

Controversies in the Foundations of Analysis: Comments on Schubring's *Conflicts*

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Abstract *Foundations of Science* recently published a rebuttal to a portion of our essay it published 2 years ago. The author, G. Schubring, argues that our 2013 text treated unfairly his 2005 book, *Conflicts between generalization, rigor, and intuition*. He further argues that our attempt to show that Cauchy is part of a long infinitesimalist tradition confuses *text* with *context* and thereby misunderstands the significance of Cauchy's use of infinitesimals. Here we defend our original analysis of various misconceptions and misinterpretations concerning the history of infinitesimals and, in particular, the role of infinitesimals in Cauchy's mathematics. We show that Schubring misinterprets Proclus, Leibniz, and Klein on non-Archimedean issues, ignores the Jesuit *context* of Moigno's flawed critique of infinitesimals, and misrepresents, to the point of caricature, the pioneering Cauchy scholarship of D. Laugwitz.

Keywords Archimedean axiom · Cauchy · Felix Klein · Horn-angle · Infinitesimal · Leibniz · Ontology · Procedure

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1 Introduction

In our article Błaszczyk et al. (2013) in the *Foundations of Science*, we sought to place Cauchy's work within an uninterrupted line of an infinitesimalist tradition, and to vindicate his infinitesimal definitions of concepts like continuity, convergence, and the Dirac delta function (see also Katz and Tall (2013)). We also sought to champion the pioneering work of D. Laugwitz from the 1980s that challenges the received views on Cauchy as a precursor of the Weierstrassian *epsilon*-*delta*. We developed an approach to the history of analysis as evolving along separate, and sometimes competing, tracks. These are the A-track, based upon an Archimedean continuum; and B-track, based upon what we called a Bernoullian (i.e., infinitesimal-enriched) continuum.

1.1 Punctiform and Non-punctiform Continua

Historians often view the work in analysis from the 17th to the middle of the 19th century as rooted in a background notion of continuum that is not punctiform. This necessarily creates a tension with modern, punctiform theories of the continuum, be it the A-type set-theoretic continuum as developed by Cantor, Dedekind, Weierstrass, and others, or a B-type continuum as developed by Hewitt, Łoś, Robinson, and others. How can one escape a trap of presentism in interpreting the past from the viewpoint of set-theoretic foundations commonly accepted today, whether of type A or B?

In analyzing Cauchy's work one must be careful to distinguish between its *syntactic* aspects, i.e., procedures and inferential moves, on the one hand, and *semantic* aspects related to the actual construction of the entities such as points of the continuum, i.e., issues of the *ontology* of mathematical entities such as points. We found that in his work on Cauchy, Laugwitz was careful not to attribute modern set-theoretic constructions to work dating from before the heroic 1870s, and focused instead of Cauchy's procedures (see also Sect. 6.2).

1.2 Rebuttals

In keeping with its tradition of fostering informed debate, *Foundations of Science* recently published a response to a portion of our essay. The author, G. Schubring, argues that our attempt to show that Cauchy is part of a long infinitesimalist tradition confuses *text* with *context* and thereby misunderstands the significance of Cauchy's use of infinitesimals. Schubring urges, instead, that Cauchy's intentions must be "reconstructed according to the contemporary conceptual horizon of the related conceptual fields" (Schubring 2015, Section 3).

He further insinuates that our analysis is "contrary to a style appropriate for a reasoned scientific discussion" (ibid., Sect. 5).

We grant that we dealt too briefly with Schubring's treatment of Cauchy. Here we develop more fully our thesis that the book *Conflicts* Schubring (2005)

- (1) misinterprets both Leibniz and Felix Klein on infinitesimals;
- (2) escapes the need of a genuine hermeneutical effort by ignoring the historical *context* of Moigno's flawed critique of infinitesimals;
- (3) underestimates the role of infinitesimals in Cauchy's work;
- (4) misrepresents Laugwitz's Cauchy scholarship, e.g., by alleging that Laugwitz tried to "prove" that Cauchy used hyper-real numbers; and

(5) conflates Laugwitz's analysis with D. Spalt's.

The issue of modern interpretation of historical mathematics is a deep and important one. Much recent scholarship has argued that modern conceptions shed little light on historical mathematics. For example, scholars have criticized late 19th and early 20th century attempts to attribute a geometrical algebra to Euclid's *Elements* (e.g., Szabo 1969, Unguru 1975 and 1979). More recently the nonsequiturs and unwarranted assumptions of these critiques have been brought to light (Blåsjö 2016), and the present article attempts to accomplish something similar. It continues a program of re-evaluation of the history of mathematical analysis undertaken in Błaszczyk et al. (2013), Bair et al. (2013), Bascelli et al. (2014), and other texts. Thus, we highlight the significance of Simon Stevin's approach to undending decimals in Katz and Katz (2012b). The text Katz et al. (2013) re-evaluates Fermat's contribution to the genesis of the infinitesimal calculus via his technique of *adequity*. The articles Katz and Sherry (2012), Katz and Sherry (2013), and Sherry and Katz (2014) argue that Leibniz's theoretical strategy for dealing with infinitesimals was more soundly based than critiques by George Berkeley and others. Euler's infinitesimal analysis is defended against A-track re-writings in Kanovei et al. (2015) and Bair et al. (2016). Cauchy's infinitesimal legacy is championed in Katz and Katz (2011), Borovik and Katz (2012), Katz and Katz (2012a), Tall and Katz (2014).

