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Non-uniformizable sets of second projective level with countable cross-sections in the form of Vitali classes

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Abstract. We use a countable-support product of invariant Jensen's forcing notions to define a model of **ZFC** set theory in which the uniformization principle fails for some planar $\mathit{\Pi}^1_2$ set all of whose vertical cross-sections are countable sets and, more specifically, Vitali classes. We also define a submodel of that model, in which there exists a countable $\mathit{\Pi}^1_2$ sequence of Vitali classes P_n whose union $\bigcup_n P_n$ is not a countable set. Of course, the axiom of choice fails in this submodel.

Keywords: uniformization, forcing, Vitali classes.

§ 1. Main results

The *Vitali class* of a real number x is the set $x+\mathbb{Q}=\{x+q\colon q\in\mathbb{Q}\}$, that is, the equivalence class of x in the sense of the Vitali equivalence relation: $x\equiv y$ if and only if the difference $x-y$ is rational.[1] Every Vitali class is obviously a countable dense subset of the real line $\mathbb{R}$. Our first main theorem continues the series of investigations into the uniformization problem in modern descriptive set theory. It introduces an example of an effectively non-uniformizable $\mathit{\Pi}^1_2$ set all of whose vertical cross-sections are countable sets and even Vitali classes, which are the most elementary type of sets admitting such a result.

Theorem 1.1. *There exists a model of* **ZFC** *set theory in which it is true that there is a planar* $\mathit{\Pi}^1_2$ *set* $P\subseteq\mathbb{R}\times\mathbb{R}$ *such that all vertical cross-sections* $P_x=\{y\colon\langle x,y\rangle\in P\}$ *of* P *are Vitali classes and* P *cannot be uniformized by a* ROD *set.*

By ROD (real-ordinal definable) we mean the class of all sets definable by formulae with real numbers and ordinals as parameters. This is the largest class of sets that can be called *effectively definable* in the most general sense.

Our second main theorem provides an example of a countable $\mathit{\Pi}^1_2$ sequence of Vitali classes whose union is not a countable set. The model considered in this theorem is a *symmetric submodel* of the model in Theorem 1.1, and of course it violates the axiom of choice.

[1] As usual, $\omega\subset\mathbb{Q}\subset\mathbb{R}$ are the natural, rational and real numbers.

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Theorem 1.2. *There exists a model of* **ZF** *set theory in which it is true that there is a set $P \subseteq \omega \times \mathbb{R}$ of class $\mathit{\Pi}^1_2$ such that*

(i) *all vertical cross-sections $P_n = \{x : \langle n, x\rangle \in P\}$ are Vitali classes*;

(ii) *the union $\bigcup_n P_n$ is not a countable set or*, *equivalently*, *P cannot be uniformized by any set.*[2]

§ 2. Comments

This section contains some necessary definitions and comments on Theorems 1.1 and 1.2 along with a short introduction to the structure of the paper.

2.1. The problem of uniformization. This problem was introduced into descriptive set theory by Luzin[3] in a short note [1] and a more detailed paper [5]. According to Luzin, a planar set Q in the real number plane $\mathbb{R} \times \mathbb{R}$ is said to be *uniform* (or *single-valued*) if it intersects every vertical straight line in at most one point. This basically means that Q is the graph of a partial function $\mathbb{R} \to \mathbb{R}$. If $Q \subseteq P \subseteq \mathbb{R} \times \mathbb{R}$ for some uniform set Q whose projection to the first axis is equal to that of P, then Luzin says that the set Q *uniformizes* the set P. In other words, uniformizing a given planar set P means choosing a point q_x in every non-empty vertical cross-section P_x of P and then gathering all the chosen points q_x or, more precisely, all the pairs $\langle x, q_x\rangle$ into a single uniformizing set $Q \subseteq P$.

Clearly, the axiom of choice enables one to obtain a uniformizing subset $Q \subseteq P$ in every planar set P. But, according to Luzin, *the Hadamard problem* consists in the following question:[4] *is it possible or not to define a point set E for which we cannot name any uniformizing set E'.*

2.2. Effectively definable sets. In modern set-theoretic terminology, the words 'define' and 'name' in this citation express the existence of effectively defined sets introduced by a concrete definition or construction, as opposed to sets obtained by pure existence proofs, for example, using the axiom of choice. In these terms, the uniformization problem, in its most general form, is *to determine whether effectively definable planar sets can be uniformized by their still effectively definable* (*and uniform*) *subsets.*

As mentioned above, the largest class of effectively definable sets in modern set theory is the class ROD of real-ordinal definable sets. This consists of all

[2]To explain the equivalence in (ii), note the following. Any uniformizing set $Q \subseteq P$ provides a function $f : \omega \to \mathbb{R}$ picking an element $f(n) \in P_n$ in every cross-section P_n. Then the union $\bigcup_n P_n = \{f(n) + q : n \in \omega \wedge q \in \mathbb{Q}\}$ is a countable set. Conversely, if $U = \bigcup_n P_n$ is countable, that is, there is a function $g : \omega \xrightarrow{\text{onto}} U$, then we obtain a uniformizing set $Q \subseteq P$ by choosing an element $p_n = g(k_n)$ in every cross-section P_n, where k_n is the least index k satisfying $g(k) \in P_n$.

[3]The note [1] was not published among Luzin's papers on descriptive set theory in Volume II of his collected works [2]. However its main elements were considered, partially translated, and analyzed in detail by Uspensky in his remarkable survey [3] (§ 4, entitled 'Uniformization. The Hadamard problem'). In [1], Luzin gives a rather long citation from Hadamard's first letter in the well-known 'Five letters' [4], which can be understood to mean that Hadamard makes distinction between a pure Zermelo-style choice and a choice of elements in non-empty sets by means of a concrete effectively defined function, but considers both possibilities to be mathematically sound. This gave Luzin occasion to connect the uniformization problem with the name of Hadamard. Uspensky argues in [3], § 4, that the role of Hadamard is definitely exaggerated here, while the priority with regard to the uniformization problem and related notions belongs to Luzin himself.

[4]The italic text is quoted from [3], p. 120.

sets definable by a formula with real numbers and ordinals as parameters of the definition.

The class ROD contains the subclass OD $\subseteq$ ROD of all *ordinal-definable* sets, that is, sets definable by a formula with ordinals (but not real numbers) as parameters.

There are more special subclasses of ROD and OD: the *projective classes* $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$, and the corresponding *effective projective classes* Σ^1_n, Π^1_n and $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$; here $n \geqslant 1$. See [6], as well as [7]–[11], for more details on the projective hierarchy. Recall that at the level $n = 1$ we have $\mathbf{\Delta}^1_1$ = Borel sets, $\mathbf{\Sigma}^1_1$ = Suslin sets, or A-sets, and $\mathbf{\Pi}^1_1$ = co-Suslin sets, or CA-sets.

2.3. Classical uniformization theorems. The first results related to uniformization were obtained before 1930 by Luzin in collaboration with P. S. Novikov and Sierpiński. Presented in [5], [6], they consist in the following:

(I) every planar $\mathbf{\Sigma}^1_1$ set can be uniformized by a Borel combination of $\mathbf{\Sigma}^1_1$ sets and complementary $\mathbf{\Pi}^1_1$ sets;

(II) there exists a planar Borel set (that is, a $\mathbf{\Delta}^1_1$ set) non-uniformizable by Borel sets or even by sets in the wider class $\mathbf{\Sigma}^1_1$;

(III) every Borel set is uniformizable by a $\mathbf{\Pi}^1_1$ set;

(IV) every Borel set P is uniformizable by a Borel set provided that all vertical cross-sections of P are at most countable; the same holds for $\mathbf{\Sigma}^1_1$ sets.

The next stage of complexity is the class $\mathbf{\Pi}^1_1$ of all co-Suslin sets. The uniformization problem for $\mathbf{\Pi}^1_1$ sets remained open for some time. Luzin initially even considered it as an impossible task (see [1], § 4). It was solved by the Japanese mathematician Kondô [12], who established that

(V) every planar $\mathbf{\Pi}^1_1$ set can be uniformized by a $\mathbf{\Pi}^1_1$ set.

The key ingredient in Kondô's proof was Novikov's method (introduced in [13]) of an effective choice of a point in a non-empty $\mathbf{\Pi}^1_1$ set. Finally, a fairly easy consequence of (V) is the following item (VI):

(VI) every planar $\mathbf{\Sigma}^1_2$ set is uniformizable by a $\mathbf{\Sigma}^1_2$ set, but there is a $\mathbf{\Pi}^1_2$ set non-uniformizable by $\mathbf{\Pi}^1_2$ sets.

2.4. Uniformization in models of set theory. Independence. The results mentioned in § 2.3 left open the problem of the uniformization of $\mathbf{\Pi}^1_2$ sets by sets more complex than $\mathbf{\Pi}^1_2$. Subsequent studies showed that this problem cannot be solved in the usual sense of the *proof* of the possibility or impossibility of uniformizing all $\mathbf{\Pi}^1_2$ sets by effectively definable sets. Namely, set theorists were able to define *models* of axioms of the Zermelo–Fraenkel theory[5] **ZFC** in which ROD-uniformization of $\mathbf{\Pi}^1_2$ holds as well as ones in which it fails.

[5] The theory **ZFC**, including the axiom of choice, is regarded as the axiomatic base of contemporary set theory. The existence of a model of **ZFC** in which a certain sentence A is true means that A does not contradict the **ZFC** axioms (in other words, A is compatible with **ZFC**) and hence one cannot prove *the negation* $\neg A$ in set theory. If in addition there exists another model of **ZFC** in which $\neg A$ holds, then $\neg A$ does not contradict the **ZFC** axioms and thus A is *independent of* **ZFC** or *undecidable in* **ZFC**, that is, A cannot be neither proved nor disproved in set theory. This is the case for the continuum hypothesis [14] (the most famous instance of undecidability).

In the direction of uniformization, Gödel [15] defined a model of the **ZFC** axioms (the class $\mathbf{L}$ of *constructible sets*) in which the real line admits a Δ^1_2 well-ordering. Therefore, by picking the least element, in the sense of this well-ordering, in every vertical cross-section of a given $\mathbf{\Pi}^1_2$ set, we obtain the following assertion:

(VII) in Gödel's model $\mathbf{L}$, every planar $\mathbf{\Pi}^1_2$ set can be uniformized by a $\mathbf{\Delta}^1_3$ set.

Some other uniformization theorems for $\mathbf{\Pi}^1_2$ have been established in different models of set theory, including the Martin–Solovay–Mansfield theorem of uniformization of $\mathbf{\Pi}^1_2$ sets by $\mathbf{\Pi}^1_3$ sets ([6], § 8H.10) and uniformization theorems under the hypothesis of projective determinacy ([6], § 6C). See [16] for more recent results.

In the direction of non-uniformization we mention a result of [17], Theorem 3:

(VIII) it is compatible with **ZFC** that the set

$$P = \{\langle x, y\rangle \in \mathbb{R}^2 \colon y \notin \mathbf{L}[x]\}$$

(of class Π^1_2) is not uniformizable by a ROD set.

The class $\mathbf{L}[x]$ contains all sets constructible *relative* to a given x. Note that every vertical cross-section

$$P_x = \{y \in \mathbb{R} \colon y \notin \mathbf{L}[x]\}$$

of this set P is either empty (provided that $\mathbb{R} \subseteq \mathbf{L}[x]$) or uncountable, that is, it can never be a non-empty finite or countable set. (Incidentally, in the Solovay model [18] all cross-sections P_x are co-countable.)

2.5. Non-uniformization of $\mathbf{\Pi}^1_2$ sets with countable cross-sections. Analyzing (VIII) in view of the role of countability of cross-sections (this role can be seen by comparing (II) and (IV)), we arrive at the following natural problem.

Problem 1. Does there exist a model of the axioms of **ZFC** in which there is a ROD-non-uniformizable $\mathbf{\Pi}^1_2$ set with *countable* vertical cross-sections?

Note that (VIII) does not solve this problem since P has uncountable cross-sections. Problem 1 is obviously connected with another question once discussed at length in the well-known mathematical internet communities Mathoverflow[6] and FOM.[7]

Problem 2. Does there exist a model of **ZFC** in which there is a *countable* definable set of reals $X \neq \varnothing$ containing no OD (ordinal-definable) elements?

We solved this problem in [19]. It turns out that a generic extension of the Gödel universe $\mathbf{L}$ by means of the countable power $\boldsymbol{J}^\omega$ (with finite support) of *Jensen's minimal forcing* introduced in [20], is such a model. Here we denote Jensen's forcing by $\boldsymbol{J}$ for ease of reference. (See also § 28A in [21] about this forcing.) Using an uncountable product of forcing notions similar to $\boldsymbol{J}^\omega$, we also solved Problem 1 in the subsequent paper [22]. To be exact, in a suitable model there exists a ROD-non-uniformizable Π^1_2 set with countable cross-sections.

[6] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. `http://mathoverflow.net/questions/17608`.

[7] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. `http://cs.nyu.edu/pipermail/fom/2010-July/014944.html`

However, it was unknown whether the countable sets mentioned in Problems 1 and 2 can be Vitali classes. As far as Problem 2 is concerned, a positive answer was obtained in [23] by means of a model in which there exists *a definable Vitali equivalence class without definable elements*. This model was obtained with the help of a forcing notion $\boldsymbol{J}^{\mathrm{inv}}$, almost identical to Jensen's forcing $\boldsymbol{J}$, but different in that it is invariant under rational shifts and hence can appropriately be called *invariant Jensen forcing*. The invariance property implies that $\boldsymbol{J}^{\mathrm{inv}}$ adjoins an entire Vitali class of $\boldsymbol{J}^{\mathrm{inv}}$-generic reals rather than a single generic real, and such a Vitali class turns out to be a definable Π^1_2 set without definable elements. The same forcing notion was used in [24] to construct a model containing an OD Groszek–Laver pair of Vitali classes.

2.6. Why Vitali classes. Theorem 1.1, *the first main result* of this paper, solves Problem 1 in the affirmative, and in such a way that the vertical cross-sections of the counter-example set turn out to be Vitali classes rather than just arbitrary countable sets. To obtain such a model, we use a forcing notion of the form $\mathbb{P} = \prod_{\xi<\omega_1} P_\xi$ (see § 12), where every factor P_ξ is entirely similar to the forcing $\boldsymbol{J}^{\mathrm{inv}}$ in terms of its basic properties, but differs in the details of its construction, making all forcing notions P_ξ independent of each other in some sense. This extends the result (mentioned above) in [23] to uniformization problems.

We are especially interested in the Vitali classes in this context because they may be regarded as the simplest countable sets in $\mathbb{R}$ which do not immediately admit an effective choice of an element. Indeed, if a set $X \subseteq \mathbb{R}$ contains at least one isolated (or even one-sided isolated) point, then one such point can be chosen effectively. But a set $X \subseteq \mathbb{R}$ without even one-sided isolated points is just everywhere dense, if we leave aside closed intervals of the complementary set. And finally the Vitali classes, that is, shifts of the set of rational numbers $\mathbb{Q}$, are the most elementary and most typical countable dense sets in $\mathbb{R}$.[8]

2.7. On countable sequences of Vitali classes. ROD uniformization problems can be considered for sets $P \subseteq \omega \times \mathbb{R}$ too, but this case has its own specific nature.[9] Every such set P splits into a countable sequence of cross-sections $P_n = \{x : \langle n, x\rangle \in P\} \subseteq \mathbb{R}$. Accordingly, a uniformizing set turns out to be a countable sequence of reals, which necessarily belongs to ROD since it is effectively encoded by a real. Thus every set $P \subseteq \omega \times \mathbb{R}$ is ROD-uniformizable for basically trivial reasons.

This argument holds in **ZFC**, where the axiom of choice is powerful enough for the initial choice of a point in every non-empty cross-section P_n. The theory **ZF** does not contain the axiom of choice, and the argument above fails because of

[8] We may mention that the interest in the Vitali equivalence relation and its equivalence classes has deep roots in descriptive set theory. Already Sierpiński ([25], p. 147) and Luzin ([26], § 64) demonstrated that the set $\mathbb{R}/\mathbb{Q}$ of all Vitali classes has the property that (*) it cannot be mapped into $\mathbb{R}$ by means of a one-to-one Borel map. On the other hand, the Hartogs number of $\mathbb{R}/\mathbb{Q}$ in models without the axiom of choice can be much bigger than the continuum; see [27]. (The Hartogs number is equal to the least cardinal which does not admit a one-to-one map into a given set; see [28], Ch. 4.) Generally, the Vitali equivalence relation plays a key role in current studies of Borel equivalence relations since it is the least (in the sense of Borel reducibility [29]) such relation satsifying (*).

[9] We recall that $\omega = \{0, 1, 2, \dots\} \subset \mathbb{R}$ is the set of all natural numbers.

the following result in [17], Theorem 8:

(IX) it is compatible with the axioms of **ZF** that there exists a Π^1_2 set $P \subseteq \omega \times \mathbb{R}$ non-uniformizable by any set at all.

We can ask about countable vertical cross-sections in this case as well. This question is answered by Theorem 1.2, *the second main result* of this paper, and once again all vertical cross-sections of the counter-example turn out to be Vitali classes. Thus the hypothesis of the existence of a countable sequence of Vitali classes P_n whose union is not a countable set does not contradict the axioms of **ZF**.

