

Is Leibnizian Calculus Embeddable in First Order Logic?

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Abstract To explore the extent of embeddability of Leibnizian infinitesimal calculus in first-order logic (FOL) and modern frameworks, we propose to set aside ontological issues and focus on procedural questions. This would enable an account of Leibnizian procedures in a framework limited to FOL with a small number of additional ingredients such as the relation of infinite proximity. If, as we argue here, first order logic is indeed suitable for developing modern proxies for the inferential moves found in Leibnizian infinitesimal calculus, then modern infinitesimal frameworks are more appropriate to interpreting Leibnizian infinitesimal calculus than modern Weierstrassian ones.

Keywords First order logic · Infinitesimal calculus · Ontology · Procedures · Leibniz · Weierstrass · Abraham Robinson

1 Introduction

Leibniz famously denied that infinite aggregates can be viewed as *wholes*, on the grounds that they would lead to a violation of the principle that the whole is greater than the part. Yet the infinitary idea is latent in Leibniz in the form of a distinction between *assignable*

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and *inassignable* quantities (Child 1920, p. 153), and explicit in his comments as to the violation of Definition V.4 of Euclid's *Elements* (Leibniz 1695, p. 322). This definition is closely related to what is known since Stolz (1883) as the *Archimedean property*, and was translated by Barrow in 1660 as follows:

Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other (Euclid 1660).

Furthermore, Leibniz produced a number of results in infinitesimal calculus which, nowadays, are expressed most naturally by means of quantifiers that range over infinite aggregates. This tension leads us to examine a possible relationship between Leibnizian infinitesimal calculus and a modern logical system known as first order logic (FOL). The precise meaning of the term is clarified in Sect. 3. We first analyze several Leibnizian examples in Sect. 2.

This text continues a program of re-evaluation of the history of infinitesimal mathematics initiated in Katz and Katz (2012), Bair et al. (2013) and elsewhere.

2 Examples from Leibniz

Let us examine some typical examples from Leibniz's infinitesimal calculus so as to gauge their relationship to FOL.

2.1 Series Presentation of $\pi/4$

In his *De vera proportione* (1682), Leibniz represented $\frac{\pi}{4}$ in terms of the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This is a remarkable result, but we wish to view it as a result concerning a specific real number, i.e., a *single case*, and in this sense involving no quantification, once we add a new function symbol for a black box procedure \square called "evaluation of convergent series" (as well as a definition of π) (we will say a few words about the various implementations of \square in modern frameworks in Sect. 4).

2.2 Leibniz Convergence Criterion for Alternating Series

This refers to an arbitrary alternating series defined by an alternating sequence with terms of decreasing absolute value tending to zero, such as the series of Sect. 2.1, or the series $\sum_n \frac{(-1)^n}{n}$ determined by the alternating sequence $\frac{(-1)^n}{n}$. We will refer to such a sequence as a 'Leibniz sequence' for the purposes of this subsection. This criterion seems to be quantifying over sequences (and therefore sets), thus transcending the FOL framework, but in fact this can be handled easily by introducing a free variable that can be interpreted later according to the chosen domain of discourse.

Thus, the criterion fits squarely within the parameters of FOL + \square at level (3) (see Sect. 3). In more detail, we are not interested here in arbitrary 'Leibniz sequences' with possibly inassignable terms. Leibniz only dealt with sequences with ordinary (assignable) terms, as in the two examples given above. Each real sequence is handled in the framework

$\mathbb{R} \subset {}^*\mathbb{R}$ by the transfer principle, which asserts the validity of each true relation when interpreted over ${}^*\mathbb{R}$.

2.3 Product Rule

We have $\frac{d(uv)}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u$ and it looks like we need quantification over pairs of functions (u, v) . Here again we are only interested in natural extensions of real functions u, v , which are handled at level (3) as in the previous section.

