Notes on the equiconsistency of ZFC without the Power Set axiom and 2nd order PA *

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Abstract

We demonstrate that theories $\mathbf{Z}^-$, $\mathbf{ZF}^-$, $\mathbf{ZFC}^-$ (minus means the absence of the Power Set axiom) and $\mathbf{PA}_2$, $\mathbf{PA}_2^-$ (minus means the absence of the Countable Choice schema) are equiconsistent to each other. The methods used include the interpretation of a power-less set theory in $\mathbf{PA}_2^-$ via well-founded extensional digraphs, as well as the Gödel constructibility in the said power-less set theory.

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1 Introduction

The following theorem is **the main result** of this paper.

Theorem 1.1. *Theories* $\mathbf{PA}_2^-$, $\mathbf{PA}_2$, $\mathbf{Z}^-$, $\mathbf{ZFC}^-$, $\mathbf{ZF}^-$ *are equiconsistent.*

Here $\mathbf{PA}_2$, resp., $\mathbf{PA}_2^-$ is 2nd order Peano arithmetic with, resp., without the (countable) AC, whereas $\mathbf{Z}^-$ is Zermelo set theory without the well-orderability axiom WA, and $\mathbf{ZFC}^-$/ $\mathbf{ZF}^-$ are Zermelo–Fraenkel set theories with/without WA, and all three of them without the Power Set axiom. (See the exact definitions in Section 8.)

The theorem has been known since at least 1960s, see [10] or [2]. However apparently no self-contained and more or less complete proof has ever been published. (See a brief discussion in [9].) The intent of this paper is to finally present such a proof.

The proof consists in two distinct parts. As the first part, we consider a proxy theory $\mathbf{FTM}\boldsymbol{\omega}$, which extends $\mathbf{Z}^-$ by the existence of finite-subset closures (*i.e.*, $y \subseteq X$ finite $\implies y \in X$) and transitive closures (which $\mathbf{Z}^-$ apparently does not prove), the $\mathsf{Countability}$ axiom, and an axiom saying that any well-founded and extensional digraph on ω admits a transitive model. Clearly $\mathbf{FTM}\boldsymbol{\omega}$ is a subtheory of $\mathbf{ZF}^- + \mathsf{Countability}$.

Our first key result is an interpretation of $\mathbf{FTM}\boldsymbol{\omega}$ is $\mathbf{PA}_2^-$, obtained by using the specified digraphs as the domain of interpretation. This is a well-known method, presented say in [2], and Simpson's book [15].

The second part of the proof presents an interpretation of $\mathbf{ZFC}^-$ in $\mathbf{FTM}\boldsymbol{\omega}$, obtained via Gödel constructible sets. The main issue here is that $\mathbf{FTM}\boldsymbol{\omega}$, albeit pretty sufficient to develop constructibility, does not seem to be strong enough to prove it outright that the whole class $\mathbf{L}$ satisfies $\mathbf{ZFC}^-$. We'll be able only to show that either $\mathbf{L}$ satisfies $\mathbf{ZFC}^-$ or so does some $\mathbf{L}_\xi$.

2 Preliminaries: 2nd order arithmetic

We recall that second order Peano arithmetic $\mathbf{PA}_2$ is a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables, for natural numbers and for sets of them.

Notation 2.1. We use j, k, m, n for variables over ω and x, y, z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets. $\square$

The axioms of $\mathbf{PA}_2$ are the Peano axioms for numbers plus the following:

$\mathsf{Induction}$: $\forall x\, \big(0 \in x \wedge \forall n\, (n \in x \implies n+1 \in x) \implies \forall n\, (n \in x)\big)$.

$\mathsf{Extensionality}$ for sets: $\forall x, y\, \big(\forall k\, (k \in x \iff k \in y) \implies x = y\big)$.

Comprehension schema CA: $\exists\, x\, \forall\, k\, (k \in x \iff \Phi(k))$ — for every formula Φ in which x does not occur, and in Φ we allow parameters, that is, free variables other than k.

Countable Choice AC_ω: $\forall\, n\, \exists\, x\, \Phi(n, x) \implies \exists\, x\, \forall\, n\, \Phi(n, (x)_n))$ — for every formula Φ with parameters, where $(x)_n = \{j : \boldsymbol{b}(n, j) \in x\}$, and $\boldsymbol{b}(n, j) = 2^n(2j + 1) - 1$ is a standard bijection $\omega \times \omega \xrightarrow{\text{onto}} \omega$.

The theory $\mathbf{PA}_2$ is also known as A_2 (see *e.g.* an early survey [2]), as Z_2 (in [14] or elsewhere). See also [4, 10, 15].

We let $\mathbf{PA}_2^-$ (Z_2^- in some papers) be $\mathbf{PA}_2$ without AC_ω.

Digraph extension. It can be viewed as a certain disadvantage of $\mathbf{PA}_2^-$ that this theory doesn't explicitly treat such objects as *digraphs*, or directed graphs on numbers, *i.e.*, basically just subsets of $\omega \times \omega$. To fix this shortcoming, we define the *extended* language $\mathcal{L}(\mathbf{dPA}_2^-)$ by adjoining the type of variables $A, B, C, \dots$ for digraphs, and according elementary formulas of the form $j\, A\, k$, to the standard language $\mathcal{L}(\mathbf{PA}_2)$ of $\mathbf{PA}_2^-$. We consider the *digraph extension* $\mathbf{dPA}_2^-$ of $\mathbf{PA}_2^-$ by the following axiom and schemata:

Extensionality for digraphs: $\forall\, A, B\, \big(\forall\, j, k\, (j\ A\ k \iff j\ B\ k) \implies A = B\big)$.

Comprehension for digraphs: $\exists\, A\, \forall\, j, k\, \big(j\ A\ k \iff \Phi(j, k)\big)$ — for each formula Φ of $\mathcal{L}(\mathbf{dPA}_2^-)$ with parameters allowed, where A doesn't occur.

Lemma 2.2. *Theories $\mathbf{PA}_2^-$ and $\mathbf{dPA}_2^-$ are equiconsistent.*
Theories $\mathbf{PA}_2$ and $\mathbf{dPA}_2$ are equiconsistent as well.

Proof (sketch)**.** Arguing in $\mathbf{PA}_2^-$, let ω be the universe of numbers and $\mathbf{P}$ the universe of sets (subsets of ω). Let $D_x = \{\langle n, k\rangle : \boldsymbol{b}(n, k) \in x\}$ for $x \in \mathbf{P}$. Then the structure $\mathscr{D} = \langle \omega; \mathbf{P}; \mathbf{D} = \{D_x : x \in \mathbf{P}\}\rangle$ satisfies $\mathbf{dPA}_2^-$ by rather standard arguments. □

3 Wellfounded digraphs

Remark 3.1. We'll consider only digraphs on ω here, so that all arguments will be maintained by means of $\mathbf{dPA}_2^-$, the digraph extension of $\mathbf{PA}_2^-$. □

Let A be a digraph (on ω). We'll use the following notation, partially taken from [7, Chap. 6].

- $|A| = \mathtt{dom}\, A \cup \mathtt{ran}\, A$.
- If $u \in |A|$ then $\mathbf{Elm}_A(u) = \{j : j\, A\, u\}$, the *extension* of u by [7, p. 67], or just the set of all A-*elements* of u.

- A set $D \subseteq |A|$ is A-*transitive*, iff $k \in D \wedge j \mathbin{A} k \Longrightarrow j \in D$.
- If $u \in |A|$ then $\mathbf{Con}_A\{u\}$ consists of all elements $j \in |A|$ such that there is a finite A-chain $j = j_0 \mathbin{A} j_1 \mathbin{A} \dots \mathbin{A} j_{K-1} \mathbin{A} j_K = u$ in $|A|$ (here $K \geq 0$, and we have $j = u \in \mathbf{Con}_A\{u\}$ in case $K = 0$). This is the A-*lower cone* of u in $|A|$, including u itself.
- $v \in |A|$ is a *vertex* of A, in symbol $v = \widehat{A}$, if $|A| = \mathbf{Con}_A\{v\}$.
- $\mathbf{Elm}(A) = \mathbf{Elm}_A(\widehat{A}) \subseteq |A|$ in case $A \neq \varnothing$, but $\mathbf{Elm}(\varnothing) = \varnothing$.
- If $X \subseteq |A|$ then $A \restriction X = A \cap (X \times X)$ is the restriction of A to X.
- If $u \in |A|$ then $A \Downarrow^{\{u\}} = A \restriction \mathbf{Con}_A\{u\}$ is the *cone restriction* of A to the A-transitive set $\mathbf{Con}_A\{u\}$.

A digraph $A \subseteq \omega \times \omega$ is:

– *well-founded*, in symbol $A \in \mathrm{WF}$, if any non-empty set $X \subseteq |A|$ has an A-*minimal* element $a \in X$, that is, $\mathbf{Elm}_A(k) \cap X = \varnothing$;

– *extensional*, if $\mathbf{Elm}_A(u) = \mathbf{Elm}_A(v) \Longrightarrow u = v$ holds for all $u, v \in |A|$;

– *vertex*, if *either* $A = \varnothing$ *or* there is a unique vertex $v = \widehat{A} \in |A|$;

– $A \in \mathrm{WFEV}$, if A is well-founded, extensional and vertex;

– $A \in \mathrm{WFE}$, if A is well-founded and extensional.

Remark 3.2. (a) If $A \in \mathrm{WF}$ then A-cycles do not exist. Indeed if $u_0 \mathbin{A} u_1 \mathbin{A} u_2 \mathbin{A} \dots \mathbin{A} u_n \mathbin{A} u_0$ is a cycle then the set $U = \{u_0, u_1, u_2, \dots, u_n\}$ has no minimal element, contrary to WF. In particular cycles $u \mathbin{A} u$ are eliminated.

(b) Any $A \in \mathrm{WF}$ admits at most one vertex $\widehat{A}$: if $u \neq v$ are vertices then $u \in \mathbf{Con}_A\{v\}$ and $v \in \mathbf{Con}_A\{u\}$, leading to a cycle, contrary to (a).

(c) If $\varnothing \neq A \in \mathrm{WFE}$ then by WF there is an A-*minimal* element $m = \min A \in |A|$ with $\mathbf{Elm}_A(m) = \varnothing$, unique by the extensionality of A.

(d) If $\varnothing \neq A \in \mathrm{WFE}$ then $|A|$ contains at least two distinct elements, since $u \mathbin{A} u$ is impossible by (a), in particular, easily $\widehat{A} \neq \min A$. □

Lemma 3.3 (routine)**.** *Let* $A \in \mathrm{WFE}, c \in |A|$. *Then* $B = A \Downarrow^{\{c\}} \in \mathrm{WFEV}$, *and either* $c = \min A$ *— then* $B = \varnothing$, *or* $c \neq \min A$ *— then* $c = \widehat{B}$. □

Lemma 3.4 (WF induction)**.** *Let* $A \in \mathrm{WF}$, H *be any set, and the implication* $\mathbf{Elm}_A(u) \subseteq H \Longrightarrow u \in H$ *hold for every* $u \in |A|$. *Then* $|A| \subseteq H$.

Proof. Otherwise the set $Y = |A| \smallsetminus H \neq \varnothing$ contains a minimal element $u \in Y$, so that $\mathbf{Elm}_A(u) \subseteq H$. Then $u \in H$, which is a contradiction. □

4 Isomorphism and rigidity of digraphs

Let A, B be two digraphs. We write $A \cong B$ iff there is an A, B-isomorphism, *i.e.*, a bijection $H : |A| \xrightarrow{\text{onto}} |B|$, satisfying $u \, A \, v \iff H(u) \, B \, H(v)$.

In the sequel, we'll use the notation introduced above in Section 3.

Lemma 4.1 ($\mathbf{dPA}_2^-$, rigidity)**.** *Let* $u, v \in |A|$, $A \in \mathrm{WFE}$. *Assume that either* (i) $\mathbf{Elm}_A(u) = \mathbf{Elm}_A(v)$, *or* (ii) $A{\Downarrow}^{\{u\}} \cong A{\Downarrow}^{\{v\}}$. *Then* $u = v$.