By the way, the consistency (relative to **ZF**) of the existence of a countable sequence of countable sets with uncountable union belongs to the very first results obtained by the method of forcing [14], [17], but the result for the Vitali classes is obtained here for the first time. To comment on the difficulties of this special form, we note that the stronger classical result on the consistency of the assertion that the real line $\mathbb{R}$ itself is a union of countably many countable sets, definitely fails for the Vitali classes. Indeed, assume that $\mathbb{R} = \bigcup_{n\in\omega} P_n$, where every set P_n is a Vitali class. One easily defines a perfect set $X \subseteq \mathbb{R}$ such that the difference $x - y$ of any two reals $x \neq y$ in X is irrational, that is, in other words, the reals in X belong to different Vitali classes.[10]

But then the function $f(x)$ defined by

$$f(x) = \text{ the unique number } n \text{ satisfying } x \in P_n$$

is a bijection of the set X of the cardinality of the continuum onto a countable subset of the natural numbers, and this is obviously impossible.

2.8. Structure of the paper. The proofs of Theorems 1.1 and 1.2 are organized as follows. First, it should be mentioned that following the modern style in descriptive set theory based on certain technical advantages, we shall consider the Cantor discontinuum 2^ω with a special equivalence relation E_0 (see Definition 15.2), instead of the real line $\mathbb{R}$ with the Vitali equivalence relation, in a substantial part of the proof.[11]

It is in this context that Theorems 1.1 and 1.2 will be established in §§ 16 and 18. The original statements of the theorems (as in § 1) will be obtained in §§ 17 and 18 by elementary topological arguments.

Some notions related to perfect trees in the set of all dyadic strings $2^{<\omega}$ and their finite products (multitrees) are introduced in §§ 3–6. Every set P of perfect trees

[10] To define such a set X, we first prove that if $J_1, \dots, J_n$ is a finite set of pairwise-disjoint closed intervals in $\mathbb{R}$ of the form $[a, b]$, $a < b$, and q is a rational number, then there are smaller intervals $J'_k \subseteq J_k$ such that $(q + J'_k) \cap J'_l = \varnothing$ for all $k \neq l$. (This is clear for $n = 2$, and then we argue by induction.) Using this claim, we define a system of closed intervals J_s in $\mathbb{R}$, indexed by finite dyadic strings $s = \langle k_1, \dots, k_n\rangle$ of numbers $k_i = 0, 1$ of arbitrary length $n = \mathrm{lh}(s)$ (including the 'empty' string Λ of length $\mathrm{lh}(\Lambda) = 0$) and satisfying the following conditions: (1) if a string s is extended by a string t then $J_t \subseteq J_s$, (2) if $s \neq t$ are strings of equal length then $J_s \cap J_t = \varnothing$, and (3) if $s \neq t$ are strings of equal length $\mathrm{lh}(s) = \mathrm{lh}(t) = n$ then $(q_n + J_s) \cap J_t = \varnothing$, where $\mathbb{Q} = \{q_n : n \in \omega\}$ is a pre-defined enumeration of the rational numbers. When such a system is defined, the set $X = \bigcap_n \bigcup_{\mathrm{lh}(s)=n} J_s$ is as required.

[11] *Translator's note.* In keeping with widespread practice of modern descriptive set theoretic papers in English, points of 2^ω will be called *reals* below, except in §§ 17 and 18, where the true real numbers in the real line $\mathbb{R}$ will be considered.

$T \subseteq 2^{<\omega}$ closed under truncation of trees at strings may be regarded as a forcing notion adding a P-generic point $x \in 2^\omega$.

We argue in the Gödel constructible universe $\mathbf{L}$. The forcing notion for proving Theorem 1.1 is defined in § 12 as the product $\mathbb{P} = \prod_{\xi<\omega_1} P_\xi$ with finite support, where every factor P_ξ (similar to the forcing notion $\boldsymbol{J}^{\rm inv}$ in [23]; see § 2.6 above) is itself defined as a union of the form $P_\xi = \bigcup_{\alpha<\omega_1} Q_\xi^\alpha$. The terms Q_ξ^α in this common inductive construction are countable sets of perfect *Silver trees* in $\mathbf{L}$, pre-dense in P_ξ. Any $\mathbb{P}$-generic extension of $\mathbf{L}$ will serve as a required model for Theorem 1.1.

The main issue in the construction of the forcing notions P_ξ, quite similar to the definition of Jensen forcing, is to make every successor 'layer' Q_ξ^α *generic* in some sense over the already defined 'layers' Q_ξ^γ, $\gamma < \alpha$. This involves a fairly complicated construction in §§ 9–11 using the technique of splitting and fusion of perfect trees. Another aspect of the construction, which was invented in [23] and has no analogues in the Jensen forcing construction, is to make every 'layer' Q_ξ^α invariant under the same homeomorphism group (in fact, $2^{<\omega}$ with componentwise addition mod 2) that induces the relation E_0. This ensures that every forcing notion P_ξ adds an entire E_0-class of generic reals rather than a single real as in [20].

The main properties of the forcing notion $\mathbb{P} = \prod_{\xi<\omega_1} P_\xi$ and $\mathbb{P}$-generic extensions are established in §§ 12–15. The key property is that, whenever $G \subseteq \mathbb{P}$ is a generic filter, if $\xi < \omega_1$ then the real $x_\xi[G]$ and those reals $x \in 2^\omega$ that are E_0-equivalent to $x_\xi[G]$ are the only P_ξ-generic reals over $\mathbf{L}$ in $\mathbf{L}[G]$ (Lemma 15.3).

The concluding §§ 16–18 complete the proofs of the main theorems.

§ 3. Trees and splitting

By $2^{<\omega}$ we denote the set of all strings (finite sequences) of numbers 0, 1 (*dyadic strings*), including *the empty string* Λ. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\frown i$ is the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that the string t extends s (including the case $s = t$), while $s \subset t$ means proper extension. The length of a string s is denoted by $\operatorname{lh}(s)$. We define $2^n = \{s \in 2^{<\omega} \colon \operatorname{lh}(s) = n\}$ (strings of length n).

Every string $s \in 2^{<\omega}$ *acts* on 2^ω so that $(s\cdot x)(k) = x(k) + s(k) \pmod 2$ whenever $k < \operatorname{lh}(s)$, and just $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then we put $s \cdot X = \{s \cdot x \colon x \in X\}$.

If $s \in 2^m$, $t \in 2^n$ and $m \leqslant n$, then we similarly define a string $s \cdot t \in 2^n$ by $(s\cdot t)(k) = t(k) + s(k) \pmod 2$ when $k < m$ and $(s\cdot t)(k) = t(k)$ when $m \leqslant k < n$. But if $m > n$ then we put $s \cdot t = (s{\restriction}n) \cdot t$. In both cases, $\operatorname{lh}(s \cdot t) = \operatorname{lh}(t)$. If $T \subseteq 2^{<\omega}$ then we put $s \cdot T = \{s \cdot t \colon t \in T\}$.

A set $T \subseteq 2^{<\omega}$ is a *tree* if, for all strings $s \subset t$ in $2^{<\omega}$, $t \in T$ implies that $s \in T$. Every non-empty tree $T \subseteq 2^{<\omega}$ contains the empty string Λ. If $T \subseteq 2^{<\omega}$ is a tree and $u \in T$, then we define $T{\restriction}_u = \{t \in T \colon u \subseteq t \vee t \subseteq u\}$ (the *truncation of T at u*).

Lemma 3.1. *If $T \subseteq 2^{<\omega}$ is a tree, $\sigma \in 2^{<\omega}$ and $u \in T$, then $\sigma \cdot (T{\restriction}_u) = (\sigma \cdot T){\restriction}_{\sigma\cdot u}$.*

By $\mathbf{PT}$ we denote the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a non-empty tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ whenever it has no endpoints and no isolated branches. In this case there is a longest string $s \in T$ such that $T = T{\restriction}_s$; it is denoted by $s = \operatorname{stem}(T)$ (the *stem* of T). Then we have both $s^\frown 1 \in T$ and $s^\frown 0 \in T$.

If $T \in \mathbf{PT}$, then the set $[T] = \{a \in 2^\omega \colon \forall\, n(a{\restriction}n \in T)\}$ of all *branches of* T is a perfect set in 2^ω. Note that $[S] \cap [T] = \varnothing$ if and only if $S \cap T$ is finite.

Definition 3.2. A tree $T \in \mathbf{PT}$ is called a *Silver tree* and we write $T \in \mathbf{ST}$) if there is an infinite sequence of strings $u_k = u_k(T) \in 2^{<\omega}$ such that T consists of all strings of the form

$$s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \cdots {}^\frown u_n{}^\frown i_n$$

and their substrings, where $n < \omega$ and $i_k = 0, 1$.

Then $\operatorname{stem}(T) = u_0$ and $[T]$ consists of all sequences

$$a = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \cdots \in 2^\omega,$$

where $i_k = 0, 1\ \ \forall\, k$. We put

$$\operatorname{spl}_n(T) = \operatorname{lh}(u_0) + 1 + \operatorname{lh}(u_1) + 1 + \cdots + \operatorname{lh}(u_{n-1}) + 1 + \operatorname{lh}(u_n).$$

In particular, $\operatorname{spl}_0(T) = \operatorname{lh}(u_0)$. Hence $\operatorname{spl}(T) = \{\operatorname{spl}_n(T) \colon n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of T.

Example 3.3. If $s \in 2^{<\omega}$, then the tree $T[s] = \{t \in 2^{<\omega} \colon s \subseteq t \vee t \subset s\}$ belongs to $\mathbf{ST}$, $\operatorname{stem}(T[s]) = u_0(T[s]) = s$, and $u_k(T[s]) = \Lambda$ for all $k \geqslant 1$. We note that $T[\Lambda] = 2^{<\omega}$ and $T[\Lambda]{\restriction}_s = (2^{<\omega}){\restriction}_s = T[s]$ for all $s \in 2^{<\omega}$.

Lemma 3.4. *Suppose that $T \in \mathbf{ST}$. Then the following assertions hold.*

(i) *If $u \in T$, then $T{\restriction}_u \in \mathbf{ST}$.*

(ii) *If $s \in 2^{<\omega}$, then $s \cdot T \in \mathbf{ST}$, $\operatorname{spl}(T) = \operatorname{spl}(s \cdot T)$, $u_k(s \cdot T) = s \cdot u_k(T)$.*

(iii) *If a set $X \subseteq 2^\omega$ is open and $X \cap [T] \neq \varnothing$, then there is a string $s \in T$ such that $T{\restriction}_s \subseteq X$.*

(iv) *If $h \in \operatorname{spl}(T)$ and $u, v \in 2^h \cap T$, then $T{\restriction}_v = (u \cdot v) \cdot T{\restriction}_u$ and $(u \cdot v) \cdot T = T$.*

(v) *If $h \in \operatorname{spl}(T)$ and $u \in 2^h \cap T$, then $T \subseteq \bigcup_{t \in 2^h} (t \cdot T{\restriction}_u)$.*

Proof. (iii) Let $a \in X \cap [T]$. Then obviously $\{a\} = \bigcap_n T{\restriction}_{a{\restriction}n}$. It follows by compactness that $T{\restriction}_{a{\restriction}n} \subseteq X$ for some n. We put $s = a{\restriction}n$. The other claims are clear. □

§ 4. Splitting Silver trees

The simple splitting of a tree $T \in \mathbf{PT}$ consists of the subtrees

$$T(\to 0) = T{\restriction}_{\operatorname{stem}(T)^\frown 0} \quad \text{and} \quad T(\to 1) = T{\restriction}_{\operatorname{stem}(T)^\frown 1}\,,$$

so that $[T(\to i)] = \{x \in [T] \colon x(h) = i\}$, where $h = \operatorname{lh}(\operatorname{stem}(T))$. Clearly, $T(\to i) \in \mathbf{PT}$ and if $T \in \mathbf{ST}$, then $T(\to i) \in \mathbf{ST}$ as well, and

$$u_0(T(\to i)) = u_0(T)^\frown\langle i\rangle^\frown u_1(T), \qquad u_k(T(\to i)) = u_{k+1}(T) \quad \text{for all} \quad k \geqslant 1,$$

and $\operatorname{spl}(T(\to i)) = \operatorname{spl}(T) \setminus \{\operatorname{spl}_0(T)\}$.

Splitting can be iterated. That is, if $s \in 2^n$, then we define

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2)) \cdots (\to s(n-1)).$$

We separately put $T(\to \Lambda) = T$ for the empty string Λ.

Example 4.1. Under the assumptions of Example 3.3 we have $T[s] = (2^{<\omega})(\to s) = (2^{<\omega})\restriction_s$ for every string s. Generally, if $T \in \mathbf{ST}$ and $2^n \subseteq T$, then $T(\to s) = T\restriction_s$ for all $s \in 2^n$.

Lemma 4.2. *Suppose that $T \in \mathbf{ST}$, $n < \omega$ and $h = \mathrm{spl}_n(T)$. Then the following assertions hold.*

(i) *If $s \in 2^n$, then $T(\to s) \in \mathbf{ST}$, $\mathrm{lh}(\mathrm{stem}(T(\to s))) = h$ and there exists a string $u[s] \in 2^h \cap T$ such that $T(\to s) = T\restriction_{u[s]}$.*

(ii) *Conversely, if $u \in 2^h \cap T$, then there is a string $s[u] \in 2^n$ such that $T\restriction_u = T(\to s[u])$.*

(iii) *Therefore $\{T\restriction_u : u \in 2^h \cap T\} = \{T(\to s) : s \in 2^n\}$ and $\{T\restriction_u : u \in T\} = \{T(\to s) : s \in 2^{<\omega}\}$.*

(iv) *If $s \neq t \in 2^n$, then $T = \bigcup_{s \in 2^n} T(\to s)$ and $[T(\to s)] \cap [T(\to t)] = \varnothing$.*

Proof. (i) We define the required string by

$$u[s] = u_0(T)^\frown\langle s(0)\rangle^\frown \cdots {}^\frown u_{n-1}(T)^\frown\langle s(n-1)\rangle^\frown u_n(T).$$

(ii) We define $s = s[u] \in 2^n$ by $s(k) = u(\mathrm{spl}_k(T))$ for all $k < n$.

(iii) To prove the second equality, let $u \in T$. Then $\mathrm{spl}_{n-1}(T) < \mathrm{lh}(u) \leqslant \mathrm{spl}_n(T)$ for some n. By Definition 3.2 there exists a (unique) string $v \in 2^h \cap T$, where $h = \mathrm{spl}_n(T)$, satisfying $T\restriction_u = T\restriction_v$. It remains to refer to (ii).

The proof of item (iv) is elementary. □

Given $T, S \in \mathbf{ST}$ and $n \in \omega$, we write $S \subseteq_n T$ (the tree S *n-refines* T) if $S \subseteq T$ and $\mathrm{spl}_k(T) = \mathrm{spl}_k(S)$ for all $k < n$. In particular, $S \subseteq_0 T$ is equivalent to $S \subseteq T$. By definition, if $S \subseteq_{n+1} T$, then $S \subseteq_n T$ (and $S \subseteq T$).

Lemma 4.3. *If $T \in \mathbf{ST}$, $n < \omega$, $s_0 \in 2^n$ and $U \in \mathbf{ST}$, $U \subseteq T(\to s_0)$, then there exists a unique tree $T' \in \mathbf{ST}$ satisfying $T' \subseteq_n T$ and $T'(\to s_0) = U$.*

Proof. Let $h = \mathrm{spl}_n(T)$. We pick a string $u_0 = u[s_0] \in 2^h$ by Lemma 4.2(i). Following Lemma 3.4(iv), we define the required tree T' so that $T' \cap 2^h = T \cap 2^h$ and if $u \in T \cap 2^h$, then $T'\restriction_u = (u \cdot u_0) \cdot U$. Then in particular $T'\restriction_{u_0} = U$. □

Lemma 4.4. *Let $\cdots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ be an infinite decreasing sequence of trees in $\mathbf{ST}$. Then the following assertions hold.*

(i) $T = \bigcap_n T_n \in \mathbf{ST}$.

(ii) *If $n < \omega$ and $s \in 2^{n+1}$, then $T(\to s) = T \cap T_n(\to s) = \bigcap_{m \geqslant n} T_m(\to s)$.*

Proof. Note that $\mathrm{spl}(T) = \{\mathrm{spl}_n(T_n) : n < \omega\}$; this easily implies both claims. □

§ 5. ST-forcings

A set $P \subseteq \mathbf{ST}$ is called *a forcing by Silver trees*, or an ST-*forcing*, if it satisfies the following requirements.

(A) If $u \in T \in P$, then $T\restriction_u \in P$.

(B) If $T \in P$ and $\sigma \in 2^{<\omega}$, then $\sigma \cdot T \in P$.

Remark 5.1. Every ST-forcing P may be regarded as a forcing notion (a set of forcing conditions) ordered in such a way that if $T \subsetneq T'$, then T is a stronger 'condition'. Then the forcing P adjoins a real $x \in 2^\omega$. To be more precise, if a set $G \subseteq P$ is P-generic over a given model or universe M (it is assumed that $P \in M$) then the intersection $\bigcap_{T \in G}[T]$ contains a single real $x = x[G] \in 2^\omega$, and this real satisfies $M[G] = M[x[G]]$ and $G = \{T \in P : x \in [T]\}$. Reals $x[G]$ of this form are themselves said to be *P-generic*.

Blanket agreement 5.2. We use the letters T, S, U, V to denote trees in $2^{<\omega}$ and the letters P, Q, R to denote sets of trees, in particular, ST-forcings.