In Leibniz (1684), the product rule is expressed in terms of differentials as $d(uv) = u dv + v du$. In *Cum Prodiisset* (Leibniz 1701, pp. 46–47) Leibniz presents an alternative justification of the product rule (see Bos 1974, p. 58). Here he divides by dx and argues with differential quotients rather than differentials. Adjusting Leibniz's notation, we obtain an equivalent calculation

$$\begin{aligned} \frac{d(uv)}{dx} &= \frac{(u + du)(v + dv) - uv}{dx} \\ &= \frac{u dv + v du + dudv}{dx} \\ &= \frac{u dv + v du}{dx} + \frac{dudv}{dx} \\ &= \frac{u dv + v du}{dx}. \end{aligned}$$

Under suitable conditions the term $\frac{du dv}{dx}$ is infinitesimal, and therefore the last step

$$\frac{u dv + v du}{dx} + \frac{du dv}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}, \quad (2.1)$$

relying on a generalized notion of equality, is legitimized as an instance of Leibniz's *transcendental law of homogeneity*, which authorizes one to discard the higher-order terms in an expression containing infinitesimals of different orders.

2.4 Law of Continuity

Leibniz proposed a heuristic principle known as the *law of continuity* to the effect that

... et il se trouve que les règles du fini réussissent dans l'infini ...; et que vice versa les règles de l'infini réussissent dans le fini, ... (Leibniz 1702, pp. 93–94),

cited by Knobloch (2002), p. 67, Robinson (1966), p. 262, Laugwitz (1992), p. 145 and other scholars.

On the face of it, one can find numerous counterexamples to such a principle. Thus, finite ordinal number addition is commutative, whereas for infinite ordinal numbers, the addition is no longer commutative: $1 + \omega = \omega \neq \omega + 1$. Thus, the infinite realm of Cantor's ordinals differs significantly from the finite: in the finite realm, commutativity rules, whereas in the infinite, it does not. Thus the transfer of properties between these two realms fails.

Similarly, there are many infinitary frameworks where the law of continuity fails to hold. For example, consider the Conway–Alling surreal framework; see e.g., Alling (1985). Here one can't extend even such an elementary function as $\sin(x)$ from \mathbb{R} to the surreals.

Even more strikingly, $\sqrt{2}$ turns out to be (sur)rational; see (Conway 2001, chapter 4). The surreals don't satisfy the Peano Arithmetic. Therefore transfer from the finite to the infinite domain fails also for the domain of the surreals.

On the other hand, the combined insight of Hewitt (1948), Łoś (1955), and Robinson (1961) was that there does exist an infinitary framework where the law of continuity can be interpreted in a meaningful fashion. This is the $\mathbb{R} \subset {}^*\mathbb{R}$ framework. While it is not much of a novelty that many infinitary systems don't obey a law of continuity/transfer, the novelty is that there is one that does, as shown by Hewitt, Łoś, and Robinson, in the context of first-order logic.

Throughout the eighteenth century, Euler and other mathematicians relied on a broad interpretation of the law of continuity or, as Cauchy will call it, the *generality of algebra*. This involved manipulation of infinite series as if they were finite sums, and in some cases it also involved ignoring the fact that the series diverges. The first serious challenge to this principle emerged from the study of Fourier series when new types of functions arose through the summation thereof. Specifically, Cauchy rejected the principle of the *generality of algebra*, and held that a series is only meaningful in its radius of convergence. Cauchy's approach was revolutionary at the time and immediately attracted followers like Abel. Cauchy in 1821 was perhaps the first to challenge such a broad interpretation of the law of continuity, with a possible exception of Bolzano, whose work dates from only a few years earlier and did not become widely known until nearly half a century later. For additional details on Cauchy see Katz and Katz (2011, 2012), Borovik and Katz (2012) and Bascelli et al. (2014). For Euler see Kanovei et al. (2015) and Bascelli et al. (2016).

2.5 Non-examples: EVT and IVT

It may be useful to illustrate the scope of the relevant results by including a negative example. Concerning results such as the extreme value theorem (EVT) and the intermediate value theorem (IVT), one notices that the proofs involve procedures that are not easily encoded in first order logic. These nineteenth century results (due to suitable combinations of Bolzano, Cauchy, and Weierstrass) arguably fall outside the scope of Leibnizian calculus, as do infinitesimal foundations for differential geometry as developed in Nowik and Katz (2015).

