Izvestiya: *Mathematics* **67**:1 55–76

Izvestiya RAN: *Ser. Mat.* **67**:1 59–82 DOI 10.1070/IM2003v067n01ABEH000418

Some new results on Borel irreducibility of equivalence relations

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Abstract. We prove that orbit equivalence relations (ERs, for brevity) of generically turbulent Polish actions are not Borel reducible to ERs of a family which includes Polish actions of S_∞ (the group of all permutations of $\mathbb{N}$) and is closed under the Fubini product modulo the ideal Fin of all finite sets and under some other operations. We show that T_2 (an equivalence relation called the *equality of countable sets of reals*) is not Borel reducible to another family of ERs which includes continuous actions of Polish CLI groups, Borel equivalence relations with $\mathbf{G}_{\delta\sigma}$ classes and some ideals, and is closed under the Fubini product modulo Fin. These results and their corollaries extend some earlier irreducibility theorems of Hjorth and Kechris.

Introduction

Classification problems for different types of mathematical structure have been at the centre of interest in descriptive set theory for the last 15 years. Suppose that X is a class of mathematical structures, identified modulo an equivalence relation E. This can be, for example, countable groups modulo the isomorphism relation, or unitary operators on a fixed space $\mathbb{C}^n$ modulo conjugacy, or probability measures on a fixed Polish space modulo the identification of measures having the same null sets, or, for instance, reals modulo Turing reducibility. (These examples are taken from the book [6] and the survey [12], where many more examples are given.) Suppose that Y is another class of mathematical structures, identified modulo an equivalence relation F. The classification problem is to decide whether there is a *definable*, or *effective*, injection $\Theta\colon X/\mathsf{E} \to Y/\mathsf{F}$. Such a map Θ may be regarded as a classification of objects in X in terms of objects in Y in a way which respects the quotients over E and F. Its existence can be a result of great importance, for instance when the objects in Y are of a simpler mathematical structure than those in X.

In many cases, it turns out that the classes of structures X and Y can be regarded as Polish spaces (that is, separable complete metric spaces), so that E, F become

Results in this paper were presented at LC'1999 (Utrecht), LC'2000 (Paris), the Lumini meeting in set theory (2000), Caltech and UCLA seminars, and P. S. Novikov's memorial conference (Moscow, 2001).

The first author was supported by the grants DFG 101/10-1, NSF DMS 96-19880.

AMS 2000 *Mathematics Subject Classification.* 03E15, 54E50.

Borel relations or, more generally, analytic relations (as sets of pairs), and the reduction maps are usually required to be Borel.[1] In this case the problem can be studied by the methods of descriptive set theory, where it takes the following form. Let E, F be Borel or analytic equivalence relations on Polish spaces X, Y respectively. Does there exist a Borel *reduction* of E to F, that is, a Borel map $\vartheta\colon X \to Y$ satisfying $x \mathsf{E} x' \iff \vartheta(x) \mathsf{F} \vartheta(x')$ for all $x, x' \in X$? If such a map ϑ exists, then E is said to be *Borel reducible* to F. The studies of Borel and analytic equivalence relations (ERs, for brevity) in terms of Borel reducibility by the methods of descriptive set theory revealed a remarkable structure of reducibility and irreducibility theorems between ERs of different types. (We cite the surveys [2], [5]–[7], [12].) Our paper lies in this direction.

Our main theorem, Theorem 1, establishes the Borel irreducibility of a large class of ERs to another class. *Class* 1 consists of ERs induced by *generically turbulent* Polish actions.[2] Hjorth [6] proved that ERs of this class are not Borel reducible to ERs of *Class* 2, which consists of ERs induced by Polish actions of S_∞, the group of all permutations of $\mathbb{N}$. (There is another form of this result: generically turbulent ERs are not classifiable by countable structures; see the comments in § 3.3.)

A possible proof of Hjorth's theorem is as follows. First, if the given ER is Borel reducible to an ER in Class 2, then it is also Borel reducible, at least on a comeagre set, to an ER in *Class* 3,[3] which consists of all ERs that can be obtained from equalities on Polish spaces by iterating the operation of countable power E^∞. Second (this involves Hjorth's turbulence theory), no ER of Class 1 is Borel reducible to an ER of Class 3, even on a comeagre set. Our Theorem 1 generalizes the second part. We consider *Class* 4, which contains all ERs that can be obtained from equalities on Polish spaces by the following operations: 1) countable union of ERs in the same space (if the result is an ER); 2) Fubini product $\prod_{k\in\mathbb{N}} \mathsf{E}_k/\operatorname{Fin}$ modulo the ideal Fin of all finite subsets of $\mathbb{N}$; 3) the countable power E^∞ (see § 1.2 for precise definitions).

Of course, Class 4 includes Class 3, but it also contains many other ERs, especially those defined by Fubini products. For example, it contains all ERs induced by generalized Fréchet ideals, indecomposable ideals, and Weiss ideals (see § 1.2).

Theorem 1. *ERs of Class* 1 *are not Borel reducible* (*or even reducible by Baire measurable functions*) *to ERs of Class* 4.

The proof (§ 2) involves induction on the construction of ERs of Class 4 by the operations indicated. The proof is based on the techniques of turbulence theory. In particular, the key step consists in proving that all ERs of Class 1 are generically ergodic with respect to ERs of Class 4 (Theorem 6). As an application of this result, we derive the theorem of Hjorth mentioned above by a few rather simple arguments in § 3.

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[1] That is, they have Borel graphs. One can also consider Baire measurable maps and reductions satisfying certain algebraic requirements [2], as well as Δ^1_2 and more complicated reductions [9], [10]. However, these are not considered in this paper.

[2] That is, all orbits (modulo first category) and even local orbits are somewhere dense (see Definition 5).

[3] This was introduced essentially by Friedman [3], [4].

Among the inhabitants of Class 1 we have ERs of the form $x \mathsf{E}_{\mathcal{I}} y$ if and only if $x \Delta y \in \mathcal{I}$ for all $x, y \in \mathcal{P}(\mathbb{N})$, where $\mathcal{I}$ is an ideal on $\mathbb{N}$. Any ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is obviously an Abelian group with the symmetric difference Δ as the group operation, and $\mathsf{E}_{\mathcal{I}}$ is induced by the shift action of $\mathcal{I}$ on $\mathcal{P}(\mathbb{N})$ by Δ. Kechris [11] has shown that this action is turbulent provided that $\mathcal{I}$ is a Borel P-ideal,[4] with a few exceptions mentioned below. This enables us to prove the following result (see § 3.1) as a corollary of Theorem 1.

Theorem 2. *If $\mathcal{Z}$ is a non-trivial*[5] *Borel P-ideal on $\mathbb{N}$, then $\mathsf{E}_{\mathcal{Z}}$ is not Borel reducible to an ER of Class* 4 *unless $\mathcal{Z}$ is the ideal* Fin *of finite sets or a trivial variation of* Fin, *or $\mathcal{Z}$ is isomorphic to $\mathcal{I}_3 = 0 \times \mathrm{Fin}$ via a bijection between the underlying sets.*

The Borel P-ideals form a widely studied class, which includes, for instance, Fin, the ideal $\mathcal{I}_3 = 0 \times \mathrm{Fin}$ of all sets $x \subseteq \mathbb{N}^2$ such that every cross-section $(x)_n = \{k : \langle n, k\rangle \in x\}$ is finite, and *trivial variations* of Fin, that is, ideals of the form $\{x \in \mathcal{P}(\mathbb{N}) : x \cap W \in \mathrm{Fin}\}$, where $W \subseteq \mathbb{N}$ is infinite (see § 1.3), as well as summable ideals, density ideals, and many others (see [2], [15]). It is easy to see that Class 4 contains the ERs $\mathsf{E}_0 = \mathsf{E}_{\mathrm{Fin}}$ and $\mathsf{E}_3 = \mathsf{E}_{\mathcal{I}_3}$ (induced by the ideals Fin and $\mathcal{I}_3$), and the ERs induced by trivial variations of Fin. Thus the exclusion of Fin, $\mathcal{I}_3$, and trivial variations of Fin in Theorem 2 is necessary and fully motivated.

We note that a weaker form of Theorem 2 (with Class 3 instead of Class 4) was essentially proved by Kechris [11]. A very particular result was announced in [3]: let $\mathsf{E}_{\mathcal{Z}_0}$ be the ER induced by the null density ideal $\mathcal{Z}_0$. Then $\mathsf{E}_{\mathcal{Z}_0}$ is not Baire reducible to any ER of Class 3 (a proof is given in [4]).

The final section, § 4, represents an attempt to obtain results in the opposite direction: ERs of Class 4 are not Borel reducible to, say, turbulent ERs or other ERs of a different nature. This area is relatively less developed, and perhaps the only known theorem of this sort was proved by Hjorth [5]: let T_2 be the ER on countable sequences of the reals defined by $\{x_n\} \mathsf{T}_2 \{y_n\}$ if and only if $\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$.[6] Then T_2 is not Borel reducible to any ER induced by a continuous action of a Polish group which admits a compatible complete left-invariant metric (a CLI *group*; examples include Polish *Abelian* groups). It might be expected that T_2 is not even Borel reducible to any Borel action of a Borel Abelian group. But this problem is still open, even with respect to the shift Δ-actions of Borel ideals.

One approach to this problem is connected with the following condition (introduced implicitly in [5]) on an ER E: *for any forcing notion $\mathbb{P}$ and any $\mathbb{P}$-term $\boldsymbol{\xi}$, if $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\mathrm{left}} \mathsf{E} \boldsymbol{\xi}_{\mathrm{right}}$, then there is a real x in the ground universe such that $\mathbb{P}$ forces $x \mathsf{E} \boldsymbol{\xi}$.* We say that an ER is *pinned* if it satisfies this condition. Note that T_2 is not pinned and is not Borel reducible to any analytic pinned ER. We prove the following facts in § 4.

1) ERs induced by Polish actions of CLI groups are pinned. (Our proof is a simplification of Hjorth's proof in [5].)

[4]An ideal $\mathcal{I}$ on $\mathbb{N}$ is called a *P-ideal* if, for any sequence of sets $x_n \in \mathcal{I}$, there is a set $x \in \mathcal{I}$ such that $x_n \setminus x$ is finite for any n.

[5]That is, it contains all singletons $\{n\}$, $n \in \mathbb{N}$, and is different from $\mathcal{P}(\mathbb{N})$.

[6]T_2 is sometimes denoted by F_2 [6] and is often called *equality of countable sets of reals*. It belongs to Class 3 and is one of the most important Borel ERs.

2) Borel ERs whose equivalence classes are $\mathbf{G}_{\delta\sigma}$-sets are pinned. (This is based on an idea communicated to us by Hjorth.)

3) ERs associated with exhaustive ideals of sequences of submeasures on $\mathbb{N}$ (not all of them admitting a Polish topology compatible with Δ) are pinned.

4) Fubini products of analytic pinned ERs modulo Fin are pinned.

None of these ERs can Borel reduce T_2. In particular, all ERs induced by Fréchet ideals are pinned and do not Borel reduce T_2.

§ 1. Preliminaries

This section contains a review of the basic notation involved in the statements and proofs of Theorems 1 and 2.

1.1. Descriptive set theory. We assume some degree of knowledge of the theory of Borel and analytic sets in Polish spaces (that is, complete separable metric spaces). We recall that *analytic* sets (also known as Suslin sets, A-sets, or $\mathbf{\Sigma}^1_1$) are continuous images of Borel sets. Every Borel set is analytic, but the converse is not true (in uncountable Polish spaces).

A map f (between Borel sets in Polish spaces) is said to be *Borel* if its graph is a Borel set or, equivalently, if all f-pre-images of open sets are Borel. A map $f\colon \mathbb{X} \to \mathbb{Y}$ is said to be *Baire measurable* if all f-pre-images of open subsets of $\mathbb{Y}$ possess the Baire property in $\mathbb{X}$. (In other words, they are equal to open sets modulo meagre sets, that is, sets of the first category.) Any such map is continuous on a dense $\mathbf{G}_\delta$-set $D \subseteq \mathbb{X}$ ($\mathbb{X}$, $\mathbb{Y}$ are supposed to be Polish).

It is easy to see that superpositions of Borel maps are also Borel maps. Generally speaking, this is not true for Baire measurable maps. However, we have a useful partial result.

Lemma 3. *Let* $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ *be Polish spaces*, $f\colon \mathbb{X} \to \mathbb{Y}$ *a Baire measurable map*, *and* $g\colon \mathbb{Y} \to \mathbb{Z}$ *a Borel map. Then the superposition* $f \circ g\colon \mathbb{X} \to \mathbb{Z}$ *is Baire measurable.*

Proof. By definition, g-pre-images of open subsets of $\mathbb{Z}$ are Borel in $\mathbb{Y}$, and their f-pre-images are Borel combinations of sets having the Baire property.

1.2. Equivalence relations. We denote by $\mathsf{D}(X)$ the ER of equality on X.

Let E be an ER on a set X. The E-*class* of any element $y \in X$ is defined as $[y]_{\mathsf{E}} = \{x \in X : y \mathrel{\mathsf{E}} x\}$. A set $Y \subseteq X$ is said to be *pairwise* E-*equivalent* if $x \mathrel{\mathsf{E}} y$ holds for all $x, y \in Y$.

Let E, F be ERs on Polish spaces $\mathbb{X}$, $\mathbb{Y}$ respectively. We make the following definitions.

(i) $\mathsf{E} \leqslant_{\mathrm{B}} \mathsf{F}$ (*Borel reducibility*, sometimes denoted by $\mathbb{X}/\mathsf{E} \leqslant_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$) means that there is a Borel map $\vartheta\colon \mathbb{X} \to \mathbb{Y}$ (called a *reduction*) such that $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$.