2 Conceptual Horizons

Schubring has repeatedly emphasized the importance of context and external history (as opposed to what he calls *immanent history*). We may agree with Schubring that

The meaning [of Cauchy's infinitesimals] has to be reconstructed according to the contemporary conceptual horizon of the related conceptual fields. Thus, a genuine hermeneutical effort is inescapable. (Schubring 2015, Section 3)

Indeed, Cauchy's *contemporary conceptual horizon* needs to be taken into account. However, it is equally true that the meaning of Cauchy's infinitesimals cannot be reconstructed, as Schubring attempts to do, while at the same time ignoring the context of the broader *conceptual horizon* that surely incorporated an awareness of Archimedean and non-Archimedean issues stretching all the way back to Euclid.

We will show that Schubring misinterprets both Leibniz and Felix Klein on both the Archimedean property and infinitesimals, and ends up endorsing Moigno's flawed critique of infinitesimals; see Sects. 4.1 and 4.2. Moreover, Schubring himself ignores the contemporary historical *context* of Moigno's critique; see Sect. 8.

3 Axioms for Archimedean Quantities

The notion of infinitesimal is related to the axiom of Archimedes, i.e., Euclid's *Elements*, Definition V.4 (see (De Risi 2016, Sect. II.3) for more details). The theory of magnitudes as developed in Book V of the *Elements* can be formalized as an ordered additive semi-group M with a total order, characterized by the five axioms given below.

Beckmann (1967/1968) and (Błaszczyk and Mrówka 2013, pp. 101–122) provide detailed sources for the axioms below in the primary source (Euclid). See also (Mueller

1981, pp. 118-148) which mostly follows Beckmann's development. Axiom E1 below interprets Euclid V.4:

- E1 $(\forall x, y \in M)(\exists n \in \mathbb{N})(nx > y)$,
 E2 $(\forall x, y \in M)(\exists z \in M)(x < y \Rightarrow x + z = y)$,
 E3 $(\forall x, y, z \in M)(x < y \Rightarrow x + z < y + z)$,
 E4 $(\forall x \in M)(\forall n \in \mathbb{N})(\exists y \in M)(x = ny)$,
 E5 $(\forall x, y, z \in M)(\exists v \in M)(x : y :: z : v)$.

4 Schubring's Pseudohistory

We find multiple problems with Schubring's analysis of authors ranging from Proclus and Leibniz to Moigno and Klein. Here we present several examples.

4.1 Klein's Horn-Angles Misunderstood by Schubring

It is well known that infinitesimals form a *proper* subset of a non-Archimedean structure. Now to formulate what it might mean for a structure to be *NON-Archimedean*, one needs first to make up one's mind as to what *Archimedean* itself might mean. Surprisingly, the book Schubring (2005) provides *no formulation whatsoever* concerning the so-called Archimedean axiom¹ (the term was introduced in Stolz (1883)). More importantly, Schubring's remarks on Klein's horn-angles reveal a misconception on Schubring's part concerning non-Archimedean systems:

[Felix] Klein has shown [sic] in detail that the hornlike angles form a model of non-Archimedean quantities. (Schubring 2005, p. 17)

But is that really what Klein had shown? In Klein's approach, horn-angles form an ordered structure isomorphic to \mathbb{R}_+ (when parametrized by the curvature), which can hardly be said to be non-Archimedean. Klein goes on to form a broader structure incorporating both horn-angles and rectilinear angles. That structure is non-Archimedean (see Sect. 3 for a discussion of the term), and a horn-angle constitutes an infinitesimal within this structure. More precisely, a horn-angle is an infinitesimal *relative* to rectilinear angles. Thus, it is incorrect to assert, as Schubring does, that "hornlike angles form a model of non-Archimedean quantities." Klein writes:

Thus multiplication of the angle of a tangent circle [i.e., horn-angle] by an integer $[n]$ always yields another angle of a tangent circle, and every multiple n is necessarily smaller, by our definition, than, say, the angle of a fixed intersecting line, however large we take n . Thus the axiom of Archimedes is not satisfied; and the angles of tangent circles must be looked upon, accordingly, as actually infinitely small *with respect to* the angle of an intersecting straight line. (Klein 1925, p. 205) (emphasis added)

¹ On page 17, Schubring cites Proclus' correct reading of Euclid V.4, but as soon as Schubring attempts to paraphrase this in his own terms, he immediately gets it wrong by describing infinitesimals in terms of *incommensurability*. The term *incommensurability* is used to describe phenomena related to irrationality, including both previous occurrences of the term in Schubring (2005) on pages 12 and 16.