There are axioms in FOL for a real closed field F (e.g., real algebraic numbers, real numbers, hyperreal numbers, Conway numbers). One of these axioms formalizes the fact that IVT holds for odd degree polynomials $F[x]$. In fact, one needs infinitely many axioms like $(\forall a, b, c)(\exists x)[x^3 + ax^2 + bx + c = 0]$. Meanwhile, IVT in its full form is equivalent to the continuity axiom for the real numbers (Błaszczyk 2015).

3 What Does “First-Order” Mean Exactly?

The adjective ‘first-order’ as we use it entails limitations on quantification over sets (as opposed to elements). Now Leibniz really did not have much to say about properties of sets in general in the context of his infinitesimal calculus, and even declared on occasion that infinite totalities don't exist, as mentioned above. Note that Leibniz arguably did exploit second-order logic in areas outside infinitesimal calculus (see Lenzen 1987, 2004) but this will not be our concern here. Leibniz famously takes for granted second order logic in formulating his principle governing the *identity of indiscernibles*. While second order logic

is possibly part of Leibniz's metaphysics it is not in any obvious way part of his infinitesimal calculus.

Once we reach topics like Baire category, measure theory, Lebesgue integration, and modern functional analysis, quantification over sets becomes important, but these were not Leibnizian concerns in the kind of analysis he explored.

In fact, the term "first order logic" has several meanings. We can distinguish three levels at which a number system could have first-order properties compatible with those of the real numbers. Note that the real numbers satisfy the axioms of an ordered field as well as a completeness axiom.

- (1) An *ordered field* obeys those among the usual axioms of the real number system that can be stated in first-order logic (completeness is excluded). For example, the following commutativity axiom holds: $(\forall x, y) [x + y = y + x]$.
- (2) A real closed ordered field has all the first-order properties of the real number system, regardless of whether such properties are usually taken as axiomatic, for statements which involve the basic ordered-field relations $+$, \times , and \leq . This is a stronger condition than obeying the ordered-field axioms. More specifically, one includes additional first-order properties, such as extraction of roots (e.g., existence of a root for every odd-degree polynomial). For example, every number must have a cube root: $(\forall x)(\exists y) [y^3 = x]$, or every positive number have a square root: $(\forall x > 0)(\exists y) [y^2 = x]$.
- (3) The system could have all the first-order properties of the real number system for statements involving arbitrary relations (regardless of whether those relations can be expressed using $+$, \times , and \leq). For example, there would have to be a sine function that is well defined for infinitesimal and infinite inputs; the same is true for every real function. To do series, one needs a symbol for \mathbb{N} , so as to define transcendental entities such as π or sine. We also introduce function symbols for whatever functions we are interested in working with; say all elementary functions occurring in Leibniz as well as their combinations via composition, differentiation, and integration.

It follows from these examples that the *first order* qualification is connected with the intended domain of discourse, so that any quantifier related to objects outside the domain of discourse is qualified as not a first order one. It could be added however that all mathematical objects are, generally speaking, (represented by suitable) sets from the set theoretic standpoint, and hence all mathematical quantifiers are first-order with respect to the background set universe (superstructure).

The point with level (3) is that instead of quantifying over sequences or functions, we relate to each individual sequence or function, and make sure that it has an analogue in the extended domain. Such an analogue of f is sometimes referred to as the *natural extension* of f . Then we can say something about the extension of every standard object in our system, e.g., function, without ever being able to assert anything about *all* functions. Thus, the product rule for differentiation is proved for the assortment of functions chosen in item (3) above.

Remark 3.1 An alternative to the multitude of functional symbols would be to add a countable list of variables u, v, w, \dots meant to denote unspecified functions. The idea is to avoid quantifying over such variables, and use them as merely free variables. Then, for example, the product rule is the following statement: "if u, v are differentiable functions then the Leibniz rule holds for u and v ", with u, v being free variables.

