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# Adding unbounded and not bounded sets to a singular cardinal

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**Abstract**

This paper solves a problem (SP) attributed to Solovay in the 1960s, which is to add an unbounded sequence to a singular cardinal $\kappa$ without adding bounded subsets to $\kappa$, while preserving $\kappa^+$. The solution applies to any cofinality of the singular cardinal, with the assumption that $\kappa$ is a strong limit needed for the case of uncountable cofinality. Unsuccessful attempts to solve (SP), mostly for the countable cofinality, have appeared in published papers by various authors.

## 1 Introduction and Motivation

In their 2020 paper [2] Dobrinen, Hathaway and Prikry present a method which they attribute to an unpublished paper by Prikry in the late 1960s. Its main purpose is to solve a problem posed by Solovay in the 1960s, which is:

**Solovay's problem (SP)**: Find a forcing to add an unbounded sequence to a singular cardinal $\kappa$ without adding any bounded sequences to $\kappa$, and while preserving $\kappa^+$.

Unfortunately, we are not convinced by the arguments from [2], as the literal translation of the "usual arguments" from arboreal forcings on $\omega$ which they relay upon, fails at many points in their intended argument. One of them is the compactness argument needed for certain topological arguments about $2^\omega$ and not present in their space $N$. A simplification of the argument from [2] was intended in [9], but just like [2], it fails an attempted fusion argument[1]. These papers have a predecessor [6, §1.3] which claims to have obtained the result from an $\omega$-sequence of measurables. However, it also fails an attempted fusion argument. The difficult part in all these approaches is to obtain that no bounded sequences are added to the singular cardinal. The purpose of this paper is to define a forcing $\mathbb{P}_{\bar\kappa}[\bar{\mathcal{F}}]$ and prove

**Theorem 1.1** (**The Solution of (SP)**). *Suppose that $\kappa$ is a singular cardinal of cofinality $\theta < \kappa$ and $\bar\kappa$ is an increasing $\theta$-sequence of regular cardinals with supremum $\kappa$. In addition, if $\theta > \aleph_0$, assume that $\kappa$ is a strong limit.*

*Then the forcing $\mathbb{P}_{\bar\kappa}[\bar{\mathcal{F}}]$ adds a $\theta$-sequence to $\kappa$ without adding any bounded subsets to $\kappa$, and it preserves $\kappa^+$.*

The problem of adding a subset to a singular cardinal without adding any bounded subsets to it was addressed by a different method in 1973 by Kanovei, [8]. Forcing over a model of $\mathbf{V} = \mathbf{L}$, for any cardinal $\lambda$, he constructs a forcing extension in which no new subsets of size $< \lambda$ are added to any ordinal $(< \lambda^+)^{\mathbf{L}}$, while a subset is added to $\lambda$ ([8, Th 2]). This does not solve (SP), because it collapses $\lambda^+$.

We address (SP) by a combination of ideas from [5], [3] and some new ideas. To explain this, let us first concentrate on the situation of $\mathrm{cf}(\kappa) = \aleph_0$. In the first place we adapt the generalised perfect set forcing from [5] to a singular cardinal $\kappa$ of countable cofinality, using a sequence

[1] In an e-mail dated 04 September 2026, Heike Mildenberger claimed that she had an explanation to the argument from [9] and she reiterated it in an e-mail dated 10 November 2026. Unfortunately, it enforces the incorrect attempt of using a Cantor-Bendixon argument to address the question of not adding bounded subsets to $\kappa$, in a situation where this argument does not apply.

$\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ of regular cardinals strictly increasing to $\kappa$ and a sequence $\bar{\mathcal{F}} = \langle \mathcal{F}_n : n < \omega \rangle$ of filters where each $\mathcal{F}_{n+1}$ is a $\kappa_n$-complete filter over $\kappa_{n+1}$. The forcing, which we denote $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$, uses trees of height $\omega$ where the splittings on the level $n$ are into a set in $\mathcal{F}_n$ and along any branch the set of splitting nodes is infinite. This allows to show that $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ adds an $\omega$-sequence to $\kappa$ and preserves $\kappa^+$. The more difficult part of the argument shows that the assumptiion of the filters being sufficiently closed gives that the forcing also does not add bounded sequences to $\kappa$. Hence we obtain a solution to the problem (SP) for the countable cofinality.