(ii) $\mathsf{E} \sim_{\mathrm{B}} \mathsf{F}$ means that $\mathsf{E} \leqslant_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \leqslant_{\mathrm{B}} \mathsf{E}$ (*bi-reducibility*);

(iii) $\mathsf{E} <_{\mathrm{B}} \mathsf{F}$ means that $\mathsf{E} \leqslant_{\mathrm{B}} \mathsf{F}$ and $\mathsf{F} \nleqslant_{\mathrm{B}} \mathsf{E}$ (*strict reducibility*).

We consider the following operations over ERs on Polish spaces.

(e1) The *countable union* (if it results in an ER) and the *countable intersection* of ERs on the same space.

(e2) The *countable disjoint union* $\mathsf{F} = \bigvee_{k\in\mathbb{N}} \mathsf{F}_k$ of ERs F_k on Polish spaces $\mathbb{S}_k$ is the ER on the space $\mathbb{S} = \bigcup_k (\{k\} \times \mathbb{S}_k)$ (whose Polish topology is generated by sets of the form $\{k\} \times U$, where $U \subseteq \mathbb{S}_k$ is open) defined as follows: $\langle k, x\rangle \,\mathsf{F}\, \langle l, y\rangle$ if and only if $k = l$ and $x \,\mathsf{F}_k\, y$.[7]

(e3) The *product* $\mathsf{P} = \prod_k \mathsf{F}_k$ of ERs F_k on spaces $\mathbb{S}_k$ is the ER on $\prod_k \mathbb{S}_k$ defined as follows: $x \,\mathsf{P}\, y$ if and only if $x_k \,\mathsf{F}_k\, y_k$ for all k. In particular, if E, F are ERs on $\mathbb{X}$, $\mathbb{Y}$ respectively, then $\mathsf{P} = \mathsf{E} \times \mathsf{F}$ is defined on $\mathbb{X} \times \mathbb{Y}$ by saying that $\langle x, y\rangle \,\mathsf{P}\, \langle x', y'\rangle$ if and only if $x \,\mathsf{E}\, x'$ and $y \,\mathsf{F}\, y'$.

(e4) The *Fubini product* $\mathsf{F} = \prod_{k\in\mathbb{N}} \mathsf{F}_k/\mathcal{I}$ of ERs F_k on spaces $\mathbb{S}_k$ modulo an ideal $\mathcal{I}$ on $\mathbb{N}$ is the ER on the product space $\prod_{k\in\mathbb{N}} \mathbb{S}_k$ defined as follows: $x \,\mathsf{F}\, y$ if and only if $\{k \colon x_k \not\mathsf{F}_k\, y_k\} \in \mathcal{I}$;

(e5) The *countable power* F^∞ of an ER F on $\mathbb{S}$ is the ER on $\mathbb{S}^\mathbb{N}$ defined as follows: $x \,\mathsf{F}^\infty\, y$ if and only if $\{[x_k]_\mathsf{F} : k \in \mathbb{N}\} = \{[y_k]_\mathsf{F} : k \in \mathbb{N}\}$, so that for every k there is l such that $x_k \,\mathsf{F}\, y_l$, and for every l there is k such that $x_k \,\mathsf{F}\, y_l$.

We note that the operations (e1), (e2), (e3), (e5), and (e4) with $\mathcal{I} = \mathrm{Fin}$ always yield Borel (resp. analytic) ERs provided that the given ERs are Borel (resp. analytic).

These operations are not independent. In particular, $\bigcap_{k\in\mathbb{N}} \mathsf{F}_k$ is Borel reducible to $\prod_k \mathsf{F}_k$ via the map $x \mapsto \langle x, x, x, \dots\rangle$ and the disjoint union $\bigvee_{k\in\mathbb{N}} \mathsf{F}_k$ is reducible to $\mathsf{D}(\mathbb{N}) \times \prod_k \mathsf{F}_k$ via the map $\langle k, x\rangle \mapsto \langle k, x_0, \dots, x_{k-1}, x, x_{k+1}, \dots\rangle$, where the $x_k \in \mathbb{S}_k$ are fixed once and for all. The product $\prod_{k\in\mathbb{N}} \mathsf{F}_k$ itself is expressible in terms of the Fubini product modulo Fin. Indeed, let $f \colon \mathbb{N} \overset{\text{onto}}{\to} \mathbb{N}$ be any map such that $f^{-1}(n)$ is infinite for each n. We put $\mathsf{E}_k = \mathsf{F}_{f(k)}$. For any $x = \langle x_0, x_1, x_2, \dots\rangle \in \prod_k \mathbb{S}_k$ (where $\mathbb{S}_k$ is the domain of F_k) we put $\vartheta(x) = \langle y_0, y_1, y_2, \dots\rangle$, where $y_k = x_{f(k)}$. Then ϑ is a Borel reduction of $\prod_k \mathsf{F}_k$ to $\prod_k \mathsf{E}_k/\,\mathrm{Fin}$. However the Fubini product and the countable power are certainly not reducible to each other, and we know little about the countable union in (e1).

It follows that Class 4 (mentioned in Theorems 1 and 2) is the smallest class of ERs that contains the equalities $\mathsf{D}(\mathbb{S})$ on Polish spaces $\mathbb{S}$ and is closed under the operations (e1)–(e5) (with $\mathcal{I} = \mathrm{Fin}$ in (e4)), and all ERs of Class 4 are Borel ERs on Polish spaces.

Class 4 contains many interesting ERs. For instance, it contains the family T_α, $\alpha < \omega_1$, of ERs introduced by Friedman [4]. This family begins with $\mathsf{T}_0 = \mathsf{D}(\mathbb{N})$ (the equality relation on $\mathbb{N}$) and satisfies $\mathsf{T}_{\alpha+1} = \mathsf{T}_\alpha^\infty$ for all α and $\mathsf{T}_\lambda = \bigvee_{\alpha<\lambda} \mathsf{T}_\alpha$ for limit ordinals λ. Thus $\operatorname{dom} \mathsf{T}_1 = \mathbb{N}^\mathbb{N}$, and $x \mathsf{T}_1 y$ if and only if $\operatorname{ran} x = \operatorname{ran} y$ for $x, y \in \mathbb{N}^\mathbb{N}$. Using the map $\vartheta(x) = \chi$, where χ is the characteristic function of $\operatorname{ran} x$, we see that $\mathsf{T}_1 \leqslant_\mathrm{B} \mathsf{D}(2^\mathbb{N})$. Let us prove the converse. Given any $a \in 2^\mathbb{N}$, let $\beta(a)$ be the unique increasing bijection $\mathbb{N} \overset{\text{onto}}{\to} |a| = \{k \colon a(k) = 1\}$ if $|a|$ is infinite. If $|a| = \{k_0, \dots, k_n\}$, then we put $\beta(a)(i) = k_i$ for $i < n$ and $\beta(a)(i) = k_n$ for $i \geqslant n$. The function β shows that $\mathsf{D}(2^\mathbb{N}) \leqslant_\mathrm{B} \mathsf{T}_1$, whence $\mathsf{T}_1 \sim_\mathrm{B} \mathsf{D}(2^\mathbb{N})$. It follows easily that $\mathsf{T}_2 \sim_\mathrm{B} \mathsf{D}(2^\mathbb{N})^\infty$. The right-hand side is often taken as the definition of the ER T_2, and this is why T_2 is usually called *equality of countable sets of reals*.

1.3. Ideals. An *ideal* on a set A is any set $\varnothing \neq \mathcal{I} \subseteq \mathcal{P}(A)$ which is closed under $\cup$ and satisfies $x \in \mathcal{I} \wedge y \subseteq x \implies y \in \mathcal{I}$. Each ideal $\mathcal{I}$ determines an ER $\mathsf{E}_\mathcal{I}$ on $\mathcal{P}(A)$ as follows: $X \,\mathsf{E}_\mathcal{I}\, Y$ if and only if $X \Delta Y \in \mathcal{I}$. We note that $\mathsf{E}_\mathcal{I}$ is a Borel ER provided

[7] If $\mathbb{S}_k$ are pairwise disjoint and open in $\mathbb{S}' = \bigcup_k \mathbb{S}_k$, then we can equivalently define $\mathsf{F} = \bigvee_k \mathsf{F}_k$ on $\mathbb{S}'$ by saying that $x \,\mathsf{F}\, y$ if and only if x, y belong to the same $\mathbb{S}_k$ and $x \,\mathsf{F}_k\, y$.

that $\mathcal{I}$ is a Borel ideal. Many important ERs arise in this way, among them

$$\begin{aligned} \mathsf{E}_0 &= \mathsf{E}_{\mathrm{Fin}}, && \text{where} \quad \mathrm{Fin} = \{x \subseteq \mathbb{N} \colon x \text{ is finite}\}; \\ \mathsf{E}_1 &= \mathsf{E}_{\mathcal{I}_1}, && \text{where} \quad \mathcal{I}_1 = \mathrm{Fin} \times 0 = \big\{x \subseteq \mathbb{N}^2 \colon \{k \colon (x)_k \neq \varnothing\} \in \mathrm{Fin}\big\}; \\ \mathsf{E}_3 &= \mathsf{E}_{\mathcal{I}_3}, && \text{where} \quad \mathcal{I}_3 = 0 \times \mathrm{Fin} = \big\{x \subseteq \mathbb{N}^2 \colon \forall\, k\big((x)_k \in \mathrm{Fin}\big)\big\}. \end{aligned}$$

These three ERs belong to Class 4.[8] Ideals of the form $\{x \in \mathcal{P}(\mathbb{N}) \colon x \cap W \in \mathrm{Fin}\}$, where $W \subseteq \mathbb{N}$ is infinite and coinfinite, are called *trivial variations of* Fin. They also produce ERs of Class 4.

We write $\mathcal{I} \leqslant_{\mathrm{B}} \mathcal{J}$, $\mathcal{I} \sim_{\mathrm{B}} \mathcal{J}$, and so on if $\mathsf{E}_{\mathcal{I}} \leqslant_{\mathrm{B}} \mathsf{E}_{\mathcal{J}}$, $\mathsf{E}_{\mathcal{I}} \sim_{\mathrm{B}} \mathsf{E}_{\mathcal{J}}$, and so on, respectively.

The *Fubini product* $\prod_{k\in\mathbb{N}} \mathcal{J}_k/\mathcal{I}$ of ideals $\mathcal{J}_k$ on sets B_k over an ideal $\mathcal{I}$ on $\mathbb{N}$ is the ideal of all sets $y \subseteq B = \{\langle k, b\rangle \colon k \in \mathbb{N} \wedge b \in B_k\}$ such that the set $\{k \colon (y)_k \notin \mathcal{J}_k\}$ belongs to $\mathcal{I}$, where $(y)_k = \{b \colon \langle k, b\rangle \in y\}$ is a *cross-section* of y. (Compare with the Fubini product of ERs.) In particular, if $\mathcal{I}$, $\mathcal{J}$ are ideals on $\mathbb{N}$, B respectively, then $\mathcal{I} \otimes \mathcal{J} = \prod_{k\in\mathbb{N}} \mathcal{J}_k/\mathcal{I}$, where $\mathcal{J}_k = \mathcal{J}$ for all $k \in \mathbb{N}$. Thus $\mathcal{I} \otimes \mathcal{J}$ is the ideal of all sets $y \subseteq \mathbb{N} \times B$ such that $\{k \colon (y)_k \notin \mathcal{J}\} \in \mathcal{I}$.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is called a *P-ideal* if for any sequence of sets $x_n \in \mathcal{I}$ there is a set $x \in \mathcal{I}$ such that $x_n \setminus x \in \mathrm{Fin}$ for all n. For example, Fin and $\mathcal{I}_3$ (but not $\mathcal{I}_1$) are P-ideals.

The class of P-ideals admits different characterizations. A *submeasure* on a set A is any map $\varphi \colon \mathcal{P}(A) \to [0, +\infty]$ such that $\varphi(\varnothing) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \leqslant \varphi(x \cup y) \leqslant \varphi(x) + \varphi(y)$. A submeasure φ on $\mathbb{N}$ is *lower semicontinuous* (or l. s. c., for brevity) if $\varphi(x) = \sup_n \varphi(x \cap [0, n))$ for all $x \in \mathcal{P}(\mathbb{N})$. Solecki [15] proved that the set of all Borel P-ideals coincides with the set of all ideals of the form $\mathrm{Exh}_\varphi = \{x \in \mathcal{P}(\mathbb{N}) \colon \varphi_\infty(x) = 0\}$, where φ is a l. s. c. submeasure on $\mathbb{N}$ and $\varphi_\infty(x) = \inf_n(x \cap [n, \infty))$. He also proved that Borel P-ideals are the same as *Polishable* ideals, that is, those admitting a Polish group topology with Δ as the group operation.

Kechris [11] proved that the shift action of any Borel P-ideal $\mathcal{I}$ (except for Fin, $\mathcal{I}_3$, and trivial variations of Fin) is generically turbulent, whence the corresponding ER $\mathsf{E}_{\mathcal{I}}$ belongs to Class 1.

The *Fréchet family* is the smallest family Fr of ideals that contains Fin and is closed under Fubini products $\prod_{n\in\mathbb{N}} \mathcal{I}_n/\mathrm{Fin}$. For instance, Fr contains the *iterated Fréchet ideals* $\mathcal{J}_\alpha$, which are defined by induction on $\alpha < \omega_1$ as follows: $\mathcal{J}_0 = \mathrm{Fin}$, $\mathcal{J}_{\alpha+1} = \mathrm{Fin} \otimes \mathcal{J}_\alpha$ for all α, and $\mathcal{J}_\lambda = \prod_{\alpha<\lambda} \mathcal{J}_\alpha/\mathrm{Fin}_\lambda$ for any limit λ, where Fin_λ is the ideal of all finite subsets of λ. (A modification of this construction in [8] involves an ω-sequence cofinal in λ, fixed for any limit λ.)