Note that we use FOL in a different sense from that used in formalizing Zermelo–Fraenkel set theory (ZFC).

When we seek hyperreal proxies, following the pioneering work of Hewitt (1948) and Robinson (1966), for Leibniz’s procedural moves, the theory of *real closed fields* at level (2) is insufficient and we must rely upon level (3).

Thus, Leibniz’s series of Sect. 2.1 is expressible in FOL + \square at level (3) but FOL level (2) does not suffice since π is not algebraic. Similarly, examples in Sects. 2.2 and 2.3 need symbols for unspecified functions which are not available at level (2).

4 Modern Frameworks

Based on the examples of Sect. 2, we would like to consider the following question:

Which modern mathematical framework is the most appropriate for interpreting Leibnizian infinitesimal calculus?

The frameworks we would like to consider are

- (A) a Weierstrassian (or “epsilon-ontic”) framework in the context of what has been called since Stolz (1883) an *Archimedean* continuum, satisfying Euclid V.4 (see Sect. 2), namely the real numbers exclusively; and
- (B) a modern framework exploiting infinitesimals such as the hyperreals, which could be termed a *Bernoullian* continuum since Johann Bernoulli was the first to exploit infinitesimals (rather than “exhaustion” methods) systematically in developing the calculus.¹

The series summation blackbox \square (see Sect. 2.1) is handled differently in A and B. Framework A exploits a first-order “epsilon-ontic” formulation that works in a complete Archimedean field. Thus, the convergence of a series $\sum_i u_i$ to L would be expressed as follows:

$$(\forall \epsilon > 0)(\exists n \in \mathbb{N})(\forall m \in \mathbb{N}) \left[m \geq n \rightarrow \left| \sum_{i=1}^m u_i - L \right| < \epsilon \right].$$

(Ishiguro 1990, Chapter 5) sought to interpret Leibnizian infinitesimal calculus by means of such quantified paraphrases, having apparently overlooked Leibniz’s remarks to the effect that his infinitesimals violate Euclid V.4 (Leibniz 1695, p. 322).

Meanwhile, framework B allows for an alternative interpretation in terms of the shadow (i.e., the standard part, closely related to Leibniz’s generalized notion of equality) and hyperfinite partial sums as follows: for each infinite hypernatural H the partial sum $\sum_{i=1}^H u_i$ is infinitely close to L , i.e.,

¹ The adjective *non-Archimedean* is used in modern mathematics to refer to certain modern theories of ordered number systems properly extending the real numbers, namely various successors of Hahn (1907). In modern mathematics, this adjective tends to evoke associations unrelated to seventeenth century mathematics. Furthermore, defining infinitesimal mathematics by a negation, i.e., as *non-Archimedean*, is a surrender to the Cantor–Dedekind–Weierstrass (CDW) view. Meanwhile, true infinitesimal calculus as practiced by Leibniz, Bernoulli, and others is the base of reference as far as seventeenth century mathematics is concerned. The CDW system could be referred to as non-Bernoullian, though the latter term has not yet gained currency.

$$(\forall H) \left[H \text{ infinite} \rightarrow \sum_{i=1}^H u_i \approx L \right].$$

This is closer to the historical occurrences of the package \square as found in Gregory, Leibniz, and Euler, as we argue below.

In pursuing modern interpretations of Leibniz's work, a helpful distinction is that between ontological and procedural issues. More specifically, we seek to sidestep traditional questions concerning the ontology of mathematical entities such as numbers, and concentrate instead on the procedures, in line with Quine's comment to the effect that

Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic. (Quine 1968, p. 198)

Related comments can be found in Benacerraf (1965). If one could separate the "ontological questions" from the rest, then framework A would be more appropriate than framework B for interpreting the classical texts *if and only if* framework A provides better proxies for the procedural moves found in Leibnizian infinitesimal calculus than framework B does, and vice versa.

