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An unpublished theorem of Solovay on OD partitions of reals into two non-OD parts, revisited

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A definable pair of disjoint non-OD sets of reals (hence, indiscernible sets) exists in the Sacks and $\mathbb{E}_0$-large generic extensions of the constructible universe $\mathbf{L}$. More specifically, if $a \in 2^\omega$ is either Sacks generic or $\mathbb{E}_0$-large generic real over $\mathbf{L}$, then it is true in $\mathbf{L}[a]$ that there is a lightface Π^1_2 equivalence relation Q on the Π^1_2 set $U = 2^\omega \smallsetminus \mathbf{L}$ with exactly two equivalence classes, and both those classes are non-OD sets.

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1. Introduction

Let a *twin partition* be any partition of a given set U into two nonempty cells A and B. We refer to U as *the universe of discourse*, and each of A and B as a twin. Assume that some robust notion of definability D is chosen in advance, *e.g.*, D might be ordinal definability OD, or D might be Δ^1_1 definability, or something similar. In this context, a twin partition $U = A \cup B$ of a D-definable set U can be called D-definable in one of two senses:

- **strongly D-definable**, *i.e.*, each of the twins A and B is D-definable;
- **weakly D-definable,** meaning that the partition $\{A, B\}$ of U, considered as an unordered pair, is D-definable.

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Strong D-definability clearly implies weak D-definability. The "twin problem" for a given notion of definability D is whether the converse holds. The twin problem obviously has a positive answer provided D is some notion of definability closed under complements and the domain of discourse U contains at least one D-definable element x, then one cell of the partition consists of those x' that share the same cell of the partition as x, and the other cell is just the complementary set. This provides a trivial positive solution for the twin problem when $U = \omega$, or when U is the class of ordinals, and generally when U admits a D-definable well-ordering. Now let's focus on the case when U is a subset of the real numbers.

The twin problem admits a positive solution in the case of Δ^1_1 definability. Indeed it follows from Theorem 4.1 below that if a Δ^1_1 equivalence relation E on a Δ^1_1 set U of reals has precisely two (or even countably many) equivalence classes then each E-class is itself a Δ^1_1 set. The problem also admits a positive solution in the case of Δ^1_2 definability. This follows from the previous paragraph and the fact that if U is Δ^1_2, then U has a Δ^1_2 element (see, *e.g.*, 4E.5 in Moschovakis [20]).[a] But slightly above Δ^1_2 there is a significant obstacle, as indicated by the following theorem.

Theorem 1.1 (the Sacks part originally by Solovay[b]). *Let $a \in 2^\omega$ be either Sacks generic or $\mathbb{E}_0$-large generic*[c] *over* $\mathbf{L}$. *Then it is true in* $\mathbf{L}[a]$ *that*

(i) *there is a lightface Π^1_2 equivalence relation* Q[d] *on the Π^1_2 set $U = 2^\omega \smallsetminus \mathbf{L}$ with exactly two equivalence classes, both being non-OD sets,*
(ii) *and hence the quotient U/Q is a weakly definable, but not strongly definable, partition of the Π^1_2 set U into two non-OD sets.*

Under the assumptions of the theorem, let A, B be those equivalence classes. As the relation Q is lightface Π^1_2, the unordered pair $\{A, B\}$ is an OD set, basically, a definable set, whose two elements (disjoint non-empty pointsets $A, B \subseteq 2^\omega$ with $A \cup B = 2^\omega \smallsetminus \mathbf{L}$) are non-OD, hence, are *OD-indiscernible*. This is somewhat unexpected, especially with respect to the Sacks extensions. Indeed, the latter are seen as extremely homogeneous, with every nonconstructible real being Sacks-generic, therefore to have two distinct but indiscernible populations of nonconstructible reals in such a model looks rather surprising.

We reiterate that Π^1_2 in (i) of Theorem 1.1 is the best possible for such an example, because if E is a Σ^1_2 equivalence relation on a subsequently Σ^1_2 set $X \subseteq 2^\omega$, then X contains a Δ^1_2 real (see above), hence X/E contains an OD equivalence class.

[a] This argument holds as well in the case of Δ^1_n definability for any $n \geq 3$ under the axiom of constructibility $\mathbf{V} = \mathbf{L}$ or simply under the assumption $2^\omega \subseteq \mathbf{L}$ (all reals are constructible), because the latter implies that Δ^1_n is a basis for Σ^1_n, see 5A.4 in Moschovakis [20]. The argument also holds under the axiom of projective determinacy in the case of Δ^1_{2k+2} definability for any k, because **PD** implies that Δ^1_{2k+2} is a basis for Σ^1_{2k+2}, see 6C.6 in [20].
[b] See Section 12 on the history of the result
[c] See Section 4 on the $\mathbb{E}_0$-large forcing.
[d] In fact Q will be equal to the restriction $\mathsf{B}{\restriction}U$ of a Σ^1_2 relation B on $U = 2^\omega \smallsetminus \mathbf{L}$.