Thus, Klein's horn-angles are only infinitely small *with respect to* rectilinear angles. Schubring's imperfect understanding of even the basic issues involving infinitesimals leads to errors of judgment on his part with regard to both Leibniz and Cauchy, as we show in Sects. 4.2 and 4.3.

4.2 Leibniz's Infinitesimals Misinterpreted by Schubring

A quotation from Leibniz on infinitesimals in Schubring (2005) is accompanied by the following claim:

[Leibniz] declared two homogeneous quantities to be equal that differed only by a quantity that was arbitrarily smaller than a *finite quantity*. (Schubring 2005, p. 169) (emphasis added)

Schubring takes two homogeneous quantities x, y to be equal if their difference $x - y$ is a quantity that is arbitrarily smaller than a finite quantity, or in symbols

$$x = y \iff x - y < \text{each finite quantity.}$$

However, an infinitesimal itself is a finite quantity (i.e., it is not infinite). How can it be smaller than *each finite quantity*? In other words, how can a quantity be smaller than its own half, its own quarter, etc.? There seems to be a logical inconsistency involved in Schubring's summary of the definition of an infinitesimal. The inconsistency is identical to the one claimed by Moigno and endorsed by Schubring; see Sect. 4.3.

In fact, Leibniz is guiltless here, and it is Schubring who misinterprets Leibniz's discussion of infinitesimals. Leibniz wrote:

I agree with Euclid Book V Definition 5 that only those homogeneous quantities are comparable, of which the one can become larger than the other if multiplied by a number, that is, a *finite number*. I assert that entities, whose difference is not such a quantity, are equal. [...] This is precisely what is meant by saying that the difference is smaller than any given quantity.² (Leibniz as quoted in Schubring 2005, p. 169)

This passages deals both with comparable quantities and infinitesimals, but *finite numbers* appear only in Leibniz's characterisation of comparable quantities, not of infinitesimals. We interpret Leibniz's reference to Euclid V.5 as a reference to the Archimedean axiom³ (see Sect. 3).

The way Schubring uses *finite quantity* in his erroneous paraphrase of Leibniz has the effect of endowing infinitesimals with contradictory properties. Schubring's erroneous paraphrase is not an isolated incident, for it is consistent with his endorsement of Moigno's flawed critique of infinitesimals analyzed in Sect. 4.3.

4.3 Schubring Endorsing Moigno

It is no accident that Schubring gives a favorable evaluation of Moigno's critique of the infinitely small. Moigno wrote in the introduction to his 1840 book that infinitesimals are

² Leibniz uses the term *finite* in his paraphrase of Euclid's definition V.5 (or V.4, as discussed in footnote 3), but here he is dealing with a finite *integer* n (which, in modern terminology, is tending to infinity), so that ne always stays less than 1 thereby violating the Archimedean property, if ϵ is infinitesimal.

³ Leibniz lists number V.5 for Euclid's definition instead of V.4. In some editions of the *Elements* this definition does appear as V.5. Thus, Euclid (1660) as translated by Barrow in 1660 provides the following definition in V.V (the notation "V.V" is from Barrow's translation): *Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other.* For our interpretation of this, see Sect. 3, Axiom E1.

contradictory, arguing that it is impossible for something be less than its own half, its own quarter, etc. (see Sect. 8) Schubring endorses Moigno's critique of infinitesimals in the following terms:

[Moigno] not only fails to use [the *infiniment petits*] as a basic concept, but also even explains explicitly why they are inappropriate as such. This makes Moigno the first writer to pick apart the traditional claim in favor of their purported *simplicité* (Schubring 2005, p. 445)

Attributing such alleged *picking apart* to Moigno involves a fundamental misunderstanding on Schubring's part, closely related to his misreading of Leibniz (see Sect. 4.2). Without properly understanding infinitesimals first, a scholar can't properly evaluate historical criticisms of infinitesimals, either. Schubring similarly ignores an important aspect of the relevant *external* history, namely the Jesuit context of Moigno's remarks on infinitesimals (see Sect. 8).

Given Schubring's endorsement of Moigno's claim that infinitesimals are self-contradictory, it is no wonder that he should seek to save Cauchy's reputation by attempting to minimize the significance of infinitesimals in Cauchy's work.

5 Schubring Versus Laugwitz

Schubring's analysis of Laugwitz's Cauchy scholarship contains numerous inaccuracies and misrepresentations.

5.1 Were Cauchy's Theorems Always Correct?

Schubring comments as follows on Laugwitz's work on Cauchy's sum theorem (a series of continuous functions under suitable conditions converges to a continuous function):

[Giusti's 1984 article] spurred Laugwitz to even more detailed attempts to banish the error and confirm that Cauchy had used hyper-real numbers. On this basis, he claims, the errors vanish and the theorems become correct, or, rather, *they always were correct* (see Laugwitz 1990, 21). (Schubring 2005, p. 432) (emphasis added)

Schubring is making two separate claims here:

- (1) that Laugwitz sought to show that Cauchy used hyper-real numbers; and
- (2) that Laugwitz asserted that Cauchy's sum theorem was always correct (including its 1821 version in the *Cours d'Analyse*).