Example 5.3. The set $\mathbf{ST}$ of all Silver trees (that is, the *Silver forcing* itself) is an ST-forcing for obvious reasons. The countable set $P_{\mathrm{coh}} = \{T[s] : s \in 2^{<\omega}\}$ of all trees $T[s]$ in Example 3.3 (the *Cohen forcing*) is another example of an ST-forcing.

Lemma 5.4. *If $\varnothing \neq Q \subseteq \mathbf{ST}$, then the set*

$$P = \{\sigma \cdot (T{\restriction_u}) : u \in T \in Q \wedge \sigma \in 2^{<\omega}\} = \{\sigma \cdot (T(\to s)) : T \in Q \wedge s, \sigma \in 2^{<\omega}\}$$

is an ST*-forcing.*

Proof. To prove (A), assume that $T \in Q$ and $v \in S = \sigma \cdot (T{\restriction_u})$. Then we have $w = \sigma \cdot v \in T{\restriction_u}$ and $v = \sigma \cdot w$. It follows that $S{\restriction_v} = \sigma \cdot (T{\restriction_u}{\restriction_w})$ by Lemma 3.1, where $T{\restriction_u}{\restriction_w} = T{\restriction_u}$ for $w \subseteq u$ (resp. $T{\restriction_u}{\restriction_w} = T{\restriction_w}$ for $u \subset w$). The second equality in the displayed formula of the lemma follows from Lemma 4.2(iii). □

If $P \subseteq \mathbf{ST}$, $T \in \mathbf{ST}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to P, then we say that T is *an n-collage over P*. Let $\mathbf{Colg}_n(P)$ denote the set of all trees $T \in \mathbf{ST}$ which are n-collages over P. Let $\mathbf{Colg}(P) = \bigcup_n \mathbf{Colg}_n(P)$.

Lemma 5.5. *Suppose that $P \subseteq \mathbf{ST}$ is an* ST*-forcing and $n < \omega$. Then the following assertions hold.*

(i) *If $T \in P$ and $s \in 2^{<\omega}$, then $T(\to s) \in P$.*
(ii) $P = \mathbf{Colg}_0(P) \subseteq \mathbf{Colg}_n(P) \subseteq \mathbf{Colg}_{n+1}(P)$.
(iii) *If $\sigma \in 2^{<\omega}$, then $T \in \mathbf{Colg}_n(P)$ is equivalent to $\sigma \cdot T \in \mathbf{Colg}_n(P)$.*
(iv) *If $T \in \mathbf{ST}$ and $s_0 \in 2^n$, then $T(\to s_0) \in P$ is equivalent to $T \in \mathbf{Colg}_n(P)$.*
(v) *If $U \in \mathbf{Colg}_n(P)$, $s_0 \in 2^n$, $S \in P$ and $S \subseteq U(\to s_0)$, then there exists a tree $V \in \mathbf{Colg}_n(P)$ such that $V \subseteq_n U$ and $V(\to s_0) = S$.*
(vi) *If $U \in \mathbf{Colg}_n(P)$ and a set $D \subseteq P$ is open dense in P, then there exists a tree $V \in \mathbf{Colg}_n(P)$ satisfying $V \subseteq_n U$ and $V(\to s) \in D$ for all $s \in 2^n$.*

Recall that a set $D \subseteq P$ is *dense* in P if for every $S \in P$ there is a tree $T \in D$, $T \subseteq S$, and *open dense* if in addition $S \in D$ holds whenever $S \in P$, $T \in D$, $S \subseteq T$.

Proof of Lemma 5.5. To prove (i), make use of (A) and Lemma 4.2(i). To prove (ii), apply (i). The proof of (iii) is elementary.

(iv) If $T \in \mathbf{Colg}_n(P)$, then by definition $T(\to s_0) \in P$. In the opposite direction, let $h = \mathrm{spl}_n(T)$ and, if $s \in 2^n$, let $u[s] \in 2^h \cap T$ be a string satisfying $T(\to s) = T{\restriction_{u[s]}}$ by Lemma 4.2(i). If $T(\to s_0) \in P$ and $s \in 2^n$, then $T(\to s) = T{\restriction_{u[s]}} = (u[s] \cdot u[s_0]) \cdot T{\restriction_{u[s]}}$ by Lemma 3.4. It follows that $T(\to s) \in P$ by Condition (B). Thus $T \in \mathbf{Colg}_n(P)$.

(v) By Lemma 4.3 there exists a tree $V \in \mathbf{ST}$ satisfying $V \subseteq_n U$ and $V(\to s_0) = S$. We have $V \in \mathbf{Colg}_n(P)$ by (iv).

To prove (vi), apply (v) 2^n times for all $s \in 2^n$. □

§ 6. Multitrees and multiforcings

A *multiforcing* is any function $\mathbf{P}$ such that $|\mathbf{P}| = \operatorname{dom} \mathbf{P} \subseteq \omega_1$ and every value $\mathbf{P}(\xi)$, $\xi \in |\mathbf{P}|$, is an ST-forcing. Thus a multiforcing is a partial ω_1-sequence of ST-forcings. A multiforcing $\mathbf{P}$ is *small* if both the domain $|\mathbf{P}|$ and every forcing $\mathbf{P}(\xi)$, $\xi \in |\mathbf{P}|$, are at most countable sets.

If $\mathbf{Q}$ is another multiforcing, then we define the componentwise union $\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q}$ as the multiforcing satisfying $|\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q}| = |\mathbf{P}| \cup |\mathbf{Q}|$ and

$$(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})(\xi) = \begin{cases} \mathbf{P}(\xi) & \text{if } \ \xi \in |\mathbf{P}| \setminus |\mathbf{Q}|, \\ \mathbf{Q}(\xi) & \text{if } \ \xi \in |\mathbf{Q}| \setminus |\mathbf{P}|, \\ \mathbf{P}(\xi) \cup \mathbf{Q}(\xi) & \text{if } \ \xi \in |\mathbf{P}| \cap |\mathbf{Q}|. \end{cases}$$

Similarly, if $\langle \mathbf{P}_\alpha \rangle_{\alpha<\lambda}$ is a sequence of multiforcings, then the componentwise union $\mathbf{P} = \bigcup^{\mathbf{cw}}_{\alpha<\lambda} \mathbf{P}_\alpha$ is a multiforcing satisfying $|\mathbf{P}| = \bigcup_{\alpha<\lambda} |\mathbf{P}_\alpha|$ and $\mathbf{P}(\xi) = \bigcup_{\alpha<\lambda,\ \xi \in |\mathbf{P}_\alpha|} \mathbf{P}_\alpha(\xi)$ for all $\xi \in |\mathbf{P}|$.

A *multitree* is any function $\mathbf{T} \colon |\mathbf{T}| \to \mathbf{ST}$ where, similarly to the above, $|\mathbf{T}| = \operatorname{dom} \mathbf{T} \subseteq \omega_1$ and every value $\mathbf{T}(\xi)$, $\xi \in |\mathbf{T}|$, is a Silver tree, but we require that *the set* $|\mathbf{T}|$ *is finite.* The set of all multitrees will be denoted by $\mathbf{MT}$.

Blanket agreement 6.1. We use the boldface letters $\mathbf{T}$, $\mathbf{S}$, $\mathbf{U}$, $\mathbf{V}$ to denote multitrees, and the boldface letters $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ to denote multiforcings.

If $\mathbf{P}$ is a multiforcing, then let $\mathbf{MT}(\mathbf{P})$ be the set of all multitrees $\mathbf{T}$ such that $|\mathbf{T}| \subseteq |\mathbf{P}|$ and $\mathbf{T}(\xi) \in \mathbf{P}(\xi)$ for all $\xi \in |\mathbf{P}|$. In this case we define a perfect 'brick'

$$\begin{aligned} [\mathbf{T}] &= \{x \in 2^{|\mathbf{T}|} \colon \forall \xi \in |\mathbf{T}| \ (x(\xi) \in [\mathbf{T}(\xi)])\} \\ &= \{x \in 2^{|\mathbf{T}|} \colon \forall \xi \ \forall m \ (x(\xi){\restriction}m \in \mathbf{T}(\xi))\} \end{aligned}$$

in $2^{|\mathbf{T}|}$. The set $\mathbf{MT}(\mathbf{P})$ can be identified with the *finite-support product* $\prod_{\xi \in |\mathbf{P}|} \mathbf{P}(\xi)$; 'finite-support' means that the product contains only those elements $\mathbf{T}$ whose domain $|\mathbf{T}| \subseteq |\mathbf{P}|$ is *finite.*

The set of multitrees $\mathbf{MT}$ is ordered componentwise, that is, $\mathbf{T} \leqslant \mathbf{S}$ ($\mathbf{S}$ is a *stronger* multitree) if $|\mathbf{T}| \subseteq |\mathbf{S}|$ and $\mathbf{S}(\xi) \subseteq \mathbf{T}(\xi)$ for all $\xi \in |\mathbf{T}|$. Thus the order on multitrees is opposite to the componentwise inclusion. The weakest (and the least in the sense of $\leqslant$) 'condition' in $\mathbf{MT}$ and in every $\mathbf{MT}(\mathbf{P})$ is the multitree $\mathbf{\Lambda}$ satisfying $|\mathbf{\Lambda}| = \varnothing$. Multitrees $\mathbf{T}$, $\mathbf{S}$ in a set $P \subseteq \mathbf{MT}$ are *compatible in* P if there exists a multitree $\mathbf{U} \in P$ satisfying $\mathbf{T} \leqslant \mathbf{U}$ and $\mathbf{S} \leqslant \mathbf{U}$.

As usual, a set $D \subseteq \mathbf{MT}(\mathbf{P})$ is said to be

dense in $\mathbf{MT}(\mathbf{P})$ if

$$\forall \mathbf{T} \in \mathbf{MT}(\mathbf{P}) \ \exists \mathbf{S} \in D \ (\mathbf{T} \leqslant \mathbf{S});$$

open dense in $\mathbf{MT}(\mathbf{P})$ if in addition

$$\forall\, \mathbf{T}, \mathbf{S} \in \mathbf{MT}(\mathbf{P})\ (\mathbf{T} \leqslant \mathbf{S} \wedge \mathbf{T} \in D \Rightarrow \mathbf{S} \in D);$$

pre-dense in $\mathbf{MT}(\mathbf{P})$ if the set

$$D^{+} = \{\mathbf{T} \in \mathbf{MT}(\mathbf{P}) \colon \exists \mathbf{S} \in D\ (\mathbf{S} \leqslant \mathbf{T})\} \quad \text{is dense in } \mathbf{MT}(\mathbf{P}).$$

Remark 6.2. The set $\mathbf{MT}(\mathbf{P})$, viewed as a forcing notion with the order defined above, adds a sequence $\langle x_\xi \rangle_{\xi \in |\mathbf{P}|}$ of generic reals. Namely, suppose that $G \subseteq \mathbf{MT}(\mathbf{P})$ is an $\mathbf{MT}(\mathbf{P})$-generic set over a ground model M. We assume that $\mathbf{P} \in M$; then obviously $\mathbf{MT}(\mathbf{P}) \in M$. If $\xi \in |\mathbf{P}|$, then the set $G_\xi = \{\mathbf{T}(\xi) \colon \mathbf{T} \in G\}$ is $\mathbf{P}(\xi)$-generic over M, and accordingly the real $x_\xi[G] = x[G_\xi] \in 2^\omega$ (see Remark 5.1) is $\mathbf{P}(\xi)$-generic over M. We have $M[G] = M[\langle x_\xi[G] \rangle_{\xi \in |\mathbf{P}|}]$ in this case.

§ 7. Names and a direct forcing relation

To avoid repetitions, let $\mathbf{P}$ be a fixed multiforcing in this section. We shall introduce a suitable notational system related to names of reals in 2^ω in the context of forcing notions of the form $\mathbf{MT}(\mathbf{P})$.

Definition 7.1. An $\mathbf{MT}(\mathbf{P})$*-real name* is any indexed family $\mathbf{c} = \langle D^{\mathbf{c}}_{ni} \rangle_{n<\omega,\ i<2}$ of sets $D^{\mathbf{c}}_{ni} \subseteq \mathbf{MT}(\mathbf{P})$ such that every union of the form $D^{\mathbf{c}}_{n} = D^{\mathbf{c}}_{n0} \cup D^{\mathbf{c}}_{n1}$ is dense or at least pre-dense in $\mathbf{MT}(\mathbf{P})$, and if $\mathbf{T} \in D^{\mathbf{c}}_{n0}$ and $\mathbf{S} \in D^{\mathbf{c}}_{n1}$, then $\mathbf{T}$, $\mathbf{S}$ are incompatible in $\mathbf{MT}(\mathbf{P})$.

If a set $G \subseteq \mathbf{MT}(\mathbf{P})$ is $\mathbf{MT}(\mathbf{P})$-generic at least over the family of all sets $D^{\mathbf{c}}_{n}$, then we define the real $\mathbf{c}[G] \in 2^\omega$ such that $\mathbf{c}[G](n) = i$ if and only if $G \cap D^{\mathbf{c}}_{ni} \neq \varnothing$.

Thus every $\mathbf{MT}(\mathbf{P})$-real name $\mathbf{c}$ is an $\mathbf{MT}(\mathbf{P})$-name of a real in 2^ω in the standard forcing sense.

Example 7.2. If $\xi \in |\mathbf{P}|$, then we define an $\mathbf{MT}(\mathbf{P})$-real name $\dot{\boldsymbol{x}}_\xi$ in such a way that every set $D^{\xi}_{ni} = D^{\dot{\boldsymbol{x}}_\xi}_{ni}$ contains all multitrees $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ satisfying $|\mathbf{U}| = \{\xi\}$ and $s(n) = i$ whenever $s \in \mathbf{U}(\xi)$ and $\operatorname{lh}(s) > n$. This name $\dot{\boldsymbol{x}}_\xi$ is clearly an $\mathbf{MT}(\mathbf{P})$-real name of the generic real $x_\xi[G]$ (see Remark 6.2). To be more precise, if $G \subseteq \mathbf{MT}(\mathbf{P})$ is an $\mathbf{MT}(\mathbf{P})$-generic set, then $\dot{\boldsymbol{x}}_\xi[G] = x_\xi[G]$.

Let $\mathbf{c}$ be an $\mathbf{MT}(\mathbf{P})$-real name. We say that a multitree $\mathbf{T}$ (not necessarily in $\mathbf{MT}(\mathbf{P})$)

(a) *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, if there exists a multitree $\mathbf{S} \in D^{\mathbf{c}}_{ni}$ satisfying $\mathbf{S} \leqslant \mathbf{T}$;

(b) *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, if for every $n < \operatorname{lh}(s)$ the multitree $\mathbf{T}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;

(c) *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, if there exists a string $s \in 2^{<\omega} \setminus T$ such that $\mathbf{T}$ directly forces $s \subset \mathbf{c}$.

The definition of direct forcing is not related to any specific forcing notion, but in all three items it is compatible with every multiforcing.

Lemma 7.3. *Let* $\mathbf{P}$ *be a multiforcing,* P *an* ST*-forcing and* $\mathbf{c}$ *an* $\mathbf{MT}(\mathbf{P})$*-name. Suppose that* $S \in P$ *and* $\mathbf{T} \in \mathbf{MT}(\mathbf{P})$*. Then the following assertions hold.*

(i) *There exist a multitree* $\mathbf{S} \in \mathbf{MT}(\mathbf{P})$ *and a tree* $S' \in P$ *such that* $\mathbf{T} \leqslant \mathbf{S}$, $S' \subseteq S$ *and* $\mathbf{S}$ *directly forces* $\mathbf{c} \notin [S']$.

(ii) *If* $n < \omega$ *and* $U \in \mathbf{Colg}_n(P)$, *then there exist an* $\mathbf{S} \in \mathbf{MT}(\mathbf{P})$ *and a tree* $U' \in \mathbf{Colg}_n(P)$ *such that* $\mathbf{T} \leqslant \mathbf{S}$, $U' \subseteq_n U$ *and* $\mathbf{S}$ *directly forces* $\mathbf{c} \notin [U']$.

Proof. (i) Clearly, there is a tree $\mathbf{S} \in \mathbf{MT}(\mathbf{P})$ and a string $u \in 2^{<\omega}$ such that $\mathbf{T} \leqslant \mathbf{S}$, $\operatorname{lh}(u) > \operatorname{lh}((\operatorname{stem}(S)))$ and $\mathbf{S}$ directly forces $u \subset \mathbf{c}$. Then there is a string $v \in S$ satisfying $\operatorname{lh}(v) = \operatorname{lh}(u)$, $v \neq u$. Then the tree $S' = S{\restriction}_v$ belongs to P, $S' \subseteq S$, and by definition the multitree $\mathbf{S}$ directly forces $\mathbf{c} \notin [S']$.

