B]$. Thus $\mathbb{A}_1$ is Borel, $\mathbb{Y}_1 \subseteq \mathbb{Y}_0$ analytic, and

($\ddagger$) if $B \subseteq \mathbb{A}_1$ is Borel and $\mu(B) > 0$ then F is **not** 1-1 on B.

Subcase 2a of Case 2: $\boldsymbol{a}_0 \in \mathbb{A}_0 \smallsetminus \mathbb{A}_1$. By construction there is a Borel set $B \subseteq \mathbb{A}_0$ such that $\boldsymbol{a}_0 \in B$, $\mu(B) > 0$, and F is $1-1$ on B. Then $\boldsymbol{a}_0 \in \Delta^1_1(p, p', \boldsymbol{y}_0)$ for some $p, p' \in \mathbf{V}$ (codes for F, B), hence $\boldsymbol{a}_0 \in \mathbf{V}[\boldsymbol{y}_0] = \mathbf{V}[\boldsymbol{x}_0]$, thus (R2) holds.

Subcase 2b of Case 2: not Subcase 2a. This is the **key subcase**, and it will be considered in the two following sections.

5 The key subcase, measure construction

Here we prove that $\mathbf{V}[\boldsymbol{a}_0]$ is a random extension of $\mathbf{V}[\boldsymbol{x}_0]$. First of all, we define, in $\mathbf{V}[\boldsymbol{x}_0]$, a measure on the set $\Omega = F^{-1}[\boldsymbol{x}_0]$, with respect to which $\boldsymbol{a}_0$ itself will be random. We'll make use of the following lemma which combines effects of forcing and the Shoenfield absoluteness theorem.

Lemma 5.1. *Let* $\varphi(x)$ *be a combination of* Σ^1_1*-formulas and* Π^1_1*-formulas, by means of* $\wedge$*,* $\vee$*,* $\neg$*, and quantifiers over* ω*, and with reals in* $\mathbf{V}$ *as parameters. If* $\varphi(\boldsymbol{y}_0)$ *is true then there is a closed set* $Y \subseteq \mathbb{I}$ *of positive measure* $\lambda(Y) > 0$*, coded in* $\mathbf{V}$*, containing* $\boldsymbol{y}_0$*, and satisfying* $\varphi(y)$ *for all* $y \in Y$*.*

Proof. The set $\{y : \varphi(y)\}$ is measurable, hence, it is true in $\mathbf{V}$ that any Borel $Y_0 \subseteq \mathbb{I}$ of positive measure contains a perfect subset $Y \subseteq Y_0$ still of positive measure $\lambda(Y) > 0$, satisfying either (1) $\forall\, y \in Y \varphi(y)$ or (2) $\forall\, y \in Y \neg\, \varphi(y)$. These formulas are Π^1_2, hence absolute by Shoenfield. It follows by the randomness of $\boldsymbol{x}_0$ that there is a perfect subset $Y \subseteq \mathbb{I}$ of positive measure, containing $\boldsymbol{y}_0$ and satisfying (1) or (2). But (2) is impossible because of $\varphi(\boldsymbol{y}_0)$. □

Now suppose that $B \subseteq \mathbb{A}_1$ is a Borel set.

If $X \subseteq \mathbb{I}$ then let $B \Uparrow X = B \cap F^{-1}[X] = \{a \in B : F(a) \in X\}$.

In particular, if $x \in \mathbb{I}$ then $B \Uparrow x = \{a \in B : F(a) = x\}$, a cross-section.

Note that $\mu(B) \le \lambda(F[B])$ by Lemma 4.3, and if $X \subseteq F[B]$ is Borel then $\mu(B \Uparrow X) \le \lambda(X)$. If $X \subseteq \mathbb{I}$ is Borel then put $\lambda_B(X) = \mu(B \Uparrow X)$; λ_B is a σ-additive Borel measure on $\mathbb{I}$, concentrated on $F[B]$ and satisfying $\lambda_B(X) \le \lambda(X)$.