The situation of adding subsets to singulars of countable cofinality, although not the problem (SP) directly, was considered by the papers [1] and [10]. The paper [1] introduces a forcing called AIM forcing, which deals with the situation of a singular $\kappa$ of countable cofinality which is a limit of $\lambda$-supercompact cardinals for some $\lambda$ inaccessible $> \kappa$. It is shown in the main theorem of the paper that AIM forcing does not add bounded subsets of $\kappa$. However the forcing does not solve (SP) even under these large cardinal assumptions, since the theorem also shows that $\kappa^+$ is collapsed, and in fact $\kappa^+$ of the extension is $\lambda$. In [1] it is shown that this forcing fits the general framework of $\Sigma$-Prikry forcing introduced in that paper. It is possible that some other example of $\Sigma$-Prikry forcing might be applicable to (SP) of countable cofinality under large cardinals, although none of the presently known results fit this description. The method of $\Sigma$-Prikry forcing has some similarities to ours, in that it considers layering forcings into countably many suborders and that it works with a sequence of filters, but our forcing does not fit the axiom scheme of $\Sigma$-Prikry forcing.

Let us describe the situation of arbitrary cofinality of the singular. Levine and Mildenberger [9, Th. 3.1], who have claimed to have simplified the proof from [2], also proved that the two forcings from their paper, generalisations of Laver and Miller forcing to this context, do not solve (SP) for arbitrary cofinality. Our method does.

For this result we need to modify the definition of $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$, which we still call by the same name, since the version of $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ used for the proof of countable cofinality is a special case of the more general version. This modification is done by blending the definition of $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ from the proof of the countable case with the generalised perfect set forcing from [5].

In particular, Theorem 1.1 shows that (SP) does not require large cardinals[2].

The paper is organised as follows. In §2 we explain what can be taken from [2] for countable cofinality and why it does not solve (SP), as one cannot prove that the forcing does not add bounded subsets. In §2 we introduce a more complex forcing $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ which does solve (SP) for countable cofinality. In §3 we recall the Generalised Perfect Set forcing from [5] and in §5 we use it to define a version of the forcing $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ which works for any cofinality of the singular cardinal. Hence we prove Theorem 1.1 and solve (SP).

## 2 A Forcing Attributed to Prikry

We present the forcing we believe is intended in Dobrinen et al. [2][3]. The approach taken here is to make the forcing as close as possible to the ordinary perfect set forcing on $\omega$. Our argument for solving (SP) depends on the modification given by introducing $[\bar{\mathcal{F}}]$. We do not see how the original forcing of Prikry could have been made to work and we explain that in this section as a motivation.

Suppose that $\kappa$ is a singular cardinal with $\mathrm{cf}(\kappa) = \aleph_0$. Let $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ be an increasing sequence of regular cardinals with $\kappa_0 = 0$, $\kappa_1 \geq \aleph_0$ and $\sup_{n<\omega} \kappa_n = \kappa$. We shall work with

[2]In an e-mail dated 07 September 2026, Moti Gitik claimed that the argument for the countable cofinality would need an $\omega$-sequence of measurable cardinals.

[3]The authors of [2] work with a strong limit cardinal even in the case $\mathrm{cf}(\kappa) = \aleph_0$, but we have not found the use for it.

subtrees of the structure

$$T^* = \bigcup_{n<\omega} (\{n\} \times {}^{<\omega}\kappa_n),$$

where the elements of $T^*$ are partially ordered by the following relation:

$$(n,s) <^* (m,t)$$

if $n < m$ and $s \subseteq t$ or $n = 0$ and $s = \langle\rangle$ is the empty sequence. Hence, $T^*$ is a tree of height $\omega$ and cardinaliy $\kappa$. Notice that its definition depends on $\bar{\kappa}$, so we should in fact denote it by $T^*_{\bar{\kappa}}$. We omit $_{\bar{\kappa}}$ if the $\bar{\kappa}$ we mean is clear from the context.

**Definition 2.1.** Suppose that $T \subseteq T^*$ is a tree.

(1) We say that a node $(n,s) \in T$ is *terminal* if $|\{t : (n,s) <_T (n+1,t)\}| = 0$, otherwise the node is *non-terminal*. The non-terminal node is *splitting in* $T$ if $|\{t : (n,s) <_T (n+1,t)\}| > 1$, otherwise the non-terminal node $(n,s)$ is a non-splitting node. A splitting node $(n,s)$ is *fully-splitting* if $|\{t : (n,s) <_T (n+1,t)\}| = \kappa_{n+1}$.

(2) We say that a tree $T \subseteq T^*$ is $\bar{\kappa}$-*perfect* if

- every node in $T$ is either non-splitting or fully-splitting,
- every node in $T$ has a fully-splitting extension.

(In particular, $T$ has no terminal nodes).

(3) The forcing notion $\mathbb{P} = \mathbb{P}_{\bar{\kappa}}$ consists of all perfect subtrees of $T^*_{\bar{\kappa}}$ ordered by reverse inclusion.

