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Vladimir Kanovei, Karin U. Katz, Mikhail G. Katz & Mary Schaps

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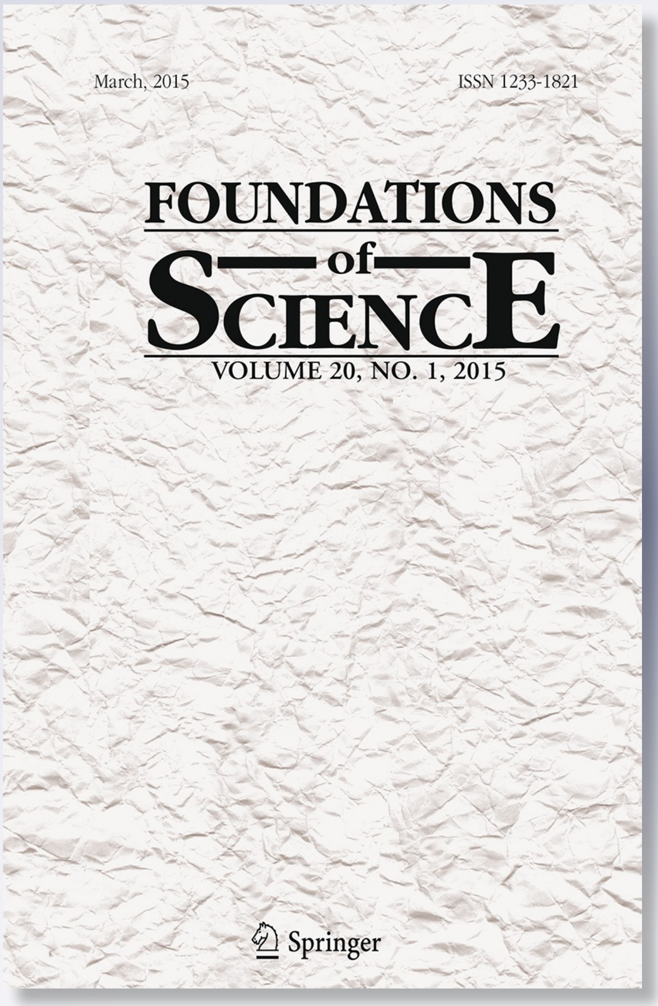

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Proofs and Retributions, Or: Why Sarah Can't *Take* Limits

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Abstract The small, the tiny, and the infinitesimal (to quote Paramedic) have been the object of both fascination and vilification for millenia. One of the most vitriolic reviews in mathematics was that written by Errett Bishop about Keisler's book *Elementary Calculus: an Infinitesimal Approach*. In this skit we investigate both the argument itself, and some of its roots in Bishop George Berkeley's criticism of Leibnizian and Newtonian Calculus. We also explore some of the consequences to students for whom the infinitesimal approach is congenial. The casual mathematical reader may be satisfied to read the text of the five act play, whereas the others may wish to delve into the 130 footnotes, some of which contain elucidation of the mathematics or comments on the history.

Keywords Errett Bishop · Abraham Robinson · Infinitesimals · Hyperreals · Intuitionists · Petard

For 'tis the sport to have the engineer
Hoist with his own petard.
Hamlet, Act 3, Scene 4.

1 Act I: Reinventing the Wheel

[*A college classroom. In the left hand corner of the board is written "Calculus 101, Section 13, Instructor, Allen Class". In the middle of the board appears the formula*

V. Kanovei
IPPI, Moscow, Russia
e-mail: kanovei@rambler.ru

V. Kanovei
MIIT, Moscow, Russia

K. U. Katz · M. G. Katz (✉) · M. Schaps
Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel
e-mail: katzmik@macs.biu.ac.il

M. Schaps
e-mail: mschaps@macs.biu.ac.il

> 0.999999999999999999999999999999999999999... = 1 *The instructor is gathering up his notes, and one student, Sarah, is lingering near the instructor's desk, holding a thick calculus book in her hand.*]

Allen: What is your name? Do you have a question?

Sarah: I'm Sarah. I was a bit puzzled by today's class on infinite decimals. [*Putting the book down on the desk, she points to the dots.*] What do you mean to happen exactly after *nine, nine, nine* when you write *dot, dot, dot*, Dr. Class?[1]

Allen: From your mouth to the rector's ears, Sarah![2] But I'm afraid that I don't have my Ph.D. yet. You can call me Al. What I mean is just that $0.9,\ 0.99,\ 0.999, \ldots$ gets closer and closer to the *limit* 1. If you were studying to be a mathematician, they would teach you the *rigorous* notion of limit.

Sarah: If I were taking math, I might be too preoccupied with the *rigors* of obtaining a degree to ask probing questions! My sister Ruth is studying mathematics, and she is in a tizzy about something she calls the Axiom of Choice.

Allen: The Axiom of Choice is just that, an axiom. She doesn't have to accept it. Of course, she will lose most of modern mathematics that way. [*Aside*] Including my thesis topic in Noncommutative Geometry.[3] [*To Sarah*] Have I succeeded in answering your "probing" question?

Sarah: Not quite. When do the nines stop?

Allen: When I was a student at M.I.T., we had something called the Infinite Corridor. You kept walking and walking and there always seemed to be more classrooms. Just imagine that you are walking down an infinite corridor in which there are always more nines. .999... is zero, point, followed by an infinite number of 9s. [*Writes formula on the board*] The identity $.\underbrace{99\ldots9}_{n} = 1 - \frac{1}{10^n}$ becomes $.999\ldots = 1$ as n increases without bound.

Sarah: What does "without bound" mean exactly? [*Writes another formula on the board*] If there is going to be an infinity of 9s there, your identity should become $.\underbrace{99\ldots9}_{\infty} = 1 - \frac{1}{10^\infty}$, not 1 !

Allen: Sarah, you are confusing numbers and cardinalities. Infinity can only be a number in abstruse systems like the hyperreals,[4] but then it is denoted H, not ∞.

Sarah: Fine, let it be H. [*Sarah corrects her formula by replacing the symbol* ∞ *by* H] Can we write $.\underbrace{99\ldots9}_{H} = 1 - \frac{1}{10^H}$ then?[5] Therefore *zero, point, followed by an infinity of* 9s is a tiny bit less than 1, right?

Allen: Wrong. As I said before, your problem is that you have not learned a rigorous notion of limit.

[1] See Heyting (1956) and Pourciau (1999) for the literary career of this character.

[2] This character made her debut in Robert Ely's study (Ely 2010). Ely was a TA in a calculus course. One of his students, Sarah, spontaneously re-invented Leibniz-style infinitesimals so as to make sense of what was going on in the calculus course (needless to say, this was taught using epsilon, delta).

[3] See note 58 for details on the foundational status of Noncommutative Geometry.

[4] The term was introduced by Hewitt (1948, p. 74). A hyperreal number H equal to its own integer part, or $H = \lfloor H \rfloor$, is called a hyperinteger. The hyperintegers form a set $\mathbb{Z}^*$. The elements of the complement $\mathbb{Z}^* \setminus \mathbb{Z}$ are called infinite hyperintegers. The hyperreals can be constructed directly out of integers; see Borovik et al. (2012).

[5] In the "semicolon" notation of Lightstone (1972), such a number would appear as the extended terminating infinite decimal $.999\ldots;\ldots9$, with the last nonzero digit appearing at infinite decimal rank H. See note 56 on interpretations of the symbol ".999...".

Sarah: Are you saying that *limits* are a way of sweeping away the very small term $\frac{1}{10^H}$? Why would one want to sweep *that* under the rug?

Allen: In the real number system, we don't have such "very small" numbers. Actually, they are called *infinitesimals*, but they only appear in abstruse number systems like the hyperreals, so they don't really exist.[6] Leibniz already talked about infinitesimals, but he was not rigorous.[7]

Sarah: I get it. The difference $1 - .999\ldots$ is infinitesimal, but since infinitesimals aren't rigorous, we declare that $.999\ldots = 1$, right? Why was Leibniz[8] talking about infinitesimals?

Allen: Not quite. Leibniz[9] was interested in velocities and slopes. He wanted to define them as a ratio $\frac{dy}{dx}$.

Sarah: How does that work?

Allen: Say, take $y = x^2$. If dx is an infinitesimal increment, Leibniz[10] formed the ratio of $(x + dx)^2 - x^2$ over dx.

Sarah: That looks like finding the slope of a line, right?

[6] See note 130 for a more nuanced view. In 1908, Felix Klein described a rivalry of two types of continua in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that "The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries (Klein 1908, p. 214). Such a different conception, according to Klein, "harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of . . . infinitely small parts" (ibid.). Thus according to Klein there is not one but two separate tracks for the development of analysis: [A] the Weierstrassian approach (in the context of an *Archimedean* continuum); and [B] the approach with indivisibles and/or infinitesimals (in the context of what could be called a *Bernoullian* continuum).