Claim (1) will be examined in Sect. 6. Here we will examine claim (2). Did Laugwitz assert that the theorems were always correct? Let us consider the relevant passage from Laugwitz's text in *Historia Mathematica*:

Rather late in his life Cauchy [1853] admitted that the statement of his theorem (but not its proof) was *incorrect*: "Au reste, il est facile de voir comment on doit modifier l'énoncé du théorème, pour qu'il n'y [ait] plus lieu à aucune exception" [Cauchy 1853, 31–32]. (Laugwitz 1987, p. 265) (emphasis added)

Thus, Laugwitz acknowledges that the 1821 formulation of the sum theorem was *incorrect* as stated, and moreover that Cauchy himself had recognized its incorrectness. This is contrary to the claim concerning Laugwitz's position made by Schubring. Schubring's

claim amounts to a misrepresentation—indeed a caricature—of Laugwitz’s position. Misrepresenting another scholar’s work is, in Schubring’s phrase, “contrary to a style appropriate for a reasoned scientific discussion.” (Schubring 2015, Sect. 5 ‘The Climax’) In addition, Schubring frequently conflates Laugwitz’s position with that of D. Spalt.⁴ Spalt’s analysis was already criticized in Katz and Katz (2011).

5.2 Didactic Component

Schubring makes the following three successive claims concerning Laugwitz’s Cauchy scholarship:

- (1) the controversy basically centers on such fundamental concepts as continuity and convergence;
- (2) Laugwitz himself talks about the ‘essentially didactic components of the infinitesimals in Cauchy’ (ibid., 18) (cf. (Laugwitz 1990, p. 18));
- (3) it is in textbooks that we find these basic terms and not in isolated research memoirs (see (Schubring 2005, p. 433)). Schubring’s claim (3) concerning ‘basic terms’ apparently refers to the terms *continuity*, *convergence*, and *infinitesimal* that he mentioned earlier in (1) and (2). In his claim (3), Schubring appears to assert that these terms are found only in Cauchy’s textbooks rather than in Cauchy’s research memoirs. Schubring claims that Laugwitz describes Cauchy’s infinitesimals as ‘essentially didactic.’ Schubring does not cite the relevant sentence from Laugwitz, who states:

Das Beispiel gibt Gelegenheit, auf die wesentlich didaktische Komponente des Infinitesimalen bei Cauchy hinzuweisen. (Laugwitz 1990, p. 18)

This can be translated as follows:

The example affords us the opportunity to point out the essentially didactic component of Cauchy’s infinitesimals.

Laugwitz’s use of the singular *Komponente* suggests that, among *other* components (i.e., aspects) of infinitesimals, there is *also* a didactic component. Schubring makes it appear as if Laugwitz views Cauchy’s infinitesimals as being *limited* to their didactic role (this interpretation is motivated by Schubring’s contention that Cauchy was forced by a curricular committee to include infinitesimals in his textbook). Schubring further reveals that he himself adheres to such a view when he claims in (3) that Cauchy’s infinitesimals are found in textbooks but not in research memoirs. We will now show that both of Schubring’s positions are untenable.

Neither Schubring’s view of Cauchy’s infinitesimals as being limited to their didactic role, nor Schubring’s interpretation of Laugwitz’s comment stand up to scrutiny. Note that Schubring’s claim (3) amounts to the extraordinary contention that Cauchy only used infinitesimals in textbooks only but not in research articles, something Laugwitz never claimed.

Was Cauchy’s use of infinitesimals indeed limited to his textbooks, as Schubring claims? Certainly not. The article Cauchy (1853) on the corrected version of the sum theorem is a research article. It deals with a property closely related to uniform convergence of series of functions, which was the cutting edge of research in analysis at the time.

⁴ Schubring repeats the performance in 2015 when he claims: “I am analysing at length the methodological approach of Laugwitz (and Spalt), which consists in attributing to Cauchy (his) own ‘universe of discourse.’” (Schubring 2015, Sect. 3). But Spalt’s approach is not identical to Laugwitz’s!

The 1853 article happens to deal with all three notions that Schubring mentions in this paragraph, namely, *continuity*, *convergence*, and *infinitesimal*. Indeed, continuity and convergence are mentioned already in Cauchy's title, while infinitesimals are exploited in the definition of continuity given already at the beginning of the article:

... une fonction u de la variable réelle x sera *continue*, entre deux limites données de x , si, cette fonction admettant pour chaque valeur intermédiaire de x une valeur unique et finie, un accroissement *infinitement petit* attribué à la variable produit toujours, entre les limites dont il s'agit, un accroissement *infinitement petit* de la fonction elle-même.⁵ Cauchy (1853) [emphasis on *infinitement petit* added]

Furthermore, the 1853 version of the sum theorem cannot even be formulated without infinitesimals, and Cauchy's proof procedure uses them in an essential way, as analyzed in (Laugwitz 1987, pp. 264–266).

