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Symmetric Matrices Whose Entries Are Linear Functions

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Abstract—There exists a large set of real symmetric matrices whose entries are linear functions in several variables such that each matrix in this set is definite at some point, that is, the matrix is definite after substituting some numbers for variables. In particular, this property holds for almost all such matrices of order two with entries depending on two variables. The same property holds for almost all matrices of order two with entries depending on a larger number of variables when this number exceeds the order of the matrix. Some examples are discussed in detail. Some asymmetric matrices are also considered. In particular, for almost every matrix whose entries are linear functions in several variables, the determinant of the matrix is positive at some point and negative at another point.

Keywords: linear algebra, symmetric matrix, semidefinite programming, Hessian matrix

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1. INTRODUCTION

We consider real matrices whose entries are linear functions in several variables. Generally speaking, these functions are not homogeneous. The most interesting case is when a matrix is symmetric and the number of independent variables is equal to the order of the matrix. An example is the Hessian matrix (matrix of second-order partial derivatives) of a polynomial of third degree in several variables. In particular, the Hessian matrix is useful in searching for a minimum of such a polynomial on a compact set [1]. However, not every symmetric $n \times n$ matrix with entries depending on n variables can be obtained in this way. For example, a diagonal 2×2 matrix $\mathrm{diag}(x_2, x_1)$ with a reverse order of variables fails to be the Hessian matrix of a polynomial in two variables x_1 and x_2.

The matrices under consideration are closely related to 3-tensors whose coordinates are numerical coefficients from the matrix entries. However, this paper deals with properties of matrices obtained when some numbers are substituted for variables. In particular, is such a matrix definite at some point? These problems are reduced to the solvability of a system of nonlinear inequalities. The latter problem has been considered in numerous publications (see, e.g., [2–4]). However, in the generic case, this problem has a high computational complexity. On the other hand, the relation between these matrices and 3-tensors is also useful for estimating computational complexity; many problems of this type are NP-hard [5].

If the Hessian matrix of a polynomial in two variables is definite at some point, then the graph of this polynomial has an elliptic point, in particular, this graph is not a ruled surface. This explains the particular interest in the properties of Hessian matrices of polynomials in the simulation of surfaces of complex geometry in computer graphics and computer-aided design systems [6].

The search for a point at which a matrix is definite is closely related to semidefinite programming problems, which are a special case of cone programming [7–10]. Graph theory is related to the close problem of symmetric positive semidefinite matrix completion in the case when the matrix entries on the main diagonal and some other entries are fixed [11]. Semidefinite programming can also be used in other combinatorial problems [12], although it is well known that an algorithmically difficult problem cannot be approximated by a low-dimensional semidefinite programming problem [13].

Special matrices whose entries are linear functions over various fields have been studied by numerous authors. Asymmetric matrices with variable (possibly coinciding) entries were considered in [14–16], where the normality property and the rank of such matrices were studied. Matrices with affinely independent columns whose entries are linear functions such that no variable occurs in two different columns were considered in [17–19] under the condition that this constraint on the rank holds for any values of the variables.

Properties that hold everywhere, except for a subset of small (e.g., zero) measure, play an important role in the development of heuristic algorithms [20–22].

In discussing the computational complexity, it can be assumed that the considered linear functions are defined over a finite extension of the field of rational numbers embedded in the field of real numbers. Then computations can be performed in computer algebra systems [23]. The definiteness of a symmetric numerical matrix can be checked using LDU decomposition produced by the corresponding routine in MathPartner. Alternatively, the check can be based on straightforward calculations of corner minors. According to the Sylvester criterion, a symmetric matrix A is positive definite if and only if its corner minors Δ_k are all positive. Then $-A$ is a negative definite matrix and its corner minors have alternating signs: the corner minors Δ_{2k} of even order are positive, while the corner minors Δ_{2k+1} of odd order are negative. The determinant of the matrix can be computed in polynomial time and its upper bound depends on the complexity of the matrix multiplication. Upper bounds for multiplication complexity were considered, for example, in the recently published works [24–27]. On the other hand, the computational complexity of multiplication and division operations over a finite extension of the ground field is connected with the complexity of the algorithm used for computing the greatest common divisor of polynomials [28].