(ii) Take an $s_1 \in 2^n$ (a string of length n). Applying (i) to the tree $S = U(\to s_1) \in P$, we get a multitree $\mathbf{S} \in \mathbf{MT}(\mathbf{P})$ with $\mathbf{T} \leqslant \mathbf{S}$ and a tree $S' \in P$, $S' \subseteq S$, such that $\mathbf{S}$ directly forces $\mathbf{c} \notin [S']$. By Lemma 5.5(v) there is a tree $U' \in \mathbf{Colg}_n(P)$ such that $U' \subseteq_n U$ and $U'(\to s_1) = S'$. Thus $\mathbf{S}$ directly forces $\mathbf{c} \notin [U'(\to s_1)]$. Consider another string $s_2 \in 2^n$ and, using the same argument, find a $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ and a tree $U'' \in \mathbf{Colg}_n(P)$ such that $\mathbf{S} \leqslant \mathbf{U}$, $U'' \subseteq_n U'$ and $\mathbf{U}$ directly forces $\mathbf{c} \notin [U'(\to s_2)]$. Continue in this way until all the strings in 2^n are exhausted. This yields the required result. □

§ 8. Generic extensions of multiforcings

As outlined in the introductory section, the forcing notion for proving our main theorems will be defined as the ω_1-union of its more elementary constituent parts (layers). The next definition lists the requirements for layers.

Let $\mathbf{ZFC}'$ be a subtheory of $\mathbf{ZFC}$ containing all the axioms except for the power-set axiom, but with a special axiom which claims the existence of $\mathcal{P}(\omega)$. (This implies the existence of the ordinal ω_1 as well as $\mathbf{PT}$ and similar sets of the cardinality of the continuum.)

Definition 8.1. Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$ and $\mathbf{P} \in \mathfrak{M}$ a multiforcing. (Then $|\mathbf{P}| \in \mathfrak{M}$ and all the sets $\mathbf{P}(\xi)$, $\xi \in |\mathbf{P}|$, and the set $\mathbf{MT}(\mathbf{P})$ belong to $\mathfrak{M}$.) We say that another multiforcing $\mathbf{Q}$ (not necessarily in $\mathfrak{M}$) is *an* $\mathfrak{M}$*-extension* of $\mathbf{P}$ if the following conditions hold.

(A) $|\mathbf{Q}| = |\mathbf{P}|$ and $\mathbf{Q}$ is a small multiforcing.

(B) If $\xi \in |\mathbf{P}|$, then $\mathbf{Q}(\xi)$ is dense in $\mathbf{Q}(\xi) \cup \mathbf{P}(\xi)$ and $\mathbf{Q}(\xi) \cap \mathbf{P}(\xi) = \varnothing$.

(C) If $\xi \in |\mathbf{P}|$ and $D \in \mathfrak{M}$ is a set such that $D \subseteq \mathbf{P}(\xi)$ is pre-dense in $\mathbf{P}(\xi)$, then the following asserions hold.

  a) D remains pre-dense in $\mathbf{P}(\xi) \cup \mathbf{Q}(\xi)$.

  b) If $U \in \mathbf{Q}(\xi)$, then $U \subseteq^{\text{fin}} \bigcup D$, that is, there exists a finite set $D' \subseteq D$ satisfying $U \subseteq \bigcup D'$.

(D) If $\xi \in |\mathbf{P}|$ and $T, T' \in \mathbf{P}(\xi)$ are incompatible trees in $\mathbf{P}(\xi)$, then T, T' remain incompatible in $\mathbf{P}(\xi) \cup \mathbf{Q}(\xi)$.

(E) If $D \in \mathfrak{M}$ is a set such that $D \subseteq \mathbf{MT}(\mathbf{P})$ is pre-dense in $\mathbf{MT}(\mathbf{P})$, then it remains pre-dense in $\mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$.

(F) Suppose that $\mathbf{c} \in \mathfrak{M}$ is an $\mathbf{MT}(\mathbf{P})$-real name, $\zeta \in |\mathbf{P}|$, the set

$$D(\sigma) = \{\mathbf{T} \in \mathbf{MT}(\mathbf{P})\colon \mathbf{T} \text{ directly forces } \mathbf{c} \notin [\sigma \cdot \mathbf{T}(\zeta)]\}$$

is dense in $\mathbf{MT}(\mathbf{P})$ whenever $\sigma \in 2^{<\omega}$ and, finally, $\mathbf{S} \in \mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$, $U \in \mathbf{Q}(\zeta)$. Then there exists a stronger multitree $\mathbf{V} \in \mathbf{MT}(\mathbf{Q})$, $\mathbf{S} \leqslant \mathbf{V}$, which directly forces $\mathbf{c} \notin [U]$.

The meaning of the last condition (F) can be explained as follows. If $\zeta < \omega_1$ and $\mathbf{c}$ is an $\mathbf{MT}(\mathbf{P})$-real name, then the extended forcing $\mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$ forces that $\mathbf{c}$ does not belong to sets of the form $[U]$, where U is a tree in $\mathbf{Q}(\zeta)$, unless $\mathbf{c}$ is a name $\dot{\boldsymbol{x}}_\zeta$ of one of the reals $\sigma \cdot x_\zeta[G]$, where $\sigma \in 2^{<\omega}$.

Theorem 8.2. *Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$ and $\mathbf{P} \in \mathfrak{M}$ a multiforcing. Then there exists a small multiforcing $\mathbf{Q}$ such that $\mathbf{Q}$ is a $\mathfrak{M}$-extension of $\mathbf{P}$ in the sense of Definition 8.1.*

The proof of the theorem is contained in the next three sections. A construction of the required multiforcing $\mathbf{Q}$ is given in § 9, and proofs of its properties follow in §§ 10, 11.

§ 9. The construction of extending multiforcing

The next definition formalizes the construction in the following section of generic multitrees by means of Lemma 4.4.

Definition 9.1. A *system* is any 'matrix' $\varphi = \langle\langle \nu^\varphi_{\xi m}, \tau^\varphi_{\xi m}\rangle\rangle_{\langle \xi,m\rangle \in |\varphi|}$, where $|\varphi| \subseteq \omega_1 \times \omega$ is finite and if $\langle \xi, m\rangle \in |\varphi|$, then:

(1) $\nu^\varphi_{\xi m} \in \omega$;

(2) $\tau^\varphi_{\xi m} = \langle T^\varphi_{\xi m}(0), T^\varphi_{\xi m}(1), \dots, T^\varphi_{\xi m}(\nu^\varphi_{\xi m})\rangle$, where every $T^\varphi_{\xi m}(n)$ is a Silver tree and $T^\varphi_{\xi m}(n+1) \subseteq_{n+1} T^\varphi_{\xi m}(n)$ for all $n < \nu^\varphi_{\xi m}$.

In this case, if $n \leqslant \nu^\varphi_{\xi m}$ and $s \in 2^n$, then we put $T^\varphi_{\xi m}(s) = T^\varphi_{\xi m}(n)(\to s)$.

If $\mathbf{P}$ is a multiforcing, then let $\mathbf{Sys}(\mathbf{P})$ denote the set of all systems φ such that $|\varphi| \subseteq |\mathbf{P}| \times \omega$ and $T^\varphi_{\xi m}(n) \in \mathbf{Colg}_n(\mathbf{P}(\xi))$ for all $\langle \xi, m\rangle \in |\varphi|$ and $n \leqslant \nu^\varphi_{\xi m}$. Then every tree $T^\varphi_{\xi m}(s)$, $s \in 2^n$, belongs to $\mathbf{P}(\xi)$.

We say that a system φ *extends* another system ψ (and write $\psi \preccurlyeq \varphi$) if $|\psi| \subseteq |\varphi|$ and for every pair $\langle \xi, m\rangle \in |\psi|$, first, $\nu^\varphi_{\xi m} \geqslant \nu^\psi_{\xi m}$ and, second, $\tau^\varphi_{\xi m}$ extends $\tau^\psi_{\xi m}$ in the sense that $T^\varphi_{\xi m}(n) = T^\psi_{\xi m}(n)$ for all $n \leqslant \nu^\psi_{\xi m}$.

Lemma 9.2. *Suppose that $\mathbf{P}$ is a multiforcing and $\varphi \in \mathbf{Sys}(\mathbf{P})$. Then the following assertions hold.*

(i) *If $\langle \xi, m\rangle \in |\varphi|$ and $n = \nu^\varphi_{\xi m}$, then the extension φ' of φ by $\nu^{\varphi'}_{\xi m} = n+1$ and $T^{\varphi'}_{\xi m}(n+1) = T^\varphi_{\xi m}(n)$ belongs to $\mathbf{Sys}(\mathbf{P})$ and $\varphi \preccurlyeq \varphi'$.*

(ii) *If $\langle \xi, m\rangle \notin |\varphi|$, then the extension φ' of φ by $|\varphi'| = |\varphi| \cup \{\langle \xi, m\rangle\}$, $\nu^{\varphi'}_{\xi m} = 0$, and $T^{\varphi'}_{\xi m}(0) = T$, where $T \in \mathbf{P}(\xi)$, belongs to $\mathbf{Sys}(\mathbf{P})$ and $\varphi \preccurlyeq \varphi'$.*

Proof. (i) We make use of the fact that $T \subseteq_n T$ for all n, T. The proof of item (ii) is elementary. □

In accordance with the hypotheses of Theorem 8.2 we now fix a countable transitive model $\mathfrak{M} \models \mathbf{ZFC}'$ and a multiforcing $\mathbf{P} \in \mathfrak{M}$.

Definition 9.3. (i) We fix a $\preccurlyeq$-increasing sequence $\mathbf{\Phi} = \langle\varphi(j)\rangle_{j<\omega}$ of systems $\varphi(j) \in \mathbf{Sys}(\mathbf{P})$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{Sys}(\mathbf{P})$, dense in $\mathbf{Sys}(\mathbf{P})$. (The density here means that for every system $\psi \in \mathbf{Sys}(\mathbf{P})$ there is a system $\varphi \in D$ with $\psi \preccurlyeq \varphi$.) In the course of this definition we construct another multiforcing $\mathbf{Q}$ on the basis of $\mathbf{\Phi}$; see item (iv) below.

(ii) In particular, $\mathbf{\Phi}$ intersects every set of the form

$$D_{\xi mn} = \{\varphi \in \mathbf{Sys}(\mathbf{P})\colon \nu^{\varphi}_{\xi m} \geqslant n\},$$

where $\xi \in |\mathbf{P}|$ and $m, n < \omega$. Indeed, all such sets are dense by Lemma 9.2 and belong to $\mathfrak{M}$. Therefore, if $\xi \in |\mathbf{P}|$ and $m < \omega$, then there is an infinite sequence

$$\cdots \subseteq_5 T^{\mathbf{\Phi}}_{\xi m}(4) \subseteq_4 T^{\mathbf{\Phi}}_{\xi m}(3) \subseteq_3 T^{\mathbf{\Phi}}_{\xi m}(2) \subseteq_2 T^{\mathbf{\Phi}}_{\xi m}(1) \subseteq_1 T^{\mathbf{\Phi}}_{\xi m}(0)$$

of trees $T^{\mathbf{\Phi}}_{\xi m}(n) \in \mathbf{Colg}_n(\mathbf{P}(\xi))$ such that for every j, if $\langle\xi, m\rangle \in |\varphi(j)|$ and $n \leqslant \nu^{\varphi(j)}_{\xi m}$, then $T^{\varphi(j)}_{\xi m}(n) = T^{\mathbf{\Phi}}_{\xi m}(n)$. If $n < \omega$ and $s \in 2^n$, then we put $T^{\mathbf{\Phi}}_{\xi m}(s) = T^{\mathbf{\Phi}}_{\xi m}(n)(\to s)$. Then $T^{\mathbf{\Phi}}_{\xi m}(s) \in \mathbf{P}(\xi)$ since $T^{\mathbf{\Phi}}_{\xi m}(n) \in \mathbf{Colg}_n(\mathbf{P}(\xi))$.

(iii) In this case, by Lemma 4.4, every set

$$U^{\mathbf{\Phi}}_{\xi m} = \bigcap_n T^{\mathbf{\Phi}}_{\xi m}(n) = \bigcap_n \bigcup_{s\in 2^n} T^{\mathbf{\Phi}}_{\xi m}(s)$$

is a tree in $\mathbf{ST}$ (not necessarily in $\mathbf{P}(\xi)$) and the same is true of the subtrees $U^{\mathbf{\Phi}}_{\xi m}(\to s)$ by Lemma 5.5(i). Moreover, Lemma 4.4 implies that

$$U^{\mathbf{\Phi}}_{\xi m}(\to s) = U^{\mathbf{\Phi}}_{\xi m} \cap T^{\mathbf{\Phi}}_{\xi m}(s) = \bigcap_{n\geqslant \mathrm{lh}(s)} T^{\mathbf{\Phi}}_{\xi m}(n)(\to s)$$

and, clearly, $U^{\mathbf{\Phi}}_{\xi m} = U^{\mathbf{\Phi}}_{\xi m}(\to \Lambda)$.

In addition, if $s, t \in 2^{<\omega}$ and $s \subseteq t$. then $T^{\mathbf{\Phi}}_{\xi m}(s) \subseteq T^{\mathbf{\Phi}}_{\xi m}(t)$ and $U^{\mathbf{\Phi}}_{\xi m}(\to s) \subseteq U^{\mathbf{\Phi}}_{\xi m}(\to t)$. But if s, t are $\subseteq$-incomparable, then

$$[U^{\mathbf{\Phi}}_{\xi m}(\to s)] \cap [U^{\mathbf{\Phi}}_{\xi m}(\to t)] = [T^{\mathbf{\Phi}}_{\xi m}(s)] \cap [T^{\mathbf{\Phi}}_{\xi m}(t)] = \varnothing.$$

(iv) If $\xi \in |\mathbf{P}|$, then the set $Q_\xi = \{\sigma \cdot U^{\mathbf{\Phi}}_{\xi m}(\to s)\colon m < \omega \wedge \sigma, s \in 2^{<\omega}\}$ is an ST-forcing by Lemma 5.4. Therefore we can define a multiforcing $\mathbf{Q}$ satisfying $|\mathbf{Q}| = |\mathbf{P}|$ and $\mathbf{Q}(\xi) = \mathbf{Q}(\xi) = Q_\xi$ for all $\xi \in |\mathbf{P}|$. □ (Definition 9.3)

We have to check that the multiforcing $\mathbf{Q}$ satisfies all the requirements of Definition 8.1. Note that 8.1(A) holds automatically by construction.

Verification of 8.1(B). Let $\xi \in |\mathbf{P}|$ and $T \in \mathbf{P}(\xi)$. To prove the density of $\mathbf{Q}(\xi)$ in $\mathbf{Q}(\xi) \cup \mathbf{P}(\xi)$, note that the set $D(T)$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P}) \cap \mathfrak{M}$ such that $T^{\varphi}_{\xi m}(0) = T$ for some m belongs to $\mathfrak{M}$ and is dense in $\mathbf{Sys}(\mathbf{P})$ by Lemma 9.2(ii). Therefore $\varphi(J) \in D(T)$ for some J by the choice of $\mathbf{\Phi}$. Then $T^{\mathbf{\Phi}}_{\xi m}(0) = T$ for some $m < \omega$. But $U^{\mathbf{\Phi}}_{\xi m}(\to \Lambda) = U^{\mathbf{\Phi}}_{\xi m} \subseteq T^{\mathbf{\Phi}}_{\xi m}(0)$ and $U^{\mathbf{\Phi}}_{\xi m} \in \mathbf{Q}(\xi)$.

To prove the equality $\mathbf{Q}(\xi) \cap \mathbf{P}(\xi) = \varnothing$, suppose that

$$U = U^{\mathbf{\Phi}}_{\xi m}(\to s) \in \mathbf{Q}(\xi).$$

(If $U = \sigma \cdot U^{\mathbf{\Phi}}_{\xi m}(\to s)$, $\sigma \in 2^{<\omega}$, then replace T by $\sigma \cdot T$.) The set $D(T, \xi, m)$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that $\langle\xi, m\rangle \in |\varphi|$ and $T \setminus T^{\varphi}_{\xi m}(n)(\to s) \neq \varnothing$, where

$n = \nu^{\varphi}_{\xi m}$, belongs to $\mathfrak{M}$ and is dense in $\mathbf{Sys}(\mathbf{P})$. But every system $\varphi(j) \in D(T, \xi, m)$ ensures that $T \setminus U^{\mathbf{\Phi}}_{\xi m}(\to s) \neq \varnothing$. This completes the verification of 8.1(B).

§ 10. Preservation of density

Here we verify conditions (C), (D), (E) of Definition 8.1 for the multiforcing $\mathbf{Q}$ in the context of Definition 9.3.

Verification of 8.1(C). Suppose that $\xi \in |\mathbf{P}|$ and the set $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}(\xi)$ is pre-dense in $\mathbf{P}(\xi)$. To verify item 8.1(C), b), assume that $U \in \mathbf{Q}(\xi)$; we have to check that $U \subseteq^{\text{fin}} \bigcup D$. By definition we have $U = \sigma \cdot U^{\mathbf{\Phi}}_{\xi m}(\to s)$, where $m < \omega$ and $s, \sigma \in 2^{<\omega}$. Here we can assume that $\sigma = \Lambda$, that is, simply $U = U^{\mathbf{\Phi}}_{\xi m}(\to s)$. (The general case reduces to this by substituting $\sigma \cdot D$ for D.) We can further assume that $s = \Lambda$, that is, $U = U^{\mathbf{\Phi}}_{\xi m}$ because $U^{\mathbf{\Phi}}_{\xi m}(\to s) \subseteq U^{\mathbf{\Phi}}_{\xi m}$. Thus let $U = U^{\mathbf{\Phi}}_{\xi m}$.