(4) Suppose that $p$ is a $\bar{\kappa}$-perfect tree and $(o,t) \in p$. We denote by $\leq_p$ the order in $p$ and define

$$\deg_p((o,t)) = |\{(m,s) : (m,s) <_p (o,t) \text{ is splitting}\}|.$$

(5) For $p, p' \in \mathbb{P}$, we let $p \leq_n p'$ if $p \leq p'$ and for every $(o,t) \in p$,

$$\deg_p((o,t)) \leq n \implies (o,t) \in p'.$$

$\blacksquare_{\text{Def.2.1}}$

**Observation 2.2.** The relations $\leq_n$ are partial orders such that for $m \leq n$ we have $\leq_n \subseteq \leq_m$.

**Definition 2.3.** (1) For a condition $p \in \mathbb{P}$ we define *the stem* $s[p]$ as the unique splitting node of $p$ which has no splitting predecessors in $p$.

(2) For $(n,s) \in p$ by $p_{(n,s)}$ we denote the subtree of $p$ given by

$$p_{(n,s)} = \{(m,t) \in p : (m,t) \leq_p (n,s) \vee (n,s) \leq_p (m,t)\}.$$

(3) If $(n,s) \in p$ we say that $n$ is the *length of* $(n,s)$ and we denote $n = \lg((n,s))$. For $k \leq n$ we define $(n,s) \restriction k$ as the unique $(k,t) \in p$ with $(k,t) \leq_p (n,s)$. $\blacksquare_{\text{Def.2.3}}$

**Lemma 2.4 (The generic function).** *Suppose that $G$ is a $\mathbb{P}_{\bar{\kappa}}$-generic filter. Let us attempt to define a function $f : \omega \to \kappa$ by*

$$f(n) = \alpha \text{ iff there is } p \in G \text{ with } (s[p] = (o,t) \wedge o > n \wedge t(n) = \alpha).$$

*Then $f : \omega \to \kappa$ is a well-defined total function, it is not an element of the ground model and* $\sup(ran(f)) = \kappa$.

*Proof.* We exhibit certain dense sets in $\mathbb{P}$. For $n < \omega$ and $\alpha < \kappa$ let

$$\mathcal{D}_{n,\alpha} = \{p \in G : (\exists o, t)[s[p] = (o,t) \wedge o > n \wedge (\exists m \in \mathrm{dom}(t))t(m) > \alpha]\}.$$

We show that $\mathcal{D}_{n,\alpha}$ is a dense set in $\mathbb{P}_{\bar{\kappa}}$. Let $m$ be such that $\alpha < \kappa_m$. Given $p \in \mathbb{P}$ we choose a fully-splitting node $(o,t) \in p$ with $o > \max\{m,n\}$. Hence, there is an extension $(o+1,t')$ of $(o,t)$ in $p$ such that $t'(o) > \alpha$. Then letting $q = q_{(o+1,t')}$ gives $q \geq p$ and $q \in \mathcal{D}_{n,\alpha}$. Hence $\mathcal{D}_{n,\alpha}$ is dense.

For $g : \omega \to \kappa$ in the ground model we let

$$\mathcal{E}_g = \{p \in G : (\exists o, t)[s[p] = (o,t) \wedge (\exists m \in \mathrm{dom}(t))t(m) \neq g(m)]\}.$$

A proof similar to the above gives that $\mathcal{E}_g$ is dense in $\mathbb{P}$. Namely, given $p \in \mathbb{P}$, we find a fully-splitting node $(o,t) \in p$ of height at least 1 and then choose an extension $(o+1,t')$ of $(o,t)$ with $t'(o) \neq g(o)$. Then defining $q$ as above gives $q \geq p$ and $q \in \mathcal{E}_g$.

The fact that $G$ is a filter gives that if $p, q \in G$ then $p, q$ are compatible, hence $s[p] \subseteq s[q]$ or vice versa. Hence $f$ is a function. By the genericity and the definition of the sets $\mathcal{D}_{n,\alpha}$ we have that $f$ is a total function on $\omega$ and that its range is unbounded in $\kappa$. Finally, by the definition of the sets $\mathcal{E}_g$ we have that it is a function which is not in the ground model. $\bigstar_{2.4}$

**Definition 2.5.** With the notation of Lemma 2.4, we denote $f = g[G]$. $\blacksquare_{\mathrm{Def.2.5}}$

**Lemma 2.6** (**Fusion Lemma**). *Suppose that $\langle p_n : n < \omega\rangle$ is a sequence of conditions in $\mathbb{P}_{\bar{\kappa}}$ such that $p_n \leq_n p_{n+1}$ for every $n < \omega$. Then $q = \bigcap_{n<\omega} p_n$ is a condition in $\mathbb{P}_{\bar{\kappa}}$ and it satisfies that for every $n < \omega$ we have $p_n \leq_n q$.*

*Proof.* Suppose that $m < \omega$ and $(o,t) \in p_m$ is such that $\deg_{p_m}((o,t)) \leq m$. By Observation 2.2 we have that $(o,t) \in p_n$ for every $n \geq m$. In particular $(o,t) \in q$.