The tempting evidence in favor of the appropriateness of a modern framework B for interpreting Leibnizian infinitesimal calculus is the presence of infinitesimals and infinite numbers in both, as well as the availability of hyperreal proxies for guiding principles in Leibniz's work such as the law of continuity as expressed in Leibniz (1701, 1702) as well as the transcendental law of homogeneity (Leibniz 1710); see Katz and Sherry (2012, 2013), Sherry and Katz (2014) and Guillaume (2014). To what extent Leibnizian infinitesimals can be implemented in differential geometry can be gauged from Nowik and Katz (2015).

The question we seek to explore is whether the limitation of working with first order logic as discussed in Sect. 3 could potentially undermine a full implementation of a hyperreal scheme for Leibnizian infinitesimal calculus.

With this in mind, let us consider Skolem's construction of nonstandard natural numbers (Skolem 1933, 1934, 1955); see (Kanovei et al. 2013, Section 3.2) for additional references. It turns out that one needs many, many nonstandard numbers in order to move from \mathbb{N} to ${}^*\mathbb{N}$, e.g., in Henkin's countable model one has

$${}^*\mathbb{N} = \mathbb{N} + (\mathbb{Z} \times \mathbb{Q}). \quad (4.1)$$

Here we use \mathbb{Q} to indicate that the galaxies are dense, so that between any pair of galaxies there is another galaxy (a galaxy is the set of numbers at finite distance from each other). Meanwhile \mathbb{Z} indicates that each galaxy other than the original \mathbb{N} itself is order-isomorphic to \mathbb{Z} rather than to \mathbb{N} , because for each infinite H the number $H - 1$ is in the same galaxy.

Leibniz arguably did not have such a perspective. In other words, one needs to build up a considerable conceptual machinery to emulate Leibniz's probably rather modest arsenal of procedural moves. That is to say, we may be able to emulate all of Leibniz moves in a modern B-framework, such as Leibniz's infinite quantities, his distinction between assignable and inassignable quantities, and his transcendental law of homogeneity. However, the B-framework also enables us to carry out many additional moves unknown to Leibniz, for instance those related to the detailed structure of ${}^*\mathbb{N}$ as in (4.1). Thus, the *difference* between the Leibnizian framework and a modern infinitesimal B-framework is *large*.

On the other hand, a Weierstrassian A-framework may not cover all the moves Leibniz may make in his framework LEI, but one might argue that the *difference* between (A) and LEI is small. Thus, one may not necessarily have $LEI \subseteq (A)$, but one might argue that the difference (A)—LEI is *small*. This may be taken as evidence that (A) and LEI are more similar to each other than (B) and LEI are. This could affect the assessment of appropriateness. Finally, could it be that neither the Weierstrassian nor the modern infinitesimal account is appropriate to cope with Leibnizian infinitesimal calculus?

5 Separating Entities from Procedures

What would it mean exactly to separate ontological problems from procedural problems? A possible approach is to attempt to account for Leibniz's procedures in a framework limited to first order logic, with a small number of additional ingredients such as the relation of *infinite proximity* and the closely related *shadow principle* for passing from a finite inassignable quantity to an assignable one (or from a finite nonstandard number to a standard one), as in $2x + dx \sqsupset 2x$.²

As far as Skolem's nonstandard extension $\mathbb{N} \subset {}^*\mathbb{N}$ is concerned, anything involving the actual construction of the number system and the entities called *numbers* would go under the heading of the *ontology* of mathematical entities. Note that *the first order theories* of \mathbb{N} and ${}^*\mathbb{N}$ are identical, as shown by Skolem (for more details see Sect. 3). In this sense, not only is one not adding *a lot*, but in fact *one is not adding anything at all* at the level of the *theory*.

What about the claim that *Leibniz did not have this perspective*? It is true that he did not have our perspective on the *ontological* issues involved in a modern construction of a suitable number system incorporating infinite numbers, but this needn't affect the *procedural* match.