[7] Recent work on Leibniz and infinitesimals includes (Katz and Sherry 2012, 2013; Sherry and Katz 2012; Tho 2012). A seminal study of Leibnizian methodology by Bos notes that Robinson's hyperreals (see Robinson 1966) provide a "preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities" (Bos 1974, p. 13). In addition to this positive assessment, the article by Bos also contains a brief Appendix 2 where Bos criticizes Robinson's reading of Leibniz. The appendix, written as it was by a fresh Ph.D. in history with apparently limited training in mathematics (not to speak of mathematical logic), contains numerous misunderstandings of the hyperreal framework. Bos's technical errors were detailed by Katz and Sherry (2013, Section 11.3), and include a misreading of Robinson's *transfer* (i.e., the transfer principle). See further in note 8.

[8] Urquhart commented that "some of Bos' criticisms of Robinson involve absurd and impossible demands—for example, his first criticism (Bos 1974, p. 83) is that Robinson proves the existence of his infinitesimals, whereas Leibniz does not!" (Urquhart 2006). See further in note 9.

[9] Bos writes: "the most essential part of non-standard analysis, namely the proof of the existence of the entities it deals with, was entirely absent in the Leibnizian infinitesimal analysis, and this constitutes, in my view, so fundamental a difference between the theories that the Leibnizian analysis cannot be called an early form, or a precursor, of non-standard analysis" (Bos 1974, p. 83). Bos's comment fails to appreciate the crucial dichotomy of mathematical practice (or *procedures*) versus mathematical ontology (or *foundations*). Leibnizian procedures exploiting infinitesimals find suitable proxies in the procedures in the hyperreal framework (see Reeder 2013 for a related discussion in the context of Euler). The relevance of such hyperreal proxies is in no way diminished by the fact that set-theoretic *foundations* of the latter ("proof of the existence of the entities", as Bos put it) were obviously as unavailable in the seventeenth century as set-theoretic foundations of the real numbers. See further in note 10.

[10] In the context of Bos's discussion of "present-day standards of mathematical rigor", Bos writes: ". . . it is understandable that for mathematicians who believe that these present-day standards are final, nonstandard analysis answers positively the question whether, after all, Leibniz was right" (Bos 1974, p. 82, item 7.3). The context of the discussion makes it clear that Bos's criticism targets Robinson. If so, Bos's criticism suffers from a strawman fallacy, for Robinson specifically wrote that he does not consider set theory to be the foundation of mathematics, and being a formalist does not subscribe to the view attributed to him by Bos that "present-day standards are final". See further in note 11.

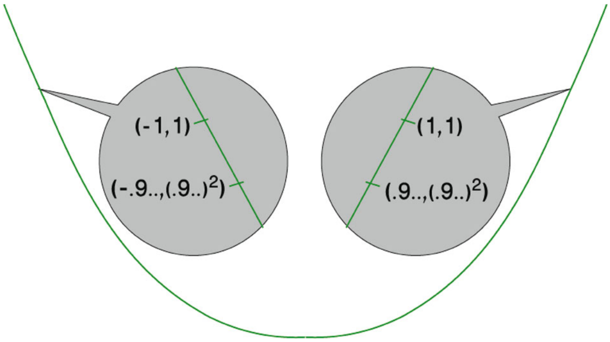

Fig. 1 Differentiating $y = f(x) = x^2$ at $x = 1$ yields $\frac{\Delta y}{\Delta x} = \frac{f(.9..)-f(1)}{.9..-1} = \frac{(.9..)^2-1}{.9..-1} = \frac{(.9..-1)(.9..+1)}{.9..-1} = .9.. + 1 \approx 2$. Here $\approx$ is the relation of being infinitely close. Hyperreals of the form .9.. are discussed in Ely (2010), Katz and Katz (2010a) and Katz and Katz (2010b)

Allen: Roughly. Then Leibniz[11] simplified the resulting expression to $2x + dx$.

Sarah: Is that the right answer?

Allen: Not exactly. Leibniz[12] then simplified $2x + dx$ to $2x$ because dx is negligible, which wasn't rigorous.[13]

Sarah: Can you give an example with numbers?

Allen: Say, at $x = 1$, the ratio would be $(1+dx)^2 - 1^2$ over dx. This simplifies to $2+dx$. So the slope at $x = 1$ should be 2.

Sarah: I have an idea! [*Writes down a calculation on the board*] If we choose .9.. − 1 for dx, the ratio is $(.9..)^2 - 1^2$ over .9.. − 1, which simplifies to .9.. + 1. That should evaluate to 2 if you evaluate .9.. to 1, the way you like to do.

Allen [*Startled*]: Don't try to reinvent the wheel. Mathematics has a perfectly adequate treatment of the slope in terms of limits.[14]

Sarah: Sorry. But I think *my way* I finally understood slope.[15]

Allen: [*red in the face, erasing a lot of nines in the middle but leaving three at each end*] Listen, Sarah, I am going to write down the formal definition of limit, but don't tell anyone that I explained this now. Last year I put it on the board and some chemistry students complained to their department Chair. It hurt my student evaluations, which will damage my

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[11] Robinson expressed his position on the status of set theory as follows: "an infinitary framework such as set theory ... cannot be regarded as the ultimate foundation for mathematics" (Robinson 1969, p. 45; see also Robinson 1966, p. 281). Furthermore, contrary to Bos's claim, Robinson's achievement was not to show that "Leibniz was right" (see note 10), but rather to provide hyperreal proxies for the inferential procedures commonly found in Leibniz as well as Euler and Cauchy. Leibniz's procedures, involving as they do infinitesimals and infinite numbers, seem far less puzzling when compared to their hyperreal proxies than from the viewpoint of the traditional A-track frameworks (see note 6 on tracks A and B). See further in note 12.

[12] Some decades later, a mellower Bos distanced himself from his flawed Appendix 2 (see notes 7, 8, 9, 10, and 11) in the following terms (in response to a question from one of the authors of the present text): "An interesting question, what made me reject a claim some 35 years ago? I reread the appendix and was surprised about the self assurance of my younger self. I'm less definite in my opinions today—or so I think. You're right that the appendix was not sympathetic to Robinson's view. Am I now more sympathetic? If you talk about "historical continuity" I have little problem to agree with you, given the fact that one can interpret continuity in historical developments in many ways; even revolutions can come to be seen as continuous developments" (Bos 2010).

[13] See Fig. 1 and note 22 for a discussion of a generalized notion of equality in Leibniz.

[14] See note 99 on wheels, infinitesimals, and limits.

[15] Sarah will pursue the matter further; see main text at note 60.

chances of getting a tenure-track job. [*Allen writes down in symbols the formal* ϵ, *n definition of the limit of a sequence in symbols*]

Sarah: That sounds complicated. Is there a simpler way of saying this?[16]

Allen: [*quickly erases the formula*] Forget you ever saw it. And that's not the worst part. These multiple quantifier statements are murderous to negate. I'm tutoring a philosophy graduate student in logic, so I know how hard it is to explain the theory, and to calculus students we don't teach logic, we just say that the negative of "for all epsilon there is an n so that something is true" is that "there is an epsilon such that for all n it is not true".

Sarah: How do they make sense of *that*?

Allen: Students don't always want to believe us. Even Priscilla, the philosophy student, didn't want to believe me, even after I explained about the law of excluded middle. [*Looks at his watch*] Oh, no! I was supposed to meet her downstairs 15 min ago.

[*Exit Allen*]

2 Act II: Model Ghosts

[*Enter Abraham and Priscilla*]

Abraham: I think this is the classroom you were looking for, Miss. See, "Allen Class" is written on the board.

Priscilla: Thank you very much, Professor Robinson. I'm sorry I bothered you in your office.

Sarah: Are you looking for Al? It's my fault he was late to meet you.

Priscilla: No hard feelings. [*pointing at the board*] Oh, what are all those nines? *Pale Fire* had 999 lines. . .[17]

Sarah: Al and I were talking about .999 . . . He was explaining to me why infinitesimals don't exist.

Priscilla: They surely don't. I am doing my thesis on the philosophical writing of Bishop George Berkeley. When writing about the foundations of the calculus, he says that there is the leap of faith required to accept the idea of an infinitesimal!

Sarah: Why is that?

Priscilla [*Pointing to Sarah's calculation on the board*]: How are we to reconcile the start and the finish? One must interpret the symbol .999 . . . your way at the start of the slope calculation, so as to make dx nonzero, in a ghostly fashion. Meanwhile, at the conclusion of the calculation, the symbol must be interpreted Al's way, so as to make the quantity dx depart. The infinitesimals are "ghosts of departed quantities".[18] Can you eat your cake and have it, too?