Let $\Delta \in \mathfrak{M}$ be the set of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that $\langle \xi, m \rangle \in |\varphi|$ and, for every string $t \in 2^n$, where $n = \nu^{\varphi}_{\xi M}$, there is a tree $S_t \in D$ with $T^{\varphi}_{\xi m}(t) \subseteq S_t$. Then Δ is dense in $\mathbf{Sys}(\mathbf{P})$ by Lemma 5.5(vi) because D itself is pre-dense. Therefore there is an index j such that $\varphi(j) \in \Delta$. Let this be ensured by trees $S_t \in D$, $t \in 2^n$, where $n = \nu^{\varphi(j)}_{\xi m}$, so that $T^{\varphi(j)}_{\xi m}(t) \subseteq S_t \ \ \forall\, t$, and hence $T^{\varphi(j)}_{\xi m}(n) \subseteq^{\text{fin}} \bigcup D$. Then

$$U = U^{\mathbf{\Phi}}_{\xi m} \subseteq U^{\mathbf{\Phi}}_{\xi m}(n) = T^{\varphi(j)}_{\xi m}(n) \subseteq^{\text{fin}} \bigcup D.$$

To verify item 8.1(C), a) (the pre-density of D in $\mathbf{P}(\xi) \cup \mathbf{Q}(\xi)$), it suffices to prove that every $U \in \mathbf{Q}(\xi)$ is compatible in $\mathbf{P}(\xi) \cup \mathbf{Q}(\xi)$ with some $T \in D$. We have $U \subseteq \bigcup D'$, where $D' \subseteq D$ is finite; see above. There exists a tree $T \in D'$ such that $[T] \cap [U]$ has a non-empty open interior in $[U]$. There is an open set $X \subseteq 2^{\omega}$ satisfying $\varnothing \neq X \cap [U] \subseteq [T] \cap [U]$. Then by Lemma 3.4(iii) there is a string $s \in U$ satisfying $[U {\restriction_s}] \subseteq X \cap [U]$. Thus the tree $U' = U{\restriction_s}$ satisfies $[U'] \subseteq [U] \cap [T]$, whence $U' \subseteq U \cap T$. Finally, $U' \in \mathbf{Q}(\xi)$ since $\mathbf{Q}(\xi)$ is an ST-forcing.

Verification of 8.1(D). Suppose that $\xi \in |\mathbf{P}|$ and $T, T' \in \mathbf{P}(\xi)$ are incompatible trees in $\mathbf{P}(\xi)$. If $S \in \mathbf{P}(\xi)$, then $S \not\subseteq T$ or $S \not\subseteq T'$. Hence there is a subtree $S' \in \mathbf{P}(\xi)$, $S' \subseteq S$ satisfying $[S'] \cap [T] \cap [T'] = \varnothing$. Therefore the set D of all trees $S \in \mathbf{P}(\xi)$ such that $[S] \cap [T] \cap [T'] = \varnothing$ is dense in $\mathbf{P}(\xi)$. It remains to apply 8.1(C).

Verification of 8.1(E). Let $D \in \mathfrak{M}$, $D \subseteq \mathbf{MT}(\mathbf{P})$ be a pre-dense set in $\mathbf{MT}(\mathbf{P})$. We claim that it remains pre-dense in $\mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$.

Consider a multitree $\mathbf{S} \in \mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$. We have to prove that $\mathbf{S}$ is compatible in $\mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$ with a multitree $\mathbf{D} \in D$. Condition 8.1(B) has already been established, whence we can assume that $\mathbf{S} \in \mathbf{MT}(\mathbf{Q})$. Then every component $\mathbf{S}(\xi)$ of $\mathbf{S}$ ($\xi \in |\mathbf{S}|$) is equal to one of the trees $\sigma_\xi \cdot U^{\mathbf{\Phi}}_{\xi, m_\xi}(\to s_\xi)$, where $m_\xi < \omega$ and σ_ξ, $s_\xi \in 2^{<\omega}$. We can assume, for the sake of simplicity, that $\sigma_\xi = \Lambda$ for all $\xi \in |\mathbf{S}|$, that is, $\mathbf{S}(\xi) = U^{\mathbf{\Phi}}_{\xi, m_\xi}(\to s_\xi)$. (If this is not the case, then we define $\overline{\mathbf{U}} \in \mathbf{MT}(\mathbf{Q})$ for every $\mathbf{U} \in \mathbf{MT}(\mathbf{Q})$ so that $|\overline{\mathbf{U}}| = |\mathbf{U}|$, $\overline{\mathbf{U}}(\xi) = \mathbf{U}(\xi)$ for all $\xi \in |\mathbf{U}| \setminus |\mathbf{S}|$ and $\overline{\mathbf{U}}(\xi) = \sigma_\xi \cdot \mathbf{U}(\xi)$ for all $\xi \in |\mathbf{U}| \cap |\mathbf{S}|$, and replace D by the set $\overline{D} = \{\overline{\mathbf{U}} \colon \mathbf{U} \in D\}$.)

Thus we assume that $\mathbf{S}(\xi) = U^{\mathbf{\Phi}}_{\xi, m_\xi}(\to s_\xi)$ for all $\xi \in |\mathbf{S}|$.

Consider the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that there exist multitrees $\mathbf{D} \in D$ and $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$, numbers k_ξ, and strings $t_\xi \in 2^{<\omega}$, $\xi \in |\mathbf{U}|$, satisfying $\mathbf{D} \leqslant \mathbf{U}$ and the following requirements:

(1) $|\mathbf{S}| \subseteq |\mathbf{U}|$, and if $\xi \in |\mathbf{S}|$ then $k_\xi = m_\xi$;

(2) if $\xi \in |\mathbf{U}|$ then $\langle \xi, k_\xi \rangle \in |\varphi|$, $\mathrm{lh}(t_\xi) \leqslant \nu^\varphi_{\xi,k_\xi}$, $s_\xi \subset t_\xi$, $\mathbf{U}(\xi) = T^\varphi_{\xi,k_\xi}(t_\xi)$.

Lemma 10.1. *The set Δ is dense in* $\mathbf{Sys}(\mathbf{P})$.

Proof. Let $\psi \in \mathbf{Sys}(\mathbf{P})$. We need to find a system $\varphi \in \mathbf{Sys}(\mathbf{P})$ satisfying $\psi \preccurlyeq \varphi$. We can assume, by Lemma 9.2(i), that if $\xi \in |\mathbf{S}|$ then $\langle \xi, m_\xi \rangle \in |\psi|$. For otherwise add $\langle \xi, m_\xi \rangle$ to $|\psi|$ and put $\nu^\psi_{\xi,m_\xi} = 0$ and $T^\psi_{\xi,m_\xi}(0) = S$, where $S \in \mathbf{P}(\xi)$ is any tree.

We define a system $\chi \in \mathbf{Sys}(\mathbf{P})$ extending ψ so that $|\chi| = |\psi|$ and if $\xi \in |\mathbf{S}|$ (then $\langle \xi, m_\xi \rangle \in |\psi|$; see above) then $\nu^\chi_{\xi,m_\xi} = \nu^\psi_{\xi,m_\xi} + 1$, $T^\chi_{\xi,m_\xi}(\nu^\chi_{\xi,m_\xi}) = T^\psi_{\xi,m_\xi}(\nu^\psi_{\xi,m_\xi})$, and $t_\xi = s_\xi{}^\frown 0$. Then $\psi \preccurlyeq \chi$, $s_\xi \subset t_\xi$, $\mathrm{lh}(t_\xi) = \nu^\chi_{\xi,m_\xi}$. But if $\langle \xi, m \rangle \in |\psi|$ is not of the form $\langle \xi, m_\xi \rangle$, $\xi \in |\mathbf{S}|$, then we keep $\nu^\chi_{\xi m} = \nu^\psi_{\xi m}$ and $T^\chi_{\xi m}(n) = T^\psi_{\xi m}(n)$ for all $n \leqslant \nu^\psi_{\xi m}$.

Define a multitree $\mathbf{T} \in \mathbf{MT}(\mathbf{P})$ by $|\mathbf{T}| = |\mathbf{S}|$ and $\mathbf{T}(\xi) = T^\chi_{\xi,m_\xi}(t_\xi)$ for all $\xi \in |\mathbf{S}|$. As D is pre-dense, there exist multitrees $\mathbf{D} \in D$ and $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ satisfying $\mathbf{D} \leqslant \mathbf{U}$ and $\mathbf{T} \leqslant \mathbf{U}$. Then $|\mathbf{S}| = |\mathbf{T}| \subseteq |\mathbf{U}|$.

Now define a system $\varphi \in \mathbf{Sys}(\mathbf{P})$ so that $|\chi| \subseteq |\varphi|$ and if $\langle \xi, m \rangle \in |\chi|$ and $n < \nu^\chi_{\xi m}$, then $\nu^\varphi_{\xi m} = \nu^\chi_{\xi m}$ and $T^\varphi_{\xi m}(n) = T^\chi_{\xi m}(n)$. The definition of the values $T^\varphi_{\xi m}(\nu^\chi_{\xi m})$ and the definition for the domain $|\varphi| \setminus |\chi|$ are as follows.

(I) If a pair $\langle \xi, m \rangle \in |\chi|$ is not of the form $\langle \xi, m_\xi \rangle$, where $\xi \in |\mathbf{S}| = |\mathbf{T}|$, then $T^\varphi_{\xi m}(\nu^\chi_{\xi m}) = T^\chi_{\xi m}(\nu^\chi_{\xi m})$.

(II) Let $\xi \in |\mathbf{T}| = |\mathbf{S}|$, so that $\langle \xi, m_\xi \rangle \in |\chi|$. We put $k_\xi = m_\xi$. Then $\mathbf{U}(\xi) = S \subseteq T = \mathbf{T}(\xi) = T^\chi_{\xi,k_\xi}(t_\xi)$ since $\mathbf{T} \leqslant \mathbf{U}$. Recall that $T^\chi_{\xi,k_\xi}(t_\xi) = T^\chi_{\xi,k_\xi}(n)(\to t_\xi)$ (see Definition 9.1), where $n = \nu^\chi_{\xi,k_\xi}$ and the tree $U = T^\chi_{\xi,k_\xi}(n)$ belongs to $\mathbf{Colg}_n(\mathbf{P}(\xi))$, while the tree $S = \mathbf{U}(\xi)$ belongs to $\mathbf{P}(\xi)$ and satisfies $S \subseteq U(\to t_\xi)$. It follows by Lemma 5.5(v) that there is a tree $V \in \mathbf{Colg}_n(\mathbf{P}(\xi))$ satisfying $V \subseteq_n U$ and $V(\to t_\xi) = S$. We define $T^\varphi_{\xi,k_\xi}(n) = V$, so that $T^\varphi_{\xi,k_\xi}(t_\xi) = S = \mathbf{U}(\xi)$.

(III) Finally let $\xi \in |\mathbf{U}| \setminus |\mathbf{S}|$. We choose a $k_\xi < \omega$ such that $\langle \xi, k_\xi \rangle \notin |\varphi|$, add $\langle \xi, k_\xi \rangle$ to $|\varphi|$, and put $\nu^\varphi_{\xi,k_\xi} = 0$, $t_\xi = \Lambda$ and $T^\varphi_{\xi,k_\xi}(0) = \mathbf{U}(\xi)$. Then clearly $T^\varphi_{\xi,k_\xi}(t_\xi) = \mathbf{U}(\xi)$ in this case as well since $T = T(\to \Lambda)$.

We note that φ belongs to $\mathbf{Sys}(\mathbf{P})$ (as $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$) and satisfies $\psi \preccurlyeq \varphi$ (the changes in φ concern only those elements of χ that do not lie in ψ). Conditions (1), (2) above hold by construction. We conclude that $\varphi \in \Delta$. $\square$

It follows from the lemma that there exists an index j satisfying $\varphi(j) \in \Delta$. Let numbers k_ξ, strings t_ξ, and multitrees $\mathbf{D} \in D$ and $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ satisfy $\mathbf{D} \leqslant \mathbf{U}$ and conditions (1), (2) for $\varphi(j)$. We define a multitree $\mathbf{V} \in \mathbf{MT}(\mathbf{Q})$ by $|\mathbf{V}| = |\mathbf{U}|$ and $\mathbf{V}(\xi) = T^{\mathbf{\Phi}}_{\xi,k_\xi}(\to t_\xi)$ for all $\xi \in |\mathbf{U}|$. Then $\mathbf{S} \leqslant \mathbf{V}$ (since $s_{\xi k} \subset t_{\xi k}$). In addition, $\mathbf{U} \leqslant \mathbf{V}$. Indeed, if $\xi \in |\mathbf{U}|$, then $\mathbf{V}(\xi) = T^{\mathbf{\Phi}}_{\xi,k_\xi}(\to t_\xi) \subseteq T^{\mathbf{\Phi}}_{\xi,k_\xi}(t_\xi) = T^{\varphi(j)}_{\xi,k_\xi}(t_\xi) = \mathbf{U}(\xi)$. Thus $\mathbf{V}$ ensures the compatibility of $\mathbf{S}$ with $\mathbf{D} \in D$ in $\mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$.

§ 11. Avoiding trees in the extending multiforcing

We complete the proof of Theorem 8.2 in this section. It remains to check condition (F) in Definition 8.1.

Verification of 8.1(F). Assume that $\zeta \in |\mathbf{P}|$, $\mathbf{S} \in \mathbf{MT}(\mathbf{P} \cup^{\mathbf{cw}} \mathbf{Q})$, $U \in \mathbf{Q}(\zeta)$, $\mathbf{c} \in \mathfrak{M}$ is an $\mathbf{MT}(\mathbf{P})$-real name and the set

$$D(\sigma) = \{\mathbf{T} \in \mathbf{MT}(\mathbf{P}) \colon \mathbf{T} \text{ directly forces } \mathbf{c} \notin [\sigma \cdot \mathbf{T}(\zeta)]\}$$

is dense in $\mathbf{MT}(\mathbf{P})$ for every string $\sigma \in 2^{<\omega}$. We need to find a stronger multitree $\mathbf{V} \in \mathbf{MT}(\mathbf{Q})$, $\mathbf{S} \leqslant \mathbf{V}$, which directly forces $\mathbf{c} \notin [U]$.

By construction, $U \subseteq \rho \cdot U^{\mathbf{\Phi}}_{\zeta M}$, where $\tau \in 2^{<\omega}$ and $M < \omega$. Therefore it can be assumed that $U = \rho \cdot U^{\mathbf{\Phi}}_{\zeta M}$. We can also assume that $\rho = \Lambda$, that is, $U = U^{\mathbf{\Phi}}_{\zeta M}$. (Otherwise replace $\mathbf{c}$ by a name $\rho \cdot \mathbf{c}$ defined so that $|\rho \cdot \mathbf{c}| = |\mathbf{c}|$ and if $n < \omega$ and $i = 0, 1$, then $D^{\rho \cdot \mathbf{c}}_{ni} = D^{\mathbf{c}}_{n,1-i}$ provided that $n < \operatorname{lh}(\rho)$ and $\rho(n) = 1$, otherwise just $D^{\rho \cdot \mathbf{c}}_{ni} = D^{\mathbf{c}}_{ni}$. Every multitree $\mathbf{V}$ directly forces $\mathbf{c} \notin [\rho \cdot U^{\mathbf{\Phi}}_{\zeta M}]$ if and only if it directly forces $(\rho \cdot \mathbf{c}) \notin [U^{\mathbf{\Phi}}_{\zeta M}]$.) Thus we assume that $U = U^{\mathbf{\Phi}}_{\zeta M}$. The indices ζ and M are fixed from this point on. We also suppose that $\zeta \in |\mathbf{S}|$, otherwise just add ζ to $|\mathbf{S}|$ by putting $\mathbf{S}(\zeta) = T$, where $T \in \mathbf{P}(\zeta)$ is arbitrary.

In addition, as in the verification of 8.1(E) above, we can assume that $\mathbf{S} \in \mathbf{MT}(\mathbf{Q})$, and for every $\xi \in |\mathbf{S}|$ there is a number $m_\xi < \omega$ and a string $s_\xi \in 2^{<\omega}$ satisfying $\mathbf{S}(\xi) = U^{\mathbf{\Phi}}_{\xi, m_\xi}(\rightarrow s_\xi)$ and $s_\xi \neq s_\eta$ whenever $\xi \neq \eta$.

We consider the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that there exist a multitree $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$, numbers k_ξ and strings $t_\xi \in 2^{<\omega}$ (for every index $\xi \in |\mathbf{U}|$) satisfying the following conditions:

(1) $|\mathbf{S}| \subseteq |\mathbf{U}|$ and if $\xi \in |\mathbf{S}|$, then $k_\xi = m_\xi$;

(2) if $\xi \in |\mathbf{U}|$, then $\langle \xi, k_\xi \rangle \in |\varphi|$, $s_\xi \subset t_\xi$, $\operatorname{lh}(t_\xi) \leqslant \nu^\varphi_{\xi, k_\xi}$, $\mathbf{U}(\xi) = T^\varphi_{\xi, k_\xi}(t_\xi)$;

(3) $\langle \zeta, M \rangle \in |\varphi|$ and $\mathbf{U}$ directly forces $\mathbf{c} \notin [T^\varphi_{\zeta M}(\ell)]$, where $\ell = \nu^\varphi_{\zeta M}$.

Lemma 11.1. *The set Δ is dense in* $\mathbf{Sys}(\mathbf{P})$.

Proof. Let $\psi \in \mathbf{Sys}(\mathbf{P})$. We have to define a system $\varphi \in \Delta$ satisfying $\psi \preccurlyeq \varphi$. As in the verification of 8.1(E), we can assume that if $\xi \in |\mathbf{S}|$, then $\langle \xi, m_\xi \rangle \in |\psi|$ and, separately, $\langle \zeta, M \rangle \in |\varphi|$.