If $x \in \mathbb{I}$ then let $U_B(x) = \lambda_B([0,x)) = \mu(B \Uparrow [0,x))$. It is important that $U_B : \mathbb{I} \to \mathbb{I}$ is non-decreasing ($x < y \Longrightarrow U_B(x) \le U_B(y)$). We'll make use of the following collection of classical results related to monotone real functions.

Proposition 5.2 (see *e.g.* [15], Chapters I and II)**.** (i) *If $B \subseteq \mathbb{A}_1$ is a Borel set then a derivative $U'_B(x) < \infty$ exists for λ-almost all $x \in \mathbb{I}$;*

(ii) *If $B \subseteq \mathbb{A}_1$ is a Borel set and $U'_B(x) = 0$ for λ-almost all $x \in \mathbb{I}$, then $U'_B(x) = 0$ for all $x \in \mathbb{I}$;*

(iii) *if $B_0, B_1, \ldots \subseteq \mathbb{A}_1$ are 2wise-disjoint Borel, and $B = \bigcup_n B_n$, then $U_B(x) = \sum_n U_{B_n}(x)$, $\forall\, x$, and $U'_B(x) = \sum_n U'_{B_n}(x)$ for λ-almost all $x \in \mathbb{I}$.* □

Lemma 5.3. *If $C \subseteq \mathbb{A}_1$ and $X \subseteq F[C]$ are Borel sets, $\lambda(C) > 0$, and $B = C \Uparrow X$, then $U'_C(x) = U'_B(x)$ for λ-almost all $x \in X$.*

Proof. Let $A = C \smallsetminus B$, so that X and $Y = F[A]$ are disjoint sets satisfying $X \cup Y = F[C]$. Accordingly we have $U_C(x) = U_B(x) + U_A(x)$ for all $x \in \mathbb{I}$, therefore $U'_C(x) = U'_B(x) + U'_A(x)$ for λ-almost all x (those in which all three derivatives are defined). However we have $U'_A(x) = 0$ for λ-almost all $x \in X$; in fact, the equality holds for all points $x \in X$ of density 1. As required. □

Definition 5.4. Let $\Omega = f^{-1}[\boldsymbol{x}_0] = F^{-1}[\boldsymbol{y}_0]$ (a closed set, containing $\boldsymbol{a}_0$). □

Lemma 5.5. *If $P \subseteq \Omega$ is a Borel set coded in $\mathbf{V}[\boldsymbol{y}_0]$ then there is a Borel set $B \subseteq \mathbb{I}$, coded in $\mathbf{V}$ and such that $P = B \Uparrow \boldsymbol{y}_0$.*

Proof. There is a Borel set $W \subseteq \mathbb{I} \times \mathbb{A}_1$, coded in $\mathbf{V}$, such that $P = W_{\boldsymbol{y}_0} = \{a : \langle \boldsymbol{y}_0, a\rangle \in W\}$ (a cross-section). Thus $W_{\boldsymbol{y}_0} \subseteq \Omega = F^{-1}[\boldsymbol{y}_0]$. By Lemma 5.1, there is a Borel set $X \subseteq \mathbb{I}$ of positive measure $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $W_y \subseteq F^{-1}[y]$ holds for all $y \in X$. Then $B = \{a : F(a) \in X \wedge \langle F(a), a\rangle \in W\}$ is a Borel set coded in $\mathbf{V}$. Moreover $W_y = B \Uparrow y$ for all $y \in X$ by construction, in particular, $P = B \Uparrow \boldsymbol{y}_0$. □ (Lemma)

Definition 5.6. If $P \subseteq \Omega$ is a Borel set coded in $\mathbf{V}[\boldsymbol{y}_0]$ then let $\nu(P) = U'_B(\boldsymbol{y}_0)$, for any B as in the lemma. It follows from Proposition 5.2(i) that $U'_B(\boldsymbol{y}_0)$ is defined, because $\boldsymbol{y}_0$ is random over $\mathbf{V}$ by the above. □