Sarah: If you want to know how high the ball will soar when you toss it up in the air, does it matter so much if you are off by an infinitesimal amount?

[16] A view of the (ϵ, δ) approach as a nominalistic paraphrase, or reconstruction, of analysis was elaborated by Katz and Katz (2012a). They note in Katz and Katz (2012b): "If our students are being dressed to perform multiple-quantifier epsilontic logical stunts on the pretense of being taught infinitesimal calculus, it is because infinitesimals are assumed to be either metaphysically dubious or logically unsound".

[17] See Nabokov (1962) and main text at note 78.

[18] See *The Analyst* Berkeley (1734) for the famous criticism of infinitesimal calculus in terms of "the ghosts of departed quantities". Berkeley claimed calculus was based on an inconsistency of the type $(dx \neq 0) \wedge (dx = 0)$. See further in note 22. See also note 36 for Robinson's comment on Berkeley. Sherry (1987) dissected Berkeley's criticism into its metaphysical and logical components.

Priscilla: It depends on whether you want your sciences *exact*, or laced with theological mysteries. That's what Berkeley thinks.

Abraham: Never reject a good idea just because it seems not to work.[19] One can try to fix it instead. Just as the rational numbers $\mathbb{Q}$ are enlarged to a wider number system called the reals $\mathbb{R}$ so as to accomodate computations in algebra and geometry, so also the reals, in turn, can be enlarged[20] to a number system called the hyperreals $\mathbb{R}^*$ so as to accomodate computations in infinitesimal calculus and analysis. Here each real number x_0 is surrounded by a cluster of hyperreals x infinitely close to it. The standard part function "st" rounds off each finite hyperreal number to its nearest real, namely $\mathrm{st}(x) = x_0$.

Sarah: How does that help us with the slope?

Abraham: If y depends on x, and Δx is an infinitesimal increment, we denote the corresponding y-increment Δy. All that remains is to modify Leibniz's definition of the slope, by incorporating the standard part.[21] Namely, the slope by definition is $\mathrm{st}(\Delta y/\Delta x)$. It is a small price to pay for the removal of an inconsistency.[22]

Sarah: Thanks, Professor Robinson. Could you please explain this enlargement?

Abraham: You can call me Abe. It all starts with languages and models…

[*Exeunt Sarah and Abraham. Sarah has left her calculus book behind*.]

3 Act III: A Tale of Two Bishops

[*Sound of glass being shattered. Priscilla screams. Errett climbs through a window of the classroom, and drags up a neon sign after him. Looking nervously around, he sets an old-fashioned pistol on the arm of one of the desks.*]

Errett: I had to get in through the window. The fundamentalist excluded thirdists[23] at the gate fought me off. According to them, constructive mathematics is doomed to the role of a scavenger.[24] [*Eyeing Priscilla suspiciously*] Are you one of *them*? Do you also accept the law of excluded third?

Priscilla: [*Nervously, eyeing the pistol*] What does it say on your neon sign?

Errett: *Do* you, yes or no?

Priscilla: I'll have to take the *third* on this one.

Errett: [*Pointing at the neon sign*] I worked on this for eighteen years. I wrote this table up on the board at a colloquium at Stanford[25] years ago. Now I have recreated it in neon lights for all generations. [*He starts to attach the sign to the board using two-sided tape*]

Priscilla: Excuse me, I have a friend who teaches in this room. I think he needs to write on the board and, you should excuse me, erase things. Who are you, exactly?

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19 See Stewart (2009).

20 See note 130.

21 See Keisler (1986, p. 43). To define the real derivative of a real function f in this approach, one can bypass an infinite limiting process as in Weierstrass's approach. Instead, one sets $f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon)-f(x)}{\epsilon}\right)$, where ϵ is infinitesimal, yielding the standard real number in the cluster of the hyperreal argument of "st" (the derivative exists if and only if the value above is independent of the choice of the infinitesimal).

22 See Robinson (1966, p. 266). Katz and Sherry (2012, 2013) argue that the inconsistency alleged by Berkeley in Leibniz, namely $(dx \neq 0) \wedge (dx = 0)$, was not there in the first place, as Leibniz repeatedly indicated that he is working with a generalized notion of equality "up to" a negligible term. Such a principle was dubbed the *transcendental law of homogeneity* by Leibniz (1710).

23 See Richman (1996, p. 249); see note 31 below for the source of the adjective.

24 See Bishop (1968, p. 54).

25 See note 30 for details on Bishop's talk at Stanford University.

Errett: Errett Bishop. Professor Errett Bishop [*He moves the sign to the wall and employs two-sided tape on the side close to Priscilla*]

Priscilla: Are you related at all to Bishop Berkeley? Oh, I suppose not. Silly of me to ask. He lived a long time ago. [*As Errett finishes applying the tape on one side and moves to the side away from Priscilla, she gets her first look at the sign*] Does this line say that Kronecker was an atheist? I think it was Kronecker who was reported to have said "G-d made the whole numbers; all else is the work of man."[26]

Errett: No, this is not description, this is analogy. [*He points to the first line*] The typical mathematician's view of the real numbers is akin to the fundamentalist views of a true believer. I did this in blue to indicate naivete.[27]

Priscilla: Interesting. Are we in some kind of a zany theomathematical satire?

Errett: Don't try to get us bogged down in paradoxes of self-reference! I will continue. A constructive analyst's view is akin to that of an agnostic. I put those in brown to indicate a down-to-earth attitude.

Priscilla: I wonder where George Berkeley's ideas would fit in. I don't think he would like to go there. Is there no space for him between the fundamentalist and the agnostic?

Errett: In my personal life, it took me no time at all to go from one religious extreme to the other. . .[28]

Priscilla: Oh! I didn't know. . .

Errett: . . . So there can't have been any sensible position in the middle can there?[29]

Priscilla: I thought you didn't like to exclude the middle.

Errett: [*Ignoring that comment*] The last line is in red for boldness. Only Kronecker (or similar types) who denied the reality of numbers other than the rationals, qualifies for the remaining group. That's exactly what the sign[30] says:[31]

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26 See Weber (1893, p. 15).

27 Hellman (1993) introduces a dichotomy within constructivism, between *liberal* constructivism and *radical* constructivism. The former views constructivism as a *companion* to classical mathematics. The latter views constructivism as an *alternative* to classical mathematics. The "fundamentalist" comment leans toward the latter variety.

28 Based on personal conversations with Bishop, Hill (2013) indicated that there was a definite connection in Bishop's mind between his rejection of what he felt was his fundamentalist upbringing, " in a strict fundamentalist Protestant situation", and his eventual rejection of classical mathematics. In a poem he wrote around 1973, Bishop compared classical mathematics to "sawdust", and depicted Formalism as decidedly diabolical; see main text at note 106.

29 Bishop's rejection of classical mathematics followed his work on the existence of holomorphic disks; see Bishop (1965a). The work exploited nonconstructive fixedpoint theorems. As Hill (2013) relates, Bishop attempted to exhibit such a disk explicitly, and was unable to do so. He then attempted to exhibit a single *point* on such a disk, and was still unable to do so. Even an attempt to exhibit a single *coordinate* of such a point failed. Bishop was apparently struck by an allegedly fundamentalist nature of a mathematician's belief in entities he is unable to exhibit. This eventually led to his abandoning complex variable research, and switching to constructive mathematics.

30 The table (except for the colors) is a replica of the first blackboard from the Stanford University colloquium lecture (Bishop 1965b), according to an eyewitness account (Hill 2009). See further in note 31.

31 Bishop indicated in the Stanford colloquium that the target of his rebellion was a perceived *fundamentalism* (see note 28). The term *Idealism* is a euphemism employed in his later writings; see notes 81, 87, and 90. The original term remains in use in constructivist circles, as in the expression *fundamentalist excluded thirdist*; see note 23.

Religion	Mathematics
True believer (fundamentalist)	A typical mathematician's view of the real numbers
Agnostic	A constructive analyst's view of the real numbers
Atheist	Perhaps Kronecker, or similar types

Priscilla: Berkeley actually warned Isaac Newton about this. He said that this is one short way of making Infidels.[32]

4 Act IV: Kronecker's Revenge

[*Enter Sarah and Abraham, engrossed in conversation*]

Sarah: ... Yes, I understand, the whole theory is interpretable over $\mathbb{R}^*$ by the Transfer Principle...[33] [*Looking at Priscilla*] Did I leave my Calculus book here? [*Priscilla points to the front desk and Sarah goes to pick up her book*] Hey, what's this neon sign?