By definition, if $\mathcal{I} \in \mathrm{Fr}$, then $\mathsf{E}_{\mathcal{I}}$ is an ER of Class 4.

Let $\operatorname{otp} X$ be the order type of $X \subseteq \mathrm{Ord}$. For any $\alpha, \gamma < \omega_1$ we consider the set

$$\mathcal{I}^\gamma_\alpha = \{A \subseteq \alpha \colon \operatorname{otp} A < \omega^\gamma\}$$

(which is non-trivial only if $\alpha \geqslant \omega^\gamma$). This is an ideal because ordinals of the form ω^γ are not sums of pairs of smaller ordinals. These ideals are said to be

[8] To show that E_0 belongs to Class 4, we take, for all k, equality on a 2-element set as F_k in (e4). To see that E_3 belongs to Class 4, we take $\mathsf{P}_k = \mathsf{E}_0$ in (e3).

indecomposable, especially in the case when $\alpha = \omega^\gamma$. We do not know whether ideal $\mathcal{I}^\gamma_\alpha$ is really isomorphic to an ideal in Fr, but it can be shown that every $\mathcal{I}^\gamma_\alpha$ is Borel reducible to an ideal in Fr. We similarly consider the *Weiss ideals*

$$\mathcal{W}^\gamma_\alpha = \{A \subseteq \alpha \colon |A|_{\mathrm{CB}} < \omega^\gamma\}$$

(non-trivial only if $\alpha \geqslant \omega^{\omega^\gamma}$), where $|X|_{\mathrm{CB}}$ is the Cantor–Bendixon rank of the set $X \subseteq \mathrm{Ord}$ (see [2], § 1.14). They are also Borel reducible to ideals in Fr.

§ 2. Proof of Theorem 1

The proof of Theorem 1 employs the following auxiliary notions. Let E, F be ERs on Polish spaces $\mathbb{X}$, $\mathbb{Y}$ respectively. A map $\vartheta \colon \mathbb{X} \to \mathbb{Y}$ is said to be

(i) (E,F)-*invariant* if $x \mathrel{\mathsf{E}} y \implies \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;

(ii) *generically*[9] (E,F)-*invariant* if we have $x \mathrel{\mathsf{E}} y \implies \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all x, y in a comeagre set $X \subseteq \mathbb{X}$;

(iii) *generically* F-*constant* if $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ holds for all x, y in a comeagre subset of the space $\mathbb{X}$.

Finally, E is said to be *generically* F-*ergodic* (see [6], § 3.1) if every Baire measurable (E,F)-invariant function is generically F-constant.

Proposition 4. (i) *If* E *is generically* F-*ergodic and has no comeagre equivalence classes*, *then* E *is not reducible to* F *by a Baire measurable map.*

(ii) *If* E *is generically* F-*ergodic*, *then every Baire measurable generically* (E,F)-*invariant function is generically* F-*constant.*

We demonstrate below that any ER induced by a Polish turbulent action is generically F-ergodic for any ER F in Class 4.

2.1. Local orbits and turbulence. An *action* of a group $\mathbb{G}$ on $\mathbb{X}$ is any map $a \colon \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ (usually written as $a(g,x) = g \cdot x$) such that 1) $e \cdot x = x$ and 2) $g \cdot (h \cdot x) = (gh) \cdot x$. Then $\langle \mathbb{X}; a\rangle$ (as well as $\mathbb{X}$ itself) is called a $\mathbb{G}$-*space*. A continuous action of a Polish group[10] $\mathbb{G}$ on a Polish space $\mathbb{X}$ is called a *Polish action*, and $\mathbb{X}$ itself is called a *Polish* $\mathbb{G}$-*space*.

Any action a of $\mathbb{G}$ on $\mathbb{X}$ induces the *orbit ER* $\mathsf{E}^{\mathbb{X}}_a = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ on $\mathbb{X}$ as follows: $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} y$ if and only if there is $g \in \mathbb{G}$ with $y = g \cdot x$. Its equivalence classes

$$[x]_{\mathbb{G}} = [x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} = \{y \colon \exists\, g \in \mathbb{G}(g \cdot x = y)\}$$

are $\mathbb{G}$-orbits. Induced ERs of Polish actions are analytic (as sets of pairs) and sometimes even Borel ([1], § 7).

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$. If $G \subseteq \mathbb{G}$ and $X \subseteq \mathbb{X}$, then we define

$$\mathsf{R}^X_G = \{\langle x, y\rangle \in X^2 \colon \exists\, g \in G(x = g \cdot y)\},$$

and let $\sim^X_G$ denote the ER-hull of R^X_G, that is, the $\subseteq$-least ER on X such that $x \mathrel{\mathsf{R}^X_G} y \implies x \sim^X_G y$. In particular, $\sim^{\mathbb{X}}_{\mathbb{G}} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$, but we generally have $\sim^X_G \subsetneqq \mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \restriction X$. We put $\mathcal{O}(x, X, G) = [x]_{\sim^X_G} = \{y \in X \colon x \sim^X_G y\}$ for $x \in X$. This is the *local orbit* of x. In particular, $[x]_{\mathbb{G}} = [x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} = \mathcal{O}(x, \mathbb{X}, \mathbb{G})$ is the full $\mathbb{G}$-orbit of $x \in \mathbb{X}$.

[9] In this context, a property is referred to as "generic" if it holds on a comeagre domain.

[10] That is, a topological group whose underlying set is a Polish space, and the group operation and the inverse map are continuous.

Definition 5 (this version is taken from Kechris [13], § 8). Suppose that $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting continuously on $\mathbb{X}$.

1) A point $x \in \mathbb{X}$ is said to be *turbulent* if, for any open non-empty set $X \subseteq \mathbb{X}$ containing x and for any neighbourhood $G \subseteq \mathbb{G}$ (not necessarily a subgroup) of $1_{\mathbb{G}}$, the local orbit $\mathcal{O}(x, X, G)$ is somewhere dense (that is, not nowhere dense) in $\mathbb{X}$.

2) An orbit $[x]_{\mathbb{G}}$ is said to be *turbulent* if the point x is turbulent (then all points $y \in [x]_{\mathbb{G}}$ are turbulent).

3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *turbulent* and $\mathbb{X}$ is a *turbulent* Polish $\mathbb{G}$-space if all orbits are dense and meagre. The action is *generically turbulent* and $\mathbb{X}$ is a *generically turbulent* Polish $\mathbb{G}$-space if the union of all dense, turbulent, and meagre orbits $[x]_{\mathbb{G}}$ is a comeagre set.

The ERs induced by generically turbulent Polish actions comprise Class 1 in Theorem 1. To prove Theorem 1, we shall show that all ERs in Class 1 are generically F-ergodic for any F in Class 4. The proof is by induction on the construction of ERs in Class 4. There is a slight inconvenience: we have to consider a somewhat stronger property in the induction scheme.

Suppose that F is an ER on a Polish space. An action of $\mathbb{G}$ on $\mathbb{X}$ is said to be *hereditarily generically* F-ergodic if the ER $\sim_G^X$ is generically F-ergodic whenever $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a non-empty open set containing $1_{\mathbb{G}}$, and the local orbits $\mathcal{O}(x, X, G)$ are dense in X for x belonging to a comeagre subset of X. This obviously implies generic F-ergodicity provided that the action is generically turbulent.

Theorem 6. *Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a generically turbulent Polish $\mathbb{G}$-space. Then the relation $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is hereditarily generically F-ergodic. Hence, by Proposition* 4, *it is not reducible to any* ER F *in Class* 4 *by a Baire measurable map.*

2.2. Preliminaries for the proof of Theorem 6. We begin with two rather simple technical facts related to turbulence.

Lemma 7. *Under the assumptions of Theorem* 6, *suppose that $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a neighbourhood of $1_{\mathbb{G}}$, and $\mathcal{O}(x, X, G)$ is dense in X for x belonging to a comeagre subset of X. If $U, U' \subseteq X$ are non-empty open sets and $D \subseteq X$ is comeagre in X, then there are points $x \in D \cap U$ and $x' \in D \cap U'$ such that $x \sim_G^X x'$.*

Proof. Under our assumptions there are points $x_0 \in U$ and $x_0' \in U'$ with $x_0 \sim_G^X x_0'$, that is, there are elements $g_1, \dots, g_n \in G$ satisfying $x_0' = g_n g_{n-1} \dots g_1 \cdot x_0$ and $g_k \dots g_1 \cdot x_0 \in X$ for all $k \leqslant n$. Since the action under consideration is continuous, there is a neighbourhood $U_0 \subseteq U$ of x_0 such that $g_k \dots g_1 \cdot x \in X$ and $g_n g_{n-1} \dots g_1 \cdot x \in U_2$ for all $x \in U_0$. Since D is comeagre, it is clear that there is $x \in U_0 \cap D$ such that $x' = g_n g_{n-1} \dots g_1 \cdot x \in U' \cap D$. The lemma is proved.

Lemma 8. *Under the assumptions of Theorem* 6, *for any open non-empty sets $U \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ with $1_{\mathbb{G}} \in G$, there is an open non-empty set $U' \subseteq U$ such that the local orbits $\mathcal{O}(x, U', G)$ are dense in U' for x belonging to a comeagre subset of U'.*

Proof. Let $\operatorname{Int}\overline{X}$ be the interior of the closure of X. If $x \in U$ and $\mathcal{O}(x,U,G)$ is somewhere dense (in U), then the set $U_x = U \cap \operatorname{Int}\overline{\mathcal{O}(x,U,G)} \subseteq U$ is open and $\sim_G^U$-invariant. (This observation was made, for example, in [13], proof of 8.4.) Moreover, $\mathcal{O}(x,U,G) \subseteq U_x$, whence $\mathcal{O}(x,U,G) = \mathcal{O}(x,U_x,G)$. The invariance of orbits implies that the sets U_x are pairwise disjoint, and the turbulence implies that their union is dense in U. We take any non-empty U_x for the desired U'. The lemma is proved.

Our proof of Theorem 6 is by induction on the construction of ERs in Class 4 by the operations listed in § 1.2. It will occupy several subsections. We begin with the base of induction, which asserts that $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is hereditarily generically $\mathsf{D}(\mathbb{N})$-ergodic under the hypotheses of the theorem. Suppose that $X \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ are non-empty open sets, $1_{\mathbb{G}} \in G$, and the local orbits $\mathcal{O}(x,X,G)$ are dense in X for x belonging to a comeagre subset of X. We claim that $\sim_G^X$ is generically $\mathsf{D}(\mathbb{N})$-ergodic.

Indeed, consider a Baire measurable generically $(\sim_G^X, \mathsf{D}(\mathbb{N}))$-invariant map $\vartheta \colon \mathbb{X} \to \mathbb{N}$. Suppose, on the contrary, that ϑ is not generically $\mathsf{D}(\mathbb{N})$-constant. Then there are open non-empty sets $U_1, U_2 \subseteq X$, numbers $l_1 \neq l_2$ and a comeagre set $D \subseteq X$ such that $\vartheta(x) = l_1$ for all $x \in D \cap U_1$ and $\vartheta(x) = l_2$ for all $x \in D \cap U_2$. Lemma 7 yields a pair of points $x_1 \in U_1 \cap D$ and $x_2 \in U_2 \cap D$ satisfying $x_1 \sim_G^X x_2$, a contradiction.

2.3. Inductive step of the countable power. Consider a generically turbulent Polish $\mathbb{G}$-space $\mathbb{X}$ and a Borel ER F on a Polish space $\mathbb{Y}$. We assume that the action of $\mathbb{G}$ on $\mathbb{X}$ is hereditarily generically F-ergodic and assert that this action is hereditarily generically F^∞-ergodic. To prove this, we fix a non-empty open set $X_0 \subseteq \mathbb{X}$ and a neighbourhood G_0 of the element $1_{\mathbb{G}}$ in $\mathbb{G}$ such that $\mathcal{O}(x,X_0,G_0)$ is dense in X_0 for x belonging to a comeagre subset of X_0. We shall prove that any given Baire measurable $(\sim_{G_0}^{X_0}, \mathsf{F}^\infty)$-invariant function $\vartheta \colon X_0 \to \mathbb{Y}^{\mathbb{N}}$ is generically F^∞-constant. By definition, we have

$$\text{for} \quad x, x' \in X_0 \colon \quad x \sim_{G_0}^{X_0} x' \implies \forall\, k \exists\, l\big(\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(x')\big), \tag{1}$$

where $\vartheta_k(x) = \vartheta(x)(k)$. We note that ϑ is continuous on a dense $\mathbf{G}_\delta$-set $D \subseteq X_0$.

Lemma 9. *For every positive integer k and every open non-empty set $U \subseteq X_0$ there is an open non-empty set $W \subseteq U$ such that ϑ_k is generically F-constant on W.*

Proof. A simple category argument beginning with (1) yields a number l, open non-empty sets $W \subseteq U$, $Q \subseteq G_0$ and a dense (in $W \times Q$) set $P \subseteq W \times Q$ of class $\mathbf{G}_\delta$ such that $\vartheta_k(x) \,\mathsf{F}\, \vartheta_l(g \cdot x)$ holds for all $\langle x, g\rangle \in P$. We can assume that $\langle x, g\rangle \in P \implies x \in D$. Since Q is open, there is an element $g_0 \in Q$ and a neighbourhood $G \subseteq G_0$ of $1_{\mathbb{G}}$ with $G^{-1} = G$ such that $g_0 G \subseteq Q$.