We introduce an intermediate system $\chi \in \mathbf{Sys}(\mathbf{P})$ extending ψ as in the proof of Lemma 10.1, that is, $|\chi| = |\psi|$ and if $\xi \in |\mathbf{S}|$, then $\nu^\chi_{\xi, m_\xi} = \nu^\psi_{\xi, m_\xi} + 1$, $T^\chi_{\xi, m_\xi}(\nu^\chi_{\xi, m_\xi}) = T^\psi_{\xi, m_\xi}(\nu^\psi_{\xi, m_\xi})$ and $t_\xi = s_\xi{}^\frown 0$. Then $\psi \preccurlyeq \chi$, $s_\xi \subset t_\xi$, $\operatorname{lh}(t_\xi) = \nu^\chi_{\xi k_\xi}$. But if $\langle \xi, m \rangle \in |\psi|$ is not of the form $\langle \xi, m_\xi \rangle$, then we keep $\nu^\chi_{\xi m} = \nu^\psi_{\xi m}$ and $T^\chi_{\xi m}(n) = T^\psi_{\xi m}(n)$ for all n. We define a multitree $\mathbf{T} \in \mathbf{MT}(\mathbf{P})$ by putting $|\mathbf{T}| = |\mathbf{S}|$ and $\mathbf{T}(\xi) = T^\chi_{\xi, m_\xi}(t_\xi)$ for all $\xi \in |\mathbf{S}|$.

Case 1: $M \neq m_\zeta$. Let $\ell = \nu^\chi_{\zeta M}$, as in (3). We put $T = T^\chi_{\zeta M}(\ell)$; $T \in \mathbf{Colg}_\ell(\mathbf{P}(\zeta))$ by construction. By Lemma 7.3(ii) there are a multitree $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ and a tree $T' \in \mathbf{Colg}_\ell(\mathbf{P}(\zeta))$ such that $\mathbf{T} \leqslant \mathbf{U}$, $T' \subseteq_\ell T$ and $\mathbf{U}$ directly forces $\mathbf{c} \notin [T']$. Following the proof of Lemma 10.1, we define a system φ along with numbers k_ξ

and strings t_ξ such that $\psi \preccurlyeq \varphi$ and the conditions (1) and (2) hold. To fulfill (3), we also put $T^{\varphi}_{\zeta M}(\ell) = T'$ in the construction of φ from χ.

Case 2: $M = m_\zeta$. The construction in Case 1 does not work here since the last step $T^{\varphi}_{\zeta M}(\ell) = T'$ can contradict the definition $T^{\varphi}_{\zeta, m_\zeta}(\ell)$ by item (II) in the proof of Lemma 10.1. The correct argument uses the density of sets of the form $D(\sigma)$. These sets, being dense, are moreover open dense. Hence there exists a multitree $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$ satisfying $\mathbf{T} \leqslant \mathbf{U}$ and $\mathbf{U} \in D(\sigma)$ for every string $\sigma \in 2^H$, where $H = \operatorname{spl}_\ell(T^{\varphi}_{\zeta M}(\ell))$. Thus, if $\sigma \in 2^H$, then $\mathbf{U}$ directly forces $\mathbf{c} \notin [\sigma \cdot \mathbf{U}(\zeta)]$.

Furthermore, still following the proof of Lemma 10.1, we define a system φ, $\psi \preccurlyeq \varphi$, numbers k_ξ and strings t_ξ such that the conditions (1) and (2) hold. Finally, we claim that (3) holds as well. Indeed, by the choice of $\mathbf{c}$ it suffices to prove that $T^{\varphi}_{\zeta M}(\ell) \subseteq \bigcup_{\sigma \in 2^H} (\sigma \cdot \mathbf{U}(\zeta))$. To that end, we note that, by (2) and the assumption of Case 2, the inequalities

$$\mathbf{U}(\zeta) = T^{\varphi}_{\zeta M}(t_\zeta) = T^{\varphi}_{\zeta M}(\ell)(\to t_\zeta) = T^{\varphi}_{\zeta M}(\ell)\restriction_u$$

hold for some string $u \in 2^H$ by Lemma 4.2(i). The required statement $T^{\varphi}_{\zeta M}(\ell) \subseteq \bigcup_{\sigma \in 2^H} (\sigma \cdot \mathbf{U}(\zeta))$ now follows by Lemma 3.4(v). $\square$

We now complete the verification of 8.1(F). By the lemma, there exists a number j such that the system $\varphi(j)$ belongs to Δ. Thus there exists a multitree $\mathbf{U} \in \mathbf{MT}(\mathbf{P})$, along with numbers k_ξ and strings t_ξ, satisfying (1), (2) and (3) for $\varphi = \varphi(j)$. Define a multitree $\mathbf{V} \in \mathbf{MT}(\mathbf{Q})$ by putting $|\mathbf{V}| = |\mathbf{U}|$ and if $\xi \in |\mathbf{U}|$, then $\mathbf{V}(\xi) = T^{\mathbf{\Phi}}_{\xi, k_\xi}(\to t_\xi) = T^{\varphi(j)}_{\xi, k_\xi}(t_\xi)$. Then $\mathbf{S} \leqslant \mathbf{V}$ and $\mathbf{U} \leqslant \mathbf{V}$ (see the end of the verification of 8.1(E)). Finally, the multitree $\mathbf{U}$ directly forces $\mathbf{c} \notin [T]$ by (3), where $T = T^{\varphi(j)}_{\zeta M}(\ell)$. Therefore $\mathbf{V}$ directly forces $\mathbf{c} \notin [T]$ as well. However $U = U^{\mathbf{\Phi}}_{\zeta M} \subseteq T^{\varphi(j)}_{\zeta M}(\ell)$. This completes the verification of 8.1(F).

The proof of Theorem 8.2 is complete.

§ 12. The main forcing

In this section, we argue in *the constructible universe* $\mathbf{L}$. Let $\leqslant_{\mathbf{L}}$ denote the canonical well-ordering of $\mathbf{L}$.

Definition 12.1 (in $\mathbf{L}$). By induction on $\alpha < \omega_1$ we define multiforcings $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ that satisfy $|\mathbf{Q}_\alpha| = |\mathbf{P}_\alpha| = \alpha = \{\xi \colon \xi < \alpha\}$, and are *small* in the sense that all constituent sets (ST-forcings) $\mathbf{P}_\alpha(\xi)$, $\mathbf{Q}_\alpha(\xi)$, where $\xi < \alpha < \omega_1$, are countable.

The base of the induction. As $|\mathbf{Q}_0| = |\mathbf{P}_0| = 0 = \varnothing$ is required, $\mathbf{P}_0$ and $\mathbf{Q}_0$ are equal to the empty set. At the first non-trivial step, we define a multiforcing $\mathbf{P}_1$ with $|\mathbf{P}_1| = 1 = \{0\}$ by $\mathbf{P}_1(0) = P_{\text{coh}}$ (see Example 5.3.)

Step $\mathbf{P}_\alpha \to \mathbf{Q}_\alpha$. Suppose that $0 < \alpha < \omega_1$ and small multiforcings $\mathbf{P}_\gamma$, $\mathbf{Q}_\gamma$ have already been defined for $\gamma \leqslant \alpha$, resp., $\gamma < \alpha$. Let $\mathfrak{M}_\alpha$ be equal to the least (transitive) model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\mu$, $\mu < \omega_1$, containing sequences $\langle \mathbf{P}_\gamma \rangle_{\gamma \leqslant \alpha}$ and $\langle \mathbf{Q}_\gamma \rangle_{\gamma < \alpha}$ and such that $\alpha < \omega_1^{\mathfrak{M}}$ and all the sets $\mathbf{P}_\gamma(\xi)$ $(\xi < \gamma \leqslant \alpha)$ and $\mathbf{Q}_\gamma(\xi)$ $(\xi < \gamma < \alpha)$ are countable in $\mathfrak{M}$. The model $\mathfrak{M}_\alpha$ is countable and $\mathbf{P}_\alpha \in \mathfrak{M}_\alpha$, whence by Theorem 8.2 there is a small multiforcing $\mathbf{Q}$ which is an $\mathfrak{M}_\alpha$-extension of $\mathbf{P}_\alpha$. Let $\mathbf{Q}_\alpha$ be the $\leqslant_{\mathbf{L}}$-least such multiforcing $\mathbf{Q}$.

Step $\mathbf{P}_\alpha, \mathbf{Q}_\alpha \to \mathbf{P}_{\alpha+1}$. Assume that $0 < \alpha < \omega_1$, the small multiforcings $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ have already been defined and $|\mathbf{P}_\alpha| = |\mathbf{Q}_\alpha| = \alpha$. We define an auxiliary multiforcing $\boldsymbol{\kappa}_\alpha$ such that $|\boldsymbol{\kappa}_\alpha| = \{\alpha\}$ and $\boldsymbol{\kappa}_\alpha(\alpha) = P_{\text{coh}}$ (see Example 4.1.) Now let $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha \cup^{\text{cw}} \mathbf{Q}_\alpha \cup^{\text{cw}} \boldsymbol{\kappa}_\alpha$ (componentwise union; see § 6) so that $|\mathbf{P}_{\alpha+1}| = \alpha + 1$, $\mathbf{P}_{\alpha+1}(\xi) = \mathbf{P}_\alpha(\xi) \cup \mathbf{Q}_\alpha(\xi)$ for all $\xi < \alpha$ and, in addition, $\mathbf{P}_{\alpha+1}(\alpha) = P_{\text{coh}}$.

The limit step. Assume that $\lambda < \omega_1$ is a limit ordinal and the small multiforcings $\mathbf{P}_\alpha$ with $|\mathbf{P}_\alpha| = \alpha$ have been defined for all $\alpha < \lambda$. We define a multiforcing $\mathbf{P}_\lambda = \bigcup^{\mathbf{cw}}_{\alpha<\lambda} \mathbf{P}_\alpha$ so that $|\mathbf{P}_\lambda| = \lambda$ and if $\xi < \lambda$, then $\mathbf{P}_\lambda(\xi) = \bigcup_{\xi<\alpha<\lambda} \mathbf{P}_\alpha(\xi)$.

The final step. Assume that all multiforcings $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ ($\alpha < \omega_1$) have been defined and satisfy $|\mathbf{Q}_\alpha| = |\mathbf{P}_\alpha| = \alpha$. We define a multiforcing $\mathbf{P} = \mathbf{P}_{\omega_1} = \bigcup^{\mathbf{cw}}_{\alpha<\omega_1} \mathbf{P}_\alpha$ so that $|\mathbf{P}| = \omega_1$ and $\mathbf{P}(\xi) = \bigcup_{\xi<\alpha<\omega_1} \mathbf{P}_\alpha(\xi)$ for all indices $\xi < \omega_1$. We put $\mathbb{P} = \mathbf{MT}(\mathbf{P})$ and if $\alpha < \omega_1$, then $\mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$. This ends the construction.

The set $\mathbb{P} = \mathbf{MT}(\mathbf{P})$ will be the forcing notion for proving Theorem 1.1. We note that the sets $\mathbf{P}$ and $\mathbb{P}$ belong to $\mathbf{L}$ by construction. The forcing notion $\mathbb{P}$ can be identified with the finite-support product $\prod_{\xi<\omega_1} \mathbf{P}(\xi)$ in $\mathbf{L}$.

Remark 12.2. If $0 < \alpha < \omega_1$, then, by construction, the multiforcing $\mathbf{Q}_\alpha$ is an $\mathfrak{M}_\alpha$-extension of $\mathbf{P}_\alpha$ in the sense of Definition 8.1.

Lemma 12.3 (in $\mathbf{L}$). *We have* $\mathbf{P} = \bigcup^{\mathbf{cw}}_{\alpha<\omega_1} (\boldsymbol{\kappa}_\alpha \cup^{\text{cw}} \mathbf{Q}_\alpha)$ *and if* $\lambda < \omega_1$ *is a limit ordinal, then* $\mathbf{P}_\lambda = \bigcup^{\mathbf{cw}}_{\alpha<\lambda} (\boldsymbol{\kappa}_\alpha \cup^{\text{cw}} \mathbf{Q}_\alpha)$. *Accordingly,* $\mathbf{P}(\xi) = P_{\text{coh}} \cup \bigcup_{\xi<\alpha<\omega_1} \mathbf{Q}_\alpha(\xi)$ *and if* $\xi < \lambda < \omega_1$, *then* $\mathbf{P}_\lambda(\xi) = P_{\text{coh}} \cup \bigcup_{\xi<\alpha<\lambda} \mathbf{Q}_\alpha(\xi)$.

Proof. Argue by elementary transfinite induction on λ. □

The next result (routine proof omitted) reveals the definability class of the sequences introduced by Definition 12.1.

Recall that HC is the set of all *hereditarily countable* sets. On the definability classes $\varSigma^X_n$, $\varPi^X_n$, $\varDelta^X_n$ for any set X see [30], Ch. 5, § 4. Especially on the classes $\varSigma^{\text{HC}}_n$, $\varPi^{\text{HC}}_n$, $\varDelta^{\text{HC}}_n$ for $X = \text{HC}$ see [10], §§ 8, 9.

Proposition 12.4. *All three sequences* $\langle \mathbf{P}_\alpha \rangle_{\alpha<\omega_1}$, $\langle \mathbf{Q}_\alpha \rangle_{\alpha<\omega_1}$, $\langle \mathfrak{M}_\alpha \rangle_{\alpha<\omega_1}$ *belong to the definability class* $\varDelta^{\text{HC}}_1$ *in* $\mathbf{L}$.

Definition 12.5. For simplicity of notation we put $P^\alpha_\xi = \mathbf{P}_\alpha(\xi)$ and $Q^\alpha_\xi = \mathbf{Q}_\alpha(\xi)$ for $\xi < \alpha < \omega_1$, and $P_\xi = \mathbf{P}(\xi)$. Thus the following assertions hold:

(i) $\mathbf{P}_\alpha = \langle P^\alpha_\xi \rangle_{\xi<\alpha}$, $\mathbf{Q}_\alpha = \langle Q^\alpha_\xi \rangle_{\xi<\alpha}$, $\mathbf{P} = \langle P_\xi \rangle_{\xi<\omega_1}$;

(ii) $P^{\alpha+1}_\alpha = P_{\text{coh}}$, and if $\xi < \alpha$, then $P^{\alpha+1}_\xi = P^\alpha_\xi \cup Q^\alpha_\xi$;

(iii) $P^\lambda_\xi = \bigcup_{\xi<\alpha<\lambda} P^\alpha_\xi$ for limit ordinals λ, and $P_\xi = \bigcup_{\xi<\alpha<\omega_1} P^\alpha_\xi$;

(iv) $P^\lambda_\xi = P_{\text{coh}} \cup \bigcup_{\xi<\alpha<\lambda} Q^\alpha_\xi$ for all λ, and $P_\xi = P_{\text{coh}} \cup \bigcup_{\xi<\alpha<\omega_1} Q^\alpha_\xi$;

(v) $\mathbb{P}$ can be identified with the finite-support product $\prod_{\xi<\omega_1} P_\xi$.

§ 13. Density preservation and other forcing properties

Here we prove some corollaries of the results in §§ 10 and 11, and some other theorems on the forcing notion $\mathbb{P}$, including the countable antichain condition. We argue under the conditions and notation of Definition 12.1.

Lemma 13.1 (in $\mathbf{L}$). *The following assertions hold.*

(i) *If $\alpha < \omega_1$ and $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}^\alpha$ is a pre-dense set in $\mathbb{P}^\alpha$, then it remains pre-dense in $\mathbb{P}$.*

(ii) *Every set $\mathbf{MT}(\mathbf{Q}_\alpha)$ is pre-dense in $\mathbb{P}$.*

(iii) *If $\xi < \alpha < \omega_1$, then the set Q^α_ξ is pre-dense in P_ξ.*

Proof. (i) By induction on γ, $\alpha \leqslant \gamma < \omega_1$, we check that if D is pre-dense in $\mathbb{P}^\gamma = \mathbf{MT}(\mathbf{P}_\gamma)$, then it remains pre-dense in $\mathbf{MT}(\mathbf{P}_\gamma \cup^{\mathrm{cw}} \mathbf{Q}_\gamma)$ by Remark 12.2 and Conditions 8.1(E) in the definition of $\mathfrak{M}$-extension. Therefore it is also pre-dense in $\mathbb{P}^{\gamma+1} = \mathbf{MT}(\mathbf{P}_{\gamma+1})$ since by construction we have $\mathbf{P}_{\gamma+1} = \mathbf{P}_\gamma \cup^{\mathrm{cw}} \mathbf{Q}_\gamma \cup^{\mathrm{cw}} \boldsymbol{\kappa}_\gamma$, where the multiforcings $\mathbf{P}_\gamma \cup^{\mathrm{cw}} \mathbf{Q}_\gamma$ and $\boldsymbol{\kappa}_\gamma$ have disjoint domains $|\mathbf{P}_\gamma \cup^{\mathrm{cw}} \mathbf{Q}_\gamma| = \gamma$ and $|\boldsymbol{\kappa}_\gamma| = \{\gamma\}$. The limit steps, including the passage to $\mathbb{P}$ at step ω_1, are elementary.