Errett: I was just beginning to explain how the reliance on the law of excluded middle[34] obliterates the notion of positive content...[35]

Sarah: Has anyone seen Al? I was looking forward to ironing out some ghostly matters with him.

Priscilla: Oh, are you also studying the works of Berkeley?

Abraham: Berkeley's *Analyst* was a brilliant exposure of the logical inconsistencies of the foundations of the calculus in the forms then proposed.[36]

[*Ruth enters*]

Sarah: Oh, and here is my sister Ruth!

Ruth: I am going now by "Integrity Ruth." This is to display my admiration for the character "Integrity Jane" in Pourciau's play. She boldly stands up to her mathematics instructor and convinces him that, whatever the cost, the constructivist position is the honest one.[37]

Errett: Is that "Integer-ity Ruth", as in "integer"?

Ruth: No, it's "Integrity" as in honesty and "Rut' ", rhyming with *brute*. My friends call me Rutie.

Sarah: We have been discussing the use of infinitesimals in teaching calculus.

Ruth: Oh, infinitesimals. [*Wrinkles her nose in distaste*] They might be forgivable to someone like Archimedes.[38] A modern mathematician should know that to actually con-

[32] Berkeley (1734) wrote as follows of the general public: "With this bias on their Minds, they submit to your Decisions where you have no right to decide. And that this is one short way of making Infidels I am credibly informed."

[33] See Keisler (1986, p. 28) and Herzberg (2013).

[34] See notes 65, 66 below.

[35] See Richman (1996, p. 257). The obliterating comment leans toward radical constructivism; see note 27.

[36] Robinson's comment on Berkeley appeared in (Robinson 1966, p. 280).

[37] See Pourciau (1999).

[38] See Netz et al. (2001) and note 99, as well as Roquette (2010). The latter text quotes a charming definition of continuity from a 1912 calculus textbook by Kiepert [67]: "If some function is given by $y = f(x)$ then, in general, infinitely small changes of x will give rise to infinitely small changes of y." See further in note 40.

struct infinitesimals you need ultrafilters,[39] and if you deny the Axiom of Choice, they don't necessarily exist.[40]

Errett: Thank you, Rutie.

Sarah: Is it true that Archimedes already did calculus with infinitesimals?[41]

Errett: Did you say *calculus with infinitesimals*? It is hard to believe that debasement of meaning[42] could go so far.

Sarah: [*taken aback*] De-what? De basement? Is that where you found that weird sign?

Errett: Know that this infinitesimal approach is an obfuscation of the wonderful ideas of calculus.[43] The distortion culminates in Keisler.

Priscilla: My father drives a Chrysler, but it wasn't distorted until someone sideswiped him.

Errett: No, not Chrysler, Keisler. He wrote a book called *Elementary Calculus: An Infinitesimal Approach.* Fortunately Halmos, the editor of the *Bulletin*, passed it on to me to review.[44] Some people think that Halmos was just trying to increase circulation by stirring up controversy, but Halmos is my teacher and I know him better than that.[45]

Priscilla: What *was* Halmos trying to do?

Errett: I am sure he wanted to give me a platform from which to proclaim my ideas.[46] No invocation[47] of Newton and Leibniz is going to justify this infinitesimal approach,[48] and Archimedes won't help either!

Allen [*Enters the room*]: What is going on here? [*Seeing Priscilla*] Oh, there you are. Come Priscilla, we have a lot more logic to learn.

Priscilla: Would you listen to this argument, Al. I don't really understand it, but you might, and it seems relevant to my thesis.

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[39] But see notes 123 and 135.

[40] Some time after World War 1, Kiepert's textbook (see note 38) seems to have been edged out of the market by Courant's. Courant (1937, p. 101) describes infinitesimals as (1) incompatible with the clarity of ideas; (2) entirely meaningless; (3) vague mystical ideas; (4) fog which hung round the foundations; (5) hazy idea. Courant was unable to peer through the hazy mystical fog the way Robinson would. It should be kept firmly in mind that Courant's criticism *predated* Robinson's framework, unlike certain criticisms of more recent vintage.

[41] See note 99.

[42] See Bishop (1975, pp. 513–514) and note 44.

[43] See Bishop (1977, p. 208).

[44] See Bishop and Keisler (1977). Bishop's claims need to be understood in the context of his anti-fundamentalist ideology; see notes 28 and 64. Keisler (1977) asked why E. Bishop was chosen as the reviewer in the first place. In his measured reply to Bishop's vitriolic review Bishop and Keisler (1977) of the first edition of *Elementary calculus, An infinitesimal approach* Keisler (1986, 1977) asked: "... why did P. Halmos, the *Bulletin* book review editor, choose a constructivist as the reviewer?"See further in note 45.

[45] Halmos' answer to Keisler's question came in the form of an editorial pointer on p. 271 of the same issue, referring the reader to Halmos' outline of his editorial philosophy on p. 283: "As for judgments, the reviewer may ... say (or imply) what he thinks." In other words, a reviewer may use the review as a springboard for developing his own ideological agenda. See further in note 46.

[46] According to a close associate of Halmos' Ewing (2009), Halmos' strategy was to confront opposing philosophies in the goal of livening up the debate. One of his goals was to boost lagging sales that were plaguing the publisher at the time (see Halmos 1985). The bottom-line issue, combined with Halmos' own unflattering opinion of Robinson's framework as "a special tool, too special" (Halmos 1985, p. 204), apparently made the choice of Halmos' student (Bishop) as the reviewer, appealing to the editor. The result was a review Bishop and Keisler (1977) that was short on pedagogy and long on vitriol. See Katz and Katz (2011) for further details.

[47] See main text at note 128 for such an invocation.

[48] See Bishop (1977, p. 208).

Sarah: [*disappointed*] I gather some vigorous foot-work[49] may be required to argue for the infinitesimal approach. Could Fermat,[50] Leibniz,[51] Euler,[52] Cauchy[53] and Cohen[54] have all been on the wrong track? Don't infinitesimals possess a peculiar pragmatic content,[55] in helping sort out the thorny .999 . . . issue?[56]

Errett: They are not adequate in numerical meaning.[57]

Allen: Inadequate, indeed; to any nonstandard number corresponds canonically a subset of [0, 1] which is not Lebesgue measurable and hence cannot be exhibited.[58]

Sarah: "Numerical" maybe not, but "meaning" they seem to have all right!

Errett: The debasement of meaning is a schizophrenic attribute of contemporary mathematics.[59]

Sarah: At any rate, I now understood the *meaning* of slope, thanks to infinitesimals![60]

Errett: Cantor would have called them the Mexican Flu Virus of mathematics.[61]

Allen: Irremediable, alas.[62]

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49 See notes 44, 64, 65, no match for Kinbote's diligence (note 17).

50 Katz et al. (2013) reexamine Fermat's contribution to the problems of maxima and minima, tangents, and variational techniques.

51 Sherry and Katz (2012) argue that Leibniz treated infinitesimals as akin to imaginary numbers: Both are fictions, but well-founded fictions because they contribute to discovery and systematization.

52 See Bair et al. (2013), Bascelli et al. (2014) and Reeder (2013). Euler's inferential moves exploiting infinitesimals and infinite numbers find natural proxies in the hyperreal framework. Thus Euler's procedures are only "puzzling" from the viewpoint of an A-track framework (see note 6).

53 See Błaszczyk et al. (2013) and Borovik and Katz (2012).

54 Mormann and Katz (2013, p. 224, Section 2.3) argue that at a time when the followers of Cantor, Dedekind and Weierstrass and their philosophical henchmen like Russell and Carnap sought to ban infinitesimals as pseudo-concepts, Hermann Cohen and the Marburg school of neo-Kantian philosophy sought to develop the foundations of a working logic of the infinitesimal. Cohen's thought is known to have influenced A. Fraenkel, whose student A. Robinson ultimately brought the idea to full fruition. Fraenkel explicitly linked Cohen and Robinson in his memoirs: "my former student Abraham Robinson had succeeded in saving the honour of infinitesimals—although in quite a different way than Cohen and his school had imagined" (Fraenkel 1967, p. 107). Of course, to Cohen, *Logik* was a philosophical discipline akin to *philosophy of science*. According to Fraenkel, Robinson's work on infinitesimals was only an indirect offspring of the concept that Cohen and his school had in mind.

55 See Bishop (1967, p. viii). Bishop's expression "peculiar pragmatic content", connoting an alleged lack of empirical validity of classical mathematics, was analyzed by Billinge (2003, p. 179).