The next part of the proof involves forcing.[11]

We fix a countable transitive model $\mathfrak{M}$ of ZFHC, that is, ZFC minus the Power Set axiom but with the axiom which postulates that every set is hereditarily countable. We can assume that $\mathbb{X}$ is coded in $\mathfrak{M}$ in the sense that there is a set $D_{\mathbb{X}} \in \mathfrak{M}$

[11] We assume some degree of acquaintance with forcing. The lemma can be proved by purely topological arguments, but then the reasoning is not so transparent.

which is a dense (countable) subset of $\mathbb{X}$, and $d_{\mathbb{X}} \restriction D_{\mathbb{X}}$ (the distance function of $\mathbb{X}$ restricted to $D_{\mathbb{X}}$) also belongs to $\mathfrak{M}$. We also assume that $\mathbb{G}$, $\mathbb{Y}$, the action of $\mathbb{G}$ on $\mathbb{X}$, the sets G, $D_{\mathbb{X}}$, P, and the map $\vartheta \restriction D_{\mathbb{X}}$ are coded in $\mathfrak{M}$ in a similar sense.

We consider the Cohen forcing $\mathbf{C}_{\mathbb{X}}$ for $\mathbb{X}$, which consists of rational balls with centres in a fixed dense countable subset of $\mathbb{X}$, and the Cohen forcing $\mathbf{C}_{\mathbb{G}}$ for $\mathbb{G}$, which is defined similarly. As usual, $U \subseteq V$ means that U is a stronger forcing condition. Under these assumptions, the notion of Cohen generic (over $\mathfrak{M}$) points of $\mathbb{X}$ or $\mathbb{G}$ makes sense, and the set of all Cohen generic (over $\mathfrak{M}$) points of $\mathbb{X}$ is a dense $\mathbf{G}_\delta$-subset of $\mathbb{X}$ contained in D.

Claim 10 (the key point of the turbulence). *If $x, x' \in W$ are $\mathbf{C}_{\mathbb{X}}$-generic points over $\mathfrak{M}$ and $x \sim_G^W x'$, then $\vartheta_k(x) \mathsf{F} \vartheta_k(x')$.*

Proof. We argue by induction on the number $n(x, x')$, which is equal to the least n such that there are elements $g_1, \dots, g_n \in G$ satisfying

$$x' = g_n g_{n-1} \dots g_1 \cdot x \quad \text{and} \quad g_k \dots g_1 \cdot x \in W \qquad \text{for all} \quad k \leqslant n. \tag{2}$$

Suppose that $n(x, x') = 1$. Thus $x = h \cdot x'$ for some $h \in G \cap \mathfrak{M}[x, x']$.[12] Take any $\mathbf{C}_{\mathbb{G}}$-generic (over $\mathfrak{M}[x, x']$) element $g \in Q$, close enough to g_0 for $g' = gh^{-1}$ to be an element of Q. The pair $\langle x, g \rangle$ is $(\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}})$-generic over $\mathfrak{M}$ by the product forcing theorem, whence $\langle x, g \rangle \in P$ (because P is a dense $\mathbf{G}_\delta$-set coded in $\mathfrak{M}$) and $\vartheta_k(x) \mathsf{F} \vartheta_l(g \cdot x)$ by the choice of P. Moreover, g' is also $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}[x']$, so that $\vartheta_k(x') \mathsf{F} \vartheta_l(g' \cdot x')$ by the same argument. However, we have $g' \cdot x' = gh^{-1}(h \cdot x) = g \cdot x$.

We now suppose that (2) holds for some $n \geqslant 2$. Take a $\mathbf{C}_{\mathbb{G}}$-generic (over $\mathfrak{M}[x, x']$) element $g_1' \in G$, close enough to g_1 for $g_2' = g_2 g_1 g_1'^{-1}$ to be an element of G and $x^* = g_1' \cdot x$ to be an element of W. Then the point x^* is $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ (product forcing) and $n(x^*, x') \leqslant n - 1$ because $g_2' \cdot x^* = g_2 g_1 \cdot x$. The claim is proved.

To summarize, we have shown that ϑ_k is generically $(\sim_G^W, \mathsf{F})$-invariant on W (that is, invariant on a comeagre subset of W). By Lemma 8 we can also assume that the orbit $\mathcal{O}(x, W, G)$ is dense in W for x belonging to a comeagre subset of W. Then the hereditarily generic F-ergodicity implies that ϑ_k is generically F-constant on W, as required. This proves Lemma 9.

According to Lemma 9, one can find an X_0-comeagre set $Z \subseteq X_0$ and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{Y}$ such that, for any k and any $x \in Z$, there is j with $\vartheta_k(x) \mathsf{F} y_j$. We put $\eta(x) = \bigcup_{k \in \mathbb{N}} \{j : \vartheta_k(x) \mathsf{F} y_j\}$. Then, for any pair $x, x' \in Z$, the relation $\vartheta(x) \mathsf{F}^\infty \vartheta(x')$ is equivalent to $\eta(x) = \eta(x')$. Hence the invariance of ϑ implies that

$$x \sim_{G_0}^{X_0} x' \implies \eta(x) = \eta(x') \qquad \text{for all} \quad x, x' \in Z. \tag{3}$$

It remains to prove that η is constant on a comeagre subset of Z.

Suppose, on the contrary, that there are two non-empty open sets $U_1, U_2 \subseteq X_0$, a number $j \in \mathbb{N}$ and a comeagre set $Z' \subseteq Z$ such that $j \in \eta(x_1)$ and $j \notin \eta(x_2)$ for all $x_1 \in Z' \cap U_1$ and $x_2 \in Z' \cap U_2$. Then Lemma 7 yields a contradiction to (3), which can be demonstrated by the same method as at the end of § 2.2.

[12] Here $\mathfrak{M}[x, x']$ is defined as any (countable transitive) model of ZFHC containing x, x' and all sets in $\mathfrak{M}$, rather than a generic extension of $\mathfrak{M}$. The model $\mathfrak{M}[x, x']$ may contain more ordinals than $\mathfrak{M}$, but this is not essential here.

2.4. Inductive step of the Fubini product. Suppose that $\mathbb{X}$ is a generically turbulent Polish $\mathbb{G}$-space. We want to prove that the action of $\mathbb{G}$ on $\mathbb{X}$ is hereditarily generically F-ergodic, where $\mathsf{F} = \prod_k \mathsf{F}_k / \mathrm{Fin}$, F_k is a Borel ER on a Polish space $\mathbb{Y}_k$, and the action is hereditarily generically F_k-ergodic for every k. We fix an open non-empty set $X_0 \subseteq \mathbb{X}$ and a neighbourhood G_0 of the element $1_{\mathbb{G}}$ in $\mathbb{G}$ such that $\mathcal{O}(x, X_0, G_0)$ is dense in X_0 for x belonging to a comeagre subset of X_0. We shall prove that any $(\sim_{G_0}^{X_0}, \mathsf{F})$-invariant Baire measurable function $\vartheta \colon X_0 \to \mathbb{Y}$ is generically F-constant on X_0. By definition,

$$\text{for} \quad x, y \in X_0 : \quad x \sim_{G_0}^{X_0} y \implies \exists\, k_0\, \forall\, k \geqslant k_0 \big(\vartheta_k(x)\, \mathsf{F}_k\, \vartheta_k(y)\big), \tag{4}$$

where $\vartheta_k(x) = \vartheta(x)(k)$. We note that ϑ is continuous on a dense $\mathbf{G}_\delta$-set $D \subseteq X_0$.

Lemma 11. *For any non-empty open set $U \subseteq X_0$ there is a number k_0 and a non-empty open set $W \subseteq U$ such that ϑ_k is generically F-constant on W for all $k \geqslant k_0$.*

Proof. Applying (4), we easily find a number k_0, open non-empty sets $W \subseteq U$, $Q \subseteq G_0$, and a dense (in $W \times Q$) set $P \subseteq W \times Q$ of class $\mathbf{G}_\delta$ such that $\vartheta_k(x)\, \mathsf{F}\, \vartheta_k(g \cdot x)$ for all $k \geqslant k_0$ and all pairs $\langle x, g \rangle \in P$. We can assume that $\langle x, g \rangle \in P \implies x \in D$. Since Q is open, there is an element $g_0 \in Q$ and a neighbourhood $G \subseteq G_0$ of the element $1_{\mathbb{G}}$ with $G^{-1} = G$ such that $g_0 G \subseteq Q$.

Let $\mathfrak{M}$ be a model as in the proof of Lemma 9. It can be proved as in Claim 10 that if points $x, x' \in W$ are $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$, $k \geqslant k_0$ and $x \sim_G^W x'$, then $\vartheta_k(x)\, \mathsf{F}_k\, \vartheta_k(x')$. In other words, each function ϑ_k with $k \geqslant k_0$ is generically $(\sim_G^W, \mathsf{F}_k)$-invariant on W. By Lemma 8, we can assume that the orbits $\mathcal{O}(x, W, G)$ are dense in W for x belonging to a comeagre subset of W. Then the hereditarily generic F_k-ergodicity implies that all maps ϑ_k with $k \geqslant k_0$ are generically F_k-constant on W, as required. This proves the lemma.

It is clear that if W is chosen as in Lemma 11, then ϑ is generically F-constant on W. It remains to show that these constants are F-equivalent to each other. Suppose, on the contrary, that there are two non-empty open sets $W_1, W_2 \subseteq X_0$ and a pair of points $y \not\mathsf{F}\, y'$ in $\mathbb{Y}$ such that $\vartheta(x)\, \mathsf{F}\, y$ and $\vartheta(x')\, \mathsf{F}\, y'$ for x belonging to a comeagre subset of W_1 and x' belonging to a comeagre subset of W_2. A contradiction can be derived by the same method as at the end of § 2.3.

2.5. Other inductive steps. We consider the operations (e1), (e2), (e3) of § 1.2.

To treat countable unions, we suppose that $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3, \dots$ are Borel ERs on a Polish space $\mathbb{Y}$, $\mathsf{F} = \bigcup_k \mathsf{F}_k$ is an ER, and the Polish generically turbulent action of $\mathbb{G}$ on $\mathbb{X}$ is hereditarily generically F_k-ergodic for every k. We claim that the action remains hereditarily generically F-ergodic.

Indeed, we fix a non-empty open set $X_0 \subseteq \mathbb{X}$ and a neighbourhood G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$ such that the orbits $\mathcal{O}(x, X_0, G_0)$ are dense in U_0 for x belonging to a comeagre subset of X_0. We consider a $(\sim_{G_0}^{X_0}, \mathsf{F})$-invariant Baire measurable function $\vartheta \colon X_0 \to \mathbb{Y}$ which is continuous on a dense $\mathbf{G}_\delta$-set $D \subseteq X_0$. Since ϑ is invariant, it follows that for every open non-empty set $U \subseteq X_0$ there are open non-empty sets $W \subseteq U$, $Q \subseteq G_0$ and a number k such that $\vartheta(x)\, \mathsf{F}_k\, \vartheta(g \cdot x)$ for any $(\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}})$-generic (over $\mathfrak{M}$) pair $\langle x, g \rangle \in W \times Q$. We can find an element $g_0 \in Q \cap \mathfrak{M}$ and a neighbourhood

$G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $g_0 G \subseteq Q$. As in the proof of Claim 10, we have $\vartheta(x) \mathsf{F}_k \vartheta(x')$ for any pair of $\mathbf{C}_{\mathbb{X}}$-generic (over $\mathfrak{M}$) elements $x, x' \in W$ that satisfy $x \sim_G^W x'$. Then the ergodicity implies that ϑ is generically F_k-constant and, therefore, generically F-constant on W. As at the end of § 2.3, one can demonstrate that these F-constants are F-equivalent to each other. The operation of countable intersection is treated similarly.

We demonstrated in § 1.2 that the operation of countable product is reducible to the Fubini product. However, there is a simple independent argument. If F_k are ERs on spaces $\mathbb{Y}_k$, then $\mathsf{F} = \prod_k \mathsf{F}_k$ is an ER on the space $\mathbb{Y} = \prod_k \mathbb{Y}_k$. Let E be any ER on $\mathbb{X}$. A map $\vartheta \colon \mathbb{X} \to \mathbb{Y}$ is (E, F)-invariant if and only if each coordinate map $\vartheta_k(x) = \vartheta(x)(k)$ is $(\mathsf{E}, \mathsf{F}_k)$-invariant. This immediately yields the required result.

The operation of disjoint union is reducible to the product (see § 1.2).

Thus Theorem 6 and Theorem 1 are proved.

§ 3. Applications

This section contains two applications of Theorem 6. One of them is Theorem 2. The other gives rather simple arguments to show that Theorem 2 implies the theorem of Hjorth mentioned in the introduction: "turbulent" ERs are not Borel reducible to Polish actions of S_∞.

3.1. Proof of Theorem 2. We fix a non-trivial Borel P-ideal $\mathcal{Z} \subseteq \mathcal{P}(\mathbb{N})$ as in Theorem 2. By a theorem of Solecki (see § 1.3) there is a l. s. c. submeasure φ on $\mathbb{N}$ such that $\mathcal{Z} = \{x \subseteq \mathbb{N} \colon \varphi_\infty(x) = 0\}$. We put $r_k = \varphi(\{k\})$.

Lemma 12 [11]. *Suppose that* $\mathcal{Z}$ *is not equal to* Fin, *is not a trivial variation of* Fin, *and is not isomorphic to* $\mathcal{I}_3 = 0 \times \mathrm{Fin}$. *Then there is a set* $W \notin \mathcal{Z}$ *such that* $\{r_k\}_{k \in W} \to 0$.