Lemma 5.7. *$\nu(P)$ is independent of the choice of B.*

Proof. Suppose that $C \subseteq \mathbb{A}_1$ is another Borel set satisfying $P = C \upuparrows \boldsymbol{y}_0$. By Lemma 5.1, there is a Borel set $X \subseteq \mathbb{I}$ of positive measure $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $C \upuparrows y = B \upuparrows y$ holds for all $y \in X$. Then $U'_B(y) = U'_C(y)$ for λ-almost all $y \in X$ by Lemma 5.3. Therefore $U'_B(\boldsymbol{y}_0) = U'_C(\boldsymbol{y}_0)$, as $\boldsymbol{y}_0 \in X$ is random. □

Thus ν is a well-defined measure on Borel sets $P \subseteq \Omega$ in $\mathbf{V}[\boldsymbol{y}_0]$.

6 The key subcase, proof of randomness

To finalize the proof of Theorem 1.2 in Case 2b, we are going to show that $\boldsymbol{a}_0$ is ν-random over $\mathbf{V}[\boldsymbol{y}_0]$. Then it suffices to apply Lemma 3.1, to transform $\boldsymbol{a}_0$ to a "standard" λ-random real in $\mathbb{I}$. We first of all show that ν is a "good" measure.

Lemma 6.1. *In $\mathbf{V}[\boldsymbol{y}_0]$, ν is a σ-additive continuous probability measure on Ω.*

Proof. (A) To prove $\nu(\Omega) = 1$ take $B = \mathbb{A}_1$. Then $B \upuparrows \boldsymbol{y}_0 = F^{-1}[\boldsymbol{y}_0] = \Omega$. Lemma 4.3 implies

$$U_B(x) = \lambda_B([0, x)) = \mu(B \upuparrows [0, x)) = \mu(F^{-1}[[0, x)]) = \lambda([0, x)) = x\,,$$

and hence $U'_B(x) = 1$ for all x. In particular, $\nu(\Omega) = U'_B(\boldsymbol{y}_0) = 1$.

(B) Prove σ-additivity. Lemma 5.5 reduces this to the following claim: *if $\langle C_n \rangle_{n<\omega} \in \mathbf{V}$ is a sequence of Borel sets $C_n \subseteq \mathbb{A}_1$, and $(C_k \upuparrows \boldsymbol{y}_0) \cap (C_n \upuparrows \boldsymbol{y}_0) = \varnothing$ for all $k \neq n$, and $C = \bigcup_n C_n$, then $U'_C(\boldsymbol{y}_0) = \sum_n U'_{C_n}(\boldsymbol{y}_0)$.* By Lemma 5.1, there is a Borel set $X \subseteq \mathbb{I}$ with $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $(C_k \upuparrows y) \cap (C_n \upuparrows y) = \varnothing$ for all $y \in X$, $k \neq n$. The Borel sets $B_n = C_n \upuparrows X \subseteq \mathbb{A}_1$ are pairwise disjoint, and the set $B = C \upuparrows X$ satisfies $B = \bigcup_n B_n$.

Moreover, we have $U_B(x) = \sum_n U_{B_n}(x)$ for all x, and $U'_B(x) = \sum_n U'_{B_n}(x)$ for λ-almost all $x \in \mathbb{I}$ by Proposition 5.2(iii). Finally, Lemma 5.3 implies that $U'_B(x) = U'_C(x)$ and $U'_{B_n}(x) = U'_{C_n}(x)$ for all n and λ-almost all $x \in X$. It follows that $U'_C(x) = \sum_n U'_{C_n}(x)$ for λ-almost all $x \in X$, hence, $U'_C(\boldsymbol{y}_0) = \sum_n U'_{C_n}(\boldsymbol{y}_0)$ by the randomness, as required.