What about the claim that *one has to build up a considerable conceptual machinery to emulate Leibniz's probably rather modest arsenal of procedural moves; that is to say, we may be able to emulate all of Leibniz's moves in the modern framework, but it also enables us to carry out many moves that Leibniz would have never dreamt of*? As mentioned above, this is not the case at the level of first order logic.

What about the claim that *the difference between Leibniz framework and the infinitesimal framework is large*? At the procedural level this is arguably not the case.

What about the claim that *the Weierstrass framework may not cover all of Leibniz's, but the difference, (A)-LEI, is small, indicating that (A) and LEI are more similar to each other than (B) and LEI are, affecting the assessment of appropriateness*? What needs to be pointed out here is that actually the considerable distance in ontology between (A) and LEI is about the same as the distance between (B) and LEI. The Weierstrassian punctiform continuum where almost all real numbers are undefinable (so that no individual number of this sort can ever be specified, unlike π , e , etc.) is a far cry from anything one might have imagined in the seventeenth century.

² On occasion Leibniz used the notation " \sqsupset " for the relation of equality. Note that Leibniz also used our " $=$ " and other signs for equality, and did not distinguish between " $=$ " and " \sqsupset " in this regard. To emphasize the special meaning *equality* had for Leibniz, it may be helpful to use the symbol \sqsupset so as to distinguish Leibniz's equality in a generalized sense of "up to" from the modern notion of equality "on the nose."

As far as the question *Could it be that neither Weierstrass nor the infinitesimal account is appropriate to cope with Leibniz?* this is of course possible in principle. However, we are interested here in the practical issue of modern commentators missing some compelling aspects of interpretation of Leibniz's work because of a self-imposed limitation to a Weierstrassian interpretive framework.

6 A Lid on Ontology

It could be objected that one cannot escape so easily with the general argument along the lines of "Let's Ignore (ontological) Differences," or LID for short (putting a *lid* on ontology, so to speak).

The LID proceeds as follows. We start with the 'real' L, i.e., the mathematician who lived, wrote, and argued in the seventeenth century. It seems plausible to assume that L based his reasoning on a mixture of first and second order arguments, without clearly differentiating between the two.

In a reconstruction of L's arguments, one replaces the cognitive agent L by a substitute L1 who argues only in a first order framework. This entails, in particular, that L1 cannot distinguish between \mathbb{N} and $^*\mathbb{N}$.

However, it seems likely that L could distinguish the two structures, simply because he did not distinguish between the first and second order levels. In other words, the LID recommendation does not help because the distinction between first and second order does not only affect the ontology but also the epistemology of the historical agents involved.

In sum, a modern infinitesimal reconstruction of L deals with a first-order version of L, namely L1, and not with L. In line with his position on geometric algebra, Unguru (1976) could point out that L1 is a modern artefact, different from the "real" L. Therefore additional arguments are needed in favor of the hypothesis that L and L1 are epistemologically sufficiently similar, but this seems difficult. In any case, a purely ontological assumption does not suffice.

To respond to the L vs. L1 distinction, note that the tools one needs are *almost* limited to first order logic, but not quite, since one needs the shadow principle and the relation of infinite proximity. Rather than arguing that $L = L1$, we are arguing that $L = L1 + \epsilon$.

Now the difference between calculus and analysis is that in calculus one deals mostly with first order phenomena (with the proviso as in Sect. 5), whereas in analysis one starts tackling phenomena that are essentially second order, such as the completeness property i.e., existence of the least upper bound for an arbitrary bounded set, etc. It seems reasonable to assume that what they were doing in the seventeenth century was calculus rather than analysis.

As far as Unguru is concerned, he is unlikely to be impressed by interpretations of Leibnizian infinitesimals as quantified propositions or for that matter by reading Leibniz as if he had already read not only Weierstrass but also Russell à la Ishiguro, contrary to much textual evidence in Leibniz himself. We provide a rebuttal of the Ishiguro–Arthur *logical fiction* reading in Bair et al. (2016) and Bascelli et al. (2016).

