ISSN 1064-2269, Journal of Communications Technology and Electronics, 2015, Vol. 60, No. 12, pp. 1429–1431. © Pleiades Publishing, Inc., 2015.
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Easily Computable Invariants for Hypersurface Recognition

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Received December 29, 2014

Abstract—A method is proposed for the short description of an algebraic hypersurface with the help of descriptors that depend on a small number of coefficients of the corresponding polynomial form and are invariant with respect to the orthogonal transformations of the enveloping space. These invariants, which can easily be computed even for high dimensionalities, allow to compare quickly the shapes of hypersurfaces in the general position and can be used as features in applied problems of object description and recognition as well as for the solution of combinatorial problems. The transformation of real multilinear cubic forms is specially considered.

DOI: 10.1134/S1064226915120074

INTRODUCTION

Various curves and surfaces of the third degree are often used for the brief approximation and presentation of data. The Bezier plane curves are widely applied. However, the analysis of 3D images that are obtained, for example, in tomography includes the interesting study of 2D surfaces making the contour image of a 3D body [1]. Close problems appear in the analysis of electrocardiograms, though other methods of mathematical processing of data can be applied in this case [2]. When different properties of large systems are simultaneously studied, hypersurfaces appear in spaces of higher dimensionalities. The comparison of such hypersurfaces calls for quick calculation of invariants that are independent of rotations and reflections. The number of monomials of degree d depending on $n + 1$ variables equals binomial coefficient C_{n+d}^{d}. Even the number of monomials of a cubic form quickly grows with the number of variables. Therefore, a special interest is of invariants that depend on a small number of coefficients and, hence, can easily be calculated for high dimensionalities. Comparing the values of such invariant, the substantial differences of the forms of two hypersurfaces can quickly be discovered. Meanwhile, the rigorous proof of coincidence necessitates the calculation of other invariants as well, whose assembly fully determines the hypersurface accurately to the orthogonal transformations of coordinates.

Since the algebraic hypersurfaces are uniquely determined by a finite number of their points in the general position, the considered problem can be reformulated as a problem of discrete analysis. Let us denote $N = C_{n+d}^{d} - 1$ and let N points of the n-dimensional space lie on a unique hypersurface of degree d. Then, for another such set of N points, the necessary condition for the existence of the orthogonal transformation of the space that maps one set to another one is the coincidence of the invariants of hypersurfaces on which the points of the corresponding sets lie. The calculation of the hypersurface equation in a certain coordinate system is reduced to the solution of the system of linear equations. Simultaneously, the condition of the hypersurface uniqueness and the degree of this hypersurface are checked. Note that, when the uniqueness is checked, each coefficient of the equation specifying the hypersurface substantially depends on the position of each point of the set. When the coordinates of the point are integer, the coefficients of the hypersurface equation can be chosen to be integer and polynomially bounded. Therefore, the invariant of this hypersurface can serve as the certification of the invariable mutual position of the considered points. A close problem appears in geological location [3]. On the other hand, the considered method makes it possible to solve certain combinatorial problems.

Recall that a nonoriented graph corresponds to the symmetric adjacency matrix that can be considered as the matrix of coefficients of a quadratic form. Its eigenvalues are invariant of the coordinate transpositions and are the same for isomorphic graphs. This property is used in heuristic algorithms serving for checking the graph nonisomorphy, because it suffices to indicate the difference of values for any of the invariants. Analogously, a hypergraph corresponds to a form of a higher degree, and its invariants can be used for the proof of the nonisomorphy of hypergraphs. Since the permutation matrices are orthogonal, any invariants of the orthogonal group can be used to check the nonisomorphy of hypergraphs.

We suppose that the characteristic of the basic field is zero. However, many of our results remain true for finite fields of the sufficiently large characteristic too. This can be used for the solution of combinatorial problems including the construction of block error-correcting codes. The application of invariants depending on a small number of the form coefficients has an important analog over the field of the characteristic two: these are binary codes having a small density of parity tests [4].

A quadratic form corresponds to the symmetric matrix whose trace equals the sum of coefficients of the quadratic form except for multilinear terms. As is known, the trace does not change during the orthogonal transformations of coordinates. The forms of higher degrees also can be compared to the polynomials of the coefficients of nonmultilinear terms. These polynomials are invariant in presence of orthogonal transformations. Such invariants are the natural generalization of the notion of the trace.

Recall that the irreducible form of three variables over the field of complex numbers can be reduced by a linear change of coordinates to the form of monomials of two variables when the corresponding projective curve is smooth [5]. The reducibility to this form is proved in [6, 7] for a general surface. In addition, for cubics of small dimensionalities, various representations that allow predetermining their properties are known [8]. However, it is known very little for large dimensionalities.

Let us denote Δ the Laplace operator

$$\Delta f = \sum_{k=0}^{n} \frac{\partial^2 f}{\partial x_k^2}.$$

It is invariant with respect to orthogonal transformations of coordinates.