Cauchy wrote many other research articles exploiting infinitesimals in an essential way. An example is his memoir Cauchy (1832), where he expresses the length of a curve in terms of the average of its projections to a variable axis. His proof procedure involves decomposing the curve into infinitesimal portions and proving the result for each portion. This article is considered by specialists in geometric probability to be a foundational text in that field; see e.g., Hykšová et al. (2012).⁶

Thus Schubring's claim (3) to the effect that Cauchy's infinitesimals appear only in textbooks is not merely inaccurate, revealing an incomplete knowledge of the original documents on his part, but more importantly it obscures the fundamental role of infinitesimals in Cauchy's thinking.

After mentioning the *essentially didactic component of Cauchy's infinitesimals* in (Laugwitz 1990, p. 18), Laugwitz goes on to discuss the corrected sum theorem three pages later, starting at the bottom of page 21. Laugwitz was well aware that Cauchy's infinitesimals are not limited to their didactic function, contrary to Schubring's claim (2). Three years earlier, Laugwitz had discussed Cauchy's corrected sum theorem in detail in his article in *Historia Mathematica* (Laugwitz 1987, pp. 264–266).

In conclusion, Schubring both misrepresents Laugwitz's position on Cauchy's infinitesimals, and reveals his (Schubring's) own misconceptions regarding the latter.

6 Cauchy, Laugwitz, and Hyperreal Numbers

In this section we examine the relationship between Cauchy's infinitesimals and modern infinitesimals as seen by Laugwitz, and the related comments by Fraser, Grabiner, and Schubring.

⁵ Translation: "A function u of a real variable x will be *continuous* between two given bounds on x if this function, taking for each intermediate value of x a unique finite value, an infinitely small increment given to the variable always produces, between the bounds in question, an infinitely small increment of the function itself."

⁶ Another example of Cauchy's use of infinitesimals in research is his foundational text on elasticity Cauchy (1832) where "un élément infinitement petit" is exploited on page 302. The article is mentioned in (Freudenthal 1971, p. 378).

6.1 Schubring–Grabiner Spin on Cauchy and Hyperreals

Schubring's opposition to Laugwitz's interpretation of Cauchy found expression in the following comment, already quoted in Sect. 5.1:

[Giusti's article] spurred Laugwitz to even more detailed attempts to banish the error and *confirm that Cauchy had used hyper-real numbers*. On this basis, he claims, the errors vanish and the theorems become correct, or, rather, they always were correct (see Laugwitz 1990, 21). (Schubring 2005, p. 432) (emphasis added)

The matter of Schubring's misrepresentation of Laugwitz's position with the regard to the correctness of the statement of Cauchy's theorems was already dealt with in Sect. 5.1. In this section, we will examine Schubring's contention that Laugwitz claimed that Cauchy used the hyperreals. In this passage, Schubring is *referring to* the article Laugwitz (1990), but he is most decidedly not *quoting* it. In fact, there is no mention of the hyperreals on page 21 in Laugwitz (1990), contrary to Schubring's claim. What we do find there is the following comment:

The "mistakes" show rather, as *experimenta crucis* that one must understand Cauchy's terms/definitions [*Begriffe*], in the spirit of the motto,⁷ in an infinitesimal-mathematical sense. (Laugwitz 1990, p. 21) (translation ours)

We fully endorse Laugwitz's comment to the effect that Cauchy's procedures must be understood in the sense of infinitesimal mathematics, rather than paraphrased to fit the *epsilon* mode. Note that we are dealing with an author, namely Laugwitz, who published Cauchy studies in the leading periodicals *Historia Mathematica* Laugwitz (1987) and *Archive for History of Exact Sciences* Laugwitz (1989).⁸ The idea that Laugwitz would countenance a claim that Cauchy "had used hyper-real numbers" whereas both the term *hyper-real* and the relevant construction were not introduced by E. Hewitt until 1948 (Hewitt 1948, p. 74), strikes us as far-fetched. Meanwhile, in a colorful us-against-"them" circle-the-wagons passage, J. Grabiner opines that

[Schubring] effectively rebuts the partisans of nonstandard analysis who wish to make Cauchy one of *them*, using the work of Cauchy's disciple the Abbé Moigno to argue for Cauchy's own intentions. (Grabiner 2006, p. 415). (emphasis added)

Grabiner's comment betrays insufficient attention to the procedure/ontology distinction, while her endorsement of Moigno's critique of infinitesimals is as preposterous as Schubring's. Schubring indeed uses the work of Moigno in an apparent attempt to refute infinitesimals (see Sect. 4.3). Moigno's confusion on the issue of infinitesimals is dealt with in Sect. 8.