Proof. We put $U_n = \bigl\{k \colon r_k \leqslant \frac{1}{n}\bigr\}$ and, separately, $U_0 = \mathbb{N}$. Then $U_{n+1} \subseteq U_n$ for all n. We claim that $\inf_{m \in \mathbb{N}} \varphi(U_m) > 0$. For otherwise, a set $x \subseteq \mathbb{N}$ belongs to $\mathcal{Z}$ if and only if $x \setminus U_n$ is finite for every n. If the set $N = \{n \colon U_n \setminus U_{n+1}$ is infinite$\}$ is empty, then we easily see that $\mathcal{Z} = \mathcal{P}(\mathbb{N})$. If $N \neq \varnothing$ is finite, then $\mathcal{Z}$ is either Fin (when $U_n = \varnothing$ for almost all n) or a trivial variation of Fin (when U_n is non-empty for all n). Finally, if N is infinite, then $\mathcal{Z}$ is isomorphic to $0 \times \mathrm{Fin}$. (For instance, suppose that all sets $D_n = U_n \setminus U_{n+1}$ are infinite. In this case, $x \in \mathcal{Z}$ if and only if $x \cap D_n$ is finite for all n.) Thus we always get a contradiction to the hypotheses of the lemma.

Hence there is $\varepsilon > 0$ such that $\varphi(U_m) > \varepsilon$ for all m. Since φ is l. s. c., we can define an increasing sequence of numbers $n_1 < n_2 < n_3 < \cdots$ and, for every l, a finite set $w_l \subseteq U_{n_l} \setminus U_{n_{l+1}}$ such that $\varphi(w_l) > \varepsilon$. Then $W = \bigcup_l w_l \notin \mathcal{Z}$ and, obviously, $\{r_k\}_{k \in W} \to 0$. The lemma is proved.

Since $\mathsf{E}_{\mathcal{Z} \restriction W} \leqslant_{\mathrm{B}} \mathsf{E}_{\mathcal{Z}}$, the following lemma is now sufficient for Theorem 2.

Lemma 13. *Suppose that* $\mathcal{Z}$, φ, r_k *are as above, and* $\{r_k\} \to 0$. *Then the shift action of* $\mathcal{Z}$ *on* $\mathcal{P}(\mathbb{N})$ *is generically turbulent.*

Proof. $\mathcal{Z}$ is a Polish group (with group operation Δ) in the topology τ induced by the metric $r(x, y) = \varphi(x \Delta y)$. The action of $\mathcal{Z}$ by means of Δ on the space $\mathcal{P}(\mathbb{N})$

(which is here identified with $2^{\mathbb{N}}$ and is endowed with the product topology) is continuous. It remains to verify the turbulence.

Let $x \in \mathcal{P}(\mathbb{N})$. It is easy to see that the orbit $[x]_{\mathcal{Z}} = \mathcal{Z}\Delta x$ is dense and meagre. Hence it suffices to verify that x is a turbulent point. We consider an open set $X \subseteq \mathcal{P}(\mathbb{N})$ containing x, and let G be a τ-neighbourhood of $\varnothing$ (the neutral element of the group $\mathcal{Z}$). We may assume that $X = \{y \in \mathcal{P}(\mathbb{N})\colon y \cap [0,k) = u\}$ for some k, where $u = x \cap [0,k)$, and $G = \{g \in \mathcal{Z}\colon \varphi(g) < \varepsilon\}$ for some $\varepsilon > 0$. We claim that the local orbit $\mathcal{O}(x, X, G)$ is somewhere dense (that is, not nowhere dense) in X.

Indeed, let $l \geqslant k$ be big enough to ensure that $r_n < \varepsilon$ for all $n \geqslant l$. We put $v = x \cap [0,l)$ and claim that $\mathcal{O}(x, X, G)$ is dense in $Y = \{y \in \mathcal{P}(\mathbb{N})\colon y \cap [0,l) = v\}$. Consider an open set $Z = \{z \in Y\colon z \cap [l,j) = w\}$, where $j \geqslant l$, $w \subseteq [l,j)$. Let z be the unique element of Z with $z \cap [j,+\infty) = x \cap [j,+\infty)$. Then $x\Delta z = \{l_1, \dots, l_m\} \subseteq [l,j)$. Each $g_i = \{l_i\}$ belongs to G by the choice of l (indeed, $l_i \geqslant l$). Moreover, the element $x_i = g_i \Delta g_{i-1} \Delta \dots \Delta g_1 \Delta x = \{l_1, \dots, l_i\}\Delta x$ belongs to X for every $i = 1, \dots, m$, and $x_m = z$. Thus $z \in \mathcal{O}(x, X, G)$, as required.

This proves the lemma and Theorem 2.

3.2. Irreducibility to actions of the group of all permutations of ℕ. We recall that S_∞ is the group of all permutations of $\mathbb{N}$ (that is, one-to-one maps of $\mathbb{N}$ onto $\mathbb{N}$) with superposition as the group operation. A compatible Polish metric on S_∞ can be defined by $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, where d is the usual Polish metric on $\mathbb{N}^{\mathbb{N}}$, that is, $d(x,y) = 2^{-m-1}$, where m is the smallest number with $x(m) \neq y(m)$.

Hjorth proved in 1995 that turbulent ERs are not Borel reducible to ERs induced by Polish actions of S_∞. The proof (as, for example, in [6], [13]) is quite complicated. In particular, it contains references to some model theoretic facts and methods such as Scott's analysis. We include a simplified proof based on the following theorem. The argument will still be lengthy because we outline the proofs of some auxiliary results in order to make the exposition accessible to a reader not experienced in special topics related to group actions and model theory.

Theorem 14. *Let* E *be an ER induced by a Polish action. Suppose that* E *is reducible by a Baire measurable map to an ER induced by a Polish action of* S_∞. *Then* E *is also reducible to one of the ERs* T_γ *by a Baire measurable map.*[13] *Hence, by Theorem* 1, *such an ER* E *cannot be induced by a generically turbulent Polish action.*

3.3. Classifiability by countable structures. Isomorphism relations of various classes of countable structures are amongst those induced by Polish actions of S_∞. Indeed, suppose that $\mathcal{L} = \{R_i\}_{i \in I}$ is a countable relational language, that is, $\operatorname{card} I \leqslant \aleph_0$, and each R_i is an m_i-ary relational symbol. Put[14] $\mathrm{Mod}_{\mathcal{L}} = \prod_{i \in I} \mathcal{P}(\mathbb{N}^{m_i})$. This is the space of $\mathcal{L}$-*structures* on the underlying set $\mathbb{N}$. The *logic action* $j_{\mathcal{L}}$ of S_∞ on $\mathrm{Mod}_{\mathcal{L}}$ is defined as follows. If $x = \{x_i\}_{i \in I} \in \mathrm{Mod}_{\mathcal{L}}$ and $g \in S_\infty$,

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[13] We cannot claim Borel reducibility because any ER Borel reducible to T_γ is Borel itself (since all the ERs T_γ are Borel) and, on the other hand, even ERs of the form $\cong_{\mathcal{L}}$ are generally non-Borel (although they are analytic).

[14] $X_{\mathcal{L}}$ is often used to denote $\mathrm{Mod}_{\mathcal{L}}$.

then $y = j_{\mathcal{L}}(g, x) = g \cdot x = \{y_i\}_{i \in I} \in \mathrm{Mod}_{\mathcal{L}}$, where we have

$$\langle k_1, \ldots, k_{m_i} \rangle \in x_i \iff \langle g(k_1), \ldots, g(k_{m_i}) \rangle \in y_i$$

for all $i \in I$ and $\langle k_1, \ldots, k_{m_i} \rangle \in \mathbb{N}^{m_i}$. Then $\langle \mathrm{Mod}_{\mathcal{L}}; j_{\mathcal{L}} \rangle$ is a Polish S_∞-space. The $j_{\mathcal{L}}$-orbits in $\mathrm{Mod}_{\mathcal{L}}$ are exactly the isomorphism classes of $\mathcal{L}$-structures, which is the reason for denoting the associated equivalence relation $\mathsf{E}_{j_{\mathcal{L}}}^{\mathrm{Mod}_{\mathcal{L}}}$ by $\cong_{\mathcal{L}}$. Of course, all ERs of the form $\cong_{\mathcal{L}}$ are analytic.

Hjorth ([6], § 2.38) defines an ER E to be *classifiable by countable structures* if there is a countable relational language $\mathcal{L}$ such that $\mathsf{E} \leqslant_{\mathrm{B}} \cong_{\mathcal{L}}$.

Theorem 15 [1]. *Any ER induced by a Polish action of S_∞ is classifiable by countable structures.*

Thus all ERs induced by Polish actions of S_∞ (even of any closed subgroup of S_∞) are Borel reducible to very special actions of S_∞.

Proof ([6], § 6.19). Consider a Polish S_∞-space $\mathbb{X}$ with a basis $\{U_l\}_{l \in \mathbb{N}}$, and a language $\mathcal{L}$ with relations R_{lk} of arity k. For any $x \in \mathbb{X}$ we define $\vartheta(x) \in \mathrm{Mod}_{\mathcal{L}}$ by putting $\vartheta(x) \models R_{lk}(s_0, \ldots, s_{k-1})$ if and only if $s_i \neq s_j$ whenever $i < j < k$, and $g^{-1} \cdot x \in U_l$ whenever $g \in S_\infty$ satisfies $\langle s_0, \ldots, s_{k-1} \rangle \subset g$. Then ϑ reduces $\mathsf{E}_{S_\infty}^{\mathbb{X}}$ to $\cong_{\mathcal{L}}$. The theorem is proved.

3.4. Reduction to countable graphs. It might be expected that more complicated languages $\mathcal{L}$ produce more complicated ERs $\cong_{\mathcal{L}}$. However, this is not the case: it turns out that a single binary relation can code structures of any countable language. Let $\mathcal{G}$ be the language of (oriented binary) graphs, that is, $\mathcal{G}$ contains a single binary predicate, say $R(\cdot, \cdot)$.

Theorem 16. *If $\mathcal{L}$ is a countable relational language, then $\cong_{\mathcal{L}} \leqslant_{\mathrm{B}} \cong_{\mathcal{G}}$.*

In contrast to the following argument, Becker and Kechris ([1], § 6.1.4) outline a proof based on coding in terms of lattices, although the idea may actually be the same.

Proof. Let $\mathrm{HF}(\mathbb{N})$ be the set of all hereditarily finite sets over the set $\mathbb{N}$ regarded as the set of atoms, and let ε be the associated "membership" (no $n \in \mathbb{N}$ has ε-elements, $\{0,1\}$ is different from 2, and so on). Let $\simeq_{\mathrm{HF}(\mathbb{N})}$ denote the $\mathrm{HF}(\mathbb{N})$-version of $\cong_{\mathcal{G}}$. In other words, if $P, Q \subseteq \mathrm{HF}(\mathbb{N})^2$, then $P \simeq_{\mathrm{HF}(\mathbb{N})} Q$ means that there is a bijection b of the set $\mathrm{HF}(\mathbb{N})$ onto itself such that $Q = b \cdot P = \{\langle b(s), b(t) \rangle : \langle s, t \rangle \in P\}$. Obviously, $(\cong_{\mathcal{G}}) \sim_{\mathrm{B}} (\simeq_{\mathrm{HF}(\mathbb{N})})$. Thus it remains to prove that $\cong_{\mathcal{L}} \leqslant_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any language $\mathcal{L}$.

We define an action $\circ$ of S_∞ on $\mathrm{HF}(\mathbb{N})$ as follows: $g \circ n = g(n)$ for $n \in \mathbb{N}$ and, by ε-induction, $g \circ (\{a_1, \ldots, a_n\}) = \{g \circ a_1, \ldots, g \circ a_n\}$ for all $a_1, \ldots, a_n \in \mathrm{HF}(\mathbb{N})$. If $g \in S_\infty$, then $a \mapsto g \circ a$ is an ε-isomorphism of $\mathrm{HF}(\mathbb{N})$.

Lemma 17. *Suppose that $X, Y \subseteq \mathrm{HF}(\mathbb{N})$ are ε-transitive subsets of $\mathrm{HF}(\mathbb{N})$, the sets $\mathbb{N} \setminus X$ and $\mathbb{N} \setminus Y$ are infinite, and $\varepsilon \restriction X \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$. Then there is a permutation $f \in S_\infty$ such that $Y = f \circ X = \{f \circ s : s \in X\}$.*

Proof. The hypothesis $\varepsilon \restriction X \cong_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$ implies that there is an ε-isomorphism $\pi : X \overset{\text{onto}}{\to} Y$. It is easy to see that $\pi \restriction (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto

$Y_0 = Y \cap \mathbb{N}$. Hence there is $f \in S_\infty$ such that $f \restriction X_0 = \pi \restriction X_0$, and then we have $f \circ s = \pi(s)$ for any $s \in X$. The lemma is proved.