1. RESULTS

Theorem 1. *A cubic form is given. There is such a nontrivial algebraic expression of its coefficients which is invariant to orthogonal transformations of coordinates and independent of the coefficients of multilinear monomials.*

Proof. We apply the Laplace operator to the form

$$f = \sum_{k=0}^{n} \alpha_k x_k^3 + \sum_{j \neq k} \beta_{jk} x_j^2 x_k + \sum_{0 \le i < j < k \le n} \gamma_{ijk} x_i x_j x_k,$$

to obtain the invariant linear form

$$\frac{1}{2}\Delta f = \sum_{k=0}^{n} \Big(3\alpha_k + \sum_{j \neq k} \beta_{jk}\Big) x_k.$$

Squaring this form, applying again the Laplace operator, and rejecting the number multiplier, we obtain the invariant numerical expression

$$\sum_{k=0}^{n} \Big(3\alpha_k + \sum_{j \neq k} \beta_{jk}\Big)^2,$$

which is independent of coefficients γ_{ijk}. The theorem is proved.

This method can be applied to obtain also other invariants that do not depend on coefficients γ_{ijk}, for example, $\Delta\Delta\Delta(f^2)$. The linear combination of this invariant and the invariant described in Theorem 1 gives the invariant

$$\frac{\Delta\Delta\Delta(f^2) - 6\Delta((\Delta f)^2)}{48} = 6\sum_{k=0}^{n} \alpha_k^2 + 2\sum_{j \neq k} \beta_{jk}^2 + \sum_{0 \le i < j < k \le n} \gamma_{ijk}^2.$$

This method can easily be extended to higher degree forms. Applying m times the Laplace operator to the form of degree $2m + 1$, we obtain an invariant linear form. Next, applying the Laplace operator to the square of this linear form, we obtain a scalar invariant. On the other hand, applying the Laplace operator to the quadratic form, we obtain the doubled trace of its matrix. Similarly, applying m times the Laplace operator to a form of degree $2m$, we also obtain an invariant that depends only on a small number of its coefficients.

We apply the described invariant to analyze the cubic forms

$$f = \sum_{k=0}^{n} \alpha_k x_k^3 + \sum_{0 \le i < j < k \le n} \gamma_{ijk} x_i x_j x_k,$$

where all the coefficients $\beta_{jk} = 0$ of the terms containing two variables vanish.

The cubic hypersurface in $\mathbb{P}^n$ that is specified by the form of the kind

$$f = \sum_{k=0}^{n-1} \alpha_k x_k^3 + \sum_{0 \le i < j < k \le n} \gamma_{ijk} x_i x_j x_k,$$

i.e., when $\alpha_n = 0$, is a singular one. Really, at the point with the homogeneous coordinates (0: ... :0: 1), form f and all of its first derivatives are zero. Hence, if a smooth cubic hypersurface is specified by an indicated form (when $\beta_{jk} = 0$), then, $\alpha_k \neq 0$ for each index k. For singular hypersurfaces, the number of zero coefficients α_k may depend on the choice of coordinates. Let us denote ε a root of the polynomial $\varepsilon^2 + \varepsilon + 1$. Then,

$$(x + y + z)(x + \varepsilon y + \varepsilon^2 z)(x + \varepsilon^2 y + \varepsilon z) = x^3 + y^3 + z^3 - 3xyz.$$

In the case of orthogonal transformations of the real number field, the situation changes.

Theorem 2. *A multilinear cubic form is given. If, after the orthogonal transformation of coordinates, the obtained form again does not contain monomials that depend on two variables, it also does not contain monomials that depend on one variable, i.e., the form remains to be multilinear.*

Proof. The multilinear cubic form invariant that is determined in the proof of Theorem 1 equals zero. If all coefficients β_{jk} remain to be zero in the case of the orthogonal transformation of coordinates, then, the sum of squares $\sum_{k=0}^{n}\alpha_k^2$ equals zero. This is possible only when each α_k is zero. This means that the form remains to be multilinear.

3. DISCUSSION

Another method of calculation of invariants is the computation of eigenvalues of symmetrical tensors [9, 10] or, speaking more accurately, E-eigenvalues according to the terminology from [9]. Note that in studies [9, 11], another definition of eigenvalues that are noninvariant under orthogonal transformations is also discussed.

The determinant of a matrix is expressed in terms of the traces of its degrees. Analogously, using the suggested generalization of the trace, we can obtain another invariants for higher degree forms (or the corresponding symmetric tensors). Then, the trace that depends only on a small number of the form coefficients can easily be calculated. Hence, some other invariants can be calculated. At the same time, calculation of them on the basis of eigenvalues is an algorithmically hard problem because of the exponentially large number of different eigenvalues for forms of degree three and for forms of higher degrees [10].

Although the invariant that we have considered may coincide for forms belonging to different orbits of an orthogonal group, the probability of random coincidence can be estimated from above with the help of the Schwartz–Zippel lemma [12]. Since the invariant has the degree two, this probability is small even in the case when the coefficients of the form are chosen from a small set of values. If coefficients of the form are chosen from the set $\{0, ..., m - 1\}$, this probability is no more than $2/m$.

ACKNOWLEDGMENTS

The authors are grateful to I.V. Latkin for useful remarks.

This study was supported by the Russian Science Foundation, project no. 14-50-00150.

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Translated by I. Efimova