Contrary to Schubring's claim, Laugwitz did not attribute 20th century number systems to Cauchy. Rather, Laugwitz sought to understand Cauchy's inferential moves in terms of their modern proxies. May we suggest that Schubring's mocking misrepresentation of Laugwitz's position is, again, "contrary to a style appropriate for a reasoned scientific discussion." (Schubring 2015, Sect. 5 'The Climax')

⁷ Here Laugwitz is referring to Cauchy's motto to the effect that "Mon but principal a été de concilier la rigueur, dont je m'étais fait une loi dans mon *Cours d'analyse* avec la simplicité que produit la consideration directe des quantités infiniment petites."

⁸ The fact that Laugwitz had published articles in leading periodicals does not mean that he could not have said something wrong. However, it does suggest the existence of a strawman aspect of Schubring's claims against him.

6.2 What Charge is Laugwitz Indicted on Exactly?

In the abstract of his 1987 article in *Historia Mathematica*, Laugwitz is careful to note that he interprets Cauchy's sum theorem "with his [i.e., Cauchy's] own concepts":

It is shown that the famous so-called errors of Cauchy are correct theorems when interpreted with his own concepts. (Laugwitz 1987, p. 258)

In the same abstract, Laugwitz goes on to emphasize:

No assumptions on uniformity or on nonstandard numbers are needed. (emphasis added)

Indeed, in section 7 on pages 264–266, Laugwitz gives a lucid discussion of the sum theorem in terms of Cauchy's infinitesimals, with not a whiff of modern number systems. In particular this section does not mention the article Schmieden and Laugwitz (1958). In a final section 15 entitled "Attempts toward theories of infinitesimals," Laugwitz presents a rather general discussion, with no specific reference to the sum theorem, of how one might formalize Cauchyan infinitesimals in modern set-theoretic terms. A reference to Schmieden and Laugwitz (1958) appears in this final section only. Thus, Laugwitz carefully distinguishes between his analysis of Cauchy's procedures, on the one hand, and the ontological issues of possible implementations of infinitesimals in a set-theoretic context, on the other.

Alas, all of Laugwitz's precautions went for naught. In 2008, he became a target of damaging innuendo in the updated version of *The Dictionary of Scientific Biography*. Here C. Fraser writes as follows in his article on Cauchy:

Laugwitz's thesis is that certain of Cauchy's results that were criticized by later mathematicians are in fact valid *if one is willing to accept certain assumptions* about Cauchy's understanding and use of infinitesimals. These assumptions reflect a theory of analysis and infinitesimals that was worked out by Laugwitz and ...Schmieden during the 1950s.⁹ (Fraser 2008, p. 76) (emphasis added)

Fraser and Schubring both claim that Laugwitz's interpretation of Cauchy depends on assumptions that reflect a modern theory of infinitesimals. While Schubring charges Laugwitz with relying on Robinson's hyperreals (see Sect. 6), Fraser's indictment is based on the Omega-theory of Schmieden–Laugwitz. Both Schubring and Fraser are off the mark, as we showed above.

7 Of Infinitesimals and Limits

Schubring claims to distance himself from scholars like Grabiner, who attribute to Cauchy a conceptual framework in the tradition of the *great triumvirate* of Cantor, Dedekind, and Weierstrass (see (Boyer 1949, p. 298)):

I am criticizing historiographical approaches like that of Judith Grabiner where one sees epsilon-delta already realized in Cauchy (Schubring 2015, Section 3).

⁹ Fraser repeats the performance in 2015 when he claims that "Laugwitz, ... some two decades following the publication by Schmieden and him of the Ω -calculus commenced to publish a series of articles arguing that their non-Archimedean formulation of analysis is well suited to interpret Cauchy's results on series and integrals." (Fraser 2015, p. 27) What Fraser fails to mention is that Laugwitz specifically separated his analysis of Cauchy's *procedures* from attempts to account *ontologically* for Cauchy's infinitesimals in modern terms.

Yet, when Schubring explains the so-called ‘compromise concept’ that he attributes to Cauchy, the infinitely small is governed by a conception of limits in the context of ordinary (real) values. While Schubring is not guilty, like Grabiner, of ignoring the role of infinitesimals in Cauchy’s thinking, it is clear that Schubring believes that it is the limit concept, rather than the infinitesimal concept that is primary in this relationship. A more detailed analysis of such a Cauchy–Weierstrass tale appears in Borovik and Katz (2012).

Schubring’s section (6.4), discussing the connection between limits and the infinitely small, reveals that Schubring is a cheerleader for the triumvirate team. Thus, he writes that the infinitely small is “*subjugated to...the limit concept*” (Schubring 2005, p. 454) (emphasis added); furthermore, “the *infiniment petits* represent only a subconcept of the *limite* concept in Cauchy’s textbooks” (Schubring 2005, p. 455). Both claims seem to be based on Cauchy’s remark that “a variable of this type has zero as a limit.” However, this comment in Cauchy is not truly a part of the definition of an infinitesimal, but rather a consequence of the definition which precedes it.