To finish the proof, we need to prove that $q$ is a perfect subtree of $T^*$. Let $(o,t) \in q$ and let $n = \deg_q((o,t))$. Let $(o',t')$ be a fully-splitting extension of $(o,t)$ in $p_{n+1}$ and of the shortest height. In particular, the $\deg_{p_{n+1}}((o'+1,t'')) = n+1$, for any $(o'+1,t'') \geq_{p_{n+1}} (o',t')$. By the definition of $\leq_{n+1}$ we have that $(o',t'),(o'+1,t'') \in q$ for any such $t''$. Hence, $(o',t')$ is a fully-splitting extension of $(o,t)$ in $q$. Therefore, every node in $q$ has a fully-splitting extension.

A similar argument proves that every splitting node in $q$ is fully-splitting. Namely, suppose that $(o,t)$ is a splitting node in $q$ and let $n = \deg_q((o,t))$. Then in particular $(o,t)$ is a splitting node in $p_{n+1}$ and all its $p_{n+1}$-immediate successors have degree $n+1$ in $p_{n+1}$. Therefore, they continue being immediate successors of $(o,t)$ in $q$, and since $(o,t)$ is fully-splitting in $p_{n+1}$, it continues being fully splitting in $q$. $\bigstar_{2.6}$

The following property 2.7(1) was called *Mastering* $\kappa$ in [3]. It is typical of the variants of the Perfect Set forcing and is used for the preservation of $\kappa^+$.

**Lemma 2.7** (**Mastering $\kappa$ and Preservation of $\kappa^+$ Lemma**). *(1) Suppose that $n < \omega$ and $p \Vdash$ "$\underset{\sim}{x}$ is an ordinal". Then there is $r \geq_{n+1} p$ and a set $X$ of size $\leq \kappa$ such that $r \Vdash$ "$\underset{\sim}{x} \in X$".*
*(2) $\mathbb{P}$ preserves $\kappa^+$.*

*Proof.* (1) Let $\{s_\alpha : \alpha \leq \alpha^*\}$ be an enumeration of all $s$ with $(m,s) \in p$ for some $m$ and such that and $\deg_p((m,s)) = n$. So $|\alpha^*| \leq |p| \leq \kappa$. For each $s_\alpha$ let $m_\alpha < \omega$ be such that $(m_\alpha, s_\alpha) \in p$. For each $\alpha$ and $i < \kappa_{m_\alpha+1}$ such that $(m_\alpha+1, s_\alpha \frown i) \in p$, let $r_{\alpha,i} \geq p_{(m_\alpha+1, s_\alpha \frown i)}$ be some condition that decides the value of $\underset{\sim}{x}$, say $\gamma_{\alpha,i}$. Let $X = \{\gamma_{\alpha,i} : \alpha < \alpha^*, i < \kappa\}$, so $|X| \leq \kappa$.

Now let $r = \bigcup_{\alpha<\alpha^*, i<\kappa} r_{\alpha,i}$ and observe that $r \in \mathbb{P}$ and $r \geq_{n+1} p$. Suppose that $q \geq r$ and $q$ decides the value of $\underset{\sim}{x}$. Then $q$ must be compatible with some $r_{\alpha,i}$ and hence the value decided by $q$ for $\underset{\sim}{x}$ must be one among the $\gamma_{\alpha,i}$'s. Therefore $r \Vdash$ "$\underset{\sim}{x} \in X$".
(2) The argument of (1) shows that $\kappa^+$ is preserved, because suppose otherwise, that $\kappa^+$ is collapsed to $\kappa$. In particular, the forcing provides a cofinal $\omega$-sequence in $(\kappa^+)^M$. Let $\underset{\sim}{\tau}$ be a $\mathbb{P}$-name for such a function. By induction on $n < \omega$ we construct a sequence $\langle (p_n, X_n) : n < \omega\rangle$ such that $p_0 = \emptyset_{\mathbb{P}}$ and for every $n < \omega$:

- $p_n \leq_{n+1} p_{n+1}$ and
- $p_{n+1} \Vdash$ "$\tau(n) \in X_n$",
- $|X_n| \leq \kappa$.