Given a set of matrices with entries depending on parameters ξ_1, ..., ξ_m, some property $\Phi(\xi_1,\ldots,\xi_m)$ holds for almost every matrix of this set if there exists a polynomial $p(\xi_1,\ldots,\xi_m)$ not identically zero such that, for each set of values of ξ_1, ..., ξ_m, if the property $\Phi(\xi_1,\ldots,\xi_m)$ does not hold, then the polynomial vanishes: $p(\xi_1,\ldots,\xi_m) = 0$. A property that holds for almost every set of parameter values does not hold only on a nowhere dense set of measure zero. For example, for symmetric 2×2 matrices

$$\begin{pmatrix} \xi_1 & \xi_3 \\ \xi_3 & \xi_2 \end{pmatrix}$$

with three entries being independent parameters ξ_1, ξ_2, and ξ_3, almost every matrix is nonsingular, since the polynomial $p(\xi_1,\xi_2,\xi_3) = \xi_1\xi_2 - \xi_3^2$ vanishes on singular matrices. However, in the general case, the set of collections of values for which the property Φ does not hold can be a proper subset of the zero set of the polynomial p.

For matrices whose entries are linear functions in n variables, the set of parameters is the set of numerical coefficients of all these functions. In the general case, an $n\times n$ matrix of this kind depends on $n^2(n+1)$ independent parameters. If this matrix is additionally assumed to be symmetric, then it depends only on $\frac{1}{2}n(n+1)^2$ independent parameters.

Let $\mathrm{diag}(\ell_1,\ldots,\ell_n)$ denote a diagonal matrix whose entries on the main diagonal are ℓ_1, ..., ℓ_n in the indicated order. For example, almost every 2×2 matrix of the form $\mathrm{diag}(\xi_1x_1+\xi_3, \xi_2x_2+\xi_4)$ is not identically zero, since this is equivalent to the fact that the value of the polynomial $\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2$ in the parameters ξ_1, ξ_2, ξ_3, and ξ_4 is nonzero. However, almost every matrix of this type is equal to a zero matrix at some point, since, if the value of the polynomial $\xi_1^2+\xi_2^2$ is nonzero, then the coordinates of this point are the numbers $x_1 = -\xi_3/\xi_1$ and $x_2 = -\xi_4/\xi_2$.

2. RESULTS

Theorem 1. *For almost every $n\times n$ matrix A whose entries are linear functions in n variables, there exists a point at which the determinant of A is positive and there exists another point at which the determinant of A is negative. The same is true for almost every symmetric $n\times n$ matrix of the considered form.*

Proof. In the general case, the determinant $\det(A)$ does not vanish identically, but vanishes at some point. To find this point, it suffices to indicate a point P at which all entries of the first row are simultaneously zero. The point P solves a system of n inhomogeneous linear algebraic equations with n unknowns. In the general case, this system is nondegenerate and has a unique solution. Indeed, for this to be true, it is sufficient that the determinant of an auxiliary $n\times n$ matrix made up of the coefficients of the linear terms of those functions that are the entries of the first row of the original matrix A be nonzero. This determinant is equal to a polynomial in the numerical parameters determining the matrix A. Therefore, it is nonzero for almost every matrix A. Since we consider only one row, this argument does not change in the case of symmetric matrices of this type.