Note also that under $\mathbf{V} = \mathbf{L}$ nothing like (i) of Theorem 1.1 is possible because of the basis result mentioned in Footnote a.

2. On some indiscernible pairs of pointsets in different models

Models of **ZF** or **ZFC** containing OD indiscernible pairs of (non-OD) disjoint sets of reals are well-known. Such is *e.g.* any **Sacks** × **Sacks** extension $\mathbf{L}[a, b]$ of $\mathbf{L}$, where an OD pair of non-OD sets consists of the $\mathbf{L}$-degrees $[a]_{\mathbf{L}} = \{x \in 2^\omega : \mathbf{L}[x] = \mathbf{L}[a]\}$ and $[b]_{\mathbf{L}}$ of the Sacks reals a, b, and in this case $\{[a]_{\mathbf{L}}, [b]_{\mathbf{L}}\}$ = all minimal $\mathbf{L}$-degrees, an OD set in the model $\mathbf{L}[a, b]$, see [7] and also [3, 5]. Another model with an OD pair of *countable* disjoint non-OD sets of reals is defined in [6]. Yet those examples fail to fulfill the property that the union of the two sets is equal to the whole domain of nonconstructible reals.

Generally, OD-indiscernible pairs (not necessarily OD-definable pairs) of disjoint sets of reals can be extracted from early works on Cohen forcing. In particular, if $\langle a, b\rangle$ is a Cohen-generic, over $\mathbf{L}$, pair of $a, b \in 2^\omega$, then the $\mathbf{L}$-degrees $[a]_{\mathbf{L}}$ and $[b]_{\mathbf{L}}$ are OD-indiscernible in $\mathbf{L}[a, b]$ [5], and so are the countable $\mathbb{E}_0$ equivalence classes $[a]_{\mathbb{E}_0}$, $[b]_{\mathbb{E}_0}$ (essentially by Feferman [4]).

Speaking of *countable* OD-indiscernible sets of reals, another relevant model is a J^ω-generic extension $\mathbf{L}[\langle a_k\rangle_{k<\omega}]$ of $\mathbf{L}$, first considered in [3], where J is the Jensen forcing of [10], J^ω is the finite-support product, and $\langle a_k\rangle_{k<\omega}$ is a sequence of J-generic reals. It follows from some results in [12] that the sets $A = \{a_{2k} : k < \omega\}$ and $B = \{a_{2k+1} : k < \omega\}$ are OD-indiscernible in this model. Another model with an even more relevant example can be defined by means of an invariant version of J, which adjoins a $\mathbb{E}_0$-equivalence class $[a]_{\mathbb{E}_0}$ of generic reals. This set $[a]_{\mathbb{E}_0}$ is a countable lightface Π^1_2 set in $\mathbf{L}[a]$, and the subrelation $\mathbb{E}_0^{\mathsf{even}}$ (see Example 3.2) splits $[a]_{\mathbb{E}_0}$ into two non-empty $\mathbb{E}_0^{\mathsf{even}}$-equivalence classes, OD-indiscernible in $\mathbf{L}[a]$. See [14] on this example.

Note that all the above examples are disjoint sets of nonconstructible reals, but very far from covering the whole set $2^\omega \smallsetminus \mathbf{L}$ of all nonconstructible reals in the models considered. In this sense, Theorem 1.1 contains a certain novelty.

On the other hand, it is established in [13] that, in some models of **ZFC**, including the Sacks extension of the constructible universe $\mathbf{L}$, it is true that any countable OD (ordinal-definable) *set of reals* necessarily consists of OD elements. A similar result in much more general setting is known from Theorem 4.8 in [1] under a strong large cardinal hypothesis.

3. Outline of the proof

To prove Theorem 1.1, the required equivalence relation will be obtained as the union of an increasing transfinite sequence $\langle \mathsf{B}_\alpha\rangle_{\alpha<\omega_1}$ of **countable** Borel equivalence relations. The sequence is defined in $\mathbf{L}$, the ground universe. The following is the principal definition related to this construction.

Definition 3.1. A *double-bubble pair*, DBP for brevity, is a pair of *countable* Borel equivalence relations $\langle \mathsf{B}, \mathsf{E}\rangle$ on 2^ω, such that each E-class is the union of a pair of distinct B-classes.