Proof. Case (i) holds by the extensionality of A. To prove case (ii) let $H : \mathbf{Con}_A\{u\} \xrightarrow{\text{onto}} \mathbf{Con}_A\{v\}$ be an isomorphism. We claim that $H(k) = k$ for all $k \in D = \mathbf{Con}_A\{u\}$ — then $v = H(u) = u$, as required. In virtue of Lemma 3.4, we have to show $H(k) = k$ provided (*) $H(j) = j$ for all $j \in \mathbf{Elm}_A(k)$. However (*) implies $\mathbf{Elm}_A(k) = \mathbf{Elm}_A(m)$, where $m = H(k)$. Therefore $m = k$ by the extensionality of A, as required. □

Definition 4.2. Given two digraphs A, B, define a map H_{AB} as follows. If $u \in |A|$, $v \in |B|$, and $A{\Downarrow}^{\{u\}} \cong B{\Downarrow}^{\{v\}}$ then put $H_{AB}(u) = v$. (This is well-defined since the uniqueness of v holds by Lemma 4.1.) □

Corollary 4.3 ($\mathbf{dPA}_2^-$, of Lemma 4.1)**.** *Assume that* $A, B \in \mathrm{WFE}$ *are non-empty. Then* $X = \mathtt{dom}\, H_{AB} \subseteq |A|$ *is* A*-transitive,* $Y = \mathtt{ran}\, H_{AB} \subseteq |B|$ *is* B*-transitive, and* H_{AB} *is a bijection* $X \xrightarrow{\text{onto}} Y$ *which witnesses* $A \restriction X \cong B \restriction Y$, *so that* $u \, A \, v \iff H_{AB}(u) \, B \, H_{AB}(v)$, *for all* $u, v \in X$. □

Lemma 4.4 ($\mathbf{dPA}_2^-$)**.** *Let* $A, B \in \mathrm{WFEV}$. *We have* $A \cong B$ *provided*

$$\left.\begin{array}{l} \forall\, k \in \mathbf{Elm}(A)\, \exists\, j \in \mathbf{Elm}(B)\, (A{\Downarrow}^{\{k\}} \cong B{\Downarrow}^{\{j\}}), \quad \textit{and} \\ \forall\, j \in \mathbf{Elm}(B)\, \exists\, k \in \mathbf{Elm}(A)\, (A{\Downarrow}^{\{k\}} \cong B{\Downarrow}^{\{j\}}). \end{array}\right\} \tag{1}$$

Proof. By (1), the sets $X = \mathtt{dom}\, H_{AB} \subseteq |A|$, $Y = \mathtt{ran}\, H_{AB} \subseteq |B|$ either satisfy $X = |A|$, $Y = |B|$ — and then $A \cong B$, or $X = |A| \smallsetminus \{u\}$ and $Y = |B| \smallsetminus \{v\}$, where $u = \widehat{A}$, $v = \widehat{B}$. In the 'or' case, H_{AB} witnesses $A \restriction X \cong B \restriction Y$. Let H' be the extension of H_{AB} by $H'(u) = v$. Then H' witnesses that $A{\Downarrow}^{\{u\}} \cong B{\Downarrow}^{\{v\}}$, hence still $u \in X$, which is a contradiction. □

5 Set theoretic structure of digraphs in 2nd-order PA

Arguing in $\mathbf{dPA}_2^-$, we consider the collection WFEV of all well-founded, extentional, vertex digraphs $A \subseteq \omega \times \omega$. Note that the isomorphism relation $\cong$ between digraphs in WFEV (see Section 4) is naturally formalized in $\mathbf{dPA}_2^-$ because partial maps $\omega \to \omega$ are just digraphs of special kind.

Definition 5.1 ($\mathbf{dPA}_2^-$)**.** If $A, B \in \mathrm{WFEV}$ then define $A \mathrel{\widetilde{\in}} B$ iff $A \cong B{\Downarrow}^{\{k\}}$ for some $k \in \mathbf{Elm}(B)$. The set theoretic structure $\mathbb{S} = \langle \mathrm{WFEV}\,; \cong, \widetilde{\in} \rangle$ is considered in $\mathbf{dPA}_2^-$. The truth of an $\in$-formula Φ (with parameters in WFEV) in $\mathbb{S}$, in symbol $\mathbb{S} \models \ulcorner\Phi\urcorner$, is understood in the sense of interpreting $=, \in$ as resp. $\cong, \widetilde{\in}$, and relativizing the quantifiers to WFEV. □

Note that $A \mathrel{\widetilde{\in}} \varnothing$ fails for each A, whereas $\varnothing \mathrel{\widetilde{\in}} B$ iff $(\min B) \in \mathbf{Elm}(B)$.

Lemma 5.2 ($\mathbf{dPA}_2^-$)**.** *$\mathbb{S}$ is a well-defined structure: $\cong$ is an equivalence relation on* WFEV*, $\widetilde{\in}$ is a binary relation on* WFEV *invariant w.r.t. $\cong$.*

Proof. To prove the invariance, let $A, B, C \in \mathrm{WFEV}$ and $A \cong B \mathrel{\widetilde{\in}} C$. Let bijections $g : |A| \xrightarrow{\text{onto}} |B|$, $h : |B| \xrightarrow{\text{onto}} C{\Downarrow}^{\{k\}}$ witness resp. $A \cong B$, $B \mathrel{\widetilde{\in}} C$, where $k \in \mathbf{Elm}(C)$. The superposition of g and h witnesses $A \mathrel{\widetilde{\in}} C$. □

Lemma 5.3 ($\mathbf{dPA}_2^-$)**.** *Let $A, B \in \mathrm{WFEV}$ and $u, v \in |B|$. Then:*

(i) $B{\Downarrow}^{\{u\}} \cong B{\Downarrow}^{\{v\}}$ *iff* $u = v$ — by Lemma 4.1;

(ii) $B{\Downarrow}^{\{u\}} \mathrel{\widetilde{\in}} B{\Downarrow}^{\{v\}}$ *iff* $u \mathbin{B} v$ *iff* $u \in \mathbf{Elm}_B(v)$;

(iii) $A \mathrel{\widetilde{\in}} B{\Downarrow}^{\{v\}}$ *iff* $A \cong B{\Downarrow}^{\{u\}}$ *for some* $u \in \mathbf{Elm}_B(v)$;

(iv) $\mathbb{S} \models \ulcorner A \subseteq B \urcorner$ *iff* $\forall\, j \in \mathbf{Elm}(A)\, \exists\, k \in \mathbf{Elm}(B)\, (A{\Downarrow}^{\{j\}} \cong B{\Downarrow}^{\{k\}})$;

(v) $\mathbb{S} \models \ulcorner B$ *is transitive*$\urcorner$ (see Remark 8.2) *iff* $\mathbf{Elm}(B) = |B| \smallsetminus \{v\}$.

Proof. (ii) If $B{\Downarrow}^{\{u\}} \mathrel{\widetilde{\in}} B{\Downarrow}^{\{v\}}$ then $B{\Downarrow}^{\{u\}} \cong B{\Downarrow}^{\{w\}}$ for some $w \in \mathbf{Elm}_B(v)$ by Definition 5.1. We conclude by (i) that $w = u$, as required.

(iii), (iv), (v) Pretty similar arguments are left to the reader. □

Definition 5.4. Let $B \in \mathrm{WFEV}$ and $j_1, \dots, j_n, p \in |B|$. If $\mathbf{Elm}_B(p) = \{j_1, \dots, j_n\}$ then we write $p = \{j_1, \dots, j_n\}_B$ (by extensionality, p is unique if exists). By Kuratowski, put $\langle j, k \rangle_B = \{\{j, j\}_B, \{j, k\}_B\}_B$, if defined. □

Lemma 5.5 ($\mathbf{dPA}_2^-$)**.** *Let $A, U, V \in \mathrm{WFEV}$, $p \in |A|$, $P = A{\Downarrow}^{\{p\}}$. Then:*

(i) *if $j, k \in |A|$ and $p = \{j, k\}_A$, $U = A{\Downarrow}^{\{j\}}$, $V = A{\Downarrow}^{\{k\}}$, then $\mathbb{S} \models \ulcorner P = \{U, V\} \urcorner$, and the same for $\langle \cdot, \cdot \rangle$ instead of $\{\cdot, \cdot\}$;*

(ii) *if $\mathbb{S} \models \ulcorner P = \{U, V\} \urcorner$, then there exist elements $j, k \in |A|$ such that $p = \{j, k\}_A$, $U \cong A{\Downarrow}^{\{j\}}$, $V \cong A{\Downarrow}^{\{k\}}$, and the same for $\langle \cdot, \cdot \rangle$.*

Proof. A routine extension of Lemma 5.3(i),(ii),(iii). □

Coding integers. Define, for any $n \in \omega$, a digraph $\underline{n} \in \mathrm{WFEV}$ corresponding to n in $\mathbb{S}$, and a digraph $\underline{\omega}$ which corresponds to ω:

$$\left.\begin{array}{rcl} \underline{n} &=& \{\langle 2j, 2k\rangle : j < k \le n\} \text{ – for all } n, \text{ in particular } \underline{0} = \varnothing = 0; \\ \underline{\omega} &=& \{\langle 2j, 2k\rangle : j < k < \omega\} \cup \{\langle 2j, 1\rangle : j < \omega\}. \end{array}\right\} \quad (2)$$

Only even numbers are involved here, hence $|\underline{\omega}|$, $|\underline{n}|$ are co-infinite.

Lemma 5.6 ($\mathbf{dPA}_2^-$, routine)**.** $\underline{\omega} \in \mathrm{WFEV}$, $\mathbf{Elm}(\underline{\omega}) = \{2n : n < \omega\}$, $\widehat{\underline{\omega}} = 1$, $\underline{\omega}\Downarrow^{\{2n\}} = \underline{n}$ *for all* n, $\mathbb{S} \models \ulcorner \underline{\omega} = \{$*all natural numbers*$\}\urcorner$.

Each digraph $\underline{n}$ *belongs to* WFEV, $k < n \iff \underline{k} \mathrel{\widetilde{\in}} \underline{n}$.

If $n < \omega$ *then* $\mathbb{S} \models \ulcorner \underline{n}$ *is the number* $1 + \ldots + 1\urcorner$ *(*n *times* 1*).* □

6 Several digraph constructions

We'll use these constructions in our study of the structure of $\mathbb{S}$ in $\mathbf{dPA}_2^-$. We begin with **bijective images**.

Lemma 6.1 ($\mathbf{dPA}_2^-$)**.** (i) *Let* $A \in \mathrm{WFEV}$. *If* $f : |A| \xrightarrow{\text{onto}} Z \subseteq \omega$ *is 1–1 then* $f[A] = \{\langle f(j), f(k)\rangle : j\, A\, k\} \in \mathrm{WFEV}$, $|f[A]| = Z$, $A \cong f[A]$ *via* f.

(ii) *If* $A \cong B$ *via* $H : |A| \xrightarrow{\text{onto}} |B|$ *then* $H[\mathbf{Elm}(A)] = \mathbf{Elm}(B)$, *and if* $x \in \mathbf{Elm}(A)$ *and* $y = H(x) \in \mathbf{Elm}(B)$ *then* $H[A\Downarrow^{\{x\}}] = B\Downarrow^{\{y\}}$.

(iii) *There is a 1–1 map* $f : |A| \to \omega$ *such that* $|f[A]|$ *is co-infinite in* ω.

Proof. To prove (iii) put $f(k) = 2k$ for all $k \in |A|$. □

Multi-restriction. The operation $A\Downarrow^{\{u\}}$ as in Section 3 can be somewhat generalized as follows. If $A \in \mathrm{WFEV}$, $v = \widehat{A}$, $\varnothing \ne X \subseteq |A|$, then put

$$A \Downarrow X = \textstyle\bigcup_{j \in X} A\Downarrow^{\{j\}} \cup (X \times \{v\}). \quad (3)$$

Corollary 6.2 ($\mathbf{dPA}_2^-$, of Lemma 5.3)**.** *Let* $A \in \mathrm{WFEV}$, $\varnothing \ne X \subseteq |A|$. *Then* $B = A \Downarrow X \in \mathrm{WFEV}$, *still* $\widehat{B} = \widehat{A}$, $|B| = \bigcup_{j \in X} \mathbf{Con}_A\{j\} \bigcup \{\widehat{B}\}$, $X = \mathbf{Elm}(B)$, *and* $A\Downarrow^{\{k\}} = B\Downarrow^{\{k\}}$ *for all* $k \in |B| \smallsetminus \{\widehat{B}\}$.