7 Robinson, Cassirer, Nelson

7.1 Robinson on Second-Order Logic

The following quote is from Robinson's *Non-standard Analysis*:

The axiomatic systems for many algebraic concepts such as groups or fields are formulated in a natural way within a first order language.... However, interesting parts of the *theory* of such a concept may well extend beyond the resources of a first order language. Thus, in the theory of groups statements regarding subgroups, or regarding the existence of subgroups of certain types will, in general, involve quantification with respect to sets of individuals... (Robinson 1966, section 2.6, p. 19)

This appears to amount to a claim that the local ontology may indeed be often formulated in first-order terms, while the global ontology is deeply infected by second-order concepts. The latter may typically involve objects and arguments qualified as second or higher order with respect to the former, which nevertheless are of the first-order type when considered as related to the background set universe.

7.2 Ernst Cassirer

Does the equation $L = L1 + \epsilon$ not amount to an underestimation of the historical Leibniz? Is it reasonable to assume that he only invented the calculus, and not analysis? According to Cassirer, the basic concepts of analysis were deeply soaked with philosophy, i.e., for Leibniz mathematical and philosophical concepts were intimately related:³

Leibniz himself asserted that the new analysis has sprung from the innermost source of philosophy, and he assigned to both regions [i.e., analysis and philosophy] the task to confirm and to elucidate each other.⁴ (Cassirer 1902, p. xi)

If $L = L1 + \epsilon$, i.e., if the historical Leibniz was mainly dealing with *calculus*, this may appear hardly compatible with Cassirer's perspective; see Mormann and Katz (2013). This impression would be, however, a misunderstanding. In order to forestall it, it merits being pointed out that developing the calculus was a great mathematical achievement of philosophical relevance. It is only today that the term *calculus* possesses a connotation of routine undergraduate mathematics, but not in the nineteenth century.

As far as Cassirer is concerned, Leibniz was indeed doing *analysis* as l'Hôpital called it. It is not even sure Cassirer was aware of the more advanced analysis. Leibnizian calculus only seems "trivial" from the standpoint of properly twentieth century mathematics. It is an advance in understanding when we make a distinction between Leibnizian calculus and analysis. We don't mean to diminish Leibniz's greatness by this distinction, nor do we suggest that Cassirer was *wrong*. He was merely using the term *analysis* in its seventeenth–nineteenth century sense rather than the sense in which we use it today.

³ In support of this claim, Cassirer refers here in particular to Gottfried Wilhelm Leibniz, *Die philosophischen Schriften*, hrg. von Carl Immanuel Gerhardt, 7 Bde., Berlin 1875–1890, Bd. VII, S. 542. (Cassirer 1902, p. xi)

⁴ In the original: "Leibniz selbst hat es ausgesprochen, daß die neue Analysis aus dem innersten Quell der Philosophie geflossen ist, und beiden Gebieten die Aufgabe zugewiesen, sich wechselseitig zu bestätigen und zu erhellen."

7.3 Edward Nelson

As far as the passage from Robinson (1966) is concerned, we find the following comment at the end of the paragraph:

The following framework for higher order structures and higher order languages copes with these and similar cases. It is rather straightforward and suitable for our purposes. (Robinson 1966, section 2.6, p. 19)

Robinson then proceeds to develop a solution, which roughly corresponds to level (3) as outlined in Sect. 3. The claim that we are dealing with first order logic plus *standard part* is in a sense a mathematical theorem, undermining the contention that “this seems hardly compatible, etc.”

Edward Nelson demonstrated that infinitesimals can be found within the ordinary real line itself in the following sense. Nelson finds infinitesimals in the real line by means of enriching the language through the introduction of a unary predicate *standard* and an axiom schemata (of Idealization), one of most immediate instances of which implies the existence of infinitely large integers and hence nonzero infinitesimals; see Nelson (1977). This is closely parallel to the dichotomy of *assignable* vs. *inassignable* in Leibniz, whereas Carnot spoke of *quantités désignées* (Carnot 1797; Barreau 1989, p. 46). Thus, we obtain infinitesimals as soon as we assume that (1) there are assignable (or standard) reals, that obey the same rules as all the reals, and (2) there are reals that not assignable.