A DBP $\langle \mathsf{B}', \mathsf{E}'\rangle$ *extends* $\langle \mathsf{B}, \mathsf{E}\rangle$, in symbol $\langle \mathsf{B}, \mathsf{E}\rangle \preccurlyeq \langle \mathsf{B}', \mathsf{E}'\rangle$, if $\mathsf{B} \subseteq \mathsf{B}'$, $\mathsf{E} \subseteq \mathsf{E}'$, and for any $x, y \in 2^\omega$, if $x \mathrel{\mathsf{E}} y$ but $x \not\mathrel{\mathsf{B}} y$ then we still have $x \not\mathrel{\mathsf{B}'} y$. □

Example 3.2. An elementary example consists of the equivalence relation $\mathbb{E}_0$, defined on 2^ω so that $x \mathrel{\mathbb{E}_0} y$ iff the set $\Delta(x,y) = \{k : x(k) \neq y(k)\}$ is finite, and its subrelation $\mathbb{E}_0^{\mathtt{even}}$, defined so that $x \mathrel{\mathbb{E}_0^{\mathtt{even}}} y$ iff the set $\Delta(x,y)$ has a finite *even* number of elements; $\langle \mathbb{E}_0^{\mathtt{even}}, \mathbb{E}_0\rangle$ is a DBP. □

Thus the extension of DBPs assumes that the equivalence classes of the original equivalence relations are merged in countable groups, in such a way that the two B-classes within the same E-class are not merged. In particular, given a Borel set $X \subseteq 2^\omega$ and a Borel map $f : X \to 2^\omega$, we will have to extend a given DBP $\langle \mathsf{B}, \mathsf{E}\rangle$ to a DBP $\langle \mathsf{B}', \mathsf{E}'\rangle$ such that $x \mathrel{\mathsf{E}'} f(x)$ for all x. This construction needs to overcome some obstacles explained in the beginning of Section 5, which we circumvent with the help of some *canonization theorems* of descriptive set theory, related to the two forcing notions of Theorem 1.1.

Anyway, to prove Theorem 1.1, we'll define a certain $\preccurlyeq$-increasing sequence $\langle\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle\rangle_{\alpha<\omega_1}$ of DBPs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ in $\mathbf{L}$, the ground universe, starting with the pair $\langle \mathbb{E}_0^{\mathtt{even}}, \mathbb{E}_0\rangle$ and eventually corralling all suitable Borel maps, and then the relation $\mathsf{B}{=}\bigcup_\alpha \mathsf{B}_\alpha$ will lead to the equivalence relation Q required. This takes some effort.

4. Canonization results used in the proof

Here we present some well-known results of modern descriptive set theory involved in the proof of Theorem 1.1. We begin with the Silver Dichotomy theorem and a canonization corollary. See *e.g.* Theorem 2.2 in [21] or Section 10.1 in [11] for a proof of the "moreover" lightface version of Theorem 4.1.

Theorem 4.1 (Silver's Dichotomy [22])**.** *Suppose that* E *is a* $\mathbf{\Pi}^1_1$ *equivalence relation on an uncountable Borel set* $X \subseteq 2^\omega$*. Then either* E *has at most countably many equivalence classes, or there exists a perfect partial* E*-transversal*[e]*.*

If moreover X *is lightface* Δ^1_1 *and* E *is lightface* Π^1_1 *then all equivalence classes are lightface* Δ^1_1 *in the "either" case.* □

Corollary 4.2. *Suppose that* E *is a* $\mathbf{\Pi}^1_1$ *equivalence relation on a Borel set* $X \subseteq 2^\omega$*. Then there is a perfect set* $Y \subseteq X$ *such that* E *coincides on* Y *with:*

– *either* (I) *the total equivalence* TOT *making all reals equivalent*;

[e] A *partial transversal* is a set of pairwise inequivalent elements. A *full* transversal requires that in addition it has a non-empty intersection with any equivalence class in a given domain.

– *or* (II) *the equality, so that* Y *is a partial* E*-transversal.*

If in addition E *is a countable*[f] *equivalence relation then* (I) *is impossible.*

Proof. In the "or" case of Theorem 4.1 we have (II). In the "either" case pick an uncountable equivalence class C and let $Y \subseteq C$ be any perfect set. □

Corollary 4.3. *If* $X \subseteq 2^\omega$ *is a perfect set, and* $f : X \to 2^\omega$ *a Borel map, then there is a perfect set* $Y \subseteq X$ *such that* $f \restriction Y$ *is a bijection or a constant.*

Proof. This is a well-known fact, of course, yet it immediately follows from Corollary 4.2. Indeed define a Borel equivalence relation E on X such that $x \mathrel{\mathsf{E}} y$ iff $f(x) = f(y)$. Apply Corollary 4.2. □

Now we recall some definitions and results related to $\mathbb{E}_0$-large sets. A Borel set $X \subseteq 2^\omega$ is called $\mathbb{E}_0$*-large* if $\mathbb{E}_0 \restriction X$ is still a non-smooth[g] equivalence relation. For instance 2^ω itself is $\mathbb{E}_0$-large, while any Borel partial $\mathbb{E}_0$-transversal is not. If $\mathbf{u} = \langle u_n^i \rangle_{n<\omega, i=0,1}$ is a double sequence of strings $u_n^i \in 2^{<\omega}$, satisfying $\mathtt{lh}(u_n^0) = \mathtt{lh}(u_n^1) \ge 1$ and $u_n^0 \ne u_n^1$ for all n, then we call $\mathbf{u}$ *a* $\mathbb{E}_0$*-matrix*, let