In addition, if $X \subseteq \mathbf{Elm}(A)$ *then* $\mathbb{S} \models \ulcorner B \subseteq A\urcorner$ *by Lemma 5.3(iv).* □

Finite-closure. Say that a digraph $A \in \mathrm{WFEV}$ is *finite-closed*, if for any $j_1, \ldots, j_n \in |A| \smallsetminus \{\widehat{A}\}$ there is an element $\{j_1, \ldots, j_n\}_A \in |A| \smallsetminus \{\widehat{A}\}$ (in terms of Definition 5.4). Then $\langle j, k\rangle_A = \{\{j, j\}_A, \{j, k\}_A\}_A \in |A| \smallsetminus \{\widehat{A}\}$ exists as well. The following is routine by Lemmas 5.3, 5.5

Lemma 6.3 ($\mathbf{dPA}_2^-$). *If a digraph $A \in \mathrm{WFEV}$ is finite-closed then*

$$\mathbb{S} \models \ulcorner A \text{ is a finite-closed set: } \forall x_1, \dots, x_n \in A\,(\{x_1, \dots, x_n\} \in A)\urcorner. \qquad \square$$

Lemma 6.4 ($\mathbf{dPA}_2^-$). *If $\varnothing \neq A \in \mathrm{WFEV}$, $|A| \subseteq \omega$ is co-infinite, and $v = \widehat{A}$, then there is a finite-closed digraph $B \in \mathrm{WFEV}$ satisfying:*

(i) *$A \subseteq B$ (hence $|A| \subseteq |B|$), still $v = \widehat{B}$, and $\mathbf{Elm}_B(v) = |B| \smallsetminus \{v\}$;*

(ii) *if $k \in |A| \smallsetminus \{v\}$ then $B{\Downarrow}^{\{k\}} = A{\Downarrow}^{\{k\}}$.*

Proof. Consider any $j_1, \dots, j_n \in |A| \smallsetminus \{v\}$. If $\mathbf{Elm}_A(k) = \{j_1, \dots, j_n\}$ for some $k \in |A| \smallsetminus \{v\}$ then let $A' = A \cup \big((|A| \smallsetminus \{v\}) \times \{v\}\big)$.

If there is no such k then take any $k \notin |A|$ not yet involved and put

$$A' = A \cup (\{j_1, \dots, j_n\} \times \{k\}) \cup \{\langle k, v\rangle\} \cup \big((|A| \smallsetminus \{v\}) \times \{v\}\big).$$

In both cases, $A' \in \mathrm{WFEV}$, $\mathbf{Elm}_{A'}(k) = \{j_1, \dots, j_n\}$, and still $v = \widehat{A'}$.

As $|A| \subseteq \omega$ is co-infinite, properly iterating this procedure (ω steps) in $\mathbf{dPA}_2^-$ yields a finite-closed digraph $B \in \mathrm{WFEV}$ satisfying (i),(ii). $\square$

7 Assembling digraphs

The next lemma will allow us to combine a pair of digraphs in a single digraph. Recall Definition 5.4.

Lemma 7.1 ($\mathbf{dPA}_2^-$). *Let $A, B \in \mathrm{WFEV}$. There is a digraph $P \in \mathrm{WFEV}$, such that $|P|$ is co-infinite, $\mathbf{Elm}(P) = \{a, b\}$, $A \cong P{\Downarrow}^{\{a\}}$, $B \cong P{\Downarrow}^{\{b\}}$ — note that then $\mathbb{S} \models \ulcorner P = \{A, B\}\urcorner$.*

Proof. We can w.l.o.g. assume from the beginning that

($*$) $|A|$ is co-infinite, $|B| \cap |A| = \varnothing$, and $|B| \cup |A|$ is still co-infinite.

Indeed, if $|A|$ is not co-infinite, then replace A with $f[A] \cong A$, where $f(k) = 2k$, $\forall k$. Then replace B with $g[B] \cong B$, where $g(k) = 4k+1$, $\forall k$.

Now, if $A = B = \varnothing$ then $P = \{\langle 0, 1\rangle\}$ works with $a = b = 0$.

If $B = \varnothing$, $A \neq \varnothing$, $a = \widehat{A}$, $b = \min A$, and $p \notin |A|$ is arbitrary, then $P = A \cup \{\langle a, p\rangle, \langle b, p\rangle\}$ works. Ditto $A = \varnothing$, $B \neq \varnothing$.

Consider **the main case:** $A, B \neq \varnothing$ and ($*$) holds. Put $a = \widehat{A}$, $b = \widehat{B}$.

We may try to define $P = A \cup B \cup \{\langle a, c\rangle, \langle b, c\rangle\}$, where $c \notin |A| \cup |B|$ — so that $\widehat{P} = c$ and $\mathbf{Elm}(P) = \{a, b\}$. Yet this does not immediately work since such a digraph P is not necessarily extensional.

To fix this obstacle, let's employ the bijection $H = H_{BA}$ (Definition 4.2) to identify H-connected elements. Let $Y = \mathtt{dom}\, H$ and $X = \mathtt{ran}\, H$ (transitive subsets of resp. $|B|$, $|A|$ by Corollary 4.3, disjoint by (∗)). Define a bijection $h : |B| \xrightarrow{\text{onto}} W = (|B| \smallsetminus Y) \cup X$ by

$$h(v) = \begin{cases} H(v)\,, & \text{in case} \quad v \in Y, \\ v\,, & \text{in case} \quad v \in |B| \smallsetminus Y. \end{cases}$$

Let $B' = h[B]$ be the h-image of B, so that $v' \mathbin{B} v \iff h(v) \mathbin{B'} h(v')$.

Then $B' \in \mathrm{WFEV}$, $|B'| = W$, $B \cong B'$. Note that the digraphs B' and A coincide on the common domain $|B'| \cap |A| = X$, which is a transitive subset in both $|B'|$ and $|A|$, whereas $(|A| \smallsetminus X) \cap (|B'| \smallsetminus X) = \varnothing$ by (∗) above. Therefore if $u \in |B'|$ then $\mathbf{Elm}_{B'}(u) \subseteq |B'|$ (*i.e.*, all B'-elements of u belong to B'), and similarly, if $u \in |A|$ then $\mathbf{Elm}_A(u) \subseteq |A|$.

We claim that the digraph $Q = A \cup B'$ is extensional.

Assume that $u, v \in |Q|$ and $\mathbf{Elm}_Q(u) = \mathbf{Elm}_Q(v)$; prove $u = v$.

If $u, v \in |B'|$ then $\mathbf{Elm}_Q(u) = \mathbf{Elm}_{B'}(u)$ and $\mathbf{Elm}_Q(v) = \mathbf{Elm}_{B'}(v)$ by the above; now the extensionality of B' implies $u = v$. Ditto $u, v \in |A|$.

Now assume $v \in |B'| \smallsetminus X = |B| \smallsetminus Y$ and $u \in |A| \smallsetminus X$. Then $\mathbf{Elm}_Q(v) = \mathbf{Elm}_{B'}(v) \subseteq |B'|$ and $\mathbf{Elm}_Q(u) = \mathbf{Elm}_A(u) \subseteq |A|$. Therefore, as $\mathbf{Elm}_Q(u) = \mathbf{Elm}_Q(v)$, we have $\mathbf{Elm}_Q(u) = \mathbf{Elm}_Q(v) \subseteq |B'| \cap |A| = X$, and hence the corresponding transitive closures are included to X except for their vertices, *i.e.* $\mathbf{Con}_{B'}\{v\} \smallsetminus \{v\} = \mathbf{Con}_A\{u\} \smallsetminus \{u\} \subseteq X$. Thus $B' \Downarrow^{\{v\}} \cong A \Downarrow^{\{u\}}$ via the map sending v to u and identical on the common part $\mathbf{Con}_{B'}\{v\} \smallsetminus \{v\} = \mathbf{Con}_A\{u\} \smallsetminus \{u\}$. It follows that $B \Downarrow^{\{v\}} \cong A \Downarrow^{\{u\}}$, hence $u \in X$, $v \in Y$ by construction. But this contradicts the choice of u, v. The contradiction ends the proof of the extensionality of Q. We conclude that $Q \in \mathrm{WFE}$.

Finally note that $|Q|$ is co-infinite by (∗). Let $c = \min(\omega \smallsetminus |Q|)$ and $P = Q \cup \{\langle a, c\rangle, \langle b', c\rangle\} \in \mathrm{WFEV}$, where $b' = h(b)$. Then $\widehat{P} = c = \{a, b'\}_P$, $\mathbf{Elm}(P) = \{a, b'\}$, $A = P \Downarrow^{\{a\}}$, and $B \cong P \Downarrow^{\{b'\}}$ by construction. □

Corollary 7.2 ($\mathbf{dPA}_2^-$)**.** *Let* $A, B \in \mathrm{WFEV}$*. There is a finite-closed digraph* $P \in \mathrm{WFEV}$ *and* $a, b \in \mathbf{Elm}(P)$*, such that* $\mathbf{Elm}(P) = |P| \smallsetminus \{\widehat{P}\}$*, and* $A \cong P \Downarrow^{\{a\}}$*,* $B \cong P \Downarrow^{\{b\}}$ *— that is,* $A, B \mathrel{\tilde{\in}} P$*.*

Proof. Lemma 7.1 yields $Q \in \mathrm{WFEV}$ with $A \cong Q \Downarrow^{\{a\}}, B \cong Q \Downarrow^{\{b\}}$, $a, b \in \mathbf{Elm}(Q)$. Use Lemma 6.4 to extend Q to a digraph P required. □

The next lemma extends the constructions above to represent functions from the ground $\mathbf{dPA}_2^-$ universe in $\mathbb{S}$.

Lemma 7.3 ($\mathbf{dPA}_2^-$). *Assume that* $A, B \in \mathrm{WFEV}$, $X = \mathbf{Elm}(A)$, $Y = \mathbf{Elm}(B)$, *and* $f : X \xrightarrow{\text{onto}} Y$. *Then there is a digraph* $F \in \mathrm{WFEV}$ *such that*:

(i) $\mathbb{S} \models \ulcorner F$ *is a map* $A \xrightarrow{\text{onto}} B\,\urcorner$, *and if* f *is 1–1 then* $\mathbb{S} \models \ulcorner F$ *is 1–1*$\urcorner$;

(ii) *if* $x \in X$ *and* $y = f(x) \in Y$ *then* $\mathbb{S} \models \ulcorner F(U) = V\,\urcorner$, *where* $U = A{\Downarrow}^{\{x\}} \mathrel{\tilde{\in}} A$, $V = B{\Downarrow}^{\{y\}} \mathrel{\tilde{\in}} B$.

Proof. By Corollary 7.2, there is a finite-closed digraph $P \in \mathrm{WFEV}$ and $a, b \in \mathbf{Elm}(P)$, such that $A \cong P{\Downarrow}^{\{a\}}$, $B \cong P{\Downarrow}^{\{b\}}$. Lemma 6.1(ii) implies that the claim of the theorem is $\cong$-invariant, hence we can w.l.o.g. assume that simply $A = P{\Downarrow}^{\{a\}}$, $B = P{\Downarrow}^{\{b\}}$. Then $X = \mathbf{Elm}_P(a)$, $Y = \mathbf{Elm}_P(b)$.

Put $W = \{\langle x, f(x)\rangle_P : x \in X\}$ (Definition 5.4). The digraph $F = P{\Downarrow}W$ belongs to WFEV by Corollary 6.2. We claim that F is as required.

Indeed assume that $C \in \mathrm{WFEV}, C \mathrel{\tilde{\in}} F$. Then $C \cong F{\Downarrow}^{\{z\}} = P{\Downarrow}^{\{z\}}$ for some $z \in \mathbf{Elm}(F) = W$ by Corollary 6.2. By construction $z = \langle x, f(x)\rangle_P$ for some $x \in X$, hence $\mathbb{S} \models \ulcorner C = \langle U, V\rangle\urcorner$ by Lemma 5.5(i), where $U = P{\Downarrow}^{\{x\}}$, $V = P{\Downarrow}^{\{f(x)\}}$. However $U \mathrel{\tilde{\in}} A$ since $x \in X = \mathbf{Elm}(A)$, and $V \mathrel{\tilde{\in}} B$ by similar reasons. We conclude that $\mathbb{S} \models \ulcorner F \subseteq A \times B$ is a function$\urcorner$.

Conversely let $U \mathrel{\tilde{\in}} A'$, thus $U \cong A{\Downarrow}^{\{x\}} = P{\Downarrow}^{\{x\}}$ for some $x \in X$. Then $z = \langle x, f(x)\rangle_P \in W$, hence $C = P{\Downarrow}^{\{z\}} \mathrel{\tilde{\in}} F$ and $\mathbb{S} \models \ulcorner C = \langle U, V\rangle\urcorner$ by Lemma 5.5, where $V = P{\Downarrow}^{\{f(x)\}}$. We conclude that $\mathbb{S} \models \ulcorner \mathtt{dom}\, F = A\,\urcorner$, and $\mathbb{S} \models \ulcorner \mathtt{ran}\, F = B\,\urcorner$ because $\mathtt{ran}\, f = Y$. □

8 Preliminaries: power-less set theories

The power-less set theory $\mathbf{ZFC}^-$ is a subtheory of $\mathbf{ZFC}$ obtained so that:

(I) the Power Set axiom PS is excluded — the upper minus in $\mathbf{ZFC}^-$ symbolizes the absence of PS;

(II) the usual set-theoretic Axiom of Choice AC of $\mathbf{ZFC}$ is removed (as it does not work properly without PS), and instead the *well-orderability axiom* WA is added, which claims that every set can be well-ordered;

(III) Separation Sep is preserved, but the Replacement schema Repl (too weak in the absence of PS) is substituted with the *Collection* schema:

$$\mathsf{Coll} : \forall X\, \big(\forall x \in X\, \exists y\, \Phi(x, y) \Longrightarrow \exists Y\, \forall x \in X\, \exists y \in Y\, \Phi(x, y)\big).$$

Note that $\mathsf{Coll} + \mathsf{Sep} \Longrightarrow \mathsf{Repl}$.