(ii) The set $\mathbf{MT}(\mathbf{Q}_\alpha)$ is dense in $\mathbf{MT}(\mathbf{P}_\alpha \cup^{\mathrm{cw}} \mathbf{Q}_\alpha)$ by Remark 12.2 and Condition 8.1(B). Therefore it is pre-dense in $\mathbb{P}^{\alpha+1}$ (see the proof of (i)) and $\mathbf{MT}(\mathbf{Q}_\alpha) \in \mathfrak{M}_{\alpha+1}$. It remains to refer to (i).

(iii) Take $T \in P_\xi$. We define a multitree $\mathbf{T} \in \mathbb{P}$ by the conditions $|\mathbf{T}| = \{\xi\}$ and $\mathbf{T}(\xi) = T$. It follows from (ii) that $\mathbf{T}$ is compatible in $\mathbb{P}$ with a multitree $\mathbf{S} \in \mathbf{MT}(\mathbf{Q}_\alpha)$. Then $T = \mathbf{T}(\xi)$ is compatible in P_ξ with the tree $T' = \mathbf{S}(\xi) \in Q^\alpha_\xi$. □

Corollary 13.2 (in $\mathbf{L}$). *If $\xi < \alpha < \omega_1$ and $T, T' \in P^\alpha_\xi$ are incompatible trees in P^α_ξ, then T, T' are incompatible in P_ξ. Therefore if $\mathbf{T}, \mathbf{T}' \in \mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$ are incompatible multitrees in $\mathbb{P}^\alpha$, then $\mathbf{T}$, $\mathbf{T}'$ are incompatible in $\mathbb{P}$.*

Proof. Assume that $T, T' \in P^\alpha_\xi$ are incompatible in P^α_ξ. Applying Remark 12.2 and Condition 8.1(D) at successor steps, one proves by induction on γ that if $\alpha < \gamma \leqslant \omega_1$, then the trees T, T' are incompatible in P^γ_ξ. □

We need the following lemma to prove the countability of antichains.

Lemma 13.3 (in $\mathbf{L}$). *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$, then the set $\mathscr{O}_X$ of all ordinals $\alpha < \omega_1$ such that the structure $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$ and $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ is unbounded in ω_1.*

Generally, if $X_n \subseteq \mathrm{HC}$ for all n, then the set $\mathscr{O}$ of all ordinals $\alpha < \omega_1$ such that the structure $\big\langle \mathbf{L}_\alpha; \langle X_n \cap \mathbf{L}_\alpha \rangle_{n<\omega} \big\rangle$ is an elementary submodel of $\big\langle \mathbf{L}_{\omega_1}; \langle X_n \rangle_{n<\omega} \big\rangle$ and $\langle X_n \cap \mathbf{L}_\alpha \rangle_{n<\omega} \in \mathfrak{M}_\alpha$ is unbounded in ω_1.

Proof. Let $\alpha_0 < \omega_1$. There exists a countable elementary submodel M of $\langle \mathbf{L}_{\omega_2}; \in \rangle$ containing α_0, ω_1, X and such that the set $M \cap \mathbf{L}_{\omega_1}$ is transitive. We consider the Mostowski collapse map $\phi \colon M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$. Let $\alpha = \phi(\omega_1)$. Then $\alpha_0 < \alpha < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\alpha$ by the choice of M. We conclude that $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$. In addition, α is uncountable in $\mathbf{L}_\lambda$. Therefore $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. It follows that $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ since $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$ by construction.

The general claim is proved similarly. □

Corollary 13.4. *The forcing notion $\mathbb{P}$ satisfies the countable antichain condition in $\mathbf{L}$. Therefore $\mathbb{P}$-generic extensions preserve cardinals.*

Proof. Consider a maximal antichain $A \subseteq \mathbb{P} = \mathbf{MT}(\mathbf{P})$. By Lemma 13.3 there is an ordinal α such that the structure $\langle \mathbf{L}_\alpha; \mathbb{P}', A' \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; \mathbb{P}, A\rangle$, where $\mathbb{P}' = \mathbb{P} \cap \mathbf{L}_\alpha$ and $A' = A \cap \mathbb{P}^\alpha$, and in addition $\mathbb{P}', A' \in \mathfrak{M}_\alpha$. By the elementariness we have $\mathbb{P}' = \mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$ and $A' = A \cap \mathbb{P}^\alpha \in \mathfrak{M}_\alpha$, and A' is a maximal antichain, hence a pre-dense set in $\mathbb{P}^\alpha$. But then A' remains pre-dense, and hence a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 13.1. We conclude that $A = A'$, that is, A is countable. □

§ 14. The generic extension

In this section, we consider some properties of $\mathbb{P}$-generic extensions $\mathbf{L}[G]$ of $\mathbf{L}$ obtained by adjoining a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ to $\mathbf{L}$. The arguments involve the forcing notion $\mathbb{P}$ defined in $\mathbf{L}$ and other objects introduced in Definition 12.1. However the arguments, generally speaking, will no longer be relativized to $\mathbf{L}$. Therefore the first uncountable ordinal in $\mathbf{L}$ will be denoted by $\omega_1^{\mathbf{L}}$ rather than ω_1.

Corollary 14.1. *If $\alpha < \omega_1^{\mathbf{L}}$ and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, then the set $G' = G \cap \mathbb{P}^\alpha$ is $\mathbb{P}^\alpha$-generic over $\mathfrak{M}_\alpha$.*

Proof. All multitrees in G' are pairwise compatible in $\mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$ by Corollary 13.2. Furthermore, if $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}^\alpha$ is a dense set in $\mathbb{P}^\alpha$, then it is pre-dense in $\mathbb{P}$ by Lemma 13.1. Therefore $G \cap D \neq \varnothing$ and $G' \cap D \neq \varnothing$. □

If $\Delta \subseteq \omega_1^{\mathbf{L}}$, then we put $\mathbb{P}{\restriction}\Delta = \{\mathbf{T} \in \mathbb{P} \colon |\mathbf{T}| \subseteq \Delta\}$.

Lemma 14.2. *Assume that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}}$ and $\Delta' = \omega_1^{\mathbf{L}} \setminus \Delta$. Then $\mathbb{P}$ is equal to the product $(\mathbb{P}{\restriction}\Delta) \times (\mathbb{P}{\restriction}\Delta')$. If $G \subseteq \mathbb{P}$ is a generic set over $\mathbf{L}$, then the set $G{\restriction}\Delta = \{\mathbf{T} \in G \colon |\mathbf{T}| \subseteq \Delta\}$ is $(\mathbb{P}{\restriction}\Delta)$-generic over $\mathbf{L}$.*

Assume that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}}$. As in Definition 7.1, we define *a $(\mathbb{P}{\restriction}\Delta)$-real name* to be any indexed family $\mathbf{c} = \langle D^{\mathbf{c}}_{ni}\rangle_{n<\omega,\ i<2}$ of sets $D^{\mathbf{c}}_{ni} \subseteq \mathbb{P}{\restriction}\Delta$ such that every union $D^{\mathbf{c}}_n = D^{\mathbf{c}}_{n0} \cup D^{\mathbf{c}}_{n1}$ is pre-dense in $\mathbb{P}{\restriction}\Delta$ and if $\mathbf{T} \in D^{\mathbf{c}}_{n0}$ and $\mathbf{S} \in D^{\mathbf{c}}_{n1}$, then the multitrees $\mathbf{T}$, $\mathbf{S}$ are incompatible in $\mathbb{P}{\restriction}\Delta$ or, equivalently, in $\mathbb{P}$. A name is *countable* if all sets $D^{\mathbf{c}}_{ni}$ are at most countable.

If $G \subseteq \mathbb{P}{\restriction}\Delta$ is a generic set, then we define a real $\mathbf{c}[G] \in 2^\omega$ by letting $\mathbf{c}[G](n) = i$ if and only if $G \cap D^{\mathbf{c}}_{ni} \neq \varnothing$.

Lemma 14.3. *Suppose that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}}$. If $G' \subseteq \mathbb{P}{\restriction}\Delta$ is a generic set over $\mathbf{L}$ and $x \in 2^\omega \cap \mathbf{L}[G']$, then there exists a $(\mathbb{P}{\restriction}\Delta)$-real name $\mathbf{c} \in \mathbf{L}$ such that $x = \mathbf{c}[G']$ and $\mathbf{c}$ is countable in $\mathbf{L}$.*

Proof. Without the countability condition, the existence of such a name is one of the basic forcing properties (see for example Lemma 2.5 in [30], Ch. 4). To convert an arbitrary real name into a countable one, note that the forcing notion $\mathbb{P}{\restriction}\Delta$ inherits the countable antichain condition in $\mathbf{L}$ from $\mathbb{P} = \mathbf{MT}(\mathbf{P})$, which has this property by Corollary 13.4. □

Definition 14.4 (generic reals). Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathbf{L}$. Note that $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$ by Corollary 13.4.

If $\xi < \omega_1^{\mathbf{L}}$, then the set $G_\xi = \{\mathbf{T}(\xi) \colon \mathbf{T} \in G\}$ is P_ξ-generic over $\mathbf{L}$. It follows that the intersection $X_\xi = \bigcap_{T \in G_\xi} [T]$ contains the single element $x_\xi = x_\xi[G] \in 2^\omega$ and this element is a P_ξ-generic real over $\mathbf{L}$.

The following lemma (an excerpt from the product forcing theorem) also reflects the product structure of the forcing notion $\mathbb{P} = \prod_{\xi<\omega_1^{\mathbf{L}}} P_\xi$. The lemma involves the notation of Definition 14.4.

Lemma 14.5. *If $\zeta < \omega_1^{\mathbf{L}}$, then the following assertions hold:*

(i) *$x_\zeta[G] \notin \mathbf{L}[G\restriction\Delta_\zeta]$, where $\Delta_\zeta = \omega_1^{\mathbf{L}} \setminus \{\zeta\}$, and*

(ii) *the real $x_\zeta[G]$ is not $(\{G\restriction\Delta_\zeta\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$.*

Proof. Claim (i) is a part of the product forcing theorem; it reflects the product structure of the forcing notion $\mathbb{P} = \prod_{\xi<\omega_1^{\mathbf{L}}} P_\xi$. To prove (ii), suppose to the contrary that $\vartheta(x)$ is a formula containing ordinals and the set $G\restriction\Delta_\zeta$ as parameters, and a multitree $\mathbf{T} \in G$ $\mathbb{P}$-forces that $x_\zeta[G]$ is the only real $x \in 2^\omega$ satisfying $\vartheta(x)$. Let $T = \mathbf{T}(\zeta)$ and $s = \operatorname{stem}(T)$, so that T contains both of the strings $s^\frown 0$ and $s^\frown 1$. Then either $s^\frown 0 \subset x_\xi[G]$ or $s^\frown 1 \subset x_\xi[G]$; we assume that $s^\frown 0 \subset x_\xi[G]$.

Let $n = \operatorname{lh}(s)$ and $\sigma = 0^{n\frown}1$. All three strings $s^\frown 0$, $s^\frown 1$, σ belong to 2^{n+1}, $s^\frown 1 = \sigma \cdot s^\frown 0$, and $\sigma \cdot T = T$ by Lemma 3.4(iv). We extend the action of σ to multitrees by $\sigma \cdot \mathbf{S} = \mathbf{S}'$, where $|\mathbf{S}'| = |\mathbf{S}|$, $\mathbf{S}'(\zeta) = \sigma \cdot \mathbf{S}(\zeta)$, and $\mathbf{S}'(\xi) = \mathbf{S}(\xi)$ for all $\xi \in |\mathbf{S}'| = |\mathbf{S}|$, $\xi \neq \zeta$. But the forcing notions P_ξ and $\mathbb{P}$ are invariant under the action of σ. Thus the set $G' = \sigma \cdot G$ is still $\mathbb{P}$-generic over $\mathbf{L}$, and $\mathbf{T} = \sigma \cdot \mathbf{T} \in G'$. We conclude that it is true in $\mathbf{L}[G'] = \mathbf{L}[G]$ that $x' = x_\xi[G'] = \sigma \cdot x_\xi[G]$ is the only real satisfying $\vartheta(x')$. But $x' \neq x$. $\square$

§ 15. Definability of generic reals

We continue to argue in terms of Definitions 12.1 and 14.4. The goal of this section is to study the nature of P_ξ-generic reals $x \in 2^\omega$ in $\mathbb{P}$-generic extensions of $\mathbf{L}$.

Lemma 15.1. *In any transitive model of* **ZF** *extending* $\mathbf{L}$, *it is true that if $\xi < \omega_1^{\mathbf{L}}$ and $x \in 2^\omega$, then the real x is P_ξ-generic over $\mathbf{L}$ if and only if x belongs to the set $Z_\xi = \bigcap_{\xi<\alpha<\omega_1^{\mathbf{L}}} \bigcup_{T\in Q_\xi^\alpha} [T]$.*

Proof. By Lemma 13.1(iii) all sets of the form Q_ξ^α are pre-dense in P_ξ, and hence every P_ξ-generic real belongs to Z_ξ. On the other hand, every maximal antichain $A \in \mathbf{L}$, $A \subseteq P_\xi$ is countable in $\mathbf{L}$ by Corollary 13.4. Therefore $A \subseteq P_\xi^\alpha$ and $A \in \mathfrak{M}_\alpha$ for some index α, $\xi < \alpha < \omega_1^{\mathbf{L}}$. But then every tree $T \in Q_\xi^\alpha$ satisfies $T \subseteq^{\text{fin}} \bigcup A$ by Remark 12.2 and Condition 8.1(C), b). We now conclude that $\bigcup_{T\in Q_\xi^\alpha} [T] \subseteq \bigcup_{S\in A} [S]$. $\square$

The next lemma claims that $\mathbb{P}$-generic extensions contain no P_ξ-generic reals except for the real $x_\xi[G]$ itself and all the reals connected to it in the sense of a well-known equivalence relation.

Definition 15.2. The equivalence relation E_0 is defined on 2^ω by saying that $x \mathrel{\mathsf{E}_0} y$ if and only if $\exists s \in 2^{<\omega} (y = s \cdot x)$. Clearly, $x \mathrel{\mathsf{E}_0} y$ if and only if the set $\{n \colon x(n) \neq y(n)\}$ is finite. The E_0-class

$$[x]_{\mathsf{E}_0} = \{y \in 2^\omega : x \mathrel{\mathsf{E}_0} y\} = \{y \in 2^\omega : \exists s \in 2^{<\omega} (y = s \cdot x)\}$$

of any real $x \in 2^\omega$ is a countable set.

Lemma 15.3. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over* $\mathbf{L}$, $\zeta < \omega_1^{\mathbf{L}}$ *and* $x \in \mathbf{L}[G] \cap 2^\omega$. *Then x is a P_ζ-generic real over* $\mathbf{L}$ *if and only if* $x \mathsf{E}_0 x_\zeta[G]$.

Proof. In the easy direction, the real $x_\zeta[G]$ is P_ζ-generic. But P_ζ is an ST-forcing. Hence, by definition, it is closed under the action $s \cdot T$ of any string $s \in 2^{<\omega}$. We conclude that every real of the form $s \cdot x_\zeta[G]$ is P_ζ-generic as well.

Now for the hard direction. Assume that $x \in \mathbf{L}[G] \cap 2^\omega$ and $\neg(x \mathsf{E}_0 x_\zeta[G])$. By Lemma 14.3, there exists a $\mathbb{P}$-real name $\mathbf{c} \in \mathbf{L}$, countable in $\mathbf{L}$ and such that $\mathbb{P}$ as a whole (that is, every 'condition' $\mathbf{T} \in \mathbb{P}$) forces $\mathbf{c} \neq \sigma \cdot x_\zeta[\underline{G}]$ for any string $\sigma \in 2^{<\omega}$. As the name $\mathbf{c}$ is countable, there is an ordinal α, $\zeta < \alpha < \omega_1^{\mathbf{L}}$, such that $\mathbf{c} \in \mathfrak{M}_\alpha$ and every set $D^{\mathbf{c}}_{ni}$ satisfies $D^{\mathbf{c}}_{ni} \subseteq \mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$ for all n, i.

We now claim that the premise of Condition 8.1(F) holds, that is, the set

$$D(\sigma) = \{\mathbf{T} \in \mathbf{MT}(\mathbf{P})\colon \mathbf{T} \text{ directly forces } \mathbf{c} \notin [\sigma \cdot \mathbf{T}(\zeta)]\}$$

is dense in $\mathbb{P}$ for every $\sigma \in 2^{<\omega}$. Indeed, let $\sigma \in 2^{<\omega}$ and let $\mathbf{S}$ be an arbitrary tree in $\mathbb{P}$. Then $\mathbf{S}$ forces $\mathbf{c} \neq \sigma \cdot x_\zeta[\underline{G}]$ by the assumption above. It follows that there is a stronger 'condition' $\mathbf{T} \in \mathbb{P}$, $\mathbf{S} \leqslant \mathbf{T}$, and a pair of strings $u \neq v$ in $2^{<\omega}$ of equal length $\mathrm{lh}(u) = \mathrm{lh}(v) = n$ such that $\mathbf{T}$ forces (and then also directly forces) $u \subset \mathbf{c}$ and $v \subset \sigma \cdot x_\zeta[\underline{G}]$. As $v \subset \sigma \cdot x_\zeta[\underline{G}]$ is forced, the tree $T = \mathbf{T}(\zeta)$ satisfies $v \subset \sigma \cdot \mathrm{stem}(T)$. Therefore, since the strings $u \neq v$ have equal length, $\mathbf{T}$ directly forces $\mathbf{c} \notin [\sigma \cdot \mathbf{T}(\zeta)]$, that is, $\mathbf{T} \in D(\sigma)$, completing the proof of density.