$$x_{\mathbf{u}}^a = u_0^{a(0)} {}^\frown u_1^{a(1)} {}^\frown u_2^{a(2)} {}^\frown \ldots {}^\frown u_n^{a(n)} {}^\frown \cdots \in 2^\omega .$$

for any $a \in 2^\omega$, and define a *canonical* $\mathbb{E}_0$*-large* set $\mathbb{X}_{\mathbf{u}} = \{x_{\mathbf{u}}^a : a \in 2^\omega\}$. Each canonical $\mathbb{E}_0$-large set $\mathbb{X}_{\mathbf{u}}$ is perfect, and $\mathbb{E}_0$-large since the map $a \mapsto x_{\mathbf{u}}^a$ is a Borel reduction of $\mathbb{E}_0$ into $\mathbb{E}_0 \restriction \mathbb{X}_{\mathbf{u}}$. On the other hand, it is known (see *e.g.* Section 7.1 in [15]) that each (Borel) $\mathbb{E}_0$-large set $X \subseteq 2^\omega$ contains a canonical $\mathbb{E}_0$-large subset $Y \subseteq X$.

Also if $\mathbf{v} = \langle v_n^i \rangle_{n<\omega, i=0,1}$ is another $\mathbb{E}_0$-matrix, then we define a map $h_{\mathbf{uv}} : \mathbb{X}_{\mathbf{u}} \xrightarrow{\text{onto}} \mathbb{X}_{\mathbf{v}}$ such that $h_{\mathbf{uv}}(x_{\mathbf{u}}^a) = x_{\mathbf{v}}^a$ for all $a \in 2^\omega$. Note that $h_{\mathbf{uv}}$ is a homeomorphism and $\mathbb{E}_0$*-isomorphism*, in the sense that $x \mathrel{\mathbb{E}_0} y \iff h_{\mathbf{uv}}(x) \mathrel{\mathbb{E}_0} h_{\mathbf{uv}}(y)$, for all $x, y \in \mathbb{X}_{\mathbf{u}}$. Maps of the form $h_{\mathbf{uv}}$ will be called *canonical* $\mathbb{E}_0$*-large maps*.

Theorem 4.4 (Theorem 7.1 in [15], or else [19])**.** *Suppose that* E *is a Borel equivalence relation on* 2^ω*, and* $X \subseteq 2^\omega$ *is a* $\mathbb{E}_0$*-large set. Then there is a canonical* $\mathbb{E}_0$*-large set* $Y \subseteq X$ *such that* E *coincides on* Y *with:*

– *either* (I) *the total equivalence relation* TOT;
– *or* (II) *the relation* $\mathbb{E}_0$;
– *or* (III) *the equality.*

[f] An equivalence relation is *countable* iff all its equivalence classes are at most countable.
[g] An equivalence relation E on a Borel set X is *smooth* if there is a Borel map $f : X \to 2^\omega$ such that we have $x \mathrel{\mathsf{E}} y$ iff $f(x) = f(y)$ for all $x, y \in X$. The equivalence relation $\mathbb{E}_0$ is *non-smooth*, meaning that such a Borel f does not exist. See Example 6.5 in [17].

In addition, if E *is a countable equivalence relation then* (I) *is impossible, while if* $\mathbb{E}_0 \subseteq \mathsf{E}$ *then* (III) *is impossible.* □

Corollary 4.5. *If* $X \subseteq 2^\omega$ *is a Borel* $\mathbb{E}_0$*-large set, and* $Z \subseteq X$ *a Borel set, then there is a canonical* $\mathbb{E}_0$*-large set* $Y \subseteq X$ *such that* $Y \subseteq Z$ *or* $Y \cap Z = \varnothing$*.*

Proof. Define a Borel equivalence relation E on X such that $x \mathsf{E} y$ iff $x, y \in Z$ or $x, y \in X \smallsetminus Z$. Apply Theorem 4.4. As E has just two equivalence classes, only (I) is possible. □

Corollary 4.6. *If* $X \subseteq 2^\omega$ *is a Borel* $\mathbb{E}_0$*-large set, and* $f : X \to 2^\omega$ *a Borel map, then there exists a canonical* $\mathbb{E}_0$*-large set* $Y \subseteq X$ *such that* $f \restriction Y$ *is a bijection or a constant.*

Proof. Define a Borel equivalence relation E on X such that $x \mathsf{E} y$ iff $f(x) = f(y)$. Apply Theorem 4.4. We have to prove that (II) is impossible. Suppose to the contrary that $\mathsf{E} = \mathbb{E}_0$ on a canonical $\mathbb{E}_0$-large set $Y \subseteq X$. In other words, we have $f(x) = f(y)$ iff $x \mathbb{E}_0 y$ for all $x, y \in Y$. Thus f is a Borel reduction of $\mathbb{E}_0 \restriction Y$ to the equality, which contradicts the assumption that Y is $\mathbb{E}_0$-large. □

As **forcing notions**, both the set **Sacks** of all perfect sets (*the Sacks forcing*) and the set $\mathbf{P}_{\mathbb{E}_0}$ of all canonical $\mathbb{E}_0$-large sets (*the* $\mathbb{E}_0$*-large forcing*) adjoin reals of minimal degree, preserve $\aleph_1$, are bounding, and have some other remarkable properties, see *e.g.* Section 7.1 in [15], or 2.3.10 in [25], including the following.