56 The standard real decimal $.999\ldots = 1$ is defined as the *limit* of the sequence $(.9, .99, .999\ldots)$. The class of the same sequence in the ultrapower $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ gives a hyperreal that falls infinitesimally short of 1, providing an alternative interpretation closer to student intuitions; see (Katz and Katz 2010b, a). The hyperreal $h = [(.9, .99, .999, \ldots)]$, represented by the sequence $(.9, .99, .999, \ldots)$, is an infinite terminating string of 9s, with the last nonzero digit occurring at a suitable infinite hypernatural rank N. The latter is represented by the sequence listing all the natural numbers $(1, 2, 3, \ldots)$, and $h = 1 - \frac{1}{10^N}$.

57 See notes 69 and 72.

58 See Connes (1995, p. 6207). Note that Connes' Noncommutative Geometry relies on nonconstructive foundational material such as free ultrafilters, Dixmier trace, and the Continuum Hypothesis. For an analysis of Connes' critique, see Kanovei et al. (2013) as well as Katz and Leichtnam (2013). See also http://mathoverflow.net/questions/57072/a-remark-of-connes.

59 See Bishop (1973/1985, p. 1).

60 This is a follow-up of the discussion in main text at note 15.

61 G. Cantor had convinced himself of the impossibility of a rigorous justification of infinitesimals, and referred to them as the "cholera bacillus" of mathematics; see Meschkowski (1965, p. 505), Dauben (1995, p. 353), Dauben (1996, p. 124), Ehrlich (2006).

62 See Connes (2001, p. 16).

Abraham: Err... Errett, your analysis appears to be more vigorous than accurate.[63] By your standard, all of classical mathematics would fall somewhere in the range between "obfuscation" and "debasement of meaning".[64] Aren't you criticizing apples[65] for not being oranges?[66] I was hoping to smooth things over at the *Workshop*[67] in a fashion that's meaningful...

Errett: Meaningful mathematics is mathematics that predicts the results of certain finitely performable computations within the set of integers.[68] Namely, if we perform certain computations within the set of positive integers, we shall get certain results.[69] Lacking numerical meaning, mathematics becomes a game.[70]

Abraham: Your definition of *meaning* appears to be too narrow. For example, it fails to encompass the meaningful infinitesimal procedures[71] developed by Leibniz for the calculus.

Errett: The scandal of classical mathematics is its deficiency in numerical meaning.[72] Meaning resides in constructive mathematics, based on logic that's necessarily intuitionistic.

Abraham: Ah! Intuitionism... Intuitionist Bertus Brouwer went beyond Kronecker, by developing a theory of the continuum which...[73]

Errett: Brouwer's bugaboo[74] has been compulsive speculation about the nature of the continuum.[75] His fear seems to have been that, unless he personally intervened to prevent it, the continuum would turn out to be discrete.[76] The result was Brouwer's semimystical theory of the continuum.[77]

63 Robinson (1968, p. 921) characterized Bishop's "attempt to describe the philosophical and historical background of [the] remarkable endeavor" of the constructive approach to mathematics, as "more vigorous than accurate".

64 Bishop failed to acknowledge in his essay Bishop and Keisler (1977) that his criticism of Keisler's textbook based on infinitesimals was motivated by Bishop's foundational preoccupation with the extirpation of the law of excluded middle (see notes 44 and 66).

65 Bishop's criticisms apply in equal measure to all of classical mathematics, relying as it does on classical logic (see note 66). Feferman (2000) made a related point in the following terms: "[Bishop] called non-constructive mathematics 'a scandal', particularly because of its 'deficiency in numerical meaning'."

66 Classical logic incorporates the law of excluded middle, unlike intuitionistic logic, favored by constructivists.

67 See Dauben (1996, p. 132). A. Robinson had been scheduled as keynote speaker at the Workshop on the evolution of modern mathematics in 1974 (see Birkhoff 1975) but did not live to deliver his lecture. In a last-minute change, the organizers replaced his lecture in the section on foundations, by Bishop's.

68 See Bishop (1968, p. 53). This text was reviewed for MathSciNet by R. L. Goodstein, who commented on the text's "curiously old-fashioned air", and "avoidance of the concept of an algorithm and apparent ignorance of almost everything that has been done in constructive mathematics in the past thirty years" (Goodstein 1970).

69 See Bishop (1975, p. 3).

70 See Bishop (1967, p. viii).

71 On procedures *vs* ontology see note 9.

72 See Bishop (1967, p. ix).

73 See Robinson (1968, p. 920).

74 An imagined fear or threat, or a fear presumed larger than it really is.

75 See Bishop (1967, p. 6).

76 See Bishop (1967, p. 6).

77 See Bishop (1967, p. 10).

Abraham: "Semimystical theory"? I see... "the moon's an arrant thief, And her pale fire she snatches from the sun[78] ..." Well, Intuitionist Arend Heyting, who has shown a consistently constructivist attitude,[79] felt that non-standard analysis is a standard model of important mathematical research...[80]

Errett: Heyting? Brouwer's movement was killed partly by compromises of Brouwer's disciples with the viewpoint of idealism.[81]

Abraham: I understand. Kolmogorov and the Russian school of intuitionism[82] interpreted it in terms of recursivity...

Errett: Many mathematicians who think they know something about the constructive point of view, have in mind a dinky formal system or, just as bad, confuse constructivism with recursive function theory.[83]

Abraham: Let's see now, Brouwer's disciples like Heyting and the Russian constructivists would be your natural allies. Why are you attacking them?

Sarah: What *do* intuitionists agree on?

Abraham: Perhaps you feel that mathematics ought not to be founded on a Platonic[84] mind-independent mathematical universe?[85]

Errett: Mathematics ought not to be founded on such speculative metaphysics.[86] I call it Idealism.[87]

Abraham: Interesting. Intuitionist Hermann Weyl's *Crisis* essay about the shaky foundations of mathematics[88] seems to have succeeded in creating a genre of its own.[89]

Errett: Weyl? He suppressed his constructivist convictions, claiming applications to physics as justification.[90]

Abraham: Ah, physics. How would you characterize the difference between physics and the natural sciences, on the one hand, and mathematics, on the other?

Errett: Big difference. Even in another universe, with another biology and another physics, mathematics will remain in essence the same as ours ... The primary concern of mathematics is number, and this means the positive integers.[91]

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78 See *Timon of Athens* (Shakespeare 1623), act IV, scene iii, and note 17.

79 See Robinson (1968, p. 921).

80 See Heijting (1973, p. 136). Maddy (Maddy 1989, pp. 1121–1122) quotes Heyting somewhat out of context, implying that Heyting is a radical constructivist; see note 27. However, a closer examination reveals conclusively that Heyting is a liberal constructivist, who declared that "intuitionistic mathematics is no longer isolated from classical mathematics... The two subjects become more and more intertwined" (Heijting 1973, p. 135).

81 See Bishop (1967, p. ix). See note 31 above for the connotation of the term *Idealism* in Bishop's ideology.

82 See Kolmogorov (2006).

83 See Bishop (1967, p. 6). See Goodstein's comment in note 68.

84 See Maddy (1989) for an analysis of mathematical Platonism in relation to other doctrines.

85 See (Billinge 2003, p. 183).

86 See Billinge (2003, p. 183).

87 See Bishop (1967, pp. 3–4).

88 See Weyl (1921), a seminal Intuitionist text.

89 Half a century later, E. Bishop penned his own "crisis" essay Bishop (1975). See also Novikov (2002b, a).

90 See Bishop (1967, p. 10) where one reads: "Weyl ... suppressed his constructivist convictions [and] expressed the opinion that idealistic mathematics finds its justification in its applications to physics".

91 See Bishop (1967, pp. 1–2).

Abraham: Most interesting. Even with an allowance for the heady days of the Sputnik era,[92] the idea of a completed infinity of natural numbers claimed to transcend our biological senses, strikes me as a Platonist position.

Errett: [*shocked*] Me? Platonist?

Abraham: What do you make of the existence results for soap films and soap bubbles[93] in the calculus of variations?

Errett: Existence results for soap bubbles are meaningless since they rely on the extreme value theorem...

Sarah: Meaningless?

Errett: Indeed, the extreme value theorem is false[94] in a constructive world.[95] Very possibly classical mathematics[96] will cease to exist as an independent discipline.[97] [*chanting*]

Computation is the heart
Of everything we prove.
Not for us the velvet wisdom
Of a softer love.[98]

Sarah: [*in agitation*] Cease to exist? Meaningless? I have a strong sentimental attachment to soap bubbles. To me they exist as much as your neon sign. They are even as real as this toy gun! [*bangs the back of the chair which has the gun on its desk arm*]

Errett: [*Ducking*] Careful! The pistol is loaded!! [*The gun falls off the chair arm onto the seat and goes off*]

92 The race to conquer space between the United States and the Soviet Union in the 60s, captured the popular imagination and formed the historical backdrop for Bishop's *another universe* comment.