See [1, 5, 6] for a comprehensive account of main features of $\mathbf{ZFC}^-$.

See [7], [16], [3, Sect. 2] or elsewhere for different but equivalent formulations of Collection, as *e.g.* the following form in [7, Chap. 6]:

$$\mathsf{Coll}' : \ \forall\, X\, \exists\, Y\, \forall\, x \in X\, \big(\exists\, y\, \varphi(x,y) \Longrightarrow \exists\, y \in Y\, \varphi(x,y)\big)\,.$$

This is apparently stronger than Coll above, yet in fact Coll' is a consequence of Coll, for let $\Phi(x,y) := \varphi(x,y) \vee \big(y = 0 \wedge \neg\, \exists\, y\, \varphi(x,y)\big)$ in Coll.

- $\mathbf{ZF}^-$ is $\mathbf{ZFC}^-$ without the well-orderability axiom WA;
  $\mathbf{Z}^-$ is $\mathbf{ZF}^-$ without the Collection schema Coll.

Definition 8.1. Now we introduce a useful intermediate theory.

Let $\mathbf{FTM}\boldsymbol{\omega}$ be defined as $\mathbf{Z}^-$ plus the following four additional axioms:

Finite-closure, FinClo: for any set X there is a *finite-closed* superset $Y \supseteq X$, *i.e.*, it is required that if $x \subseteq Y$ is finite then $x \in Y$;

Transitive superset, TrSups: for any X there is a transitive superset $Y \supseteq X$.

Countability: $\forall\, x\, \exists\, f\, (f : \omega \text{ onto } x)$, *i.e.*, every set x is at most countable.

Mostowski Collapse, MostColl: any digraph $A \in \mathrm{WFE}$ admits a transitive set X and a bijection $\boldsymbol{\mu} : |A|$ onto X, satisfying (*) $j\, A\, k$ iff $\boldsymbol{\mu}(j) \in \boldsymbol{\mu}(k)$, for all $j, k \in |A|$. □

Remark 8.2 ($\mathbf{Z}^-$)**.** (1) X is *transitive* if $\forall\, x\, \forall\, y\, (x \in y \in X \Longrightarrow x \in X)$.

(2) If a set X is *finite-closed* as in FinClo then X also is closed under ordered pairs $\langle x, y\rangle$ and generally tuples $\langle x_1, \dots, x_n\rangle$ of any length.

(3) The *transitive closure* $\mathrm{TC}(X)$, resp., the *finite closure* $\mathrm{FC}(X)$, of X is the $\subseteq$-least transitive, resp., finite-closed superset $Y \supseteq X$ (if exists).

(4) Dealing with digraphs A in WFE *etc.* in set-theoretic environment, we'll not assume that $A \subseteq \omega \times \omega$ by default. We'll explicitly indicate this assumption whenever need be. □

Proposition 8.3 (not used below)**.** $\mathbf{FTM}\boldsymbol{\omega} \subseteq \mathbf{ZF}^- + \mathsf{Countability}$. □

See *e.g.* [7, Theorem 6.15] for a proof of $\mathsf{MostColl}$ in $\mathbf{ZF}$.

Lemma 8.4 ($\mathbf{FTM}\boldsymbol{\omega}$)**.** *For any set X, the finite-closure $\mathrm{FC}(X)$ and the transitive closure $\mathrm{TC}(X)$ do exist.*

Proof. The axiom FinClo implies that there is a finite-closed superset $Y \supseteq X$. Then $\mathrm{FC}(X) = \{y \in Y : \pi(y)\}$ is a set by Separation, where $\pi(y)$ says:

> *there is a finite tree T with the root Λ, and a function $f : T \to Y$, such that $f(t) \in X$ for all terminal nodes $t \in T$, $f(t) = \{f(t_1), \dots, f(t_n)\}$ for each non-terminal node t with successors $t_1, \dots, t_n$, and $f(\Lambda) = y$.*

Accordingly, $\mathrm{TC}(X) = \{y \in Y : \tau(y)\}$ is a set, where $Y \supseteq X$ is a transitive superset and $\tau(y)$ says: *there is a finite sequence* $y = y_n \in y_{n-1} \in \ldots \in y_1 \in y_0 \in X$ *of sets* $y_i \in Y$. (Including the case $n = 0$ and $y \in X$.) □

Lemma 8.5 ($\mathbf{FTM}\boldsymbol{\omega}$). *Let* U, V, *be any sets. Then* $U \times V$ *is a set.*

Proof. $X = U \cup V = \bigcup\{U, V\}$ is a set by $\mathbf{Z}^-$. If $Y \supseteq X$ is a finite-closed superset then easily $U \times V \subseteq Y$ is a set by Separation. □

Thus $\mathbf{FTM}\boldsymbol{\omega}$ proves the existence of Cartesian products. Note that $\mathbf{Z}^-$ does not prove even the existence of $\omega \times \omega$!

9 Digraph structure satisfies the intermediate theory

In this section, theory $\mathbf{FTM}\boldsymbol{\omega}$ is interpreted in the digraph second order arithmetic $\mathbf{dPA}_2^-$ by means of the structure $\mathbb{S} = \langle \mathrm{WFEV}; \cong, \widetilde{\in} \rangle$. This is a version of the interpretation defined in [2], § 5, especially Theorem 5.11, or in [15], Definition VII.3.10 ff, or elsewhere. Kreisel [10], VI(a)(ii), attributed this and similar interpretations to the type of *crude* results.

Theorem 9.1 ($\mathbf{dPA}_2^-$). *The structure* $\mathbb{S}$ *satisfies* $\mathbf{FTM}\boldsymbol{\omega}$.

Proof. $\mathbb{S}$ is a well-defined structure by Lemma 5.2.

Arguing in $\mathbf{dPA}_2^-$, let us check all the relevant axioms.

Extensionality. Given digraphs $A, B \in \mathrm{WFEV}$ satisfying $C \,\widetilde{\in}\, A \iff C \,\widetilde{\in}\, B$ for all $C \in \mathrm{WFEV}$, we easily get (1) of Lemma 4.4, and then $A \cong B$.

Union. Given a digraph $A \in \mathrm{WFEV}$, we put $X = \mathbf{Elm}(A) = \{k : k \, A \, \widehat{A}\}$ (all A-elements of the vertex), $Y = \bigcup_{k \in X} \mathbf{Elm}_A(k)$ (all their A-elements). Then $B = A \Downarrow Y \in \mathrm{WFEV}$ by Corollary 6.2, and $\mathbb{S} \models \ulcorner B = \bigcup A \urcorner$.

Regularity. Given a digraph $A \in \mathrm{WFEV}$, let $X = \mathbf{Elm}(A)$. As A is extensional, there is $k \in X$ such that $\neg (j \, A \, k)$ holds for all $j \in X$. The digraph $B = A \Downarrow^{\{k\}}$ then satisfies $B \,\widetilde{\in}\, A$, and at the same time A, B have no common $\widetilde{\in}$-elements, as required.

Infinity. Make use of $\omega \in \mathrm{WFEV}$ and Lemma 5.6.

Separation. Assume that $A \in \mathrm{WFEV}$, $X = \mathbf{Elm}(A)$, and $\Phi(x)$ is an $\in$-formula with parameters in WFEV and x as the only free variable. Digraphs of the form $A_u = A \Downarrow^{\{u\}}$, $u \in X$, belong to $\mathbb{S}$ and are the only (modulo $\cong$) $\widetilde{\in}$-elements of A in $\mathbb{S}$. Now let $Y = \{u \in X : \mathbb{S} \models \ulcorner \Phi(A \Downarrow^{\{u\}}) \urcorner\}$. Then $B = A \Downarrow Y \in \mathrm{WFEV}$ by Corollary 6.2, and $\mathbb{S} \models \ulcorner B = \{x \in A : \Phi(x)\} \urcorner$.

Indeed assume that $C \in \mathrm{WFEV}$, $C \,\widetilde{\in}\, A$, and $\mathbb{S} \models \ulcorner \Phi(C) \urcorner$. Then $C \cong A \Downarrow^{\{u\}}$ for some $u \in X$, by Lemma 5.3, so that $\mathbb{S} \models \ulcorner \Phi(A \Downarrow^{\{u\}}) \urcorner$, and hence $u \in Y$. It follows that $C \cong B \Downarrow^{\{u\}} = A \Downarrow^{\{u\}}$, and finally $C \mathrel{\widetilde{\in}} B$. The proof of the inverse implication is similar.

Pair. Make use of Lemma 7.1 and Lemma 5.5.

Finite-closure, FinClo (Definition 8.1). Use Lemmas 6.4, 6.3.

Transitive superset. To prove TrSups of Definition 8.1, let $A \in \text{WFEV}$, $v = \widehat{A}$. We define $B = A \cup \{\langle u, v\rangle : u \in |A| \smallsetminus \{v\}\}$. Then $B \in \text{WFEV}$ and $\mathbb{S} \models \ulcorner A \subseteq B$ and B is transitive$\urcorner$ by Lemma 5.3(iv),(v).

Countability. Let $B \in \text{WFEV}$ and $Y = \mathbf{Elm}(B)$. We consider the digraph ω satisfying Lemma 5.6, in particular $\mathbf{Elm}(\omega) = \{2n : n < \omega\}$, hence there is a map $f : \mathbf{Elm}(\omega) \xrightarrow{\text{onto}} Y$. By Lemma 7.3, there is a digraph $F \in \text{WFEV}$ such that $\mathbb{S} \models \ulcorner F$ is a map $\omega \xrightarrow{\text{onto}} B\urcorner$, and hence $\mathbb{S} \models \ulcorner B$ is at most countable$\urcorner$.

Now MostColl of Definition 8.1. Suppose that $A \in \text{WFE}$, $u = \widehat{A}$, and

1*. $\mathbb{S} \models \ulcorner A \in \text{WFE}\urcorner$, thus A is a digraph which codes a digraph in $\mathbb{S}$.

In particular $\mathbb{S} \models \ulcorner A$ consists of ordered pairs$\urcorner$. It follows by Definition 5.1 and Lemma 5.5(ii) that for each $p \in \mathbf{Elm}(A)$ there are (unique) elements $j = j_p$ and $k = k_p$ in $|A|$ such that $p = \langle j, k\rangle_A$. Consider the digraph

$$\left.\begin{aligned} E &= \{\langle j_p, k_p\rangle : p \in \mathbf{Elm}(A)\} \\ &= \{\langle j, k\rangle \in |A| \times |A| : \langle j, k\rangle_A \text{ exists and belongs to } \mathbf{Elm}(A)\} \\ &= \{\langle j, k\rangle \in |A| \times |A| : \\ &\qquad\qquad \mathbb{S} \models \ulcorner\langle J, K\rangle \in A\urcorner, \text{ where } J = A\Downarrow^{\{j\}}, K = A\Downarrow^{\{k\}}\} \end{aligned}\right\} \tag{4}$$

in our ground $\mathbf{dPA}_2^-$ universe. (Lemma 5.5 is used to deduce the last equality in (4) from the 2nd one.) Put $X = |E| = \bigcup_{p \in \mathbf{Elm}(A)} \{j_p, k_p\} \subseteq |A|$, and

$$R = A \Downarrow X = \bigcup_{n \in X} (A \Downarrow^{\{n\}}) \,\cup\, (X \times \{u\}), \tag{5}$$

so that $R \in \text{WFEV}$, $A \Downarrow^{\{n\}} = R \Downarrow^{\{n\}}$ for all $n \in X$, $u = \widehat{R}$, $X = \mathbf{Elm}(R)$ — by Corollary 6.2, and clearly (*) $\mathbb{S} \models \ulcorner R = |A|\urcorner$ by construction. (Claim (*) absolutely does not imply $R = |A|$ in the ground $\mathbf{dPA}_2^-$ universe.)

We claim that $E \in \text{WFE}$ (not necessarily $\in \text{WFEV}$). Indeed to see that E is WF, suppose that $\varnothing \neq Y \subseteq X = |E|$. Then $Q = A \Downarrow Y \in \text{WFEV}$, $A \Downarrow^{\{n\}} = Q \Downarrow^{\{n\}}$ for all $n \in Y$, still $u = \widehat{Q}$, $Y = \mathbf{Elm}_R(u)$, and $\mathbb{S} \models \ulcorner Q \subseteq R\urcorner$, by Corollary 6.2. We conclude by 1* that some $m \in Y$ satisfies:

$$\mathbb{S} \models \ulcorner\langle A \Downarrow^{\{n\}}, A \Downarrow^{\{m\}}\rangle \notin A\urcorner, \text{ or equivalently by (4), } \langle n, m\rangle \notin E,$$

for all $n \in Y$. This proves the WF property of E.