Lemma 4.7. *Both* ***Sacks*** *and* $\mathbf{P}_{\mathbb{E}_0}$ *satisfy the property of continuous reading of names, i.e., if* $a \in 2^\omega$ *is* ***Sacks****-generic or* $\mathbf{P}_{\mathbb{E}_0}$*-generic over* $\mathbf{L}$*, and* $x \in 2^\omega \cap \mathbf{L}[a]$*, then* $x = f(a)$*, where* $f : 2^\omega \to 2^\omega$ *is a continuous map coded in* $\mathbf{L}$*.*

Proof. In the Sacks case, given $X \in$ **Sacks** and a name t for a real in 2^ω, we define, in $\mathbf{L}$, a *splitting scheme* $\langle X_u \rangle_{u \in 2^{<\omega}}$ of conditions $X_u \in$ **Sacks**, satisfying

(i) $X_\Lambda \subseteq X$ (Λ is the empty string),

(ii) $X_{u^\frown i} \subseteq X_u$ and $\mathtt{stem}(X)^\frown i \subseteq \mathtt{stem}(X_{u^\frown i})$ for all $i = 0, 1$ and $u \in 2^{<\omega}$, where $r = \mathtt{stem}(Y) \in 2^{<\omega}$ (the stem) is the largest string satisfying $r \subset y$ for all $y \in Y$ — it follows that $X_{u^\frown 0} \cap X_{u^\frown 1} = \varnothing$,

(iii) each X_u decides the value of $t(n)$, where $n = \mathtt{lh}(u)$ (the *length*).

Then $Y = \bigcap_n \bigcup_{\mathtt{lh}(u)=n} X_u \subseteq X$ is a perfect set, and the map $f_0 : Y \to 2^\omega$, defined by $f_0(x)(n) = i$ iff X_u **Sacks**-forces $t(n) = i$, where u is the only string with $\mathtt{lh}(u) = n$ such that $x \in X_u$, is continuous. In $\mathbf{L}$, let $f : 2^\omega \to 2^\omega$ be the continuous extension of f_0. Then Y forces that t is equal to $f(a_G)$, where a_G is the name for the **Sacks**-generic real.

The above is a standard construction present, in this or another way, in many papers on the Sacks forcing and related forcing notions. Its more sophisticated version, designed in [8] (see also, *e.g.*, [11], pp. 129–130) yields a canonical $\mathbb{E}_0$-large set $Y = \bigcap_n \bigcup_{\mathtt{lh}(u)=n} X_u \subseteq X$, and hence proves the lemma for $\mathbf{P}_{\mathbb{E}_0}$. Another proof for the $\mathbf{P}_{\mathbb{E}_0}$ case can be derived from two results in Zapletal [26], namely, that $\mathbf{P}_{\mathbb{E}_0}$ has the bounding property (pp. 214–215) and that the bounding property implies continuous reading of names (Theorem 3.3.2). □

5. Corralling maps, Sacks case

The following general result will be applied below in some key arguments.

Lemma 5.1. *If $R \subseteq 2^\omega \to 2^\omega$ is a Borel relation and all vertical and horizontal cross-sections of R are countable, then the least equivalence relation E satisfying $R \subseteq \mathsf{E}$ (the equivalence hull of R) is countable and Borel.*

Proof. That E is countable (*i.e.*, all E-classes are countable) is clear, so it remains to check the Borelness. Note that $x \mathrel{\mathsf{E}} y$ is equivalent to:

($\star$) there exists a string $\vec{z} = \langle x = z_0, z_1, \dots, z_n = y\rangle$ of reals x_i satisfying $\langle x_i, x_{i+1}\rangle \in R$ or $\langle x_{i+1}, x_i\rangle \in R$ for all $i < k$.

It follows from the condition of countable cross-sections that for any pair of reals x, y there exist at most countably many strings $\vec{z}$ satisfying ($\star$). Therefore E as a set of pairs is equal to the projection of a Borel set with countable cross-sections. (We skip details related to coding of strings of reals by reals themselves.) But any such a projection is a Borel set, see *e.g.* 18.10 in Kechris [16]. □

Now a principal definition follows.