We use (1) at successor stages. Let $q$ be the fusion of $\langle p_n : n < \omega \rangle$ and $X = \bigcup_{n<\omega} X_n$. Then $X \in M$ is a set of size $\leq \kappa$ and therefore $\alpha = \sup(X) < \kappa^+$, while $q \Vdash$ "$\sup(\mathrm{ran}(\tau)) \leq \alpha$". This is in contradiction with the choice of $\tau$. $\bigstar_{2.7}$

We can go an and prove more properties of $\mathbb{P}$, but the difficulty of this approach without any modifications is that there is no reason to believe that no bounded sequences are added to $\kappa$. To start resolving that problem, in §3 we pass to the modified version using Generalised Perfect Set forcing, as announced in the introduction. The modification will allow us to have closure properties on the orders $\leq_{n+1}$, which will control the subsets added to $\kappa_n$.

# 3 Generalised Perfect Set Forcing at a singular cardinal and the solution to (SP) for countable cofinality

We use the same notation from §2. In this section we prove the case of Theorem 1.1 for singular cardinals of countable cofinality.

We start by modifying the forcing $\mathbb{P}_{\bar{\kappa}}$. Now we also choose a sequence $\bar{\mathcal{F}} = \langle \mathcal{F}_n : n < \omega \rangle$, where $\mathcal{F}_0 = \emptyset$ and each $\mathcal{F}_{n+1}$ is a $(< \kappa_n)$-complete filter on $\kappa_{n+1}$. The forcing we define will be denoted $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$. The conditions in $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ are defined similarly to those in $\mathbb{P}_{\bar{\kappa}}$, except that we modify what we mean by *fully-splitting*:

**Definition 3.1.** (1) We say that a node $(n, t)$ in a perfect tree $T \subseteq T^*$ is *fully-splitting for* $\bar{\mathcal{F}}$ if

$$\{t'(n) : (n, t) <_T (n + 1, t')\} \in \mathcal{F}_{n+1}.$$

(2) The conditions, the order and the orders $\leq_n$ are defined in $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ in the same way as in $\mathbb{P}_{\bar{\kappa}}$, except that the requirement on fully-splitting is everywhere replaced by fully-splitting for $\bar{\mathcal{F}}$. $\blacksquare_{\text{Def.3.1}}$

**Observation 3.2.** Lemma 2.4, Lemma 2.6 and Lemma 2.7 hold with $\mathbb{P}_{\bar{\kappa}}$ replaced by $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$, using the same proofs.

In the rest of the paper, we shall continue to use the notation $f = g[G]$ as in the above. From this point on we keep $\bar{\kappa}$ and $[\bar{\mathcal{F}}]$ fixed and use $\mathbb{P}$ to denote $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$.

Next, in Lemma 3.5 we prove that $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ does not add any bounded subsets to $\kappa$, so completing the countable cofinality case of the proof of Theorem 1.1.

We need to introduce certain auxiliary orders, which are not the same as the $\leq_n$ we used in the fusion argument.

**Definition 3.3.** (1) For $p, q \in \mathbb{P}$ we write $\equiv_n$ for the equivalence relation which holds if $p, q$ have the same first $n$-levels.
(2) For $p, q \in \mathbb{P}$, we write $p \preceq_n q$ if $p \leq q$ and and $p \equiv_n q$.

$\blacksquare_{\text{Def.3.3}}$

**Lemma 3.4.** *For every $n < \omega$, the order $\preceq_n$ is $(< \kappa_n)$-closed in $\mathbb{P}$.*

*Proof.* Suppose that $\zeta^* < \kappa_n$ and $\bar{p} = \langle p_\zeta : \zeta < \zeta^* \rangle$ is a increasing in $\preceq_n$. Then $q = \bigcap_{\zeta<\zeta^*} p_\zeta$ is a common extension of $\bar{p}$, because all $p_\zeta$ have the same first $n$ levels, and the splitting on higher levels come from filters which are at least $(< \kappa_n)$-complete. $\bigstar_{3.4}$

**Lemma 3.5** (**Main Lemma for Theorem 1.1, countable cofinality**). *The forcing $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ does not add bounded subsets to $\kappa$.*

*Proof.* Let $\mathbb{P}$ denote $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$ and suppose that $\underset{\sim}{X}$ is a $\mathbb{P}$-name for a subset of $\kappa_n$ for some $n$. Let $p \in P$ and $\alpha < \kappa_n$. We first construct $q = q_\alpha(p)$ such that

- $p \preceq_{n+1} q$ and
- whenever $r, r' \geq q$ are such that $\lg(s[r]), \lg(s[r']) \geq n+1$ and $r, r'$ decide if $\alpha \in \underset{\sim}{X}$, then

$$s[r] \restriction (n+1) = s[r'] \restriction (n+1) \implies [r \Vdash \text{“}\alpha \in \underset{\sim}{X}\text{”} \iff r' \Vdash \text{“}\alpha \in \underset{\sim}{X}\text{”}].$$