Similar arguments prove that E is extensional. Thus indeed $E \in \mathrm{WFE}$.

Note that $X = |E| \subseteq |A| \smallsetminus \{u\}$. Consider the extended digraph $\mathcal{E} = E \cup (|E| \times \{u\})$; $\mathcal{E} \in \mathrm{WFEV}$ with vertex $\widehat{\mathcal{E}} = u$, $\mathbf{Elm}(\mathcal{E}) = X$, and $|\mathcal{E}| = X \cup \{u\}$. Note that $\mathbb{S} \models \ulcorner \mathcal{E}$ is transitive$\urcorner$ by Lemma 5.3(v).

Now let $f(x) = x$ for all $x \in X = \mathbf{Elm}(R) = \mathbf{Elm}(\mathcal{E})$. By Lemma 7.3, there is a digraph $F \in \mathrm{WFEV}$ such that:

2^*. $\mathbb{S} \models \ulcorner F$ is a bijection $R \xrightarrow{\text{onto}} \mathcal{E}\,\urcorner$.

3^*. If $x \in X$ then $\mathbb{S} \models \ulcorner F(J) = U\,\urcorner$, where $J = R{\Downarrow}^{\{x\}}$, $U = \mathcal{E}{\Downarrow}^{\{x\}}$.

Lemma 9.2. $\mathbb{S} \models \ulcorner\,\forall\, J, K \in R\,\big(J \mathrel{A} K \iff F(J) \in F(K)\big)\urcorner$.

Proof. Indeed let $J, K \in \mathrm{WFEV}$, and suppose that $J, K \mathrel{\tilde{\in}} R$. By definition (5) this implies $J \cong A{\Downarrow}^{\{j\}}$ and $K \cong A{\Downarrow}^{\{k\}}$ for some $j, k \in X$. We also put $U = \mathcal{E}{\Downarrow}^{\{j\}}$ and $V = \mathcal{E}{\Downarrow}^{\{k\}}$, so that $\mathbb{S} \models \ulcorner F(J) = U \wedge F(K) = V\urcorner$ by 3^*. The goal is to prove $\mathbb{S} \models \ulcorner(J \mathrel{A} K) \iff (U \in V)\urcorner$.

We may note that $\mathbb{S} \models \ulcorner J \mathrel{A} K\urcorner$ is equivalent to $\langle j, k\rangle \in E$ by (4), whereas $\mathbb{S} \models \ulcorner U \in V\urcorner$ is equivalent to $U \mathrel{\tilde{\in}} V$, and then to $\langle j, k\rangle \in \mathcal{E}$ by Lemma 5.3(ii). Thus the goal can be rewritten as the equivalence $(\langle j, k\rangle \in E) \iff (\langle j, k\rangle \in \mathcal{E})$. But the equivalence is obvious since $\mathcal{E}$ extends E by an irrelevant element (the vertex). □ (Lemma)

We conclude from 2^*, 3^*, and Lemma 9.2 that $\mathbb{S} \models$

$\ulcorner$the transitive set $\mathcal{E}$ and $F : R = |A| \xrightarrow{\text{onto}} \mathcal{E}$ witness $\mathsf{MostColl}$ for $A\urcorner$.

This completes the proof of the axiom $\mathsf{MostColl}$ of Definition 8.1 in $\mathbb{S}$, and subsequently Theorem 9.1 as a whole. □ (Theorem 9.1)

Corollary 9.3. *Theories* $\mathbf{PA}_2^-$, $\mathbf{dPA}_2^-$, $\mathbf{FTM}\boldsymbol{\omega}$ *are equiconsistent.*

Proof. Use Theorem 9.1 and Lemma 2.2. □

Remark 9.4. Under the full $\mathbf{PA}_2$ (with AC_ω), the structure $\mathbb{S}$ accordingly satisfies $\mathsf{Collection}$, in addition to $\mathbf{FTM}\boldsymbol{\omega}$, that is, satisfies $\mathbf{ZFC}^-$. To prove this extension, one has to establish suitable generalizations of Lemma 7.1 and Corollary 7.2 for infinite sequences of digraphs in WFEV instead of just pairs A, B of them. It follows that the equiconsistency of $\mathbf{PA}_2$ and $\mathbf{PA}_2^-$ is pretty sufficient for Theorem 1.1 as a whole. However we don't know how to prove the equiconsistency of $\mathbf{PA}_2$ and $\mathbf{PA}_2^-$ without making use of structures similar to $\mathbb{S}$ and corresponding set theories, explicit or implicit. □

10 Second reduction: outline

Corollary 9.3 is the first part of the proof of Theorem 1.1. The remainder of the proof involves the ideas and technique of Gödel's constructibility, and the goal will be the equiconsistency of $\mathbf{FTM}\boldsymbol{\omega}$ and $\mathbf{ZF}^-$.

We may note that the $\mathbf{FTM}\boldsymbol{\omega}$ ordinals are no less rich than the $\mathbf{ZF}^-$ ordinals, in particular, any countable well-ordering is isomorphic to an ordinal by $\mathsf{MostColl}$ of Definition 8.1. (On the contrary, theory $\mathbf{Z}^-$ does not even prove the existence of $\omega+\omega$.) Thus we may hope that the constructible sets behave in $\mathbf{FTM}\boldsymbol{\omega}$ similar to those in $\mathbf{ZF}^-$. And indeed *the class* $\mathbf{L} = \bigcup_{\alpha\in\mathsf{Ord}} \mathbf{L}_\alpha$ *of all Gödel-constructible sets* is well-defined in $\mathbf{FTM}\boldsymbol{\omega}$ and has a number of general properties known from $\mathbf{ZF}$ or $\mathbf{ZF}^-$ — *mutatis mutandis*, of course.

The final goal of the remainder will be the following theorem.

Theorem 10.1 ($\mathbf{FTM}\boldsymbol{\omega}$)**.** *Class* $\mathbf{L}$ *or one of* $\mathbf{L}_\alpha$*,* α *limit, satisfies* $\mathbf{ZFC}^-$.

It is immediately clear that, by Corollary 9.3, **Theorem 10.1 implies the equiconsistency result of Theorem 1.1.** However Theorem 10.1 has the additional advantage of giving a *transitive standard* (*i.e.*, with the true membership) model $\mathbf{L}$ or $\mathbf{L}_\alpha$ of $\mathbf{ZFC}^-$ within $\mathbf{FTM}\boldsymbol{\omega}$, a theory that only rather mildly extends $\mathbf{Z}^- + \mathsf{Countability}$.

The next lemma will be useful in the course of the proof of the theorem. Let a *class-function* be a (definable) class which satisfies the standard definition of a function (*e.g.*, consists of ordered pairs, *etc.*).

Lemma 10.2 ($\mathbf{FTM}\boldsymbol{\omega}$)**.** *If* F *is a class-function,* $X = \mathtt{dom}\, F$ *any set, and the image* $R = F[X] = \{F(x) : x \in X\}$ *is transitive then* R *and* F *are sets.*

Proof. By $\mathsf{Countability}$ we can w.l.o.g. assume that $X \subseteq \omega$. We can also assume that $F \restriction X$ is 1–1, for otherwise replace X by the set

$$X' = \{k \in X : \forall\, j \in X\, (j < k \implies F(j) \neq F(k))\}.$$

Then the digraph $A = \{\langle j, k\rangle : j, k \in X \wedge F(j) \in F(k)\}$ is WFE as isomorphic to $\in\restriction R$. On the other hand, by $\mathbf{FTM}\boldsymbol{\omega}$, A is isomorphic to $\in\restriction Y$ where Y is a transitive *set*. It follows that Y and R are $\in$ isomorphic, and hence $R = Y$ is a set. Finally $F \restriction X \subseteq X \times R$ is a set by Separation. □

Note the role of the axiom of $\mathsf{Countability}$ in the proof. It is an open question whether the lemma holds in $\mathbf{FTM}\boldsymbol{\omega}$ sans $\mathsf{Countability}$.

11 Gödel operations in the intermediate theory

Definition 11.1 (**FTM**$\boldsymbol{\omega}$)**.** Consider the following list of Gödel operations.

$$\begin{array}{rcl}
\mathcal{F}_0(x,y) &=& \{x,y\},\\
\mathcal{F}_1(x,y) &=& x\times y,\\
\mathcal{F}_2(x,y) &=& x\smallsetminus y,\\
\mathcal{F}_3(x,y) &=& \mathtt{dom}\,x=\{u\colon\exists\,v\,(\langle u,v\rangle\in x)\},\\
\mathcal{F}_4(x,y) &=& \{\langle u,v\rangle\in x\colon u\in v\},\\
\mathcal{F}_5(x,y) &=& \{\langle a,b,c\rangle\colon\langle b,c,a\rangle\in x\},\\
\mathcal{F}_6(x,y) &=& \{\langle a,b,c\rangle\colon\langle c,b,a\rangle\in x\},\\
\mathcal{F}_7(x,y) &=& \{\langle a,b,c\rangle\colon\langle a,c,b\rangle\in x\},\\
\mathcal{F}_8(x,y) &=& x\cup y,\\
\mathcal{F}_9(x,y) &=& \{\langle u,v\rangle\in x\colon u=v\}.
\end{array}\qquad\Box$$

Operations $\mathcal{F}_0-\mathcal{F}_7$ are copied from [8, Sect. 10] whereas $\mathcal{F}_8$ and $\mathcal{F}_9$ are added for technical reasons. Note that $\mathcal{F}_3-\mathcal{F}_7$ and $\mathcal{F}_9$ are unary operations; we treat them as binary to ensure the uniformity of processing.

Lemma 11.2 (**FTM**$\boldsymbol{\omega}$)**.** *If x,y are sets and $i\le 9$ then $\mathcal{F}_i(x,y)$ is a set.*

Proof. $\mathcal{F}_0(x,y)=\{x,y\}$ is a set by the $\mathbf{Z}^-$ Pair axiom.

$\mathcal{F}_1(x,y)=x\times y$ is a set, see Lemma 8.5.

$\mathcal{F}_2(x,y)=x\smallsetminus y=\{a\in x\colon a\notin y\}$ is a set by Separation. Ditto $\mathcal{F}_4$.

$\mathcal{F}_8(x,y)=\bigcup\{x,y\}$, and use the Pair and Union axioms.

$\mathcal{F}_3(x,y)=\mathtt{dom}\,x\subseteq Y$, where $Y\supseteq x$ is any transitive superset (axiom TrSups of Definition 8.1, hence $\mathcal{F}_3(x,y)$ is a set by Separation. Ditto $\mathcal{F}_9$.

As for $\mathcal{F}_5$, let $Y\supseteq x$ be any transitive superset, and $Z\supseteq Y$ be any finite-closed superset (here FinClo of Definition 8.1 is used). Thus

$$\langle a,b,c\rangle\in x\implies a,b,c\in Y,\quad\text{whereas}\quad a,b,c\in Y\implies\langle b,c,a\rangle\in P.$$

We conclude that $\langle a,b,c\rangle\in x\implies\langle b,c,a\rangle\in P$, hence $\mathcal{F}_5(x,y)$ is a set by Separation. Ditto $\mathcal{F}_6,\mathcal{F}_7$. $\Box$

Fix $n\ge 3$. Let Π_n be the set of all **permutations** of $\{1,\dots,n\}$. Let each $\pi\in\Pi_n$ naturally act on n-tuples by $\pi\cdot\langle u_1,\dots,u_n\rangle=\langle u_{\pi(1)},\dots,u_{\pi(n)}\rangle$. Let

$$f_5\cdot\langle a,b,c\rangle=\langle c,a,b\rangle,\quad f_6\cdot\langle a,b,c\rangle=\langle c,b,a\rangle,\quad f_7\cdot\langle a,b,c\rangle=\langle a,c,b\rangle$$

— then $\mathcal{F}_i(x,y) = \{f_i \cdot \langle a,b,c\rangle : \langle a,b,c\rangle \in X\}$, for $i = 5,6,7$.

Recall that in set theory any n-tuple $\vec{u} = \langle u_1,\dots,u_n\rangle$ is represented as a triple as $s = \langle \underbrace{\langle u_1,\dots,u_{n-2}\rangle}_a, \underbrace{u_{n-1}}_b, \underbrace{u_n}_c \rangle$, see *e.g.* [8]. Let each f_i act on n-tuples exactly in the sense of this representation, so that for instance $f_5 \cdot \langle u_1,\dots,u_n\rangle = \langle u_n, u_1,\dots,u_{n-2},u_{n-1}\rangle$.

The next lemma is similar to the penultimate lemma on p. 102 in [13].