Returning to the proof of Theorem 16, we first claim that $\cong_{\mathcal{G}(m)} \leqslant_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $m \geqslant 3$, where $\mathcal{G}(m)$ is the language with a single m-ary predicate. Indeed, we observe that $\langle i_1, \dots, i_m\rangle \in \mathrm{HF}(\mathbb{N})$ whenever $i_1, \dots, i_m \in \mathbb{N}$. We put $\Theta(x) = \{\vartheta(s) \colon s \in x\}$ for every $x \in \mathrm{Mod}_{\mathcal{G}(m)} = \mathcal{P}(\mathbb{N}^m)$, where $\vartheta(s) = \mathrm{TC}_\varepsilon(\{\langle 2i_1, \dots, 2i_m\rangle\})$ for each $s = \langle i_1, \dots, i_m\rangle \in \mathbb{N}^m$ and, finally, if $X \subseteq \mathrm{HF}(\mathbb{N})$, then $\mathrm{TC}_\varepsilon(X)$ is the smallest ε-transitive set $T \subseteq \mathrm{HF}(\mathbb{N})$ with $X \subseteq T$. It follows easily from Lemma 17 that $x \cong_{\mathcal{G}(m)} y$ is equivalent to $\varepsilon \restriction \Theta(x) \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction \Theta(y)$. This completes the proof that $\cong_{\mathcal{G}(m)} \leqslant_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathcal{L}'} \leqslant_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$, where $\mathcal{L}'$ is the language with infinitely many binary predicates. In this case $\mathrm{Mod}_{\mathcal{L}'} = \mathcal{P}(\mathbb{N}^2)^{\mathbb{N}}$, so we can assume that every $x \in \mathrm{Mod}_{\mathcal{L}'}$ is given by $x = \{x_n\}_{n \geqslant 1}$ with $x_n \subseteq (\mathbb{N} \setminus \{0\})^2$ for all n. We put $\Theta(x) = \{s_n(k,l) \colon n \geqslant 1 \wedge \langle k,l\rangle \in x_n\}$ for any such x, where $s_n(k,l) = \mathrm{TC}_\varepsilon\bigl(\{\{\dots\{\langle k,l\rangle\}\dots\}, 0\}\bigr)$ with $n+2$ pairs of braces $\{\,,\,\}$. Then Θ is a continuous reduction of $\cong_{\mathcal{L}'}$ to $\simeq_{\mathrm{HF}(\mathbb{N})}$. The theorem is proved.

3.5. Proof of Theorem 14. The proof (a version of the argument in [4]) is based on Scott's analysis.

We define a family $\equiv^\alpha_{st}$ of Borel binary relations on $\mathcal{P}(\mathbb{N}^2)$, where $\alpha < \omega_1$ and $s, t \in \mathbb{N}^{<\omega}$, as follows:

(i) $A \equiv^0_{st} B$ if and only if $A(s_i, s_j) \iff B(t_i, t_j)$ for all $i, j < \operatorname{lh} s = \operatorname{lh} t$;

(ii) $A \equiv^{\alpha+1}_{st} B$ if and only if $\forall\, k \exists\, l (A \equiv^\alpha_{s^\wedge k, t^\wedge l} B)$ and $\forall\, l \exists\, k (A \equiv^\alpha_{s^\wedge k, t^\wedge l} B)$;

(iii) if $\lambda < \omega_1$ is limit, then $A \equiv^\lambda_{st} B$ means that $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$.

By definition, we put $\langle s, A\rangle \equiv^\alpha \langle t, B\rangle$ if and only if $A \equiv^\alpha_{st} B$. Then an induction over α shows that each $\equiv^\alpha$ is a Borel ER on $\mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2)$ and $\equiv^\beta \subseteq \equiv^\alpha$ for $\alpha < \beta$.

Consider an ER $\mathsf{E} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ induced by a Polish action of a Polish group $\mathbb{G}$ on a Polish space $\mathbb{X}$ which is reducible to a Polish action of S_∞ by a Baire measurable map. According to Theorems 15, 16 and Proposition 3 there is a Baire measurable reduction $\vartheta \colon \mathbb{X} \to \mathcal{P}(\mathbb{N}^2)$ of E to $\cong_{\mathcal{G}}$. This reduction is continuous on a dense $\mathbf{G}_\delta$-set $D_0 \subseteq \mathbb{X}$. We recall that the relation $A \cong_{\mathcal{G}} B$ for $A, B \subseteq \mathbb{N}^2$ means that there is a function $f \in S_\infty$ such that $A(k,l) \iff B\bigl(f(k), f(l)\bigr)$ for all k, l. We easily prove by induction on α that $\cong_{\mathcal{G}} \subseteq \equiv^\alpha_{st}$, where $t = f \circ s$. In particular, $\cong_{\mathcal{G}} \subseteq \equiv^\alpha_{\Lambda\Lambda}$, where Λ is the empty sequence. Since ϑ is a reduction, the equivalence $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \cong_{\mathcal{G}} \vartheta(y)$ holds for all x, y. Our goal is to find a dense $\mathbf{G}_\delta$-set $D \subseteq D_0$ and an ordinal $\alpha < \omega_1$ such that

(∗) the implication $x \mathrel{\not\mathsf{E}} y \implies \vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ folds for all $x, y \in D$.

To find D, we fix a countable transitive model $\mathfrak{M}$ of ZFHC (see above). We assume that $\mathbb{X}$, the group $\mathbb{G}$, its action on $\mathbb{X}$, the set D_0, and the function $\vartheta \restriction D_0$ are coded in $\mathfrak{M}$ in the same sense as in the proof of Lemma 9. We claim that the set D of all Cohen generic (over $\mathfrak{M}$) points of $\mathbb{X}$ (a dense $\mathbf{G}_\delta$-subset of $\mathbb{X}$ included in D_0) satisfies (∗).

Indeed, take $x, y \in D$. We first consider the case when $\langle x, y\rangle$ is a Cohen generic pair over $\mathfrak{M}$. If $x \mathrel{\not\mathsf{E}} y$, then $\vartheta(x) \ncong_{\mathcal{G}} \vartheta(y)$ by the choice of ϑ. Hence the Mostowski absoluteness theorem yields that $\vartheta(x) \ncong_{\mathcal{G}} \vartheta(y)$ in $\mathfrak{M}[x, y]$. Therefore, arguing relatively to the model $\mathfrak{M}[x, y]$ (which is still a model of ZFHC, see § 2.3), we find an ordinal $\alpha \in \mathrm{Ord}^{\mathfrak{M}} = \mathrm{Ord}^{\mathfrak{M}[x,y]}$ with $\vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$. Moreover, since the Cohen forcing satisfies CCC (the countable antichain condition), there is an ordinal $\alpha \in \mathfrak{M}$ such that $\vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ for all Cohen generic (over $\mathfrak{M}$) pairs $\langle x, y\rangle \in D^2$ with $x \not\mathsf{E}\, y$. It remains to show that this also holds when $x, y \in D$ satisfy $x \not\mathsf{E}\, y$ but do not form a Cohen generic pair.

Consider a Cohen generic (over $\mathfrak{M}[x,y]$) element $g \in \mathbb{G}$. We easily see that the point $z = g \cdot x \in \mathbb{X}$ is Cohen generic over $\mathfrak{M}[x,y]$ (because the action is continuous). Moreover, $x \mathsf{E}\, z$, whence $y \not\mathsf{E}\, z$. However y is Cohen generic over $\mathfrak{M}$, and z is generic over $\mathfrak{M}[y]$. Therefore the pair $\langle y, z\rangle$ is Cohen generic over $\mathfrak{M}$, and we get $\vartheta(z) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ by the above. On the other hand, $\vartheta(x) \equiv^{\alpha}_{\Lambda\Lambda} \vartheta(z)$ holds because $x \mathsf{E}\, z$. We finally obtain $\vartheta(x) \not\equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$, as required by $(*)$.

To conclude, we have $x \mathsf{E}\, y \iff \vartheta(x) \equiv^{\alpha}_{\Lambda\Lambda} \vartheta(y)$ for all $x, y \in D$. In this case we can easily redefine ϑ on the complement of D in $\mathbb{X}$ in such a way that the equivalence holds for all $x, y \in \mathbb{X}$. In other words, the improved ϑ is a Baire measurable (because $\vartheta \upharpoonright D$ is continuous and D is a dense $\mathbf{G}_\delta$-set) reduction of E to $\equiv^{\alpha}_{\Lambda\Lambda}$.

The following result completes the proof of the theorem.

Proposition 18. *Every ER $\equiv^{\alpha}$ is Borel reducible to some T_γ.*

Proof. We have $\equiv^0 \leqslant_{\mathrm{B}} \mathsf{T}_0$ since $\equiv^0$ has countably many equivalence classes, all of which are open-and-closed sets. To carry out the step $\alpha \mapsto \alpha+1$, we note that the map $\langle s, A\rangle \mapsto \{\langle s^\wedge k, A\rangle\}_{k\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^{\alpha})^\infty$. As for the limit step, consider a limit ordinal $\lambda = \{\alpha_n : n \in \mathbb{N}\}$ and put $\mathsf{R} = \bigvee_{n\in\mathbb{N}} \equiv^{\alpha_n}$. Hence R is the ER on $\mathbb{N} \times \mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2)$ defined as follows: $\langle m, s, A\rangle \mathsf{R} \langle n, t, B\rangle$ if and only if $m = n$ and $A \equiv^{\alpha_m}_{st} B$. Then the map $\langle s, A\rangle \mapsto \{\langle m, s, A\rangle\}_{m\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\lambda}$ to R^∞.

§ 4. Pinned ERs and the irreducibility of T_2

This section contains a theorem showing that the ER T_2 (equality of countable sets of reals) is not Borel reducible to ERs belonging to the family of *pinned* ERs. This family includes, for instance, continuous actions of CLI groups, some ideals (not necessarily Polishable) and ERs having $\mathbf{G}_{\delta\sigma}$ equivalence classes, and is closed under the Fubini product modulo Fin. The definition of this family is based on a rather metamathematical property extracted from Hjorth's paper [5].

4.1. Pinned ERs. First of all, if X is an analytic set in the universe $\mathbb{V}$ of all sets (in particular, this applies when X is Borel), and $\mathbb{V}^+$ is a generic extension of the universe $\mathbb{V}$, then $X^\sharp$ will denote the result of the sequence of operations contained in the definition of X but applied in $\mathbb{V}^+$. This is well defined by the Schoenfield absoluteness theorem, and we easily see that $X = X^\sharp \cap \mathbb{V}$.

For instance, if E is an analytic ER on a Polish space $\mathbb{X}$ in the universe $\mathbb{V}$, then $\mathsf{E}^\sharp$ is an analytic ER on $\mathbb{X}^\sharp$ by the Schoenfield absoluteness theorem. If $x \in \mathbb{X}$ (hence $x \in \mathbb{V}$), then the E-class $[x]_{\mathsf{E}} \subseteq \mathbb{X}$ of x (defined in $\mathbb{V}$) is included in a unique $\mathsf{E}^\sharp$-class $[x]_{\mathsf{E}^\sharp} \subseteq \mathbb{X}^\sharp$ (in $\mathbb{V}^+$). The classes $[x]_{\mathsf{E}^\sharp}$ with $x \in \mathbb{X}$ belong to a wider category of $\mathsf{E}^\sharp$-classes, which admit a description from the point of view of the universe $\mathbb{V}$.

Definition 19. Assume that $\mathbb{P}$ is a notion of forcing in $\mathbb{V}$. A *virtual* E-*class* is any $\mathbb{P}$-term $\boldsymbol{\xi}$ such that $\mathbb{P}$ forces $\boldsymbol{\xi} \in \mathbb{X}^\sharp$ and $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\text{left}} \mathsf{E}^\sharp \boldsymbol{\xi}_{\text{right}}$.[15] A virtual class is *pinned* if there is a point $x \in \mathbb{X}$ (in $\mathbb{V}$) which pins it in the sense that $\mathbb{P}$ forces $x \mathsf{E}^\sharp \boldsymbol{\xi}$. Finally, an analytic ER E is *pinned* if, for any forcing notion $\mathbb{P} \in \mathbb{V}$, all virtual E-classes are pinned.

Let $\boldsymbol{\xi}$ be a virtual E-class, and let $\mathbb{V}^+$ be any extension of the universe $\mathbb{V}$. If U and V are generic subsets of $\mathbb{P}$, then $x = \boldsymbol{\xi}[U]$ and $y = \boldsymbol{\xi}[V]$ belong to $\mathbb{X}^\sharp$ and satisfy $x \,\mathsf{E}^\sharp\, y$. Hence $\boldsymbol{\xi}$ induces an $\mathsf{E}^\sharp$-class in the extension. If $\boldsymbol{\xi}$ is pinned, then this class contains an element in the ground universe $\mathbb{V}$. In other words, pinned virtual classes induce $\mathsf{E}^\sharp$-equivalence classes of the form $[x]_{\mathsf{E}^\sharp}$, $x \in \mathbb{V}$, in extensions of the universe $\mathbb{V}$.

We prove below that T_2 is not pinned. Moreover, T_2 is not Borel reducible to any pinned analytic ER. We also give a simplified proof of Hjorth's theorem that continuous actions of Polish CLI groups never induce pinned ERs, introduce a family of pinned ERs associated with $\mathbf{F}_{\sigma\delta}$ ideals, show that a Borel ER is pinned if all its equivalence classes are $\mathbf{G}_{\delta\sigma}$, and prove that the class of all pinned analytic ERs is closed under the Fubini product modulo Fin.

4.2. Pinned ERs and T_2. We recall that, modulo $\sim_{\text{B}}$, the relation T_2 is the ER on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $x \,\mathsf{T}_2\, y$ if and only if $\operatorname{ran} x = \operatorname{ran} y$.

Lemma 20. *T_2 is not pinned. If E, F are analytic ERs with $\mathsf{E} \leqslant_{\text{B}} \mathsf{F}$ and F is pinned, then E is also pinned. Hence T_2 is not Borel reducible to a pinned analytic ER.*

Proof. To prove that T_2 is not pinned, we consider, in $\mathbb{V}$, the forcing notion $\mathbb{P} = \operatorname{Coll}(\mathbb{N}, 2^{\mathbb{N}})$, which induces a generic map $f \colon \mathbb{N} \overset{\text{onto}}{\to} 2^{\mathbb{N}}$. ($\mathbb{P}$ consists of all functions $p \colon u \to 2^{\mathbb{N}}$, where $u \subseteq \mathbb{N}$ is finite.) The $\mathbb{P}$-term $\boldsymbol{\xi}$ for the set $\operatorname{ran} f = \{f(n) \colon n \in \mathbb{N}\}$ is a virtual T_2-class, but it is not pinned because $2^{\mathbb{N}}$ is uncountable in the ground universe $\mathbb{V}$.