(C) To prove that ν is continuous, suppose to the contrary that $z_0 \in \Omega$ and $\nu(\{z_0\}) > 0$. By definition there is a Borel set $C \subseteq \mathbb{A}_1$, coded in $\mathbf{V}$ and satisfying $C \upuparrows \boldsymbol{y}_0 = \{z_0\}$ and $U'_C(\boldsymbol{y}_0) > 0$. By Lemma 5.1, there is a Borel set $X \subseteq \mathbb{I}$ with $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $C \upuparrows y$ is a singleton and $U'_C(y) > 0$ for all $y \in X$. Let $B = C \upuparrows X$. Then $B \upuparrows \boldsymbol{y}_0 = \{z_0\}$, $B \upuparrows y$ is a singleton for all $y \in X$, and $U'_B(y) > 0$ for λ-almost all $y \in X$, by Lemma 5.3.

It follows that $U_B(1) > 0$, hence $\mu(B) = U_B(1) > 0$. Moreover, by the singleton condition, the preimage $F^{-1}[y] \cap B = B \upharpoonright\upharpoonright y$ is a singleton for all $y \in F[B] \subseteq X$. But this contradicts the Case 2b assumption. $\square$

Lemma 6.2. *$\boldsymbol{a}_0$ is ν-random over $\mathbf{V}[\boldsymbol{y}_0]$.*

Proof. Assume that $P \subseteq \Omega$ is a Borel set, coded in $\mathbf{V}[\boldsymbol{y}_0]$, and $\nu(P) = 0$; we have to prove that $\boldsymbol{a}_0 \notin P$. By definition there is a Borel set $C \subseteq \mathbb{A}_1$, coded in $\mathbf{V}$ and satisfying $P = C \upharpoonright\upharpoonright \boldsymbol{y}_0$ and $U'_C(\boldsymbol{y}_0) = 0$. By Lemma 5.1, there is a closed (here, this is more suitable than Borel) set $X \subseteq \mathbb{I}$ of positive measure $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $U'_C(y) = 0$ for all $y \in X$.

Let $B = C \upharpoonright\upharpoonright X$. Then $P = B \upharpoonright\upharpoonright \boldsymbol{y}_0$, and $U'_B(y) = 0$ for λ-almost all $y \in X$ by Lemma 5.3. Note that $F[B] \subseteq X$, thus $U_B(x)$ is a constant inside any open interval disjoint with X. Thus $U'_B(y) = 0$ for all $y \in \mathbb{I} \smallsetminus X$, hence overall $U'_B(y) = 0$ for λ-almost all $y \in \mathbb{I}$. This implies $U_B(x) = 0$ for all $x \in \mathbb{I}$ by Proposition 5.2(ii). Therefore $\lambda_B(\mathbb{I}) = \mu(B) = 0$ by construction. We conclude that $\boldsymbol{a}_0 \notin B$, by the μ-randomness of $\boldsymbol{a}_0$. Then $\boldsymbol{a}_0 \notin P = B \upharpoonright\upharpoonright \boldsymbol{y}_0$, as required. $\square$

$\square$ (Theorem 1.2)

Corollary 6.3. *If x, y are reals in an $\aleph_1$-random extension $N = \mathbf{L}[\langle a_\xi \rangle_{\xi<\omega_1}]$ of $\mathbf{L}$, then y belongs to a random extension of $\mathbf{L}[x]$ inside N.*

Proof. We have $x \in N_\alpha = \mathbf{L}[\langle a_\xi \rangle_{\xi<\alpha}]$ and $y \in N_\beta$, for some $\alpha < \beta < \omega_1$. The model N_α is equal to a simple extension of $\mathbf{L}$ by one random real. Thus, by Theorem 1.2, either $N_\alpha = \mathbf{L}[x]$ or N_α is a random extension of $\mathbf{L}[x]$. In addition, N_β is a random extension of N_α. This implies the result required. $\square$

7 Proof of the localization lemma

Proof (Lemma 2.3)**.** Let $\mathbb{1}$ be the weakest element of any forcing considered, and $\dot{x} = \{\mathbf{1}\} \times x$ be the canonical name for any set x in the ground set universe $\mathbf{V}$. Let $\mathbf{R}$ be the random forcing and $\Vdash_{\mathbf{R}}$ be the associated forcing relation.