Lemma 11.3. *For any permutation* $\pi \in \Pi_n$ *there exists a combination* f_π *of operations* f_5, f_6, f_7 *satisfying* $f_\pi \cdot \langle u_1,\dots,u_n\rangle = \pi \cdot \langle u_1,\dots,u_n\rangle$.

Proof. It clearly suffices to define, for any $j = 1,\dots,n-1$, a combination h_j of f_5, f_6, f_7 satisfying

$$h_j \cdot \langle u_1,\dots,u_{j-1},u_j,u_{j+1},u_{j+2}\dots,u_n\rangle = \\ \langle u_1,\dots,u_{j-1},u_{j+1},u_j,u_{j+2}\dots,u_n\rangle.$$

For that purpose, apply f_5, $(n-j-1)$ times, then f_7, then again f_5, $(j+1)$ times, getting the transformation required as follows:

$$\langle u_{j+2},u_{j+3}\dots,u_n,u_1,\dots,u_{j-1},u_j,u_{j+1}\rangle \to \\ \langle u_{j+2},u_{j+3}\dots,u_n,u_1,\dots,u_{j-1},u_{j+1},u_j\rangle \to \\ \langle u_1,\dots,u_{j-1},u_{j+1},u_j,u_{j+2},u_{j+3},\dots,u_n\rangle. \qquad \square$$

12 Gödel closures

We have to be careful in definitions of this notion in $\mathbf{FTM}\boldsymbol{\omega}$, in the absence of Collection/Replacement. We proceed following the proof of Lemma 8.4.

We argue in $\mathbf{FTM}\boldsymbol{\omega}$. Given a set Y and a class-function Ψ with $\mathtt{dom}\,\Psi = Y$, consider the image $\Psi[Y] = \{\Psi(y) : y \in Y\}$, not assumed to necessarily be a set here. Let P_Y consist of all pairs $p = \langle D, f\rangle$, where D is a finite tree of dyadic tuples, with the root Λ (the empty tuple), f is a function, $\mathtt{dom}\, f = T$, $f(d) \in Y$ for each terminal node $d \in D$, and $f(d) \in \{0,1,\dots,9\}$ for non-terminal nodes. Note that P_Y is a set by FinClo and Sep in $\mathbf{FTM}\boldsymbol{\omega}$.

If $p = \langle D, f\rangle \in P_Y$ then define $p_\Psi[d]$ for $d \in D$ by induction, setting $p_\Psi[d] = \Psi(f(d))$ for each terminal node $d \in D$, and $p_\Psi[d] = \mathcal{F}_i(a,b)$ for each non-terminal node d, where $i = f(d)$, $a = p_\Psi[d^\frown 0]$, $b = p_\Psi[d^\frown 1]$, and $d^\frown 0, d^\frown 1$ are the immediate successors of d in D. Finally let $[p]_\Psi = p_\Psi[\Lambda]$.

Definition 12.1 ($\mathbf{FTM}\boldsymbol{\omega}$)**.** Under the assumptions and notation above, put $\mathrm{GC}(\Psi[Y]) = \{[p]_\Psi : p \in P_Y\}$, *the Gödel closure* of $\Psi[Y]$; $\Psi[Y] \subseteq \mathrm{GC}(\Psi[Y])$.

We say that $\Psi[Y]$ is *Gödel-closed* if $\Psi[Y] = \mathrm{GC}(\Psi[Y])$. $\square$

(I) As we have the operation $\mathcal{F}_0(x,y) = \{x,y\}$, the collection $\mathrm{GC}(\Psi[Y])$ is transitive in case $\Psi[Y]$ is transitive.

(II) As we have the operations $\mathcal{F}_0$ and $\mathcal{F}_8(x,y) = x \cup y$, the collection $\mathrm{GC}(\Psi[Y])$ is finite-closed (*i.e.*, contains every finite subset).

(III) If $\Psi(y) = y, \forall y$, then $\Psi[Y] = Y$, hence Definition 12.1 provides $\mathrm{GC}(Y)$ for any set Y.

Corollary 12.2 ($\mathbf{FTM}\boldsymbol{\omega}$). *Under the assumptions and notation of Definition 12.1, suppose that* $\mathrm{GC}(\Psi[Y])$ *is transitive. Then*:

— $\mathrm{GC}(\Psi[Y])$ *is a set,*

— *therefore* $\Psi[Y]$ *and* Ψ *itself are sets as well by Separation,*

— $\mathrm{GC}(\Psi[Y])$ *is the least Gödel-closed superset of* $\Psi[Y]$.

In particular, (1) *if* Y *is any set and* $\mathrm{GC}(Y)$ *is transitive then* $\mathrm{GC}(Y)$ *is a set,* (2) *if* Y *is transitive then so is* $\mathrm{GC}(Y)$, (3) *if* Y *is any set then* $\mathrm{GC}(Y)$ *is a set, and* (4) $\mathrm{GC}(Y)$ *is a finite-closed set.*

Proof. Apply Lemma 10.2 for $X = P$ and $F(p) = [p]$ as in Definition 12.1. The particular case is handled by (I)–(III). To prove (3), let $Y' \supseteq Y$ be a transitive superset by TrSups of $\mathbf{FTM}\boldsymbol{\omega}$. Then $\mathrm{GC}(Y')$ is transitive by (2) and a set by (1), and finally $\mathrm{GC}(Y) \subseteq \mathrm{GC}(Y')$ is a set by Separation. □

13 Definability in Gödel-closed domains

The next lemma will be of principal importance in the development of Gödel constructibility under the axioms of $\mathbf{FTM}\boldsymbol{\omega}$ in the remainder.

Lemma 13.1 ($\mathbf{FTM}\boldsymbol{\omega}$). *If* U *is a transitive and Gödel-closed set or class,* $X \in U$ *is transitive and Gödel-closed, and a set* $Y \subseteq X$ *is* $\in$*-definable over* X, *possibly with parameters* $p \in X$. *Then* $Y \in U$.

Proof. The proof runs essentially along the lines of the proof of [8, Lemma 31], or of [13, Theorem on p. 103]. We begin with the parameter-free case.

Given an $\in$-formula $\varphi(x_1, \dots, x_n)$, we have to prove that

$$Y = Y_\varphi = \{\langle x_1, \dots, x_n\rangle \in X^n : \varphi^X(x_1, \dots, x_n)\} \in U, \tag{6}$$

where φ^X is the relativization to X. This goes on by induction. Suppose that $\varphi(x_1, \dots, x_n)$ is $x_i \in x_j$, $1 \le i \ne j \le n$. (We allow dummy variables.) By Lemma 11.3, it can be w.l.o.g. assumed that $i = 1$ and $j = 2$. Then

$$Y = Y_1 \times X^{n-2} = \mathcal{F}_1(Y_1, \underbrace{\mathcal{F}_1(X, \mathcal{F}_1(X, \mathcal{F}_1(X, \ldots (\mathcal{F}_1(X, X)) \ldots)))}_{n-2 \text{ times } X}),$$

where

$$Y_1 = \{\langle x_1, x_2 \rangle \in X^2 \colon x_1 \in x_2\} = \mathcal{F}_4(X^2, X) = \mathcal{F}_4(\mathcal{F}_1(X, X), X) \in U.$$

This implies $Y \in U$ since U is Gödel-closed.

The case of φ being $x_i = x_j$ is handled with $\mathcal{F}_9$ instead of $\mathcal{F}_4$.

The inductive step $\neg$ is easy: $Y_{\neg\varphi} = X^n \smallsetminus Y_\varphi$, and use $\mathcal{F}_2$.

Similarly, $Y_{\varphi\wedge\psi} = Y_\varphi \cap Y_\psi$, and we can use $\mathcal{F}_2$ as $A \cap B = A \smallsetminus (A \smallsetminus B)$. In this argument, it is assumed that φ, ψ have the same list of free variables, which can be easily arranged by suitably adding dummy variables to all elementary subformulas of φ and ψ.

Let finally $\varphi(x_1, \ldots, x_n)$ be $\exists\, x_{n+1}\, \psi(y, x_1, \ldots, x_n, x_{n+1})$, and accordingly $Y_\psi \subseteq X^{n+1}$ belongs to U. Then $Y_\varphi = \mathtt{dom}\, Y_\psi = \mathcal{F}_3(Y_\psi, Y_\psi) \in U$.

This ends the parameter-free case.

Now consider the case of parameters allowed. Given an $\in$-formula $\varphi(x_1, \ldots, x_n, p_0)$, where $p_0 \in X$ is a parameter, we have to prove that

$$Y = Y_\varphi = \{\langle x_1, \ldots, x_n, p_0 \rangle \in X^n \colon \varphi^X(x_1, \ldots, x_n, p_0)\} \in U. \tag{7}$$

It follows from the above that the parameter-free definable set

$$Z = \{\langle x_1, \ldots, x_n, p \rangle \in X^{n+1} \colon \varphi^X(x_1, \ldots, x_n, p)\}$$

belongs to U. So does the set $P = X^n \times \{p_0\}$ (operations $\mathcal{F}_0, \mathcal{F}_1$). Then the set $Q = Z \cap P = Z \smallsetminus (Z \smallsetminus P)$ (operation $\mathcal{F}_2$) belongs to U as well. Finally $Y = \mathtt{dom}\, Q \in U$ (operation $\mathcal{F}_3$), and this completes the proof. □

14 Constructible sets in the intermediate theory

We make use of the constructible hierarchy as presented in [8]:

$$\left.\begin{array}{lcl} \mathbf{L}_0 & = & \varnothing, \\ \mathbf{L}_\alpha & = & \mathrm{GC}(\{\mathbf{L}_\gamma \colon \gamma < \alpha\}) \ \text{ for all } \ \alpha \geq 1, \\ \mathbf{L} & = & \bigcup_{\alpha \in \mathrm{Ord}} \mathbf{L}_\alpha \ = \text{ all constructible sets.} \end{array}\right\} \tag{8}$$

First of all we have to prove that this construction is legitimate in $\mathbf{FTM}\boldsymbol{\omega}$. The standard reference to **ZF** transfinite recursion does not work as it is based on the Collection/Replacement schemata. Therefore we have to legitimize construction (8) in $\mathbf{FTM}\boldsymbol{\omega}$ at a more basic level.

For that purpose, let's say that a set f is a λ-constructing sequence, λ-*ConSeq* for brevity, where $\lambda \in \mathrm{Ord}$, if f is a function, $\mathtt{dom}\, f = \lambda$, $f(0) = \varnothing$, and $f(\alpha) = \mathrm{GC}(\{\mathbf{L}_\gamma \colon \gamma < \alpha\})$ for all $1 \leq \alpha < \lambda$. Thus $\mathbb{f}_\lambda = \langle \mathbf{L}_\alpha \rangle_{\alpha < \lambda}$ is the only λ-ConSeq (if exists) in the $\mathbf{FTM}\boldsymbol{\omega}$ universe.

Lemma 14.1 (**FTM$\boldsymbol{\omega}$**). *If* $\lambda \in \mathrm{Ord}$ *then a* λ*-ConSeq exists (as a set) and is unique. In addition*:

(i) *if* $\alpha \in \mathrm{Ord}$ *then* $\mathbf{L}_\alpha$ *is a transitive Gödel-closed* **set**;

(ii) $\mathbf{L}$ *is a transitive Gödel-closed class*;

(iii) *if* $\gamma < \alpha$ *then* $\mathbf{L}_\gamma \in \mathbf{L}_\alpha$ *and* $\mathbf{L}_\gamma \subseteq \mathbf{L}_\alpha$.

Proof. The ordinals are well-ordered in **FTM$\boldsymbol{\omega}$** by the Regularity axiom, hence the usual proof of the uniqueness of λ-ConSeq holds. To prove the existence, *suppose to the contrary that* λ*-ConSeq exist not for all* $\lambda \in \mathrm{Ord}$, and let λ be the least ordinal of such kind. (Recall that Ord is well-ordered.) Thus a unique α-ConSeq $\mathbb{f}_\alpha$ exists for every $\alpha < \lambda$, and $\alpha \le \beta \Longrightarrow \mathbb{f}_\alpha \subseteq \mathbb{f}_\beta$.

Case 1: $\lambda = \alpha + 1$. Then $\mathbb{f}_\alpha$ is a set by the choice of λ, and $\mathbf{L}_\alpha = \mathrm{GC}(\mathbb{f}_\alpha)$ is a set as well by Corollary 12.2, hence $\mathbb{f}_\lambda = \mathbb{f}_\alpha \cup \{\langle \alpha, \mathbf{L}_\alpha \rangle\}$ is a set and a λ-ConSeq, which is a contradiction.