A similar argument shows that, for almost every matrix A, there exists a straight line L passing through the point P such that the entries of the first row of A at each point of L vanish identically, except for the first entry of the first row. Let y denote the value of this entry. In the general case, when y is not identically zero, the value y determines a point on L. In particular, the zero value $y = 0$ corresponds to the point P. The value of the determinant of A at points on L is equal to $y\det(B)$, where the $(n-1)\times(n-1)$ matrix B is obtained from A by deleting the first row and the first column. Since B is independent of the entries of the first row of A, it is independent of the choice of the point P and the straight line L. Therefore, for almost every matrix A with a fixed first row, the submatrix B is nonsingular at the point P. Therefore, in a sufficiently small neighborhood of P on the straight line L, the determinant $\det(B)$ is nonzero and does not change its sign. However, in this neighborhood, the determinant $\det(A) = y\det(B)$ changes its sign depending on the sign of y. Therefore, in the neighborhood of P, the determinant of A takes both positive and negative values. The same argument holds in the case of symmetric matrices of this type.

Theorem 2. *For almost all sets of linear functions $\ell_0, \ldots, \ell_n$ in n variables $x_1, \ldots, x_n$ and for any $n\times n$ matrix M, the matrix*

$$\ell_0 M + \operatorname{diag}(\ell_1, \ldots, \ell_n)$$

is definite at some point. In particular, almost every symmetric 2×2 matrix whose entries are linear functions in two variables is definite at some point.

Proof. Without loss of generality, we may assume that none of the linear functions $\ell_0, \ldots, \ell_n$ is a linear combination of the others. Otherwise, the $(n+1)\times(n+1)$ matrix of their coefficients is singular, i.e., its determinant—a polynomial of degree $n+1$ in these coefficients—is zero. Moreover, it may be assumed that these linear functions are $\ell_1 = x_1, \ldots, \ell_n = x_n$, and some $\ell_0(x_1, \ldots, x_n)$ that does not vanish at the origin. Since the hyperplane $\ell_0 = 0$ is not incident to the origin, it contains a point of the first orthant, all of whose coordinates are positive, or a point of the opposite orthant, all of whose coordinates are negative. At this point, the matrix $\ell_0 M + \operatorname{diag}(x_1, \ldots, x_n)$ is diagonal and definite.

Each symmetric 2×2 matrix can be decomposed into the sum $\ell_0 J_2 + \operatorname{diag}(\ell_1, \ell_2)$, where J_2 denotes the exchange matrix

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following example shows that determining whether there exists a point at which the matrix from Theorem 2 is definite is not always a trivial task.

Example. Consider a matrix depending on a numerical parameter μ, namely,

$$\mu(x_1 + x_2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}.$$

Its determinant is equal to the polynomial $\mu x_1^2 + \mu x_2^2 + (1 + 2\mu)x_1 x_2$. If $\mu > -1/4$ and the values of both x_1 and x_2 are equal to a positive number ($x_1 = x_2 > 0$), then the matrix is positive definite. However, if $\mu = -1$, then the determinant is $-(x_1^2 + x_1 x_2 + x_2^2) \leqslant 0$. In this case, there is no point at which the matrix is definite.

Remark 1. Under the conditions of Theorem 2, if the linear functions $\ell_0, \ldots, \ell_n$ are linearly independent (i.e., none of them is an identically linear combination of the other functions with numerical coefficients), then the explicit computation of a point at which the matrix is definite is reduced to solving a linear programming problem. If the functions $\ell_0, \ldots, \ell_n$ are defined over a finite extension of the rational number field, then all necessary computations, including the check of linear independence of the functions, can be executed in polynomial time. Theorem 2 is also valid for matrices whose entries are linear functions in variables whose number is larger than the matrix order.

Theorem 3. *For almost all sets of linear functions $\ell_1, \ldots, \ell_n$ in n variables $x_1, \ldots, x_n$ and for any numerical $n \times n$ matrix M, the matrix $M + \operatorname{diag}(\ell_1, \ldots, \ell_n)$ is positive definite at some point and negative definite at another point.*

Proof. By analogy with the proof of Theorem 2, in the general case for any number α, the system of linear equations $\ell_1 = \alpha, \ldots, \ell_n = \alpha$ has a solution. For sufficiently large values of α at the point corresponding to the solution, we obtain a diagonally dominant matrix. Therefore, it is positive definite. For negative α with a sufficiently large absolute value, the corresponding numerical matrix is negative definite.