93 The field is known as Plateau's problem.

94 See Beeson (1985, p. 22) and Katz et al. (2014).

95 See Hellman (1998, pp. 426–427 and p. 432).

96 See note 65 on Bishop's quarrel with classical mathematics.

97 See Bishop (1968, p. 54).

98 See Bishop (1973/1985, p. 14).

[*Boom!!!*[99] *A bullet*[100] *bounces off the wall and hits the billboard, smashing it to smithereens. A sharp-edged splinter bearing the letters* Kronecker[101] *is flung dangerously toward Errett*]

Errett: [*feebly*] The work of eighteen years! [*collapses into an armchair*]

Priscilla: Quickly, Al, call an ambulance from your office. [*They exit. The other four bend over Errett*]

[*Sound of ambulance siren approaching*]

Paramedic: [*panting as he runs in with a stretcher*] Who smashed all these test tubes?!

Sarah: Sorry! Seems as though he can't take lim... Err, can't take the thought of the infinitesimal genie out of its bottle.

Paramedic: You mean, the bacillus[102] out of its test tube? [*leaning over Errett's armchair*] Are you OK, or *not*??

Errett: [*through gritted teeth*] Aaargh! Another excluded thirdist...[103]

Paramedic: [*taken aback*] Hmm... I sensed there was one Bishop too many in this comedy! [*places Errett on the stretcher*]

Ruth: [*pensively*] I wonder if a bubble burst has numerical meaning, constructively speaking...

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99 *Reinventing the wheel* (see main text at note 14) is a metaphor that fits the history of the calculus with uncanny accuracy. The recent work of Netz et al. (see Netz et al. 2001; Netz et al. 2002, pp. 118–119) on the Archimedes Codex reveals that not only have elements of integral calculus been invented by Archimedes, but that Archimedes based his arguments on infinitary concepts involving infinite sums in his *Method*. As vintage wine that only improves with age, the infinitary idea of the calculus lay dormant for over a millennium, making a comeback in the seventeenth century. Two centuries later, the *reinvention* of the calculus wheel on the basis of ϵ, δ and limits was completed by Weierstrass. The Weierstrassian track did not displace the infinitary approach, in spite of the ever-rising stridency of the anti-infinitesimal rhetoric, ranging from the *cholera bacillus* of Cantor (see note 61 for references) to the *entirely meaningless hazy fog* of Courant (see Courant 1937, p. 101 and note 38).
As a vintage wine, the infinitary idea endured several decades of post-Weierstrassian scorn, before being clarified by A. Robinson (see note 130 for Robinson's comments on infinitary processes). The infinitary idea persevered through teetotaller (expression used by Keisler 1977; see note 44) vitriol (term used by Dauben 1996, p. 139 to describe Bishop's condemnation of Robinson's framework) of a reluctant guru (expression used by Halmos 1985, p. 162), enjoyed a refreshing endorsement by the intuitionist Heyting (who particularly appreciated Robinson's insight into the Dirac delta function; see Katz and Katz 2011; Katz and Tall 2013 for a discussion), and has passed the reality check of thousands of publications in economics, engineering, mathematics, and physics.
Getting back to Archimedes, an argument in the proof of Proposition 14 from *Method*, "derives a proportion of solids and areas, from a proportion of areas and lines, based on on a rule of summation of proportion[, namely,] Lemma 11" (Netz et al. 2002, p. 119). Archimedes compares a pair of infinite sums that are "equal in multitude" (*ísos plethei*), i.e. equal in the *number* of summands. The argument derives the equality of the infinite sums, from the equality of the corresponding summands. Netz *et al* conclude that Archimedes was explicitly calculating with infinitely great numbers.

100 A gun should have been hung on the wall in the first act and fired in the third, according to Chekhov's rule, which we have bent.

101 The splinter will complete its task in the main text at note 119. Russian-speaking readers may be reminded of Pushkin's visionary quip to a fellow revolutionary, "K Chaadayevu": "our names will be inscribed on the splinters of autocracy." Cf. http://www.poetarium.info/pushkin/chaad.htm.

102 See note 61 for references on the Cantor bacillus.

103 See note 23 for an explanation of the term.

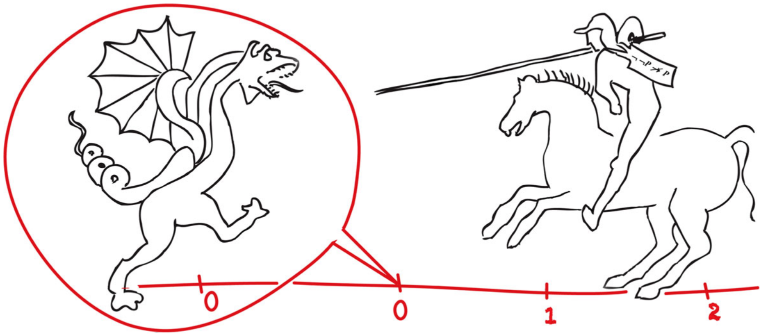

Fig. 2 Errett Bishop's attempted slaying of the infinitesimal. See note 105 for an explanation of the formula $\neg\neg P \not\rightarrow P$

Errett: *Et tu, Rutie*? With my last croak,[104] I will say: the negative of the negative of P, is different from P. . .[105]

Sarah: What is he saying?

Paramedic: [*panting as he lifts the stretcher*] Negatives? Perhaps "P" is for "picture"? Was he into photography? Why is the negative of the negative different from the original photo? Perhaps it comes out grainier?. . .

Errett: [*chanting as in a trance*] Formalism.

The devil is very neat. It is his pride
To keep his house in order. Every bit
Of trivia has its place. He takes great pains
To see that nothing ever does not fit.
And yet his guests are queasy. All their food,
Served with a flair and pleasant to the eye,
Goes through like sawdust. Pity the perfect host!
The devil thinks and thinks and he cannot cry.[106]

Priscilla: [*overawed*] Wow! I now see that the integers were created by Errett; everything else is the work of the D. . ..

Allen: [*interrupting*] This is the end of the rope for being "explicit".[107]

Paramedic: By "his own petard",[108] indeed. . . [*The paramedic and Allen run out with the stretcher, with Priscilla and Ruth running alongside.*]

5 Act V: Number Systems Tomorrow

[*Wailing of ambulance siren tapers off into the distance. A number of other mathematicians, attracted by the commotion, crowd into the room.*]

104 See Richman (1987).

105 In intuitionistic logic, $\neg\neg P$ does not imply P, or in symbols $\neg\neg P \not\rightarrow P$; see Fig. 2.

106 See Bishop (1973/1985, p. 14).

107 See Connes (2007).

108 See epigraph.

Sarah: That was no imaginary conversation between the leaders of two opposing schools![109] He did not take too kindly, or gently,[110] to the novelty of the hyperreal number system, did he?

Abraham: Ah, novel number systems... The dynamic evolution of mathematics is an ongoing process not only at the summit, but also at the more basic level of our number systems.[111]

Sarah: I take it you are referring to the hyperreals. But aren't they somehow less *real* than the ordinary real numbers?

Abraham: The infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers, as both standard irrational numbers and non-standard numbers are introduced by certain infinitary processes.[112]

Sarah: How can you place irrationals and hyperreals on the same level? The diagonal of the square has length $\sqrt{2}$. The area of the circle is π. Can the hyperreals match such concreteness?[113]

Abraham: When I mentioned the standard irrationals, I was referring to a generic, necessarily undefinable, irrational.[114] Of course, specific irrationals arising in algebra and geometry can be as concrete as the rationals.[115]

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109 Bishop's essay Bishop (1975) is cast in the form of an imaginary conversation between Hilbert and Brouwer. G. Birkhoff described non-standard analysis and constructivism as *two opposing schools*; see Dauben (1996, p. 132).

110 See Katz and Katz (Katz and Katz (2011), Section 3).

111 See Dauben (1995, p. 461) quoting A. Robinson's acceptance speech for the Brouwer medal.

112 See note 130.

113 To be more precise, one could ask whether a nonstandard object can be uniquely defined (as e.g., $\sqrt{2}$ or π are uniquely defined by their usual definitions). In other words, is there a formula $\phi(x)$ such that 1) $\exists! x\ \phi(x)$, and 2) such an x is nonstandard. The answer tends to be in the negative, at least in IST (Internal Set Theory; see note 123) and related theories (e.g., Hrbáček 1978). With IST as the background nonstandard setup, there are two "degrees" of the negative answer.
First, if ϕ is assumed to be an internal formula (namely, no occurrence of the st-ness predicate) then the fact that its unique solution is standard is an elementary consequence of the Transfer Principle of IST.
Furthermore, if ϕ is not necessarily internal, then the Transfer argument does not work, but a much less trivial argument (see 3.4.16 in Kanovei and Reeken 2004) yields the standardness of the unique solution of ϕ anyway.