8 Indivisibles Banned by the Jesuits

Most scholars agree that infinitesimal analysis was a natural outgrowth of the indivisibilist techniques as developed by Galileo and Cavalieri, and in fact Galileo’s work may be closer to the infinitesimal techniques of their contemporary Kepler.¹⁰ Indivisibles were perceived as a theological threat and opposed on doctrinal grounds in the 17th century (Feingold 2003). The opposition was spearheaded by clerics and more specifically by the Jesuits. Tracts opposing indivisibles were composed by Jesuits Paul Guldin, Mario Bettini, and André Tacquet (Redondi 1987, p. 291). P. Mancosu writes:

Guldin is taking Cavalieri to be composing the continuum out of indivisibles, a position rejected by the Aristotelian orthodoxy as having atomistic implications. ... Against Cavalieri’s proposition that “all the lines” and “all the planes” are magnitudes - they admit of ratios - Guldin argues that “all the lines ... of both figures are infinite; but an infinite has no proportion or ratio to another infinite.” (Mancosu 1996, p. 54)

Tacquet for his part declared that the method of indivisibles “makes war upon geometry to such an extent, that if it is not to destroy it, it must itself be destroyed.” Alexander (2014)

In 1632 (the year Galileo was summoned to stand trial over heliocentrism) the Society’s Revisors General led by Jacob Bidermann banned teaching indivisibles in their colleges (Festa 1990, 1992, p. 198). Referring to this ban, Feingold notes:

Six months later, General Vitelleschi formulated his strong opposition to mathematical atomism in a letter he dispatched to Ignace Cappon in Dole: “As regards the opinion on *quantity made up of indivisibles*, I have already written to the Provinces many times that it is in no way approved by me and up to now I have allowed nobody to propose it or defend it. If it has ever been explained or defended, it was done without my knowledge. Rather, I demonstrated clearly to Cardinal Giovanni de Lugo himself that I did not wish our members to treat or disseminate that opinion.” (Feingold 2003, p. 28–29) (emphasis added)

¹⁰ Note that the term *infinitesimal* itself was not coined until the 1670s, by either Mercator or Leibniz; see (Leibniz 1699, p. 63).

Indivisibles were placed on the Society's list of *permanently* banned doctrines in 1651 (Hellyer 1996). In the 18th century, most Jesuit mathematicians adhered to the methods of Euclidean geometry (to the exclusion of the new infinitesimal methods):

...le grand nombre des mathématiciens de [l'Ordre] resta jusqu'à la fin du XVIII^e siècle profondément attaché aux méthodes euclidiennes. (Bosmans 1927, p. 77)

Echoes of such bans were still heard in the 19th century. Thus, in 1840, Moigno wrote:

In effect, either these magnitudes, smaller than any *given* magnitude, still have substance and are *divisible*, or they are simple and *indivisible*: in the first case their existence is a chimera,¹¹ since, necessarily greater than *their half*, *their quarter*, etc., they are not actually less than any *given* magnitude; in the second hypothesis, they are no longer mathematical magnitudes, but take on this quality, this would renounce the idea of the continuum divisible to infinity, a necessary and fundamental point of departure of all the mathematical sciences ... (as quoted in Schubring 2005, p. 456) [emphasis added]

It is to be noted that Moigno formulates his objections to infinitesimals specifically in the terminology of indivisible/divisible which had been obsolete for nearly two centuries. Moigno saw a contradiction where there is none. Indeed, an infinitesimal is smaller than any *assignable*, or *given* magnitude; and has been so at least since Leibniz.¹² Thus, infinitesimals need not be less than "their half, their quarter, etc.," because the latter are not given/assignable.

In fact, Moigno's dichotomy is reminiscent of the dichotomy contained in a critique of Galileo's indivisibles penned by Moigno's fellow Jesuit Orazio Grassi some three centuries earlier. P. Redondi summarizes it as follows:

As for light - composed according to Galileo of indivisible atoms, more mathematical than physical - in this case, logical contradictions arise. Such indivisible atoms must be finite or infinite. If they are finite, mathematical difficulties would arise. If they are infinite, one runs into all the paradoxes of the separation to infinity which had already caused Aristotle to discard the atomist theory ... (Redondi 1987, p. 196).

This criticism appeared in the first edition of Grassi's book *Ratio ponderum librae et simbellae*, published in Paris in 1626. According to Redondi, this criticism of Grassi's

exhumed a discounted argument, copied word-for-word from almost any scholastic philosophy textbook. ... The Jesuit mathematician [Paul] Guldin, great opponent of the geometry of indivisibles, and an excellent Roman friend of [Orazio] Grassi, must have dissuaded him from repeating such obvious objections. Thus the second edition of the *Ratio*, the Neapolitan edition of 1627, omitted as superfluous the whole section on indivisibles. (Redondi 1987, p. 197).