Case 2: λ is limit. We consistently define $\mathbf{L}_\alpha = \mathbb{f}_{\alpha+1}(\alpha)$ for all $\alpha < \lambda$. Then $\mathbb{f}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{f}_\alpha = \langle \mathbf{L}_\alpha \rangle_{\alpha<\lambda}$ would be a λ-ConSeq, yet we don't know yet that it is a set. To prove this fact, let's prove (i) by induction below λ.
Thus assume that $\alpha < \lambda$ and let all $\mathbf{L}_\gamma, \gamma < \alpha$ be transitive sets.

Subcase 2A: $\alpha = \gamma + 1$. Then, as $\mathbf{L}_\gamma = \mathrm{GC}(\{\mathbf{L}_\xi : \xi < \gamma\})$, we have $\mathbf{L}_\alpha = \mathrm{GC}(\mathbf{L}_\gamma \cup \{\mathbf{L}_\gamma\})$. But $\mathbf{L}_\gamma$ is transitive by the inductive hypothesis, hence so is $Y = \mathbf{L}_\gamma \cup \{\mathbf{L}_\gamma\}$, and then so is $\mathbf{L}_\alpha = \mathrm{GC}(Y)$ by Corollary 12.2.

Subcase 2B: α is limit. Then, by construction,

$$\mathbf{L}_\alpha = \mathrm{GC}(\{\mathbf{L}_\gamma : \gamma < \alpha\}) = \bigcup_{\gamma<\alpha} \mathrm{GC}(\{\mathbf{L}_\xi : \xi < \gamma\}) = \bigcup_{\gamma<\alpha} \mathbf{L}_\gamma \tag{9}$$

is transitive by the inductive hypothesis.

Thus indeed all $\mathbf{L}_\alpha$, $\alpha < \lambda$, are transitive sets. Then yet again by (9) $\mathbf{L}_\lambda = \mathrm{GC}(\{\mathbf{L}_\alpha : \alpha < \lambda\}) = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha$ is transitive, and hence, by Corollary 12.2, both $\mathbf{L}_\lambda$ and $\mathbb{f}_\lambda$ are sets, which contradicts the choice of λ.

From the contradictions obtained in cases 1 and 2, we conclude that $\mathbb{f}_\lambda$ exists for all $\lambda \in \mathrm{Ord}$, and hence $\mathbf{L}_\lambda = \mathbb{f}_{\lambda+1}(\lambda)$ is well-defined, as required.

Claim (i) has been already established, (ii) is a consequence of (i), and claim (iii) is rather obvious. □

As a by-product of the proof, we can present the inductive step in (8) in the following somewhat more familiar form:

$$\mathbf{L}_{\gamma+1} = \mathrm{GC}(\mathbf{L}_\gamma \cup \{\mathbf{L}_\gamma\}), \quad \text{and} \quad \mathbf{L}_\alpha = \bigcup_{\gamma<\alpha} \mathbf{L}_\gamma \text{ for all limit } \alpha. \tag{10}$$

What kind of set theory $\mathbf{L}$ supports in **FTM$\boldsymbol{\omega}$**?

Lemma 14.2 (**FTM$\boldsymbol{\omega}$**)**.** *All axioms of* $\mathbf{Z}^-$, *minus* Sep, *hold in* $\mathbf{L}$ *and in any set* $\mathbf{L}_\lambda$, *where* $\lambda \in \mathrm{Ord}$ *is limit.*

Proof (sketch)**.** This does not differ from the full-**ZF** case. Consider *e.g.* the Union axiom. Let $X \in \mathbf{L}$, so that $X \in \mathbf{L}_\alpha$, $\alpha \in \mathrm{Ord}$. As $\mathbf{L}_\alpha$ is transitive, the union $Y = \bigcup X \subseteq \mathbf{L}_\alpha$ is definable over $\mathbf{L}_\alpha$, hence $Y \in \mathbf{L}_{\alpha+1}$ by Lemma 13.1. □

The schemata of Replacement/Collection are not necessarily true in $\mathbf{L}$ (defined in a **FTM$\boldsymbol{\omega}$** universe), as the next example shows.

Example 14.3. Arguing in the full **ZF**, let $L = \mathbf{L}_\vartheta$, where $\vartheta = (\aleph_\omega)^{\mathbf{L}}$. Let V be the extension of L by ajoining a generic sequence of (generic) maps $f_n : \omega \overset{\text{onto}}{\longrightarrow} (\aleph_n)^{\mathbf{L}}$. Then V is a model of **FTM$\boldsymbol{\omega}$**. However $(\mathbf{L})^V = L$, and Repl/Coll definitely fail in L. □

The Separation schema remains an open problem in this context. On the one hand, Sep holds (unlike Repl/Coll) in L of Example 14.3. On the other hand, we are not able to prove in **FTM$\boldsymbol{\omega}$** that Sep holds in $\mathbf{L}$.

Anyway, we conclude from Example 14.3 that the class $\mathbf{L}$ alone does not lead to the proof of Theorem 10.1 in **FTM$\boldsymbol{\omega}$**. To explore the second option in Theorem 10.1, we still need some technical developments.

15 Definability and well-ordering

Following Section 11 in [8], consider the formula:

$$C(\lambda, z) \;:=\; \lambda \in \mathrm{Ord} \wedge \exists f \left(f \text{ is a } (\lambda+1)\text{-ConSeq} \wedge z = f(\lambda)\right).$$

Let $C^K(\lambda, z)$ be the formal relativization to a given set or class K, *i.e.*, each quantifier $\exists x$ or $\forall x$ is replaced with resp. $\exists x \in K$, $\forall x \in K$.

Theorem 15.1 (**FTM$\boldsymbol{\omega}$**)**.** $\mathfrak{A}$: $\langle \mathbf{L}_\alpha \rangle_{\alpha \in \mathrm{Ord}} = \{\langle \alpha, z\rangle \in \mathbf{L} : C^{\mathbf{L}}(\alpha, z)\}$.

If $\lambda \in \mathrm{Ord}$ *is limit then* $\mathfrak{A}_\lambda$: $\langle \mathbf{L}_\alpha \rangle_{\alpha < \lambda} = \{\langle \alpha, z\rangle \in \mathbf{L}_\lambda : C^{\mathbf{L}_\lambda}(\alpha, z)\}$.

In other words, the whole hierarchy $\langle \mathbf{L}_\alpha \rangle_{\alpha \in \mathrm{Ord}}$ *is* $\in$*-definable over* $\mathbf{L}$, *whereas if* λ *is limit then* $\langle \mathbf{L}_\alpha \rangle_{\alpha < \lambda}$ *is* $\in$*-definable over* $\mathbf{L}_\lambda$.

Proof. We will establish the following:

1°. (a) If $\forall \alpha\, (\mathbb{f}_\alpha \in \mathbf{L}_{\alpha+1})$, then $\mathfrak{A}$.
(b) If λ is a limit ordinal and $\forall \alpha < \lambda\, (\mathbb{f}_\alpha \in \mathbf{L}_{\alpha+1})$, then $\mathfrak{A}_\lambda$.

2°. If λ is a limit ordinal and $\mathfrak{A}_\lambda$ holds, then $\mathbb{f}_\lambda \in \mathbf{L}_{\lambda+1}$.

3°. If $\mathbb{f}_\alpha \in \mathbf{L}_{\alpha+1}$ then $\mathbb{f}_{\alpha+1} \in \mathbf{L}_{\alpha+2}$.

Proof of 1°. We prove (a) only; the proof of (b) is similar. Lemma 14.1(ii) implies, by the definition of GC, that if $x, y \in \mathbf{L}$, then

$$y = \mathrm{GC}(x) \iff \mathrm{GC}(x) \in \mathbf{L} \wedge (y = \mathrm{GC}(x))^{\mathbf{L}}.$$

It follows (note the operation $\mathcal{F}_3$) that

$$f \text{ is a } \lambda\text{-ConSeq} \iff (f \text{ is a } \lambda\text{-ConSeq})^{\mathbf{L}}$$

for each $\lambda \in \mathbf{L}$. This implies 1°a, as required.

Proof of 2°. Indeed, $\mathfrak{A}_\lambda$ says that $\mathbb{f}_\lambda$ is a class in $\mathbf{L}_\lambda$. Then it remains to apply Lemma 13.1 with $U = \mathbf{L}_{\lambda+1}$ and $X = \mathbf{L}_\lambda$.

Proof of 3°. Indeed, we have, on the one hand, that $\mathbb{f}_{\alpha+1} = \mathbb{f}_\alpha \cup \{\langle \alpha, \mathbf{L}_\alpha \rangle\}$, and therefore $\mathbb{f}_{\alpha+1}$ is obtained from $\mathbb{f}_\alpha$ and $\mathbf{L}_\alpha$ by composing the operations $\mathcal{F}_0$, $\mathcal{F}_3$, $\mathcal{F}_8$. On the other hand, $\mathbb{f}_\alpha \in \mathbf{L}_{\alpha+1}$, and $\mathbf{L}_\alpha \in \mathbf{L}_{\alpha+1}$ by Lemma 14.1. But $\mathbf{L}_{\alpha+1} \subseteq \mathbf{L}_{\alpha+2}$ and $\mathbf{L}_{\alpha+2}$ is Gödel-closed.

It is now evident how to use Propositions 1°, 2°, 3° and transfinite induction to prove $\mathbb{f}_\alpha \in \mathbf{L}_{\alpha+1}$ for all α, $\mathfrak{A}_\lambda$ for all limit λ, and $\mathfrak{A}$. □

Now let us prove the following theorem about the well-ordering of the class $\mathbf{L}$ of all constructible sets, and of the sets $\mathbf{L}_\alpha \subseteq \mathbf{L}$.

Theorem 15.2 (**FTM$\boldsymbol{\omega}$**). *There exists a well-ordering $<_{\mathbf{L}}$ of the class $\mathbf{L}$, $\in$-definable over $\mathbf{L}$, and such that if $\lambda \in \mathrm{Ord}$ is limit, then*

(1) *$\mathbf{L}_\lambda$ is an initial segment of $\mathbf{L}$ in the sense of $<_{\mathbf{L}}$, and*

(2) *the restricted well-ordering $<^\lambda_{\mathbf{L}} = <_{\mathbf{L}} \restriction \mathbf{L}_\lambda$ is $\in$-definable over $\mathbf{L}_\lambda$.*

Proof. We prove only the first claim; the claim related $\mathbf{L}_\lambda$ is pretty similar. If $\alpha \in \mathrm{Ord}$ and $k < \omega$ then, *arguing in* $\mathbf{L}$, we define a set $M^\alpha_k \subseteq \mathbf{L}$, and an auxiliary well-ordering $\prec^\alpha_k$ of M^α_k, by induction.

Put $M^\alpha_0 = \{\mathbf{L}_\gamma : \gamma < \alpha\}$ (a definable class by Theorem 15.1). Define $\mathbf{L}_\gamma \prec^\alpha_0 \mathbf{L}_\delta$ iff just $\gamma < \delta$.

Now suppose that M^α_k and $\prec^\alpha_k$ have been defined. Put

$$M^\alpha_{k+1} = \{\langle a, b, i \rangle : a, b \in M^\alpha_k \wedge i \le 9\}.$$

Define $\prec^\alpha_{k+1}$ on M^α_{k+1} to be the lexicographical order, where the first two co-ordinates are ordered by $\prec^\alpha_k$, and the last one — by the order on integers.

If $p \in M^\alpha = \bigcup_k M^\alpha_k$ then define a set $F^\alpha(p)$ by induction on k so that

$$F^\alpha(p) = \begin{cases} \mathbf{L}_\gamma & , \text{ in case } \ p = \mathbf{L}_\gamma \in M^\alpha_0 \ (\gamma < \alpha); \\ \mathcal{F}_i(F^\alpha(a), F^\alpha(b)) & , \text{ in case } \ p = \langle a, b, i \rangle \in M^\alpha_{k+1}. \end{cases} \tag{11}$$

Thus F^α is a definable class-function $M^\alpha \xrightarrow{\text{onto}} \mathbf{L}_\alpha$ by (8). If $x \in \mathbf{L}$ then:

let $\alpha(x)$ be the least α for which $x = F^\alpha(p)$ for some $p \in M^\alpha$,

let $k(x)$ be the least k for which $x = F^{\alpha(x)}(p)$ for some $p \in M_k^{\alpha(x)}$,

let $p(x)$ be the $\prec_k^\alpha$-least $p \in M_{k(x)}^{\alpha(x)}$ satisfying $x = F^{\alpha(x)}(p)$.

Now we introduce the required ordering on $\mathbf{L}$ by putting $x <_{\mathbf{L}} y$ if

$\alpha(x) < \alpha(y)$, or

$\alpha(x) = \alpha(y) \wedge k(x) < k(y)$, or

$\alpha(x) = \alpha(y) \wedge k(x) = k(y) \wedge p(x) \prec_{k(x)}^{\alpha(x)} p(y)$.