114 Since there are only countably many definable irrationals, a generic irrational is undefinable. One could ask (following early intuitionists like Poincaré) why one needs non-definable mathematical objects at all? This question makes sense not only in the context of validation of nonstandard methods, of course.
The answer is that, first of all, by Tarski's undefinability result, the informal property of definability cannot be soundly described by a mathematical formula. One can observe that a real number x, say $x = \pi$, is definable in virtue of the mere fact that speaking of it we have in mind its canonical definition, but it turns out that we cannot form "the set of all definable reals" on the basis of Zermelo–Fraenkel axioms (with or without choice) alone. See further in note 115.

115 On the other hand, various particular types of definability do form legitimate sets of accordingly definable reals, among them:

- the set COMP of computable reals,
- the (bigger) set HYP of hyperarithmetic reals,

and many others, whose common property is that restricting the real line to one of such sets leads to a failure of basic mathematical results.
In particular, pretending that there is no real outside of COMP, one obtains the failure of the intermediate value theorem (asserting that if $a < b$, f is continuous, $f(a) < 0$, and $f(b) > 0$ then $f(x) = 0$ for a suitable x with $a < x < b$).
Pretending that there is no real outside of HYP, one obtains the failure of the Cantor principle of comparability of any pair of countable wellordered sets; see Simpson (2009). See further in main text at note 135.

Sarah: What does "definable" mean exactly when referring to a mathematical concept such as a real number?

Jules: Well, *definable* just means capable of being defined, referring to an object you can define by a mathematically meaningful sentence in English. For example, "The real number whose integer part is 17 and whose nth decimal place is 0 if n is even and 1 if n is odd" defines the real number 17.1010101 . . ., while the phrase "the capital of England" does not define a real number.

Sarah: Certainly not!

Jules: Moreover, as all mathematically meaningful sentences in English can be ordered lexicographically in a simple infinite sequence, definable reals can be presented in the form of a simple infinite sequence such as $\{x_n : n \in \mathbb{N}\}$.[116]

Alfred: It's not that simple, Monsieur. There is no mathematical formula which soundly expresses the notion of definability. That is, there is no formula $D(x)$ such that $D(x)$ is true if and only if a real x is definable. Accordingly, there is no mathematically sound way to define a sequence $\{x_n : n \in \mathbb{N}\}$ of all definable real numbers, which your essay speculatively invented.

Sarah: So there are several ways of defining definability, so to speak?

Alfred: Well put. On the other hand, as long as we stick to a suitable particular kind of definability, like the definability by means of formulas of Peano arithmetic, then such a restricted notion of definability becomes mathematically rigorous. One can then define a Jules-style sequence $\{x_n : n \in \mathbb{N}\}$ of all Peano-arithmetically definable real numbers. But such a construction is not arithmetically definable.[117]

Sarah: I see. . . Thus to define the set of all definable—in some precise sense—real numbers and place them in a Jules-style sequence, one has to employ a broader notion of definability.

Georg: Anyway, whatever method you use to form a simple infinite sequence of "concrete", definable reals, or in fact any other sequence of reals, the fact is as simple as this: the whole real line is uncountable while there is only a countable set of real numbers which appear in this sequence.[118]

Sarah: What is a countable set?

Georg: Hmm, you missed no opportunity to skip math classes, obviously. But no matter, a set X is countable if its elements can be presented as a finite or infinite sequence whose terms are indexed by the natural numbers. In other words, one has $X = \{x_k : k < k_0\}$ or $X = \{x_k : k \in \mathbb{N}\}$. Thus countability means that the set is not essentially bigger than the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of all natural numbers. Such is the set $\mathbb{N} \times \mathbb{N}$ of all pairs $\langle m, n\rangle$ of natural numbers. . .

Sarah: How is that? $\mathbb{N} \times \mathbb{N}$ looks so much bigger that $\mathbb{N}$ itself.

Georg: It's an illusion. If $k = 2^m 3^n$ then let x_k be the pair $\langle m, n\rangle$. If k does not have the form of $2^m 3^n$ then let x_k be the pair $\langle 0, 0\rangle$. Then $\mathbb{N} \times \mathbb{N} = \{x_k : k \in \mathbb{N}\}$.

Sarah: Fantastic. Are there sets that are even bigger but still countable?

Abraham: If you wish, the set of all formulas of a countable language is countable.

Sarah: A countable language is one containing countably many symbols, and a formula is a well-constructed finite sequence of symbols, right? And since to define a set of any kind, in particular a real number . . . is a real number a set by the way?

Georg: Identify it with the set of all smaller rationals and forget the problem.

116 See Richard (1964).

117 See Tarski (1936).

118 See Cantor (1892).

Sarah: . . . to define a real number means to use a formula which defines it, so we conclude that there are only countably many concrete, definable real numbers. But could it be that the whole real line is countable?

Georg: That can't be, because whatever sequence $\{x_n : n \in \mathbb{N}\}$ one may write down, one can always construct a real number x not on the list. Namely, for each n choose the nth decimal digit of x to be different from the nth digit of x_n. It follows that x differs from each x_n, and is therefore not on the list.

Sarah: Aha, I get it, the uncountable real line contains non-definable reals since there are only countably many definable ones. I see you are an expert on infinite numbers. Does inverting an infinite number give you an infinitesimal?

Georg: Did you say *Infinitesimal*?? I see that your calculus professor wasted his time on you! These infinitesimals are an abomination, paper numbers, cholera bacillus of mathematics. . .[119] [*Jumps up in anger, lands on the splinter bearing the name* Kronecker*, and screeches in pain*] Oh no, not him again! I knew he would get me in the end!

Paramedic: [*Runs in with a stretcher just in time to catch Georg's falling body, and whisks it offstage while calling out*]: Looks like it's been a rough day for the small, the tiny, and the infinitesimal. . .

Sarah: Oh dear. I see where Errett got his ideas from. . .

Jerry: Teetotallers are generally not known for fine wine-sampling. . .[120]

Sarah: Getting back to number systems, if most real numbers are undefinable, then the non-definability of hyperintegers is not such a big issue, either. Don't we already observe non-definable objects at the very foundation of real number theory?

Jerry: It's a bit more complicated than that. The "standard" real line is pretty well definable as a whole, in several equivalent ways. Which is not the case for the hyperreal line.[121]

Sarah: Why is that?

Jerry: Because there exist infinitely many (even class-many, if you know what I mean) non-isomorphic examples of the hyperreal line, equally suitable to be considered as such, so it does not seem easy to pick one of them to be *the* hyperreal line.

Saharon: Yes this looks like an obstacle, but fortunately a solution does exist. Very roughly, there is a way to "superpose" all possible hyperreal lines in a fixed cardinality, placed in a certain definable linear order, to get a concrete, definable hyperreal line.[122]

Jerry: Thus indeed "the hyperreal number system, like the real number system, can be built as an explicitly definable mathematical structure".

Sarah: Yet, even at the level of a number system as a whole, the hyperreals seem more speculative. I heard, for example, that there are uncountably many hyperintegers. How does that compare with the concreteness of the "real" integers?

Priscilla: [*Running in breathless*] Ruth went off with the ambulance! Did you mention the integers? Errett said something about them. . .

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119 See note 61 for sources for each of these epithets.

120 See Keisler (1977, p. 269).

121 See Keisler (1994).

122 See Kanovei and Shelah (2004) as well as Keisler (2007, pp. 23–31).

Abraham: I heard my friend Edward[123] is developing his own *internal* take on this. I would merely add that both standard and non-standard analysis are grounded in the classical ZFC,[124] which is their common foundational framework.

Sarah: Yet, after all is said and done about their rigor and their being grounded in ZFC, it still seems to require something of a leap of faith to accept the procedures of a hyperreal framework, doesn't it?

Abraham: It is hard for me to comment on issues of faith, but my father learned the Talmud,[125] so perhaps some of it rubbed off... I wonder what Leibniz's reception may have been...[126]

Sarah: Didn't Al say that Leibniz wasn't rigorous?

Abraham: Leibniz argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with the real numbers, but which were *to possess the same properties as the latter*...[127]

Sarah: Sounds optimistic...

Abraham: Leibniz's ideas can be fully vindicated. They lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics.[128]

Sarah: You mentioned that the dynamic evolution of mathematics goes on at the level of our number systems, as well. In what way does the dynamism of number systems manifest itself?

Abraham: Such new number systems, tomorrow, may seem to a new generation to have been around forever, as yesterday's technological innovations[129] seem to a child of today.[130]

Allen: [*Walking in, annoyed*] I am not going to use infinitesimals to teach elementary calculus because my supervisor would fire me as a TA.

Priscilla: [*Looking disappointed with him*] Allen Class, are you a coward?