Suppose that $\vartheta \colon \mathbb{X} \to \mathbb{Y}$ is a Borel reduction of E to F in $\mathbb{V}$, where $\mathbb{X} = \operatorname{dom} \mathsf{E}$ and $\mathbb{Y} = \operatorname{dom} \mathsf{F}$. We can assume that $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$. Let $\mathbb{P}$ be a forcing notion, and let $\boldsymbol{\xi}$ be a $\mathbb{P}$-term which is a virtual E-class. By the Schoenfield absoluteness theorem, $\vartheta^\sharp$ is a reduction of $\mathsf{E}^\sharp$ to $\mathsf{F}^\sharp$ in any extension of $\mathbb{V}$. Hence the $\mathbb{P}$-term $\boldsymbol{\sigma}$ for $\vartheta^\sharp(\boldsymbol{\xi})$ is a virtual F-class. Since F is pinned, there is $y \in \mathbb{Y}$ such that $\mathbb{P}$ forces $y \,\mathsf{F}^\sharp\, \boldsymbol{\sigma}$. Note that it is true in the $\mathbb{P}$-extension that $y \,\mathsf{F}^\sharp\, \vartheta^\sharp(x)$ for some $x \in \mathbb{X}^\sharp$. Hence, by Schoenfield's theorem, in the ground universe there is $x \in \mathbb{X}$ such that $y \,\mathsf{F}\, \vartheta(x)$. Clearly, $\mathbb{P}$ forces $x \,\mathsf{E}^\sharp\, \boldsymbol{\xi}$. The lemma is proved.

4.3. The Fubini product of pinned ERs. We recall that the Fubini product $\mathsf{E} = \prod_{k \in \mathbb{N}} \mathsf{E}_k / \operatorname{Fin}$ of ERs E_k on $\mathbb{X}_k$ modulo Fin is the ER on $\mathbb{X} = \prod_k \mathbb{X}_k$ defined as follows: $x \,\mathsf{E}\, y$ if and only if $x(k) \,\mathsf{E}_k\, y(k)$ for all but finitely many indices k.

Lemma 21. *The family of all analytic pinned ERs is closed under Fubini products modulo* Fin.

[15] $\boldsymbol{\xi}_{\text{left}}$ and $\boldsymbol{\xi}_{\text{right}}$ are $(\mathbb{P} \times \mathbb{P})$-terms meaning that $\boldsymbol{\xi}$ is associated respectively with the left and right factor $\mathbb{P}$ in the product forcing. Formally, $\boldsymbol{\xi}_{\text{left}}[U \times V] = \boldsymbol{\xi}[U]$ and $\boldsymbol{\xi}_{\text{right}}[U \times V] = \boldsymbol{\xi}[V]$ for any $(\mathbb{P} \times \mathbb{P})$-generic set $U \times V$, where $\boldsymbol{\xi}[U]$ is the interpretation of $\boldsymbol{\xi}$ via a generic set U.

Proof. Let E_k be pinned analytic ERs on Polish spaces $\mathbb{X}_k$. We claim that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k/\operatorname{Fin}$ is a pinned ER on $\mathbb{X} = \prod_k \mathbb{X}_k$. Consider a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-term $\boldsymbol{\xi}$ which is a virtual E-class. There is a number k_0 and "conditions" $p, q \in \mathbb{P}$ such that $\langle p, q\rangle$ $(\mathbb{P}\times\mathbb{P})$-forces $\boldsymbol{\xi}_{\mathrm{left}}(k)\,\mathsf{E}_k^\sharp\,\boldsymbol{\xi}_{\mathrm{right}}(k)$ for all $k \geqslant k_0$. As all E_k are ERs, we conclude that the "condition" $\langle p, p\rangle$ also forces $\boldsymbol{\xi}_{\mathrm{left}}(k)\,\mathsf{E}_k^\sharp\,\boldsymbol{\xi}_{\mathrm{right}}(k)$ for all $k \geqslant k_0$. Therefore, since the E_k are pinned, there is (in $\mathbb{V}$) a sequence of points $x_k \in \mathbb{X}_k$ such that p $\mathbb{P}$-forces $x_k\,\mathsf{E}_k^\sharp\,\boldsymbol{\xi}(k)$ for all $k \geqslant k_0$. Let $x \in \mathbb{X}$ be such that $x(k) = x_k$ for all $k \geqslant k_0$. (The values $x(k) \in \mathbb{X}_k$ with $k < k_0$ can be arbitrary.) Then p obviously $\mathbb{P}$-forces $x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}$.

It remains to show that each $q \in \mathbb{P}$ forces $x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}$. Assume the opposite: some $q \in \mathbb{P}$ forces that $x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}$ fails. Consider the pair $\langle p, q\rangle$ as a "condition" in $\mathbb{P}\times\mathbb{P}$. It forces $x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}_{\mathrm{left}}$ and $\neg x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}_{\mathrm{right}}$ as well as $\boldsymbol{\xi}_{\mathrm{left}}\,\mathsf{E}^\sharp\,\boldsymbol{\xi}_{\mathrm{right}}$ by the choice of E and $\boldsymbol{\xi}$. This contradiction proves the lemma.

4.4. Left-invariant actions and pinned ERs. We recall that a Polish group $\mathbb{G}$ is *complete left-invariant* (CLI, for brevity) if $\mathbb{G}$ admits a compatible left-invariant complete metric. Then $\mathbb{G}$ also admits a compatible right-invariant complete metric, and this will be used in what follows.

Theorem 22 [5]. *Let* $\mathsf{E} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ *be an ER induced by a Polish action of a* CLI *group* $\mathbb{G}$ *on a Polish space* $\mathbb{X}$. *Then* E *is pinned. Hence* T_2 *is not Borel reducible to* E.

Proof. Let $\mathbb{P}$ be a forcing notion, and let $\boldsymbol{\xi}$ be a virtual E-class. We denote by $\leqslant$ the partial order on $\mathbb{P}$. As usual, $p \leqslant q$ means that p is a stronger condition. We fix a compatible complete right-invariant metric ρ on $\mathbb{G}$. For every $\varepsilon > 0$ put $G_\varepsilon = \{g \in \mathbb{G} \colon \rho(g, 1_{\mathbb{G}}) < \varepsilon\}$. We say that $q \in \mathbb{P}$ *is of size* $\leqslant \varepsilon$ if $\langle q, q\rangle$ $(\mathbb{P}\times\mathbb{P})$-forces the existence of $g \in G_\varepsilon^\sharp$ such that $\boldsymbol{\xi}_{\mathrm{left}} = g\cdot\boldsymbol{\xi}_{\mathrm{right}}$.

Lemma 23. *If* $q \in \mathbb{P}$ *and* $\varepsilon > 0$, *then there is a condition* $r \in \mathbb{P}$ *of size* $\leqslant \varepsilon$ *such that* $r \leqslant q$.

Proof. Otherwise, for every $r \in \mathbb{P}$ with $r \leqslant q$ there is a pair of conditions $r', r'' \in \mathbb{P}$ stronger than r and such that $\langle r', r''\rangle$ $(\mathbb{P}\times\mathbb{P})$-forces that there is no $g \in G_\varepsilon^\sharp$ with $\boldsymbol{\xi}_{\mathrm{left}} = g\cdot\boldsymbol{\xi}_{\mathrm{right}}$. Let $\mathbb{V}^+$ be a generic extension of $\mathbb{V}$ in which $\mathcal{P}(\mathbb{P})\cap\mathbb{V}$ is countable. Applying the usual splitting construction in $\mathbb{V}^+$, we find an uncountable set $\mathcal{U}$ of generic sets $U \subseteq \mathbb{P}$ with $q \in U$ such that all pairs $\langle U, V\rangle$ with $U \neq V$ in $\mathcal{U}$ are $(\mathbb{P}\times\mathbb{P})$-generic (over $\mathbb{V}$). Hence there is no $g \in G_\varepsilon^\sharp$ with $\boldsymbol{\xi}[U] = g\cdot\boldsymbol{\xi}[V]$.[16]

Fix $U_0 \in \mathcal{U}$. For every $U \in \mathcal{U}$ we can take (in $\mathbb{V}^+$) an element $g_U \in G^\sharp$ such that $\boldsymbol{\xi}[U] = g_U\cdot\boldsymbol{\xi}[U_0]$. Then $g_U \notin G_\varepsilon^\sharp$ by the above. Moreover, we have $g_V g_U^{-1}\cdot\boldsymbol{\xi}[U] = \boldsymbol{\xi}[V]$ for all $U, V \in \mathcal{U}$, whence $g_V g_U^{-1} \notin G_\varepsilon^\sharp$ whenever $U \neq V$. It follows that $\rho(g_U, g_V) \geqslant \varepsilon$ since the metric is right-invariant. But this contradicts the separability of G. The lemma is proved.

We return to the proof of Theorem 22. Suppose, on the contrary, that a condition $p \in \mathbb{P}$ forces that there is no $x \in \mathbb{X}$ (in the ground universe $\mathbb{V}$) satisfying $x\,\mathsf{E}^\sharp\,\boldsymbol{\xi}$. By Lemma 23 one can find (in $\mathbb{V}$) a sequence of conditions $p_n \in \mathbb{P}$ of size $\leqslant 2^{-n}$ and closed sets $X_n \subseteq \mathbb{X}$ of $\mathbb{X}$-diameter $\leqslant 2^{-n}$ such that $p_0 \leqslant p$, $p_{n+1} \leqslant p_n$, $X_{n+1} \subseteq X_n$,

[16] $\boldsymbol{\xi}[U]$ is the interpretation of the $\mathbb{P}$-term $\boldsymbol{\xi}$ obtained by taking U as the generic set.

and p_n forces $\boldsymbol{\xi} \in X_n^\sharp$ for any n. Let x be the common point of the sets X_n in $\mathbb{V}$. We claim that p_0 forces $x \mathsf{E}^\sharp \boldsymbol{\xi}$.

For otherwise, there is $q \in \mathbb{P}$ such that $q \leqslant p_0$ and q forces $\neg x \mathsf{E}^\sharp \boldsymbol{\xi}$. Consider an extension $\mathbb{V}^+$ of $\mathbb{V}$ rich enough to contain, for any n, a generic set $U_n \subseteq \mathbb{P}$ with $p_n \in U_n$ such that each pair $\langle U_n, U_{n+1}\rangle$ is $(\mathbb{P} \times \mathbb{P})$-generic (over $\mathbb{V}$) and, in addition, $q \in U_0$. Put $x_n = \boldsymbol{\xi}[U_n]$ (an element of $\mathbb{X}^\sharp$). Then $\{x_n\} \to x$. Moreover, U_n and U_{n+1} contain p_n for each n. Since p_n has size $\leqslant 2^{-n-1}$, there is $g_{n+1} \in \mathbb{G}_\varepsilon^\sharp$ with $x_{n+1} = g_{n+1}x_n$. Thus $x_n = h_n \cdot x_0$, where $h_n = g_n \dots g_1$. However $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \leqslant 2^{-n+1}$ by the right-invariance of the metric ρ. Hence $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^\sharp$. We denote its limit by $h = \lim_{n\to\infty} h_n \in \mathbb{G}^\sharp$. Since the action is continuous, we have $x = \lim_n x_n = h \cdot x_0$. It follows that $x \mathsf{E}^\sharp x_0$ holds in $\mathbb{V}^+$, hence also in $\mathbb{V}[U_0]$. However $x_0 = \boldsymbol{\xi}[U_0]$, while $q \in U_0$ forces $\neg x \mathsf{E}^\sharp \boldsymbol{\xi}$, a contradiction.

Thus p_0 $\mathbb{P}$-forces $x \mathsf{E}^\sharp \boldsymbol{\xi}$. Then any $r \in \mathbb{P}$ also forces $x \mathsf{E}^\sharp \boldsymbol{\xi}$. Indeed, if some $r \in \mathbb{P}$ forces $\neg x \mathsf{E}^\sharp \boldsymbol{\xi}$, then the pair $\langle p_0, r\rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $x \mathsf{E}^\sharp \boldsymbol{\xi}_{\text{left}}$ and $\neg x \mathsf{E}^\sharp \boldsymbol{\xi}_{\text{right}}$, which contradicts the fact that $(\mathbb{P} \times \mathbb{P})$ forces $\boldsymbol{\xi}_{\text{left}} \mathsf{E}^\sharp \boldsymbol{\xi}_{\text{right}}$. This proves Theorem 22.

4.5. ERs with $\mathbf{G}_{\delta\sigma}$-classes. We have a non-pinned ER T_2, obviously of class $\mathbf{F}_{\sigma\delta}$. The following theorem shows that this is the simplest possible case of a non-pinned ER.