Definition 5.2. Given a set $X \subseteq 2^\omega$ and a map $f : X \to 2^\omega$, a DBP $\langle \mathsf{B}, \mathsf{E}\rangle$:

- *corralls* f if E contains f, that is, $f(x) \in [x]_{\mathsf{E}}$ for all $x \in X$;
- *positively corralls* f if B contains f, that is, $f(x) \in [x]_{\mathsf{B}}$ for all $x \in X$;
- *negatively corralls* f if $f(x) \in [x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}$ for all $x \in X$. □

Let $\langle \mathsf{B}, \mathsf{E}\rangle$ be a DBP and $f : 2^\omega \to 2^\omega$ be a 1-1 Borel map. Is there a DBP $\langle \mathsf{B}', \mathsf{E}'\rangle$ extending $\langle \mathsf{B}, \mathsf{E}\rangle$ and corralling f? The first step is obvious: merge each E-class $[x]_{\mathsf{E}}$, $x \in 2^\omega$, with $[f(x)]_{\mathsf{E}}$. To be more precise, let E' be the smallest equivalence relation including both E and the graph of f. (Note that E' is still a countable equivalence relation provided E is such and f is 1-1 or even countable-to-1, and is Borel by Lemma 5.1 provided E and f are such.) However how do we then merge B-subclasses of E-classes? One may want to use the same idea, that is, to merge each $[x]_{\mathsf{B}}$ with $[f(x)]_{\mathsf{B}}$, but this does not necessarily work. Indeed assume that $x \mathrel{\mathsf{B}} x'$, while the values $y = f(x)$ and $y' = f(x')$ are E-equivalent but not B-equivalent. If we make x, y B'-equivalent and x', y' B'-equivalent then we get $y \mathrel{\mathsf{B}'} y'$ as well since $x \mathrel{\mathsf{B}} x'$. But this contradicts the definition of extension of DBPs since $y \mathrel{\mathsf{E}} y'$ but $y \not\mathrel{\mathsf{B}} y'$.

To circumvent this difficulty, we allow the domain of the map f considered to be reduced to a smaller set (which still belongs to the forcing considered, *i.e.*, the Sacks forcing in this Section) on which $\mathsf{B}, \mathsf{E}, f$ behave exceptionally well. This is where the canonization results of Section 4 will play the key role.

Lemma 5.3. *Assume that $\langle \mathsf{B}, \mathsf{E} \rangle$ is a DBP, $X \subseteq 2^\omega$ is a perfect set, and $f : X \to 2^\omega$ is Borel*[h] *and 1-1. There exist a perfect set $Y \subseteq X$ and a DBP $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and corralls $f \restriction Y$.*

Proof. The sets $X' = \{x \in X : x \mathrel{\mathsf{E}} f(x)\}$ and $X'' = \{x \in X : x \mathrel{\not{\mathsf{E}}} f(x)\}$ are Borel, hence there is a perfect set X_0 with either $X_0 \subseteq X'$ or $X_0 \subseteq X''$. But if $X_0 \subseteq X'$ then $\langle \mathsf{B}, \mathsf{E} \rangle$ already corralls $f \restriction X_0$, and we are done. Thus we assume that $X_0 \subseteq X''$, that is, $x \mathrel{\not{\mathsf{E}}} f(x)$ for all $x \in X_0$.

By Corollary 4.2, there is a perfect set $X_1 \subseteq X_0$ such that E, B coincide with the equality on X_1. Define an equivalence relation $\widehat{\mathsf{E}}$ on X_1 such that $x \mathrel{\widehat{\mathsf{E}}} y$ iff $f(x) \mathrel{\mathsf{E}} f(y)$, and define $\widehat{\mathsf{B}}$ similarly. Consider the $\subseteq$-minimal equivalence relation F defined on 2^ω such that $\mathsf{E} \subseteq \mathsf{F}$ and if $x, y \in 2^\omega$ and $f(x) \mathrel{\mathsf{E}} y$ then $x \mathrel{\mathsf{F}} y$. Thus $\widehat{\mathsf{E}}, \widehat{\mathsf{B}}, \mathsf{F}$ are countable Borel equivalence relations on X_1. (The borelness of F holds by Lemma 5.1.) By Corollary 4.2, there is a perfect set $Y \subseteq X_1$ such that $\widehat{\mathsf{E}}, \widehat{\mathsf{B}}, \mathsf{F}$ coincide with the equality on Y, along with E, B. It follows, by the choice of X_0 and using the fact that F on Y is the equality relation, that if $x, y \in Y$ (whether equal or not) then $x \mathrel{\not{\mathsf{E}}} f(y)$.

We define the equivalence relations E', B' as follows.

If $x \in 2^\omega$ does **not** belong to *the critical domain* $\Delta = [Y \cup \{f(x) : x \in Y\}]_{\mathsf{E}}$, then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$, so such an E-class and its B-subclasses are not changed. But within Δ some classes will be merged. Namely if $x \in Y$ then we have to merge $[x]_{\mathsf{E}}$ with $[f(x)]_{\mathsf{E}}$, hence put

$$[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [f(x)]_{\mathsf{E}} \quad \text{and} \quad [x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup [f(x)]_{\mathsf{B}} \,,$$

and define the other B'-class within $[x]_{\mathsf{E}'}$ as $[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'}$.