To construct $q$, first fix $(n+1, s) \in p$. Let $m > n+1$ be the least such that $m$ is the splitting level of any condition extending $(n+1, s)$. Hence, there is a unique $o \supseteq s$ such that $(m, o)$ is a splitting node in $p$. Given that $p_{(n+1,s)}$ forces that $\underset{\sim}{X}$ is a subset of $\kappa_n$, for every $\zeta < \kappa_m$ there is a condition extending $p_{(m,o^\frown\zeta)}$ which decides if $\alpha \in \underset{\sim}{X}$. Every condition extending $p_{(m,o^\frown\zeta)}$ is of the form $p_{(k,t)}$ for some $k \geq m$ and $t \supseteq o$. So we can define the sets

$$X_0 = \{\zeta < \kappa_m : (\exists (k,t))(k \geq m \wedge t \supseteq o^\frown\zeta \wedge p_{(k,t)} \Vdash \text{“}\alpha \notin \underset{\sim}{X}\text{”}\}$$

and

$$X_1 = \{\zeta < \kappa_m : (\exists (k,t))(k \geq m \wedge t \supseteq o^\frown\zeta \wedge p_{(k,t)} \Vdash \text{“}\alpha \in \underset{\sim}{X}\text{”}\}.$$

Observe that $X_0 \cup X_1 = \{\zeta < \kappa_m : (m, o^\frown\zeta) \in p\} \in \mathcal{F}_m$. We do not claim that the sets $X_0$ and $X_1$ are disjoint, but since $\mathcal{F}_m$ is a filter, at least one of the sets $X_0$ and $X_1$ is in $\mathcal{F}_m$.

Let us suppose that $X_0 \in \mathcal{F}_m$, if not we do a similar construction using $X_1$ instead. Let $(k,t) \in p$ be such that $k$ is the least $\geq m$ exemplifying that $\zeta \in X_0$. We define $q^{(n+1,s)} = p_{(k,t)}$.

Now we choose such $q^{(n+1,s')}$ for every $s'$ such that $(n+1, s') \in p$ and define

$$q = \bigcup_{\{s' : (n+1,s') \in p\}} q^{(n+1,s')}.$$

We claim that $q$ is as required. Clearly $p \preceq_{n+1} q$ since every $(n+1, s) \in p$ is contained in $q^{(n+1,s)}$, and $q^{(n+1,s)} \subseteq p$.

Now suppose that $r, r' \geq q$ are conditions that decide if $\alpha \in \underset{\sim}{X}$ whose both stems are of length at least $n+1$, and such that $s[r] \restriction (n+1) = s[r'] \restriction (n+1) = (n+1, s)$. In particular, both $r$ and $r'$ are compatible with $q^{(n+1,s)}$ and therefore $r$ and $r'$ force that $\alpha \in X$ iff $q^{(n+1,s)}$ does.

Having shown how to construct $q_\alpha(p)$ for a fixed $p$ and $\alpha$, we proceed to define by induction on $\alpha < \kappa_n$ a sequence $\langle p_\alpha : \alpha < \kappa_n \rangle$, as follows:

- $p_0 = p$,
- $p_{\alpha+1} = q_\alpha(p_\alpha)$,
- $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$ if $0 < \delta < \kappa_n$ is a limit ordinal.

Notice that there is no problem in doing the limit steps, because the sequence is chosen to be $\prec_{n+1}$-increasing and by Lemma 3.4 this order is $(< \kappa_{n+1})$-closed. For the same reason, defining $q^* = \bigcap_{\alpha<\kappa_n} p_\alpha$, we obtain a condition in $\mathbb{P}$. This condition is an extension of $p$ and we claim that it forces that $\underset{\sim}{X}$ is an element of $M$. Once we show this, the proof of the lemma is complete, since we have effectively shown that the set of conditions that force that $\underset{\sim}{X}$ is an element of $M$, is dense.

So suppose that $r \geq q^*$ forces that $\underset{\sim}{X}$ is not an element of $M$. Without loss of generality we can extend $r$ so that its stem has length at least $n+1$. Let us denote its restriction to $n+1$ by $(n+1, s)$. Our construction was made so that

$$r \Vdash \text{“}\underset{\sim}{X} = Y = \{\alpha < \kappa_n : p_\alpha^{(n+1,s)} \Vdash \text{“}\alpha \in \underset{\sim}{X}\text{”}\}$$

and $Y$ is a set in $M$. A contradiction. $\bigstar_{3.5}$

This finishes the proof of the countable cofinality case of Theorem 1.1.