Theorem 24. *Let* E *be a Borel ER all of whose equivalence classes are* $\mathbf{G}_{\delta\sigma}$*-sets. Then* E *is pinned.*

Proof (based on an idea communicated by Hjorth). We can assume that $\operatorname{dom}\mathsf{E} = \mathbb{N}^\mathbb{N}$. It follows from a theorem of Louveau [14] that there is a Borel map γ defined on $\mathbb{N}^\mathbb{N}$ in such a way that $\gamma(x)$ is a $\mathbf{G}_{\delta\sigma}$-code of $[x]_\mathsf{E}$ for each $x \in \mathbb{N}^\mathbb{N}$, that is, for instance, $\gamma(x) \subseteq \mathbb{N}^2 \times \mathbb{N}^{<\omega}$ and

$$[x]_\mathsf{E} = \bigcup_i \bigcap_j \bigcup_{\langle i,j,s\rangle \in\gamma(x)} B_s, \quad \text{where} \quad B_s = \{a \in \mathbb{N}^\mathbb{N} : s \subset a\} \quad \text{for all} \quad s \in \mathbb{N}^{<\omega}.$$

We consider a forcing notion $\mathbb{P} = \langle \mathbb{P}; \leqslant\rangle$ and a virtual E-class $\boldsymbol{\xi}$. Then $\mathbb{P} \times \mathbb{P}$ forces $\boldsymbol{\xi}_{\text{left}} \mathsf{E}^\sharp \boldsymbol{\xi}_{\text{right}}$. Hence there is a number i_0 and a condition $\langle p_0, q_0\rangle \in \mathbb{P} \times \mathbb{P}$ which forces $\boldsymbol{\xi}_{\text{left}} \in \vartheta^\sharp(\boldsymbol{\xi}_{\text{right}})$, where $\vartheta(x) = \bigcap_j \bigcup_{\langle i_0,j,s\rangle\in\gamma(x)} B_s$ for all $x \in \mathbb{N}^\mathbb{N}$.

The key idea of the proof is to replace $\mathbb{P}$ by the Cohen forcing. Let $\mathbb{S}$ be the set of all $s \in \mathbb{N}^{<\omega}$ such that p_0 does not $\mathbb{P}$-force $s \not\subset \boldsymbol{\xi}$. We regard $\mathbb{S}$ as a forcing, and $s \subseteq t$ (that is, t is an extension of s) means that t is a stronger condition. The empty sequence Λ is the weakest condition in $\mathbb{S}$. If $s \in \mathbb{S}$, then we obviously have at least one n such that $s^\wedge n \in \mathbb{S}$. Hence $\mathbb{S}$ forces an element of $\mathbb{N}^\mathbb{N}$, whose $\mathbb{S}$-name will be $\mathbf{a}$.

Lemma 25. *The pair* $\langle \Lambda, q_0\rangle$ $(\mathbb{S} \times \mathbb{P})$*-forces* $\mathbf{a} \in \vartheta^\sharp(\boldsymbol{\xi})$.

Proof. Otherwise, some condition $\langle s_0, q\rangle \in \mathbb{S} \times \mathbb{P}$ with $q \leqslant q_0$ forces $\mathbf{a} \notin \vartheta^\sharp(\boldsymbol{\xi})$. By the definition of ϑ we can assume that

$$\langle s_0, q\rangle \quad (\mathbb{S} \times \mathbb{P})\text{-forces} \qquad \neg\exists\, s\big(\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi}) \wedge s \subset \mathbf{a}\big) \tag{5}$$

for some j_0. Since $s_0 \in \mathbb{S}$, there is a condition $p' \in \mathbb{P}$ with $p' \leqslant p_0$ which $\mathbb{P}$-forces $s_0 \subset \boldsymbol{\xi}$. By the choice of $\langle p_0, q_0\rangle$ we can assume that

$$\langle p', q'\rangle \quad (\mathbb{P}\times\mathbb{P})\text{-forces} \qquad \langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi}_{\text{right}}) \wedge s \subset \boldsymbol{\xi}_{\text{left}}$$

for suitable $s \in \mathbb{S}$ and $q' \in \mathbb{P}$, $q' \leqslant q$. This means that
1) p' $\mathbb{P}$-forces $s \subset \boldsymbol{\xi}$;
2) q' $\mathbb{P}$-forces $\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi})$.

In particular, by the above, p' forces both $s_0 \subset \boldsymbol{\xi}$ and $s \subset \boldsymbol{\xi}$. Hence we have either $s \subseteq s_0$ (when we put $s' = s_0$), or $s_0 \subset s$ (when we put $s' = s$). In both cases $\langle s', q'\rangle$ $(\mathbb{S}\times\mathbb{P})$-forces $\langle i_0, j_0, s\rangle \in \gamma(\boldsymbol{\xi})$ and $s \subset \mathbf{a}$. This contradicts (5). The lemma is proved.

We note that $\mathbb{S}$ is a subforcing of the Cohen forcing $\mathbb{C} = \mathbb{N}^{<\omega}$. Hence, by Lemma 25, there is a $\mathbb{C}$-term $\boldsymbol{\sigma}$ such that $\langle \Lambda, q_0\rangle$ $(\mathbb{C}\times\mathbb{P})$-forces $\boldsymbol{\sigma} \in \vartheta^\sharp(\boldsymbol{\xi})$ and therefore forces $\boldsymbol{\sigma} \mathsf{E}^\sharp \boldsymbol{\xi}$. By considering the forcing $\mathbb{C}\times\mathbb{P}\times\mathbb{P}$, we see that $\mathbb{C}\times\mathbb{P}$ forces $\boldsymbol{\sigma} \mathsf{E}^\sharp \boldsymbol{\xi}$. It follows that, first, $\mathbb{C}\times\mathbb{C}$ forces $\boldsymbol{\sigma}_{\text{left}} \mathsf{E}^\sharp \boldsymbol{\sigma}_{\text{right}}$ and, second, to prove the theorem it suffices to find $x \in \mathbb{N}^\mathbb{N}$ in $\mathbb{V}$ such that $\mathbb{C}$ forces $x \mathsf{E}^\sharp \boldsymbol{\sigma}$. This is our next goal.

Let $\mathbf{a}$ be a $\mathbb{C}$-name of the Cohen generic element of $\mathbb{N}^\mathbb{N}$. The term $\boldsymbol{\sigma}$ can be of a complicated nature, but it can be replaced by a term of the form $f^\sharp(\mathbf{a})$, where $f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a Borel map in the ground universe $\mathbb{V}$. It follows from the above that $f^\sharp(\mathbf{a}) \mathsf{E}^\sharp f^\sharp(\mathbf{b})$ for any $(\mathbb{C}\times\mathbb{C})$-generic (over $\mathbb{V}$) pair $\langle \mathbf{a}, \mathbf{b}\rangle \in \mathbb{N}^\mathbb{N}\times\mathbb{N}^\mathbb{N}$. We conclude that $f^\sharp(\mathbf{a}) \mathsf{E}^\sharp f^\sharp(\mathbf{b})$ for any pair of separately Cohen generic elements $\mathbf{a}, \mathbf{b} \in \mathbb{N}^\mathbb{N}$. Thus, in a generic extension of $\mathbb{V}$ where there are comeagre-many Cohen generic reals, there is a comeagre $\mathbf{G}_\delta$-set $X \subseteq \mathbb{N}^\mathbb{N}$ such that $f^\sharp(a) \mathsf{E}^\sharp f^\sharp(b)$ for all $a, b \in X$. By the Schoenfield absoluteness theorem, the statement on the existence of such a set X is true in $\mathbb{V}$ as well. Hence, in $\mathbb{V}$, there is a point $x \in \mathbb{N}^\mathbb{N}$ such that we have $x \mathsf{E} f(a)$ for a comeagre set of points $a \in \mathbb{N}^\mathbb{N}$. This is again a Schoenfield absolute property of x. Hence $\mathbb{C}$ forces $x \mathsf{E}^\sharp f^\sharp(\mathbf{a})$, as required.

Theorem 24 is proved.

4.6. A family of pinned ideals. A Borel ideal $\mathcal{I}$ is said to be *pinned* if the induced ER $\mathsf{E}_\mathcal{I}$ is pinned. Theorem 22 implies that all P-ideals are pinned because all Borel P-ideals are Polishable [15] while all Polish Abelian groups are CLI. Yet there are non-Polishable pinned ideals.

We introduce a family of such ideals here. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of lower semicontinuous (l. s. c.) submeasures on $\mathbb{N}$. We define the exhaustive ideal of the sequence of submeasures by setting

$$\operatorname{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} \colon \varphi_\infty(X) = 0\}, \quad \text{where} \quad \varphi_\infty(X) = \limsup_{i\to\infty} \varphi_i(X).$$

By Solecki's theorem [15], for any Borel P-ideal $\mathcal{I}$ there is a l. s. c. submeasure φ such that $\mathcal{I} = \operatorname{Exh}_{\{\varphi_i\}} = \operatorname{Exh}_\varphi$, where $\varphi_i(x) = \varphi(x \cap [i, \infty))$. However, for example, the non-Polishable ideal $\mathcal{I}_1 = \mathrm{Fin}\times 0$ is also of the form $\operatorname{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define $\varphi_i(x) = 0$ or $\varphi_i(x) = 1$ if $x \subseteq \{0, \dots, n-1\}\times\mathbb{N}$ or $x \nsubseteq \{0, \dots, n-1\}\times\mathbb{N}$, respectively.

Theorem 26. *All ideals of the form* $\mathrm{Exh}_{\{\varphi_i\}}$ *are pinned.*

Proof. Put $\mathcal{I} = \mathrm{Exh}_{\{\varphi_i\}}$, where all φ_i are l. s. c. submeasures on $\mathbb{N}$. We can assume that the submeasures φ_i decrease, that is, $\varphi_{i+1}(x) \leqslant \varphi_i(x)$ for each x. For otherwise, we can consider the l. s. c. submeasures $\varphi'_i(x) = \sup_{j \geqslant i} \varphi_j(x)$.

Suppose that $\mathsf{E} = \mathsf{E}_{\mathcal{I}}$ is not pinned. Then we have a forcing notion $\mathbb{P}$, a virtual E-class $\boldsymbol{\xi}$ and a condition $p \in \mathbb{P}$ which $\mathbb{P}$-forces $\neg x \,\mathsf{E}^\sharp\, \boldsymbol{\xi}$ for any $x \in \mathcal{P}(\mathbb{N})$ in $\mathbb{V}$. By definition, for any $p' \in \mathbb{P}$ and $n \in \mathbb{N}$ we can find $i \geqslant n$ and conditions $q, r \in \mathbb{P}$ with $q, r \leqslant p'$ such that $\langle q, r\rangle$ $(\mathbb{P} \times \mathbb{P})$-forces the inequality $\varphi_i(\boldsymbol{\xi}_{\mathrm{left}} \Delta \boldsymbol{\xi}_{\mathrm{right}}) \leqslant 2^{-n-1}$. Hence $\langle q, q\rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $\varphi_i(\boldsymbol{\xi}_{\mathrm{left}} \Delta \boldsymbol{\xi}_{\mathrm{right}}) \leqslant 2^{-n}$. It follows that, in $\mathbb{V}$, one can find a sequence of numbers $i_0 < i_1 < i_2 < \cdots$, a sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \cdots$ of conditions in $\mathbb{P}$ and a set $u_n \subseteq [0, n)$ for every n, such that $p_0 \leqslant p$ and

1) each p_n $\mathbb{P}$-forces $\boldsymbol{\xi} \cap [0, n) = u_n$;

2) each $\langle p_n, p_n\rangle$ $(\mathbb{P} \times \mathbb{P})$-forces $\varphi_{i_n}(\boldsymbol{\xi}_{\mathrm{left}} \Delta \boldsymbol{\xi}_{\mathrm{right}}) \leqslant 2^{-n}$.

Arguing in the universe $\mathbb{V}$, we put $a = \bigcup_n u_n$. Then $a \cap [0, n) = u_n$ for all n. We claim that p_0 forces $a \,\mathsf{E}^\sharp\, \boldsymbol{\xi}$, which will contradict the assumption above, thus proving the theorem.

If not, there is a condition $q_0 \leqslant p_0$ that forces $\neg a \,\mathsf{E}^\sharp\, \boldsymbol{\xi}$. Consider a generic extension $\mathbb{V}^+$ of the universe in which there is a sequence of $\mathbb{P}$-generic sets $U_n \subseteq \mathbb{P}$ such that, for every n, the pair $\langle U_n, U_{n+1}\rangle$ is $(\mathbb{P} \times \mathbb{P})$-generic over $\mathbb{V}$, $p_n \in U_n$ and, in addition, $q_0 \in U_0$. Then, in $\mathbb{V}^+$, the sets $x_n = \boldsymbol{\xi}[U_n] \in \mathcal{P}(\mathbb{N})$ satisfy $\varphi_{i_n}(x_n \Delta x_m) \leqslant 2^{-n}$ whenever $n \leqslant m$, by 2). It follows that $\varphi_{i_n}(x_n \Delta a) \leqslant 2^{-n}$ because $a = \lim_m x_m$, by 1). However, we have assumed that the submeasures φ_j decrease, whence $\varphi_\infty(x_n \Delta a) \leqslant 2^{-n}$. On the other hand, $\varphi_\infty(x_n \Delta x_0) = 0$ because $\boldsymbol{\xi}$ is a virtual E-class. We conclude that $\varphi_\infty(x_0 \Delta a) \leqslant 2^{-n}$ for any n. In other words, $\varphi_\infty(x_0 \Delta a) = 0$, that is, $x_0 \,\mathsf{E}^\sharp\, a$. This contradicts the choice of U_0 because $x_0 = \boldsymbol{\xi}[U_0]$ and $q_0 \in U_0$.

Question 1. Are all Borel ideals pinned? The expected answer "yes" would show that T_2 is not Borel reducible to any Borel ideal. Is any ER induced by a Borel action of a Borel CLI group pinned?

Question 2 (Kechris). Is there a $\leqslant_{\mathrm{B}}$-least non-pinned Borel ER? It was once expected that T_2 is such a one, but Hjorth has informed us that there is a strictly $\leqslant_{\mathrm{B}}$-smaller non-pinned Borel ER of a rather complicated nature.

We are grateful to Greg Hjorth, A. S. Kechris, and Su Gao for useful discussions related to the content of this paper. We are also grateful to Greg Hjorth for his kind permission to include one of his unpublished results (Theorem 24).

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Received 30/JUL/01
Translated by V. G. KANOVEI

Typeset by $\mathcal{AMS}$-TEX