Theorem 4. *For all triples of linear functions ℓ_0, ℓ_1, and ℓ_2 in two variables x_1 and x_2, if $\ell_0(0,0) \neq 0$, then the symmetric 3×3 matrix*

$$\begin{pmatrix} x_1 & \ell_0 & \ell_1 \\ \ell_0 & x_2 & \ell_2 \\ \ell_1 & \ell_2 & x_3 \end{pmatrix}$$

is definite at some point.

Proof. Let us show that there exist values of x_1 and x_2 for which the second corner minor $\Delta_2 = x_1x_2 - \ell_0^2$ is positive. The following two cases are possible. If ℓ_0 is not identically constant, then the system of two equations $\ell_0 = 0$, $x_1 = x_2$ has a nontrivial solution, since $\ell(0,0) \neq 0$. For these values of the variables, the minor Δ_2 is positive. If ℓ_0 is identically constant, then Δ_2 is positive for all sufficiently large values of x_1 and x_2.

If the values of x_1 and x_2 thus chosen are both positive, then, for sufficiently large values of x_3, the determinant of the matrix is positive as well. According to the Sylvester criterion, the matrix is positive definite. Similarly, if the values of x_1 and x_2 are both negative, then, for negative x_3 (sufficiently large in absolute value), the determinant of the matrix is negative; the matrix is negative definite.

Remark 2. In Theorem 4, the inhomogeneity of the linear function ℓ_0 is an essential condition.

Example 1. At any point, the symmetric 3×3 matrix

$$\begin{pmatrix} x_1 & x_1 + x_2 & 0 \\ x_1 + x_2 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$$

is not definite, since the second corner minor $\Delta_2 = -(x_1^2 + x_1x_2 + x_2^2)$ is never positive. However, at points of the straight line defined by the equations $x_1 = x_2 = 0$, this matrix is semidefinite. It is positive semidefinite for $x_3 > 0$ and negative semidefinite for $x_3 < 0$. For any matrix whose entries are homogeneous linear functions, all entries vanish at the origin, i.e., the matrix is semidefinite.

The linear independence of the entries on the main diagonal of the matrix also simplifies the search for a point at which the matrix of semidefinite. Otherwise, such a point may not exist even for diagonal matrices.

Example 2. The diagonal 3×3 matrix $\operatorname{diag}(x_1, x_1 + 2, -x_1 - 1)$, whose entries depend only on one variable x_1, is not semidefinite at any point. Therefore, for any functions ℓ_0, ℓ_1, and ℓ_2 in any number of variables, the symmetric 3×3 matrix

$$\begin{pmatrix} x_1 & \ell_0 & \ell_1 \\ \ell_0 & x_1 + 2 & \ell_2 \\ \ell_1 & \ell_2 & -x_1 - 1 \end{pmatrix}$$

is also not semidefinite at any point.

On the other hand, the diagonal matrix $\mathrm{diag}(x_1, x_1 + 2, -x_1 - 1)$ is the limit, as $k \to \infty$, of the sequence of diagonal matrices

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_1 + \frac{1}{k}x_2 + 2 & 0 \\ 0 & 0 & -x_1 + \frac{1}{k}x_3 - 1 \end{pmatrix},$$

each of which is positive definite at the point $(1, 0, 3k)$ and negative definite at the point $(-1, -2k, 0)$.

Similarly, any constant matrix M is the limit, as $k \to \infty$, of the sequence of matrices $M + \frac{1}{k}\mathrm{diag}(x_1, \ldots, x_1)$ with entries depending on a single variable such that each matrix is positive definite at some point and negative definite at another point.