It's quite clear that $<_{\mathbf{L}}$ is really a well-ordering of the class $\mathbf{L}$, definable over $\mathbf{L}$, with each $\mathbf{L}_\alpha$ being an initial segment of $\mathbf{L}$. The details are left to the reader. □

16 Subclass $\mathbf{L}^*$

Coming back to Theorem 10.1, we know already that the **FTM*ω*** axioms do not fix the universe $\mathbf{L}$ to necessarily satisfying Collection/Replacement by Example 14.3. To still get a model satisfying Sep and even Coll based on constructible sets in **FTM*ω***, let us consider the collection of ordinals

$$\Omega = \{\alpha \in \mathrm{Ord}\colon \forall\, \gamma \le \alpha, \gamma \ge 1\, \exists\, f \in \mathbf{L}_{\gamma+1}(f : \omega \xrightarrow{\text{onto}} \mathbf{L}_\gamma)\}.$$

This definition distinguishes ordinals α which hereditarily (*i.e.* also for all smaller ordinals) satisfy the property that $\mathbf{L}_\alpha$ is countable in $\mathbf{L}_{\alpha+1}$. Similar ideas (for instance, the notion of a gap ordinal) are known in the studies of constructibility since 1970s, see *e.g.* [11, 12].

Let $\mathbf{L}^* = \bigcup_{\alpha\in\Omega} \mathbf{L}_\alpha$.

Lemma 16.1 (**FTM*ω***). *Either* (I) $\Omega = \mathrm{Ord}$, *and then* $\mathbf{L}^* = \mathbf{L}$, *or* (II) $\Omega \in \mathrm{Ord}$ *is a limit ordinal, and then* $\mathbf{L}^* = \mathbf{L}_\Omega$.

Proof. Clearly $0 \in \Omega$. To prove that $1 \in \Omega$, define a recursive bijection $b : \omega \times \omega \times \{0, \dots, 9\} \xrightarrow{\text{onto}} \omega$ by $b(m, n, i) = 10(2^m(2n + 1) - 1) + i$. Define by induction $h(0) = \varnothing$, and if $k = b(m, n, i) > 0$ (then $m, n < k$) then put $h(k) = \mathcal{F}_i(h(m), (h(n))$. Then $\mathtt{ran}\, h = \mathbf{L}_1 =$ all sets of finite rank, and $h \in \mathbf{L}_2$ by Lemma 13.1, as h is definable over $\mathbf{L}_1$.

It remains to prove that $\alpha \in \Omega \implies (\alpha + 1) \in \Omega$ for any $\alpha \ge 1$. By the hypothesis, there is a function $f \in \mathbf{L}_{\alpha+1}$, $f : \omega \xrightarrow{\text{onto}} \mathbf{L}_{\alpha+1}$. Put $h(0) = \mathbf{L}_\alpha$, $h(2n + 2) = f(n)$ for all n, and if $k = b(m, n, i)$ (see above about b), then

$h(2k+1) = \mathcal{F}_i(h(m),(h(n))$. Thus $h : \omega \xrightarrow{\text{onto}} \mathbf{L}_{\alpha+1} = \mathrm{GC}(\mathbf{L}_\alpha \cup \{\mathbf{L}_\alpha\})$. However h is definable in $\mathbf{L}_{\alpha+1}$, hence $h \in \mathbf{L}_{\alpha+2}$ by Lemma 13.1. □

We let $<^*$ be $<_{\mathbf{L}}$ in case $\Omega = \mathrm{Ord}$, and be $<_{\mathbf{L}} \restriction \mathbf{L}_\Omega$ in case $\Omega \in \mathrm{Ord}$.

Corollary 16.2 (**FTM$\boldsymbol{\omega}$**). *All axioms of $\mathbf{Z}^-$, minus* Sep*, hold in $\mathbf{L}^*$.*

The axiom Countability := *"all sets are countable" also holds in $\mathbf{L}^*$.*

$<^*$ is a well-ordering of $\mathbf{L}^*$ definable over $\mathbf{L}^*$.

Proof. Use Lemma 14.2 and Theorem 15.2. □

17 Check the schemata

The final step in the proof of Theorems 10.1 and 1.1 will be to verify the Separation and Collection schemata in $\mathbf{L}^*$. The following lemma is the crucial step; the rest of the verification of Sep and Coll via the reflection principle will be pretty standard.

Lemma 17.1 (**FTM$\boldsymbol{\omega}$**). *Let $F : \omega \to \Omega$ be a class-function definable over $\mathbf{L}^*$. Then the collection $\mathtt{ran}\, F = \{F(j) : j < \omega\}$ is bounded in Ω, i.e., $\mathtt{ran}\, F \subseteq \rho$ for some limit $\rho \in \Omega$.*

Proof. Assume towards the contrary that $\mathtt{ran}\, F$ is unbounded. Then $\mathbf{L}^* = \bigcup_{n<\omega} \mathbf{L}_{F(n)}$. For any n, let h_n be the $<^*$-least of the functions $h \in \mathbf{L}^*$ with $\mathtt{dom}\, h = \omega$ and $\mathbf{L}_{F(n)} \subseteq \mathtt{ran}\, h$. (Such functions exist by the definition of Ω.) Now if $k = 2^n(2m+1) - 1$, then we put $G(k) = h_n(m)$.

It follows by the choice of h_n that G is a class-function from ω onto $\mathbf{L}^*$. Moreover, G is definable in $\mathbf{L}^*$ as such are F and $<^*$. We conclude that the class $\mathbf{L}^*$ is a set by Lemma 10.2 since $\mathbf{L}^* = \mathtt{ran}\, F$ is transitive, hence $\Omega \subseteq \mathbf{L}^*$ is a set as well by Sep. This immediately contradicts the case $\Omega = \mathrm{Ord}$ since Ord is not a set even in $\mathbf{Z}^-$.

This Ω is a limit ordinal. Then $\mathbf{L}^* = \mathbf{L}_\Omega$, and we have $G \in \mathbf{L}_{\Omega+1}$ by Lemma 13.1. It follows that $\Omega \in \Omega$, which is a contradiction. □

Corollary 17.2 (**FTM$\boldsymbol{\omega}$**). *Let $X \in \mathbf{L}^*$ and $G : X \to \mathbf{L}^*$ be a class-function definable over $\mathbf{L}^*$. Then the collection $\mathtt{ran}\, G = \{G(x) : x \in X\}$ satisfies $\mathtt{ran}\, G \subseteq \mathbf{L}_\rho$ for some limit $\rho \in \Omega$.*

Proof. Let $f \in \mathbf{L}^*$, $f : \omega \xrightarrow{\text{onto}} X$ by Countability. Apply Lemma 17.1 for $F(k) =$ the least $\alpha \in \Omega$ satisfying $G(f(k)) \in \mathbf{L}_\alpha$. □

Corollary 17.3 (**FTM$\boldsymbol{\omega}$**, fixed point theorem). *Let $\alpha \in \Omega$, $m < \omega$, and $G_1, \dots, G_m : \mathbf{L}^* \to \mathbf{L}^*$ are class-functions definable over $\mathbf{L}^*$, then there is a limit ordinal $\beta \in \Omega$, $\beta > \alpha$, satisfying $G_k[\mathbf{L}_\beta] \subseteq \mathbf{L}_\beta$ for all $k = 1, \dots, m$.*

Proof. Put $G(x) = \langle G_1(x), \dots, G_m(x)\rangle$. Use Corollary 17.2 to get a class-sequence $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \dots$ of ordinals in Ω satisfying $G[\mathbf{L}_{\alpha_n}] \subseteq \mathbf{L}_{\alpha_{n+1}}$. Apply Lemma 17.1 to make sure that $\beta = \sup_n \alpha_n \in \Omega$. □

An ordinal $\beta \in \Omega$ *reflects* a parameter-free formula $\varphi(x_1, \dots, x_n)$, if the equivalence $\varphi^{\mathbf{L}^*}(x_1, \dots, x_n) \Longleftrightarrow \varphi^{\mathbf{L}_\beta}(x_1, \dots, x_n)$ holds for all $x_j \in \mathbf{L}_\beta$. The following is a standard consequence of Corollary 17.3.

Theorem 17.4 (**FTM$\boldsymbol{\omega}$**, reflection)**.** *Let $\alpha \in \Omega$ and φ is a parameter-free formula then there exists a limit ordinal $\beta \in \Omega$, $\beta > \alpha$ which reflects φ and every subformula of φ.*

Proof (sketch)**.** We w.l.o.g. assume that φ does not contain $\forall$ (otherwise replace $\forall$ with $\neg\exists\neg$). Let's enumerate $\psi_1, \dots, \psi_n$ all the sub-formulas of φ (including possibly φ itself) beginning with $\exists$. If $k \le n$ then we define a class-function G_k as follows.

Assume that $k \le n$ and ψ_k is $\exists y\, \chi_k(y, x_1, \dots, x_m)$. Suppose that $p = \langle x_1, \dots, x_m\rangle \in \mathbf{L}^*$ and there exists $y \in \mathbf{L}^*$ satisfying $\chi_k^{\mathbf{L}^*}(y, x_1, \dots, x_m)$. Then let $G_k(p)$ be the $<^*$-least of these y. Otherwise let $G_k(p) = \varnothing$. Each class-function G_k is definable in $\mathbf{L}^*$ since such is the ordering $<^*$.

By Theorem 17.4, there is an ordinal $\beta \in \Omega$, $\beta > \alpha$, satisfying $G_k[\mathbf{L}_\beta] \subseteq \mathbf{L}_\beta$ for all $k = 1, \dots, n$. Now it easily goes by induction on the number of logical symbols that β reflects every subformula of φ, in particular it reflects φ itself, as required. □

Corollary 17.5 (**FTM$\boldsymbol{\omega}$**)**.** Separation *and* Collection *hold in* $\mathbf{L}^*$*.*

Therefore **ZFC$^-$** *as a whole holds in* $\mathbf{L}^*$ *by Corollary 16.2.*

Proof. Separation. Assume that $\varphi(x, y)$ is a parameter-free formula, $\alpha \in \Omega$, $p \in X = \mathbf{L}_\alpha$. We have to prove that $Y = \{x \in X : \varphi^{\mathbf{L}^*}(x, p)\} \in \mathbf{L}^*$. Let, by Theorem 17.4, an ordinal $\beta \in \Omega$, $\beta > \alpha$ reflect $\varphi(x, y)$, so that

$$Y = \{x \in X : \varphi^{\mathbf{L}_\beta}(x, p)\}.$$

Then $X \in \mathbf{L}_\beta$ by Lemma 13.1, as required.

Collection. Assume that $\varphi(x, y, z)$ is a parameter-free formula, $\alpha \in \Omega$, $p \in X = \mathbf{L}_\alpha$, and we have $\forall x \in X\, \exists y \in \mathbf{L}^*\, \varphi^{\mathbf{L}^*}(x, y, p)$. By Theorem 17.4, there exists an ordinal $\beta \in \Omega$, $\beta > \alpha$ which reflects $\exists y\, \varphi(x, y, z)$, with all its subformulas, including $\varphi(x, y, z)$, so that

$$\forall x \in X\, \exists y \in \mathbf{L}_\beta\, \varphi^{\mathbf{L}_\beta}(x, y, p), \quad \text{and} \quad \forall x \in X\, \exists y \in \mathbf{L}_\beta\, \varphi^{\mathbf{L}^*}(x, y, p),$$

as required. □

Proof (Theorem 10.1)**.** Make use of Corollary 17.5. □

Proof (Theorem 1.1)**.** Make use of Corollary 9.3 and Theorem 10.1. □

18 Conclusions and problems

In this study, the methods of second-order arithmetic and set theory were employed to giving a full and self-contained proof of the formal equiconsistency of such theories as second-order arithmetic $\mathbf{PA}_2^-$ and Zermelo–Fraenkel $\mathbf{ZFC}^-$ sans the Power Set axiom (Theorem 1.1).

The following problems arise from our study.

Problem 18.1. In spite of Example 14.3, is it true in $\mathbf{FTM}\boldsymbol{\omega}$ that $\mathbf{L}$ satisfies Separation? The positive answer implies that $\mathscr{P}(\omega) \cap \mathbf{L}$ is a model of $\mathbf{PA}_2^-$, and hence even of full $\mathbf{PA}_2$ because AC_ω is provided by $<_{\mathbf{L}}$. This would lead to an interpretation of $\mathbf{ZFC}^-$ by Remark 9.4, and would provide another proof of Theorem 1.1.

Problem 18.2. The nature of the set or class $\mathbf{L}^*$ of Section 16 remains not fully clear. In $\mathbf{ZFC}$, obviously $\Omega \subseteq \omega_1$ is an ordinal, and accordingly $\mathbf{L}^* \subseteq \mathbf{L}_{\omega_1}$ is definitely a set. What about the possibility of the equality and of the strict inclusion $\subsetneqq$ in this relation?

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