# 4 Generalised Perfect Set forcing at a regular cardinal-recall

Let us recall the definition of the forcing $\mathbb{P}(\mathcal{F})$ introduced in [5] and further studied in [3]. The forcing $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$, which is our main interest in this paper, is an adaptation to a singular cardinal of the forcing $\mathbb{P}(\mathcal{F})$. On the other hand, for uncountable $\kappa$, the forcing $\mathbb{P}(\mathcal{F})$ is the appropriate generalisation of the perfect set forcing. At $\kappa = \aleph_0$, the direct predecessor of these definitions is Grigorieff forcing from [7]. We recall the relevant definitions in order to blend them with the method of the proof of the countable cofinality case to pass to the arbitrary cofinality in Theorem 1.1.

**Definition 4.1.** Let $\kappa$ be an infinite regular cardinal and let $\mathcal{F}$ be a $(<\kappa)$-complete filter on $\kappa$. We say that a subtree $p \subseteq {}^{<\kappa}\kappa$ is *$\mathcal{F}$-perfect* if

- $\sup\{\mathrm{dom}(s) : s \in p\} = \kappa$, and
- for any $s \in p$, there is $t \supseteq s$ such that $\{\alpha < \kappa : t \frown \alpha \in p\} \in \mathcal{F}$.

We say that such $t$ as in the last property is *$\mathcal{F}$-splitting*. For $s \in p$ we define its *splitting history* as

$$\mathrm{deg}_p(s) = \{i < \mathrm{dom}(s) : (\exists t \in p)\,[t \restriction i = s \restriction i \text{ and } t(i) \neq s(i)]\}.$$

(1) The $\mathcal{F}$-perfect tree forcing $\mathbb{P} = \mathbb{P}(\mathcal{F})$ is defined as follows:

The conditions in $\mathbb{P}$ are $\mathcal{F}$-perfect subtrees $p$ of ${}^{<\kappa}\kappa$, satisfying the following additional conditions:

(o) For every $s \in p$, either $|\{\alpha < \kappa : s \frown \alpha \in p\}| = 1$, or $\{\alpha < \kappa : s \frown \alpha \in p\} \in \mathcal{F}$.

(a) [Closure of $p$] If $\delta < \kappa$ is a limit ordinal and $s \in {}^{\delta}\kappa$ is such that for all $\alpha < \delta$ we have $s \restriction \alpha \in p$, then $s \in p$.

(b) [Closure of splitting for every node] For every $s \in p$, the set $\mathrm{deg}_p(s)$ is a closed subset of $\mathrm{dom}(s)$.

The order is defined by $p \leq q$ if $p \supseteq q$.

The *stem* of $p$, denoted by $s[p]$ is defined by

$$s[p] = \bigcup\{s \in p : \text{no } t \subset s \text{ is splitting and } s \text{ is splitting}\}.$$

(Note that this is well-defined by (o) and that in fact $s[p]$ is a uniquely defined element of $p$ since there cannot be two incomparable $s$ satisfying the conditions in the definition of $s[p]$).

(2) For $\zeta < \kappa$ the order $\leq_\zeta$ is defined by letting $p \leq_\zeta q$ if

- $p \leq q$,
- for all $s \in p$ with $\mathrm{otp}(\mathrm{deg}_p(s)) < \zeta$ we have $s \in q$.

The following were proved in [3] and [4]:

**Fact 4.2.** (1) Every sequence of the form $\bar{p} = \langle p_\zeta : \zeta < \kappa\rangle$ such that $p_\zeta \leq_\zeta p_{\zeta+1}$ for each $\zeta$ and $p_\delta = \bigcap_{\zeta<\delta} p_\zeta$ for $\delta \in (0,\kappa)$ a limit ordinal, has a least upper bound which is $q = \bigcap_{\zeta<\kappa} p_\zeta$. It in addition satisfies $p_\zeta \leq_\zeta q$, for every $\zeta < \kappa$.

(2) Suppose that $\kappa > \aleph_0$. The generic function $g = g_G : \kappa \to \kappa$ added by $\mathbb{P}(\mathcal{F})$ satisfies that for every $f : \kappa \to \kappa \in M$ there is a club $C = C_f$ of $\kappa$ in $M$ such that for every $\alpha \in C$ we have $f(\alpha) < g(\alpha)$.

A sequence $\bar{p}$ as in Fact 4.2(2) is called a *fusion sequence*.