Allen: No, I am a man who has begun to think of getting married, [*Priscilla perks up*] but I am not going to do so with no income. [*Priscilla looks thoughtful*] Infinitesimals might be the way to go, but only for tenured faculty members, and not in service courses for the Chemistry Department.

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123 Nelson (1977) introduced a syntactic enrichment into set theory by means of a unary predicate "standard" (st(x), meaning "x is standard"), which allows one to detect both non-standard (i.e., infinitely large) integers within the ordinary ZFC (see note 124) integers, and infinitesimals within the ordinary ZFC reals. In particular, Nelson's internal set theory IST contains an axiom schema called Transfer, which guarantees that all standard sets obey the same mathematical rules as do all sets, standard and nonstandard combined together.

124 Zermelo-Fraenkel set theory with the Axiom of Choice.

125 See Dauben (2003, p. 243).

126 Robinson has been quoted as saying that he would like to get into Leibniz's head.

127 In a 2 feb. 1702 letter to Varignon, Leibniz formulated the *law of continuity*, described as a "souverain principe", as follows: "il se trouve que les règles du fini réussissent dans l'infini... et que vice versa les règles de l'infini réussissent dans le fini" (Leibniz, p. 350). This formulation was cited in (Robinson 1966, p. 262), and connected with the *Transfer Priciple*. To summarize: *the rules of the finite succeed in the infinite, and conversely*.

128 See Robinson (1966, p. 2).

129 See Dauben (1995, p. 461).

130 Robinson wrote: "[T]he infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, the standard irrational numbers ... both standard irrational numbers and non-standard numbers are introduced by certain infinitary processes" (Robinson 1966, p. 282). See note 99 for infinitary considerations in Archimedes.

Sarah: But the Physics Department, yes? I was thinking of majoring in Physics.

Allen: No, the physicists don't let us near their students. They think that we don't get practical fast enough.

[*Exeunt all*]

6 Epilogue

[*Some time later. Enter Sarah, Paolo, Karel, Sam*]

Sarah: There is another intriguing issue in connection with infinitesimals. A typical application of the infinitesimal approach runs as follows. We start with a problem which does not refer to either infinitesimals or infinitely large numbers at all. Then we say: let α be an infinitesimal, or: let H be an infinitely large integer. There follows a string of arguments exploiting α, or respectively, H. The final outcome is a result which again does not refer explicitly to either infinitesimals or infinitely large numbers. Leonhard Euler's famous factorization of the sine function as an infinite product is a case in point.[131] Isn't there something strange about this?

Paulo: Why is that? In order finally to prove Fermat's Last Theorem, which speaks of integers and nothing more complex than integers, Wiles employed much mathematical technology well beyond the integers.[132] Note that this non-integer technology plays a role similar to infinitesimals (or infinitely large numbers) in Euler's proof of the sine decomposition.

Sarah: True, but the gap (if there is one) between integers and other fields of non-infinitesimal mathematics appears to be narrower than the gap between the non-infinitesimal mathematics as a whole and the mathematics of infinitesimals/infinitely large numbers. It therefore interests me whether a proof like Euler's derivation of the product decomposition for sine can be rendered infinitesimal-free (and infinitely-large-number-free)?

Karel: Interesting question, Sarah. The answer happens to be affirmative. If any mathematical proposition can be proved by means of infinitesimal methods then there exists a proof of the same proposition which does not refer to infinitesimals/infinitely large numbers. This result is referred to as the *conservativity* of infinitesimal methods, and it holds with respect to both set and class background theories.[133]

Sarah: If they can always be replaced, what's the point of using infinitesimals in the first place?

Karel: The point is that an infinitesimal-style proof may often be more intuitively transparent, more appealing, and sometimes easier to follow than sans-infinitesimal proofs of the same proposition.

Sarah: There is another issue involved here. As far as I can understand Ribenboim's book (not much indeed), Wiles' proof of the Fermat's Last Theorem involves non-integer technology,[134] which includes, among others, algebraic curves, that is, objects which, while different from integers and much more complex, can be individually defined, classified, etc.

But this is not the case for an infinitely large integer i with which Euler's sine factorization proof starts. Such an i cannot be defined,[135] that is, individually chosen, as for example one

131 See Euler (1748), Kanovei (1988), McKinzie and Tuckey (1997) and Bair et al. (2013).

132 See Ribenboim (1999) as well as McLarty (2011).

133 See Hrbacek (2005), Kanovei and Reeken (2004) and Kanovei and Lyubetskii (2007).

134 For a recent analysis motivated by the foundational status of Fermat's Last Theorem, see McLarty (2010, 2011).

135 See notes 113, 114, 115.

can choose π to represent a transcendental number whenever an example of such a number is needed. But it turns out that, moreover, Euler's infinitely large integer i can be completely arbitrary, and none of its properties really matters except for, well, being infinitely large. Has this issue been analyzed, or perhaps led to further insights?

Sam: Yes, it has. In fact, it leads to the notion of Ω-*invariance*. Invariance in its various forms is a blueprint in many fields of modern mathematics. It means, basically, that if we change a detail x in a mathematical construction or argument X to another detail x', which is in a certain relation to x, then the changed construction X' will be equal or "similar" to X in a suitable sense.[136]

Something of this sort happens in the mathematics of infinitesimal methods. Let's call a formula or relation $\Phi(n)$ Ω-*invariant* (with respect to n) if either $\Phi(n)$ is true for all infinitely large integers n, or $\Phi(n)$ is false for all infinitely large integers n. In other words, $\Phi(n)$ is independent of the choice of $n \in \Omega$, where Ω stands for the set of all infinitely large integers. For instance, we may observe that all parts of Euler's sine factorization argument, presented as relations, are Ω-invariant with respect to the infinitely large integer i chosen in the beginning of the proof.

Karel: But then, by the underflow principle, Ω-invariance is equivalent to what one might call eventual invariance in the finite domain ("eventual" meaning that a sentence or formula considered holds from some unspecified natural number onward), that is, there should be an integer N such that either $\Phi(n)$ is true for all (finite) integers $n \geq N$, or $\Phi(n)$ is false for all (finite) integers $n \geq N$.

Sam: True, but your interpretation requires two quantifiers, i.e.

there exists N such that for all $n \geq N \ldots$

whereas mine needs only one quantifier

for all infinitely large N,

which is important in the fields of mathematical foundations where the number and order of quantifiers really matter.

Sarah: But... well, if a relation $\Phi(x, n)$ is Ω-invariant with respect to n for every x, then we are free to eliminate n altogether, that is, define $\Psi(x)$ to be $\Phi(x, n)$ for any infinitely large integers n. What is then the importance of the notion?

Sam: Exactly. But then you may ask how much of conventional mathematical principles you really need to legitimally claim the existence of Ψ for a given Φ as above. And the answer turns out to be nontrivial even for quite simple Ω-invariant relations Φ.[137]

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[136] See http://www.math.uni-hamburg.de/home/loewe/HiPhI/Slides/sanders.

[137] See Sanders (2014) and http://mathoverflow.net/questions/128791/can-nonstandard-analysis-be-used-to-prove-results-in-constructive-or-computable .

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Vladimir Kanovei graduated in 1973 from Moscow State University, and obtained a Ph.D. in physics and mathematics from Moscow State University in 1976. In 1986, he became Doctor of science in physics and mathematics at Moscow Steklov Mathematical Institute (MIAN). He is currently Leading Researcher at the Institute for Information Transmission Problems (IPPI), Moscow, Russia. Among his publications is the book Borel equivalence relations. Structure and classification. *University Lecture Series* **44**. American Mathematical Society, Providence, RI, 2008.

Karin U. Katz (B.A. Bryn Mawr College, '82; Ph.D. Indiana University, '91) teaches mathematics at Bar Ilan University, Ramat Gan, Israel. Her joint study with her husband, Mikhail Katz entitled "A Burgessian critique of nominalistic tendencies in contemporary mathematics and its historiography", appeared in *Foundations of Science*.

Mikhail G. Katz (B.A. Harvard University, '80; Ph.D. Columbia University, '84) is Professor of Mathematics at Bar Ilan University. Among his publications is the book Systolic geometry and topology, with an appendix by Jake P. Solomon, published by the American Mathematical Society; and the article (with A. Borovik and R. Jin) An integer construction of infinitesimals: Toward a theory of Eudoxus hyperreals, published in *Notre Dame Journal of Formal Logic*.

Mary Schaps has a Ph.D. from Harvard in Algebraic Geometry. She has also published in Deformation Theory, Group Theory and Representation Theory, as well as fiction. She is currently a Full Professor of Mathematics at Bar-Ilan University, Dean of the Faculty of Exact Sciences, and Head of the Financial Mathematics Program.