In more detail, one considers the ordinary ZFC formulated in first order logic (here the term is used in a different sense from the rest of this article), adds to it the unary predicate and the axiom schemata, and obtains a framework where calculus and analysis can be done with infinitesimals. For further discussion see Katz and Kutateladze (2015).

The passage from Robinson cited above does indicate that second order theory may often be interesting. However, in the case of the calculus/analysis as it was practiced in the seventeenth century, we are not aware of a single significant result that cannot be formulated in a system of type FOL + \square (see Sect. 2 for examples of results that can). Arguably it was calculus (rather than analysis) that Leibniz invented, in the sense that there don't appear to be any essentially second order statements there.

It may seem surprising that there could be a kind of pre-established harmony between a modern logical category, namely, first-order results, and a historical category, namely, results of seventeenth century calculus. This idea suggests further questions: does this only hold for the calculus, or is it always or often the case that a historically earlier realization of a theory covers only the first-order part of its successor. How do arithmetic and geometry behave in this respect? Would it really make sense to systematically distinguish between Leibniz and Leibniz1, Euclid and Euclid1, etc.?

7.4 Not Standalone

Let us return to the comment *In the context of Skolem's construction of nonstandard natural numbers and some related stuff, one is impressed by how many nonstandard numbers one needs to move from \mathbb{N} to ${}^*\mathbb{N}$, e.g., in Henkin's model ${}^*\mathbb{N} = \mathbb{N} + (\mathbb{Z} \times \mathbb{Q})$ and it goes without saying that Leibniz did not have this perspective, that was addressed briefly above. One could elaborate on the “impression” concerning “how many nonstandard numbers” one needs to define ${}^*\mathbb{N}$ consistently and conveniently.*

An infinitely large number say H is not a standalone object, but rather lives in a community of numbers obeying certain laws which mathematicians anticipate as a goal of

the construction of a nonstandard number system ${}^*\mathbb{N}$. Such a commitment to anticipated laws forces Skolem and others to add to \mathbb{N} a suitable entourage of H along with H itself. What are the laws involved?

Modern specialists in Nonstandard Analysis (NSA) stipulate that ${}^*\mathbb{N}$ should satisfy the axioms of Peano Arithmetic and moreover, satisfy the same sentences of the language (not necessarily consequences of the axioms) that are true in \mathbb{N} itself. This is called (the principle of) Transfer today. Mathematicians of the seventeenth (or even nineteenth) century had neither this perspective nor the tools consistently to define ${}^*\mathbb{N}$ or ${}^*\mathbb{R}$.

On the other hand, one can argue that there is no need for actually rigorously defining ${}^*\mathbb{N}$ in order to make use of its benefits. One can argue that it is sufficient to have some idea of Transfer on top of an acceptance of infinitely large numbers per se (possibly as *useful fictions*, to borrow Leibniz's expression). We have argued that the Leibnizian *Law of continuity* is closely related to the Transfer principle; see Katz and Sherry (2013).

Therefore the claim that *Leibniz had not the slightest idea of this stuff* (the "stuff" being the modern technique of building nonstandard models) is perhaps technically true, but it does not reflect all the aspects of the interrelations within the Leibniz/Weierstrass/NSA triangle.

8 Conclusion

The vast oeuvre of Leibniz is still in the process of publication. In principle a lucky scholar might one day unearth a manuscript where Leibniz tackles a property equivalent to the completeness of the reals (after all the existence of the shadow is so equivalent), involving quantification over *all* sets of the number system and therefore second-order.

However, this is unlikely in view of Leibniz's reluctance to deal with infinite collections, as mentioned above. If level (3) of first order logic is indeed suitable for developing modern proxies for the inferential moves found in Leibnizian infinitesimal calculus, as we have argued, then modern infinitesimal frameworks are more appropriate to interpreting Leibnizian infinitesimal calculus than modern Weierstrassian ones.

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