# 5 The proof of Theorem 1.1, completing the solution to (SP)

In this section we are going to adapt the proof of the countable cofinality case to singular cardinals $\kappa$ of uncountable cofinality, and so obtain the complete proof of Theorem 1.1

We start with a singular cardinal $\kappa$ with $\theta = \mathrm{cf}(\kappa) < \kappa$ arbitrary. Since we have already dealt with the countable cofinality, for notational purposes we assume $\theta > \aleph_0$. We also assume that $\kappa$ is a strong limit, which is necessary for the Mastering $\kappa$ argument in this case - see the explanation after the definition of $T^*$ below. Let us fix an increasing sequence $\bar{\kappa} = \langle \kappa_i : i < \theta \rangle$ of regular cardinals satisfying:

- $\kappa_0 = \theta$,
- $\kappa_{i+1} \geq 2^{\kappa_i}$ (this is possible, since $\kappa$ is a strong limit),
- $\kappa_\delta \geq \sup_{i<\delta} \kappa_i$ for $\delta \in (0, \theta)$ a limit ordinal and
- $\sup_{i<\theta} \kappa_i = \kappa$.

We also choose a sequence $\bar{\mathcal{F}} = \langle \mathcal{F}_{i+1} : i < \theta \rangle$, where each $\mathcal{F}_{i+1}$ is a $(< \kappa_i)$-complete filter on $\kappa_{i+1}$. The forcing we define will be denoted $\mathbb{P}_{\bar{\kappa}}[\bar{\mathcal{F}}]$.

We shall work with subtrees of the structure

$$T^* = T^*_{\bar{\kappa}} = \bigcup_{i<\theta} (\{i\} \times {}^i\kappa_i),$$

where the elements of $T^*$ are partially ordered by the following relation:

$$(i, s) <^* (j, t)$$

if $i < j$ and $s \subseteq t$ or $n = 0$ and $s = \langle\rangle$ is the empty sequence. We introduce the concept of generalised perfect subtrees of $T^*$, which is similar to what we did in Definitions 2.1 and 3.1. However, we embed into the definition the requirements from Definition 4.1(1).

In the argument for Mastering $\kappa$, similar to that of Lemma 2.7 , and hence for the preservation of $\kappa^+$ we need that $|T^*| = \kappa$. This was automatic in the case $\mathrm{cf}(\kappa) = \aleph_0$, but in general we need to assume that $\kappa$ is a strong limit.

**Definition 5.1.** Suppose that $p \subseteq T^*$ is a tree.

(1) We say that a node $(i, s) \in p$ is *terminal* if $|\{t : (i, s) <_p (i, t)\}| = 0$, otherwise the node is *non-terminal*. The non-terminal node $(i, s)$ is *splitting in* $p$ if $|\{t : (i, s) <_p (i + 1, t)\}| \geq 1$, otherwise the non-terminal node $(i, s)$ is a non-splitting node. A splitting node $(i, s)$ is *fully-splitting for* $\bar{\mathcal{F}}$ if

$$\{t(i) : (i, s) <_p (i + 1, t)\}| \in \mathcal{F}_{i+1}.$$

For $(i, s) \in p$ we define its *splitting history* as

$$\mathrm{deg}_p((i, s)) = \{j < i : (j, s \restriction j) \text{ is fully-splitting for } \bar{\mathcal{F}}\}.$$

(2) We say that a tree $p \subseteq T^*$ is $(\bar{\kappa}, \bar{\mathcal{F}})$-*perfect* if

- $\sup\{i : (\exists s)(i, s) \in p\} = \theta$,
- for any $(i, s) \in p$, there is $(j, t) >_p (i, s)$ which is fully-splitting for $\bar{\mathcal{F}}$,
- (a) [Closure of $p$] If $\delta < \theta$ is a limit ordinal and $s \in {}^\delta\kappa$ is such that for all $i < \delta$ we have $(i, s \restriction i) \in p$, then $(\delta, s) \in p$, and
- (b) [Closure of splitting for every node] For every $(i, s) \in p$, the set $\mathrm{deg}_p((i, s))$ is a closed subset of $i$.

(3) The conditions in $\mathbb{P} = \mathbb{P}_{\bar{\kappa}}[\mathcal{F}]$ are $(\bar{\kappa}, \bar{\mathcal{F}})$-perfect subtrees $p$ of $T^*$, which we order by reverse inclusion.
(4) For $p, p' \in \mathbb{P}$ and $\zeta < \theta$, we let $p \leq_\zeta p'$ if $p \leq p'$ and for every $(i,s) \in p$,

$$\deg_p((i,s)) \leq \zeta \implies (i,s) \in p'.$$

■$_{\text{Def.5.1}}$

The proof of Theorem 1.1 now proceeds exactly as in the case of countable cofinality, but using the version of the forcing $\mathbb{P}_{\bar{\kappa}}[\mathcal{F}]$ as presented here.

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