Note that the domain Δ and the subdomains $\Delta_1 = [Y]_{\mathsf{E}}$ and $\Delta_2 = [Y_2]_{\mathsf{E}}$, where $Y_2 = \{f(x) : x \in Y\}$, are Borel sets. Indeed Y_2 is Borel because f is Borel and 1-1. Further, as E is a Borel countable equivalence relation, the Lusin-Novikov theorem (see 18.10 in [16]) implies the existence of a sequence of Borel countable-to-1 maps $h_n : 2^\omega \to 2^\omega$ such that $[x]_{\mathsf{E}} = \{h_n(x) : n < \omega\}$. It follows that *e.g.* $\Delta_1 = \bigcup_n Z_n$, where each $Z_n = \{h_n(z) : z \in Y\}$ is Borel by Lemma 18.12 in [16] since Y is Borel and each h_n is Borel and countable-to-1. Now to conclude that E' is a *Borel* relation, we note that by definition $x \mathrel{\mathsf{E}'} y$ iff

– either (1) $(x, y \notin \Delta \vee x, y \in \Delta_1 \vee x, y \in \Delta_2)$ and $x \mathrel{\mathsf{E}} y$,
– or (2) $x \in \Delta_1$, $y \in \Delta_2$, and $\exists\, x' \in Y \, (x' \mathrel{\mathsf{E}} x \wedge y \mathrel{\mathsf{E}} f(x'))$,

[h] Lemmas 5.3, 5.4, 6.1, 6.2 will be applied below essentially only in the case of continuous maps f via Lemma 4.7, yet we decided to slightly increase generality in the lemmas.

– or (3) *vice versa*, $y \in \Delta_1$, $x \in \Delta_2$, and $\exists\, y' \in Y\,(y' \mathsf{E}\, y \wedge x \mathsf{E}\, f(y'))$.

Here case (1) is clearly expressed in a Borel way. The last condition in (2) can be rewritten in a Borel way by $\exists\, n\,(h_n(x) \in Y \wedge y \mathsf{E}\, f(h_n(x)))$, so (2) is Borel too. Ditto for (3). Thus indeed E' is a Borel equivalence relation.

A routine further verification shows that B' is Borel as well, and the pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a DBP which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and positively corralls $f \restriction Y$ (because we have $f(x) \in [x]_{\mathsf{B}'}$ for all $x \in Y$ simply by construction). □

Lemma 5.4. *Let $\langle \mathsf{B}, \mathsf{E} \rangle$ be a DBP, and $R, X \subseteq 2^\omega$ be perfect sets. There exist: a perfect set $Y \subseteq X$, Borel 1-1 maps $f, g : Y \to R$, and a DBP $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$, positively corralls $f \restriction Y$, and negatively corralls $g \restriction Y$.*

Proof. By Corollary 4.2, there exist perfect partial E-transversals $X_0 \subseteq X$ and $R_0 \subseteq R$. Let $R_0 = R_1 \cup R_2$ be a partition into two disjoint perfect sets. Then $[R_1]_{\mathsf{E}}$ and $[R_2]_{\mathsf{E}}$ are disjoint, hence there is a perfect set $Y \subseteq X_0$ such that $[Y]_{\mathsf{E}}$ does not intersect either $[R_1]_{\mathsf{E}}$ or $[R_2]_{\mathsf{E}}$. Let say $[Y]_{\mathsf{E}} \cap [R_1]_{\mathsf{E}} = \varnothing$.

Let $R_1 = R' \cup R''$ be a partition into two disjoint perfect sets. It follows by construction that (*) the Borel sets Y, R', R'' are pairwise disjoint and the union $\Delta = Y \cup R' \cup R''$ is a partial E-transversal. Let $f : Y \to R'$ and $g : Y \to R''$ be arbitrary Borel 1-1 maps.

We define the equivalence relations E', B' as follows.

If $x \in 2^\omega$ does **not** belong to *the critical domain* $\Delta = [Y \cup R' \cup R'']_{\mathsf{E}}$, then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$, so such a E-class and its B-subclasses are not changed. But within Δ some classes will be merged. Namely if $x \in Y$ then we have to merge $[x]_{\mathsf{E}}$ with $[f(x)]_{\mathsf{E}}$ and $[g(x)]_{\mathsf{E}}$, hence we put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [f(x)]_{\mathsf{E}} \cup [g(x)]_{\mathsf{E}}$. We further define

$$[x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup [f(x)]_{\mathsf{B}} \cup ([g(x)]_{\mathsf{E}} \smallsetminus [g(x)]_{\mathsf{B}})\,,$$

and let $([x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}) \cup ([f(x)]_{\mathsf{E}} \smallsetminus [f(x)]_{\mathsf{B}}) \cup [g(x)]_{\mathsf{B}}$ be the other B'-class within $[x]_{\mathsf{E}'}$. A routine verification using (*) shows that the relations E', B' are Borel (see the proof of Lemma 5.3), and the pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a DBP that extends $\langle \mathsf{B}, \mathsf{E} \rangle$, positively corralls $f \restriction Y$, and negatively corralls $g \restriction Y$. □

6. Corralling maps, $\mathbb{E}_0$-large case

We prove two lemmas similar to Lemma 5.3 and Lemma 5.4, yet with a bit more complex proofs.