¹¹ Moigno's *chimerical* anti-infinitesimal thread has not remained without modern French adherents; see Kanovei et al. (2013).

¹² Leibniz's dichotomy between assignable and inassignable quantity, on which his concept of infinitesimal was based, finds a rigorous mathematical treatment in the hyperreal number system (where an assignable number is a standard real number). Yet the mathematics of earlier times allowed an adequate intuitive understanding of the issue, sufficient to effectively and fruitfully use infinitesimals in mathematical practice, even though a semantic base (accounting for the ontology of a number) acceptable by modern standards was as yet unavailable. For further details on Leibniz's theoretical strategy in dealing with infinitesimals see Katz and Sherry (2012), Katz and Sherry (2013), Sherry and Katz (2014).

Alas, unlike Father Grassi, Father Moigno had no Paul Guldin to dissuade him. Schubring fails to take into account the pertinent Jesuit background of Moigno's rhetorical flourishes against infinitesimals. Schubring's endorsement (see Sect. 4.3) of Moigno's myopic anti-infinitesimal stance is nothing short of comical.

While we agree with Schubring on the importance of considering context, we observe that he fails to consider the Jesuit context of Moigno's flawed critique. Schubring seeks to exploit Moigno's *text* in trying to prove a point about Cauchy. However, escaping the need of a genuine hermeneutic effort, Schubring ignores the *context* of Moigno's still being a Jesuit at the time he wrote his 1840 *text*.

9 Gilain on Records from the Ecole

Cauchy wrote in the introduction to his 1821 *Cours d'Analyse*:

In speaking of the continuity of functions, I could not dispense with a treatment of the principal properties of infinitely small quantities, properties which serve as the foundation of the infinitesimal calculus. (translation from Bradley and Sandifer (2009))

When Cauchy writes that he was unable to dispense with infinitesimals, was he complaining about being forced to teach infinitesimals, or was he emphasizing the crucial importance of infinitesimals? (Schubring 2005, p. 436) asserts that Cauchy is complaining. Schubring bases his claim on the *context* of Gilain's analysis of teaching records from the *Ecole* in Gilain (1989). However, Gilain makes the following points:

- (1) Unlike Cauchy's later textbooks, his 1821 book was not commissioned by the *Ecole* but was rather written upon the personal request of Laplace and Poisson.
- (2) When the portion of the curriculum devoted to *Analyse Algébrique* was reduced in 1825, Cauchy insisted on placing the topic of continuous functions [and therefore also the topic of infinitesimals exploited in Cauchy's definition of continuity] at the beginning of the Differential Calculus.

Thus, Gilain writes:

...rien ne montre, dans les documents utilisés, qu'il y ait eu une hostilité de principe de la part de Cauchy à la suppression de l'analyse algébrique, en tant que partie autonome située au début du cours d'analyse. Ce qui lui importait, par contre, c'était la présence de plusieurs articles de cette partie, et les méthodes utilisées pour les présenter. ... Notons, sur deux points importants pour Cauchy, que les fonctions continues sont placées au début du calcul différentiel et que l'étude de la convergence des séries trouve sa place au voisinage de la formule de Taylor, dans le calcul différentiel et intégral.¹³ (Gilain 1989, end of § 131)

The fact that Cauchy insisted on retaining continuity/infinitesimals in 1825, as Gilain documents, indicates that the importance of infinitesimals to Cauchy goes beyond their pedagogical value. Also, the fact that the *Cours d'Analyse* was written on personal request

¹³ Translation: "Nothing indicates, in the documents used, any principled hostility on the part of Cauchy to the elimination of algebraic analysis as an autonomous part placed at the beginning of the analysis course. What was important to him, on the other hand, was the presence of several items from this part, and the methods used in presenting them. ... Note two particular points important to Cauchy, namely that continuous functions be placed at the beginning of differential calculus, and that the study of the convergence of series should find its place in the vicinity of Taylor's formula, in differential and integral calculus."

from Poisson and others, rather than being commissioned by the *Ecole*, indicates that the pressures of the type Schubring claims are unlikely to have played a role in the writing of this particular book.

Later texts were indeed commissioned by the *Ecole*, but in 1821 Cauchy was free to write as he felt, and he felt that infinitesimals were of crucial importance for analysis. In painting a dim picture of Cauchy's attitude toward infinitesimals, Schubring misjudges the historical context of the *Cours d'Analyse* and ignores the fact that decades after finishing his teaching stint at the *Ecole*, Cauchy exploits infinitesimals in an essential way in his research article Cauchy (1853).

10 Conclusion

Shoddy scholarship sometimes parades under the dual banner of *sophisticated historiographic analysis* and *genuine hermeneutical effort*. As we have shown, Schubring's "sophisticated historiographical analysis" (Schubring 2015, Section 5 'The Climax') collapses in the face of simple historical facts and straightforward mathematical arguments. Moreover, Schubring's repeated misrepresentation of Laugwitz's position is indeed "contrary to a style appropriate for a reasoned scientific discussion." (ibid.)

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