3. DISCUSSION

The semidefinite programming problem is to optimize a linear functional at the intersection of an affine space and the convex cone of positive semidefinite matrices. The interior point method is often efficient as applied to this problem, but requires knowledge of an initial point from the feasible domain. The above results allow us to establish the existence of such a point for some cases. Moreover, the proofs of Theorems 2 and 3 are constructive, i.e., under the generality assumption, from the conditions of these theorems, a point at which the matrix is definite can easily be computed using a polynomial number of arithmetic operations. On the other hand, in the general case, Theorem 2 does not guarantee the existence of a point at which the matrix is positive semidefinite.

Theorem 1 implies that almost every matrix whose entries are linear functions with a sufficiently large number of variables fails to be definite at each point. In particular, if such a matrix is positive definite at some point, then there exists another point at which it is positive semidefinite, but not definite.

Consider Hessian matrices of third-degree polynomials. For polynomials in two variables, these are symmetric 2×2 matrices to which Theorem 2 can be applied, i.e., for almost every polynomial of this kind, there exists a point at which its Hessian matrix is definite. Therefore, for almost every polynomial of third degree in two variables, its graph contains an elliptic point. In particular, this graph is not a ruled surface. Note that ruled surfaces play an important role in modeling surfaces of complex geometry. If the graph of a polynomial is a ruled surface, then the Hessian matrix of this polynomial is semidefinite at each point. On the other hand, the monkey saddle is an example of the graph of a third-degree polynomial that does not have an elliptic point and is not a ruled surface. In this case, the Hessian matrix of the corresponding polynomial vanishes at some point.

In the general case, for polynomials of third degree in three variables, a linear change of variables brings the Hessian matrix to the form

$$\mathrm{diag}(y_1, y_2, y_3) + \ell_0(y_1, y_2, y_3)M + C,$$

where $\ell_0(y_1, y_2, y_3)$ is a linear function and M and C are symmetric numerical 3×3 matrices. This statement follows from the fact that a cubic form in three variables can be decomposed into a sum of at most four cubes of linear forms [29]. If the matrix C is diagonal, then the existence of a point at which the matrix is definite can be proved by applying Theorem 2. If the matrix M is diagonal, then Theorem 3 is applicable. If the function ℓ_0 depends only on the values of y_1 and y_2, then Theorem 4 is applicable.

For almost all polynomials of the form

$$\ell_1^3(x_1, \ldots, x_n) + \ldots + \ell_n^3(x_1, \ldots) + \ell_0^3(x_1, \ldots) + (\mu\ell_0(x_1, \ldots) + \lambda)^3,$$

where each of the linear functions $\ell_0, \ldots, \ell_n$ depends on n variables, while λ and μ are numbers, a nondegenerate linear transformation of coordinates yields a polynomial with a Hessian matrix satisfying the conditions of Theorem 2 or 3. However, if the number n of variables is greater than two, then there are polynomials of third degree that cannot be represented in such a form [29].

For notational simplicity, we consider polynomials of the form

$$f = x_1^3 + \ldots + x_n^3 + \ell^3(x_1,\ldots,x_n) + (\mu\ell(x_1,\ldots,x_n) + \lambda)^3.$$

There exists a linear function $m(x_1,\ldots,x_n)$ for which the second partial derivatives of f are

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = 6\delta_{jk}x_k + 6\frac{\partial \ell}{\partial x_j}\frac{\partial \ell}{\partial x_k} m(x_1,\ldots,x_n),$$

where $\delta_{kk} = 1$ and $\delta_{jk} = 0$ if $j \neq k$. If m is a nonzero constant, then Theorem 3 is applicable. Otherwise, if $m(0,\ldots,0) \neq 0$, then Theorem 2 is applicable. In these cases, the Hessian matrix is definite at some point. Only if $m(0,\ldots,0) = 0$, such a point may not exist. However, if $m(0,\ldots,0) = 0$, then all entries of the Hessian matrix vanish at the origin. Therefore, for any polynomial of the considered form, the Hessian matrix is semidefinite at some point.

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