By Lemma 13.3 and the claim just proved, we can assume that the same ordinal α satisfies the following condition: if $\sigma \in 2^{<\omega}$, then the set $D'(\sigma) = D(\sigma) \cap \mathbb{P}^\alpha$ is dense in $\mathbb{P}^\alpha$.

It now is implied by Remark 12.2 that if $U \in Q^\alpha_\zeta$, then the set M_U of all multitrees $\mathbf{V} \in \mathbb{P}^\alpha = \mathbf{MT}(\mathbf{P}_\alpha)$ which directly force $\mathbf{c} \notin [U]$ is dense in $\mathbf{MT}(\mathbf{P}_\alpha \cup^{\mathrm{cw}} \mathbf{Q}_\alpha)$ and, therefore, pre-dense in $\mathbb{P}^{\alpha+1} = \mathbf{MT}(\mathbf{P}_{\alpha+1})$. But we have $M_U \in \mathfrak{M}_{\alpha+1}$. It follows that M_U is pre-dense in $\mathbb{P}$ by Lemma 13.1. Therefore if $U \in Q^\alpha_\zeta$, then $M_U \cap G \neq \varnothing$, whence $x \notin [U]$. In other words, $x \notin \bigcup_{U \in Q^\alpha_\zeta} [U]$. Thus $\mathbf{c}$ is not a P_ζ-generic real by Lemma 15.1, as required. □

§ 16. A non-uniformizable set

This brief section contains a key ingredient in the proof of Theorem 1.1, our first main result. It is as follows.

Lemma 16.1. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over* $\mathbf{L}$. *Then the set* $K = K[G] = \{\langle \xi, x\rangle\colon \xi < \omega_1^{\mathbf{L}} \wedge x \mathsf{E}_0 x_\xi[G]\}$ *belongs to* $\mathbf{L}[G]$ *and has the following properties in* $\mathbf{L}[G]$:

(i) $K = \{\langle \xi, x\rangle\colon \xi < \omega_1 \wedge$ *the real* $x \in 2^\omega$ *is P_ξ-generic over* $\mathbf{L}\}$;

(ii) *K belongs to the definability class* Π^{HC}_1;

(iii) *if $\xi < \omega_1$ then the cross-section* $K_\xi = \{x\colon \langle \xi, x\rangle \in K\}$ *is a* E_0-*class*;

(iv) *the set K is not* ROD-*uniformizable.*

Proof. Part (i) holds by Lemma 15.3 while (iii) holds by definition: $K_\xi = [x_\xi[G]]_{\mathsf{E}_0}$. To prove (ii), note that Corollary 13.4 implies the equality $\omega_1 = \omega_1^{\mathbf{L}}$ in $\mathbf{L}[G]$. Hence, by Lemma 15.1, the formula $\langle\xi, x\rangle \in K$ is equivalent to the assertion

$$\xi < \omega_1 \wedge \forall\, \alpha\ \big(\xi < \alpha < \omega_1 \Rightarrow \exists T \in Q^{\alpha}_{\xi}\ (x \in [T])\big).$$

Here the formula in the outer parentheses expresses a Δ^{HC}_1-relation by Proposition 12.4.

Now for the proof of part (iv). Suppose to the contrary that, in $\mathbf{L}[G]$, $R \subseteq K$ is a uniformizing ROD set. Let $r \in 2^{\omega} \cap \mathbf{L}[G]$ be a real parameter ensuring that R is $\{r\} \cup \mathbf{Ord}$-definable in $\mathbf{L}[G]$.

Corollary 13.4 implies that there exists an ordinal $\zeta < \omega_1^{\mathbf{L}}$ satisfying $r \in \mathbf{L}[G{\restriction}\zeta]$. Hence $r \in \mathbf{L}[G{\restriction}\Delta_\zeta]$, where $\Delta_\zeta = \omega_1^{\mathbf{L}} \setminus \{\zeta\}$. Therefore the only real $x \in 2^{\omega}$ with $\langle\zeta, x\rangle \in R$ is $(\{G{\restriction}\Delta_\zeta\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$. However, $R \subseteq K$. Thus we have $x\ \mathsf{E}_0\ x_\zeta[G]$. It follows that the real $x_\zeta[G]$ is itself $(\{G{\restriction}\Delta_\zeta\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$. But this contradicts Lemma 14.5(ii). □

§ 17. A non-uniformizable set in the Euclidean plane with cross-sections in the form of Vitali classes

To complete the proof of Theorem 1.1, we now transfer the set $K[G]$ to the Euclidean plane $\mathbb{R} \times \mathbb{R}$, changing E_0-classes into Vitali classes. This is a rather elementary transformation. It is involved in the proof of the next result, and has no connection with forcing and models.

Corollary 17.1. *If $K \subseteq \omega_1 \times 2^{\omega}$ satisfies conditions (ii), (iii), (iv) of Lemma 16.1 and we have $\omega_1^{\mathbf{L}} = \omega_1$, then there is a set $W \subseteq \mathbb{R} \times \mathbb{R}$ such that*

(i) *W belongs to the definability class Π^1_2;*

(ii) *if $z \in \mathbb{R}$, then the cross-section $W_z = \{x \colon \langle z, x\rangle \in W\}$ is a Vitali class;*

(iii) *the set W is not* ROD*-uniformizable.*

Proof. The construction of the required set is divided into three steps.

Step 1: transformation from $\omega_1 \times 2^{\omega}$ to the space $2^{\omega} \times 2^{\omega}$. Fix a recursive enumeration of the rational numbers, $\mathbb{Q} = \{q_n \colon n < \omega\}$. If $z \in 2^{\omega}$ then we define $Q_z = \{q_n \colon z(n) = 1\} \subseteq \mathbb{Q}$. Let $Q'_z \subseteq Q_z$ be the largest well-ordered (possibly empty) initial segment of Q_z, and let $|z| < \omega_1$ be the order type of Q'_z. Then $\{|z| \colon z \in 2^{\omega}\} = \omega_1$. We claim that the set

$$A = \{\langle z, x\rangle \in 2^{\omega} \times 2^{\omega} \colon \langle |z|, x\rangle \in K\}$$

satisfies the following conditions:

(1) A belongs to the definability class Π^1_2;

(2) if $z \in 2^{\omega}$, then the cross-section $A_z = \{x \colon \langle z, x\rangle \in A\}$ is a E_0-class;

(3) A is not ROD-uniformizable.

Indeed, A belongs to Π^{HC}_1 along with K since the map $z \mapsto |z|$ is a Δ^{HC}_1-function. Therefore A is a Π^1_2 set by the definability transfer theorem (Theorem 9.1 in [10]).

Furthermore, every cross-section A_z coincides with the corresponding cross-section K_ξ of K, where $\xi = |z|$, and hence A_z is a E_0-class.

To prove the non-uniformizability, we suppose to the contrary that A is uniformized by a ROD set $S \subseteq A$. As soon as $\omega_1^{\mathbf{L}} = \omega_1$ is assumed, for every ordinal $\xi < \omega_1$ there is a point $z \in 2^{\omega} \cap \mathbf{L}$ satisfying $|z| = \xi$. Let $z(\xi)$ be the $\leqslant_{\mathbf{L}}$-least of such points. Then

$$R = \{\langle \xi, x\rangle \in K : \langle z(\xi), x\rangle \in S\}$$

is a ROD subset of K and R uniformizes K, which contradicts the choice of K.

Thus A does indeed satisfy conditions (1), (2), (3).

Step 2: transformation to the space $\mathbb{R}\times\mathbb{R}$. Consider the set $\mathbb{X}$ of all points $x \in 2^\omega$ such that both of the sets $\{k : x(k) = 0\}$, $\{k : x(k) = 1\}$ are infinite, and let $\mathbb{Y}$ be the set of all real numbers r in the interval $0 \leqslant r \leqslant 1$ which are not dyadic rationals. Both are $\mathbf{G}_\delta$ sets in the corresponding spaces 2^ω and $\mathbb{R}$. More precisely, they are sets of the lightface class $\varPi^0_2$. The map $H(x) = \sum_{x(n)=1} 2^{-n}$ is a bijection of $\mathbb{X}$ onto $\mathbb{Y}$ and a $\varDelta^1_1$ function (in fact a recursive homeomorphism). If $x, y \in \mathbb{X}$ and $x \mathsf{E}_0 y$, then $|x - y|$ is rational. We conclude that the set

$$B = \{\langle H(x), H(y)\rangle : \langle x, y\rangle \in A\} \subseteq \mathbb{R}\times\mathbb{R}$$

satisfies the following conditions:

(1′) B belongs to the definability class $\varPi^1_2$;

(2′) if $r = H(x) \in \mathbb{Y}$, then the cross-section $B_r = \{r' : \langle r, r'\rangle \in B\}$ is a part (proper or improper) of some Vitali class;

(3′) B is not ROD-uniformizable since such a uniformization would imply the ROD-uniformizability of A via the map H.

Step 3: transformation to full Vitali classes. To expand every cross-section B_r of the set B to a full Vitali class, we define a set $W \subseteq \mathbb{R}\times\mathbb{R}$ in terms of its cross-sections W_r by putting

$$W_r = \begin{cases} \{r' + q : r' \in B_r \wedge q \text{ is rational}\} & \text{if } r \in \mathbb{Y}, \\ \mathbb{Q} \text{ (all rational numbers)} & \text{if } r \notin \mathbb{Y}. \end{cases}$$

We now use this set W to complete the proof of Corollary 17.1.

Indeed, first, we have $W = P' \cup \bigcup_{q\in\mathbb{Q}} P_q$ by construction, where

$$P' = \{\langle r, r'\rangle : r' \in \mathbb{Q} \wedge r \in \mathbb{R} \setminus \mathbb{X}\}$$

is a $\varDelta^0_3$ set (since $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{X}$ are sets in $\varSigma^0_2$) while $P_q = \{\langle r, r'+q\rangle : \langle r, r'\rangle \in B\}$, where, we recall, B is a $\varPi^1_2$ set by (1′). It follows that $W \in \varPi^1_2$ as well, that is, we get condition (i) of Corollary 17.1.

To prove requirement (ii) of Corollary 17.1, assume that $r \in \mathbb{R}$. If $r \in \mathbb{Y}$, then B_r is a part of a Vitali class by (2′) and, therefore, W_r is a Vitali class. If $r \notin \mathbb{Y}$, then obviously $W_r = \mathbb{Q}$.

Finally, to prove (iii), suppose to the contrary that W is uniformized by a ROD set $S' \subseteq W$. If $r \in \mathbb{R}$, then the cross-section S'_r consists of a single point, which we denote by $f(r)$. Thus $f(r) \in W_r$ and f belongs to ROD since S' does. Now if $r \in \mathbb{Y}$, then by definition there is a rational number $q \in \mathbb{Q}$ such that $f(r) - q \in B_r$. Using the recursive enumeration $\mathbb{Q} = \{q_n : n < \omega\}$, we let $n(r)$ be the least index n such that $g(r) = f(r) - q_{n(r)} \in B_r$. The map $g : \mathbb{Y} \to \mathbb{R}$ is obviously ROD, while its graph $S = \{\langle r, g(r)\rangle : r \in \mathbb{Y}\}$ is a ROD set uniformizing B. But this contradicts condition (3′). $\square$

Lemma 16.1 and Corollary 17.1 complete the proof of Theorem 1.1.

§ 18. A paradoxical sequence of Vitali classes

This final section contains the proof of Theorem 1.2, our second main result. The model will be a certain part of a $\mathbb{P}$-generic extension $\mathbf{L}[G]$.

Definition 18.1. Let a set $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathbf{L}$. We define $X[G] = \bigcup_{\xi<\omega_1^{\mathbf{L}}} [x_\xi[G]]_{\mathsf{E}_0}$.

Thus $X[G]$ is the union of all cross-sections

$$K[G]_\xi = \{x \colon \langle \xi, x\rangle \in K[G]\} = [x_\xi[G]]_{\mathsf{E}_0} \quad (\xi < \omega_1^{\mathbf{L}})$$

of the set $K[G]$ in Lemma 16.1. By Lemma 16.1(i), $X[G]$ is equal to the set of all points $x \in 2^\omega \cap \mathbf{L}[G]$ that are P_ξ-generic for some $\xi < \omega_1^{\mathbf{L}}$. Clearly, $X[G] \in \mathbf{L}[G]$.

The model for the proof of Theorem 1.2 will be a class of the form $\mathbf{L}(X[G])$. However, for technical reasons. it appears more convenient to introduce the model through the notions of definability rather than relative constructibility.

Definition 18.2. $\mathrm{HOD}(X[G])$ is the class of all sets $z \in \mathbf{L}[G]$ that are hereditarily $(\mathbf{Ord} \cup X[G])$-definable[12] in $\mathbf{L}[G]$.

In other words, a set $z \in \mathbf{L}[G]$ belongs to $\mathrm{HOD}(X[G])$ if it itself, all its elements, all elements of elements, and so on, are definable in $\mathbf{L}[G]$ by a formula containing ordinals and points $x \in X[G]$ (finitely many of them, of course) as parameters. Note that, among all possible parameters $x \in X[G]$, it suffices to take only points of the form $x_\xi[G]$, $\xi < \omega_1^{\mathbf{L}}$, since every point $x \in [x_\xi[G]]_{\mathsf{E}_0}$ is obviously definable with $x_\xi[G]$ as a parameter.

Lemma 18.3. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathbf{L}$. Then $\mathrm{HOD}(X[G])$ is a model for Theorem 1.2.*

Proof. By definition, the class $\mathrm{HOD}(X[G])$ is transitive and contains all ordinals and all points in $X[G]$. It follows that the sets $K[G]$ and $X[G] = \operatorname{ran} K[G]$ also belong to $\mathrm{HOD}(X[G])$ since they are definable in $\mathbf{L}[G]$ by Lemma 16.1(i). The class $\mathrm{HOD}(X[G])$ itself, as well as any class of the form $\mathrm{HOD}(X)$, where $X \subseteq 2^\omega$ is definable, is a model of **ZF** on the basis of general theorems on ordinal definability (see, for example, Theorem 24 in [28]).

Furthemore, the set $K[G] \in \mathrm{HOD}(X[G])$ belongs to the definability class $\varPi_1^{\mathrm{HC}}$ in $\mathrm{HOD}(X[G])$ for the same reasons as in the proof of Lemma 16.1(ii).

After these general remarks, we now prove that the set

$$P = \{\langle \xi, x\rangle \in K[G] \colon \xi < \omega\} \subseteq \omega \times 2^\omega$$

satisfies all the conditions of Theorem 1.2. Namely, it is true in $\mathrm{HOD}(X[G])$ that

(1) the set P is $\varPi_1^{\mathrm{HC}}$ and hence $\varPi_2^1$;

(2) all vertical cross-sections $P_n = \{x \colon \langle n, x\rangle \in P\}$ $(n < \omega)$ are E_0-equivalence classes and hence countable sets;[13]

[12] HOD is a commonly used abbreviation of *hereditarily ordinal definable*. The argument $X[G]$ is the source of additional parameters (additional to the ordinals) in formulae used to define sets.

[13] The transfer of P to $\omega \times \mathbb{R}$ and the passage to Vitali classes, as formally required by Theorem 1.2, are carried out in the same way as the corresponding transformation in § 17 dealing with the example for Theorem 1.1, and hence we skip this step.

(3) the union $\bigcup_n P_n$ is not countable, or, equivalently, P is not uniformizable by any set.

Assertions (1) and (2) easily follow from the corresponding properties of the background set $K[G]$ by Lemma 16.1. This enables us to concentrate on (3). First, both forms of (3) are equivalent. Indeed, if $U = \bigcup_n P_n$ is countable, so that there is a bijection $f\colon \omega \xrightarrow{\text{onto}} U$, then a uniformizing set $Q \subseteq P$ can be defined as the set of all pairs of the form $\langle n, f(k_n)\rangle$, where $n < \omega$ and k_n is equal to the least number k satisfying $f(k) \in P_n$. To prove the converse, let $Q \subseteq P$ be a uniformizing set. Thus $Q = \{\langle n, g(n)\rangle : n < \omega\}$, where $g\colon \omega \to 2^\omega$. Then the set $U = \bigcup_n P_n$ satisfies $U = \{s \cdot g(n) : s \in 2^{<\omega} \wedge n < \omega\}$ and hence U is countable.

We now prove the non-uniformizability claim. Suppose to the contrary that the set P is uniformized by a set $R \in \mathrm{HOD}(X[G])$, $R \subseteq P$. Then by definition there is a finite set $\Xi = \{\xi_1, \dots, \xi_k\} \subseteq \omega_1^{\mathbf{L}}$ such that R is $(\{x_{\xi_1}[G], \dots, x_{\xi_k}[G]\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$ and then, clearly, $(\{G{\restriction}\Xi\} \cup \mathbf{Ord})$-definable. Consider any number $n < \omega$, $n \notin \Xi$. The unique real $x \in 2^\omega$ satisfying $\langle n, x\rangle \in R$ is $(\{G{\restriction}\Xi\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$. However, $R \subseteq K$ and hence $x \mathrel{\mathsf{E}_0} x_n[G]$. It follows that the real $x_n[G]$ is itself $(\{G{\restriction}\Xi\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[G]$. But this contradicts Lemma 14.5(ii) since $n \notin \Xi$. This completes the proof of Lemma 18.3. □

Theorem 1.2 is proved.

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