Claim 7.1. *If $n \geq 2$ and $\varphi(\cdot)$ is a parameter-free Σ^1_n-formula, resp., Π^1_n-formula, then the set $F_\varphi = \{x : \mathbb{1} \Vdash_{\mathbf{R}} \varphi(\dot{x})\}$ is Σ^1_n, resp., Π^1_n.*

Proof. We make use of a standard Borel coding system for subsets of 2^ω. It consists of Π^1_1 sets $\mathbf{C} \subseteq 2^\omega$ and $W_+, W_- \subseteq \omega^\omega \times \omega^\omega$, and an assignment $c \mapsto \mathbf{B}_c \subseteq 2^\omega$, such that (1) $\{\mathbf{B}_c : c \in \mathbf{C}\}$ is exactly the family of all Borel sets $X \subseteq 2^\omega$, and (2) if $c \in \mathbf{C}$ and $x \in 2^\omega$ then $x \in \mathbf{B}_c$ iff $W_+(c, x)$ iff $\neg W_-(c, x)$.

To define an associated coding system for Borel maps, let $e \mapsto \langle (e)_n \rangle_{n<\omega}$ be a recursive homeomorphism $2^\omega \xrightarrow{\text{onto}} (2^\omega)^\omega$. Let $\mathbf{CF} = \{e \in 2^\omega : \forall n((e)_n \in \mathbf{C}\}$ — codes of Borel maps $f : 2^\omega \to 2^\omega$. If $e \in \mathbf{CF}$ then define a Borel map $\mathbf{F}_e : 2^\omega \to 2^\omega$ so that $\mathbf{F}_e(x)(n) = 1$ iff $x \in \mathbf{B}_{(e)_n}$, for all $x \in 2^\omega$, $n < \omega$.

If $\varphi(v_1,\dots,v_k)$ is any formula, $e_1,\dots,e_k \in \mathbf{CF}$, and $x \in \omega^\omega$, then let $\varphi(e_1,\dots,e_k)[x]$ be the formula $\varphi(\mathbf{F}_{e_1}(x),\dots,\mathbf{F}_{e_k}(x))$, and let

$$\mathbf{Forc}_\varphi = \{\langle c, e_1,\dots,e_k\rangle \in \mathbf{C} \times \mathbf{CF}^k : \mu(\mathbf{B}_c) > 0 \wedge \mathbf{B}_c \Vdash_{\mathbf{R}} \varphi(e_1,\dots,e_k)[\mathbf{a}]\},$$

where $\mathbf{a}$ is a canonical name for the random real. We assert the following.

($*$) If φ is a Π^1_1 formula then $\mathbf{Forc}_\varphi \in \Sigma^1_2$. If φ is a Σ^1_n formula, $n \ge 2$, then $\mathbf{Forc}_\varphi \in \Sigma^1_n$. If φ is a Π^1_n formula, $n \ge 2$, then $\mathbf{Forc}_\varphi \in \Pi^1_n$.

This is proved by induction. If $\varphi(v)$ is Π^1_1 then $\langle c, e\rangle \in \mathbf{Forc}_\varphi$ iff the set $X = \{x \in B_c : \neg\,\varphi(\mathbf{F}_e(x))\}$ is null, which roughly estimated to be Σ^1_2 by coverings with $\mathbf{G}_\delta$ sets. To pass $\Pi^1_n \to \Sigma^1_{n+1}$, assume that $\varphi(v_1) := \exists\, v_2\, \psi(v_1, v_2)$, ψ is Π^1_n. Then $\langle c, e_1\rangle \in \mathbf{Forc}_\varphi$ iff $\exists\, e_2 \in \mathbf{CF}\,(\langle c, e_1, e_2\rangle \in \mathbf{Forc}_\psi)$. (We make use of the fact that the random forcing admits Borel reading of names.) Thus if $\mathbf{Forc}_\psi$ is Σ^1_{n+1} then so is $\mathbf{Forc}_\varphi$. To pass $\Sigma^1_n \to \Pi^1_n$, let $\varphi(v)$ be Σ^1_n. Then

$$\langle c, e\rangle \in \mathbf{Forc}_{\neg\,\varphi} \iff \forall\, c' \in \mathbf{C}\,(\mathbf{B}_{c'} \subseteq \mathbf{B}_c \wedge \mu(\mathbf{B}_{c'}) > 0 \implies \langle c', e\rangle \notin \mathbf{Forc}_\psi).$$

Thus if $\mathbf{Forc}_\varphi$ is Σ^1_n then $\mathbf{Forc}_{\neg\,\varphi}$ is Π^1_n. This ends the proof of ($*$).

Now to prove the claim note that $x \in F_\varphi$ iff $\langle c_0, e_x\rangle \in \mathbf{Forc}_\varphi$, where $c_0 \in \mathbf{C}$ satisfies $\mathbf{B}_{c_0} = 2^\omega$, while $e_x \in \mathbf{CF}$ is such that $\mathbf{F}_{e_x}$ is the constant map $\mathbf{F}_{e_x}(a) = x$, $\forall\, a \in 2^\omega$. □ (Claim)

To finalize the proof of Lemma 2.3, we define formulas $\varphi^*(x)$ by induction. If φ is Σ^1_2 or Π^1_2 then $\varphi^* := \varphi$ works by Shoenfield. Suppose that $n \ge 2$, $\psi(x,y)$ is Π^1_n, and a Π^1_n-formula ψ^* is defined, satisfying $\psi(x,y) \iff \mathbf{L}[x,y] \models \psi^*(x,y)$ in $N = \mathbf{L}[\langle a_\xi\rangle_{\xi<\omega_1}]$ (a given $\aleph_1$-random extension). We define $\varphi^*(x)$ to be the formula $\mathbb{1} \Vdash_{\mathbf{R}} \exists\, y\,(\mathbf{L}[\dot{x}, y] \models \psi^*(\dot{x}, y))$. This is a Σ^1_{n+1}-formula by Claim 7.1, so it remains to show that $\varphi(x) \iff \mathbf{L}[x] \models \varphi^*(x)$ in N.

Assume that x is a real in N satisfying $\varphi(x)$. Thus there is a real $y \in N$ satisfying $\psi(x,y)$, or equivalently, $\mathbf{L}[x,y] \models \psi^*(x,y)$. By Corollary 6.3, y belongs to a random extension of $\mathbf{L}[x]$ inside N. Therefore, as the random forcing is homogeneous, it is true in $\mathbf{L}[x]$ that $\mathbb{1} \Vdash_{\mathbf{R}} \exists\, y\,(\mathbf{L}[\dot{x}, y] \models \psi^*(\dot{x}, y))$. In other words, $\mathbf{L}[x] \models \varphi^*(x)$.

To prove the converse, assume that $\mathbf{L}[x] \models \big(\mathbb{1} \Vdash_{\mathbf{R}} \exists\, y\,(\mathbf{L}[\dot{x}, y] \models \psi^*(\dot{x}, y))\big)$. Consider any real $z \in N$ random over $\mathbf{L}[x]$. Then $\exists\, y\,(\mathbf{L}[x,y] \models \psi^*(x,y))$ holds in $\mathbf{L}[x,z]$, so there is a real $y \in \mathbf{L}[x,z]$ satisfying $\mathbf{L}[x,y] \models \psi^*(x,y)$. Then $N \models \psi(x,y)$ by the choice of ψ^*, hence finally $N \models \varphi(x)$.

□ (Lemma 2.3 and Theorem 2.2)

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