Lemma 6.1. *Assume that $\langle \mathsf{B}, \mathsf{E} \rangle$ is a DBP, $\mathbb{E}_0 \subseteq \mathsf{E}$, $X \subseteq 2^\omega$ is a canonical $\mathbb{E}_0$-large set, and $f : X \to 2^\omega$ is Borel and 1-1. There exist a canonical $\mathbb{E}_0$-large set $Y \subseteq X$ and a DBP $\langle \mathsf{B}', \mathsf{E}' \rangle$ which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and corralls $f \restriction Y$.*

Proof. First of all, arguing as in the proof of Lemma 5.3 (but using Corollary 4.5), we get a canonical $\mathbb{E}_0$-large set $X_0 \subseteq X$ with $x \not\mathbb{E} f(x)$ for all $x \in X_0$. By Theorem 4.4, there is a canonical $\mathbb{E}_0$-large perfect set $X_1 \subseteq X_0$ such that the relations

E, B coincide with $\mathbb{E}_0$ on X_1. Define an equivalence relation $\widehat{\mathsf{E}}$ on X_1 such that $x \mathbin{\widehat{\mathsf{E}}} y$ iff $f(x) \mathrel{\mathsf{E}} f(y)$, and define $\widehat{\mathsf{B}}$ similarly. Consider the $\subseteq$-minimal equivalence relation F defined on 2^ω such that $\mathsf{E} \subseteq \mathsf{F}$ and if $x, y \in 2^\omega$ and $f(x) \mathrel{\mathsf{E}} y$ then $x \mathrel{\mathsf{F}} y$. Thus $\widehat{\mathsf{E}}, \widehat{\mathsf{B}}, \mathsf{F}$ are countable Borel equivalence relations on X_1. (F is Borel by Lemma 5.1.) By Theorem 4.4, there is a canonical $\mathbb{E}_0$-large perfect set $Y \subseteq X_1$ such that each of these three equivalence relations is either of type (I) or of type (II) on Y. However, as each E-class contains two B-classes, $\widehat{\mathsf{E}}$ has to coincide with $\widehat{\mathsf{B}}$ on Y. Finally, as $\mathsf{E} \subseteq \mathsf{F}$, we have $\mathsf{F} = \mathbb{E}_0$ on Y. It follows by the choice of X_0 that if $x, y \in Y$ (whether equal or not) then $x \not\mathrel{\mathsf{E}} f(y)$.

To conclude, $\mathsf{E} = \mathsf{B} = \mathsf{F} = \mathbb{E}_0$ on Y, and also either $\widehat{\mathsf{E}} = \widehat{\mathsf{B}}$ is the equality on Y, or $\widehat{\mathsf{E}} = \widehat{\mathsf{B}} = \mathbb{E}_0$ on Y. This leads to the following two cases.

In each case, we are going to define the equivalence relations E', B' required. If $x \in 2^\omega$ does **not** belong to *the critical domain* $\Delta = [Y \cup \{f(x) : x \in Y\}]_{\mathsf{E}}$, then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$, so such a E-class and its B-subclasses are not changed. But within Δ some classes will be merged. In particular, we are going to merge $[x]_{\mathsf{E}}$ with $[f(x)]_{\mathsf{E}}$ for any $x \in Y$.

Case 1: $\widehat{\mathsf{E}} = \widehat{\mathsf{B}}$ is the equality on Y while $\mathsf{B} = \mathsf{E} = \mathsf{F} = \mathbb{E}_0$ on Y, thus if $x, y \in Y$ then first, $x \neq y$ implies $f(x) \not\mathrel{\mathsf{E}} f(y)$ and $f(x) \not\mathrel{\mathsf{B}} f(y)$, and second, $[x]_{\mathsf{E}} \cap Y = [x]_{\mathsf{B}} \cap Y = [x]_{\mathbb{E}_0} \cap Y$. If $x \in Y$ then put

$$[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup \bigcup_{y \in Y \cap [x]_{\mathbb{E}_0}} [f(y)]_{\mathsf{E}} \quad \text{and} \quad [x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup \bigcup_{y \in Y \cap [x]_{\mathbb{E}_0}} [f(y)]_{\mathsf{B}} \,,$$

and define the other B'-class within $[x]_{\mathsf{E}'}$ as $[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'}$.

Case 2: $\mathsf{E} = \mathsf{B} = \widehat{\mathsf{E}} = \widehat{\mathsf{B}} = \mathsf{F} = \mathbb{E}_0$ on Y, that is, if $x, y \in Y$ then

$$x \mathrel{\mathbb{E}_0} y \iff x \mathrel{\mathsf{E}} y \iff x \mathrel{\mathsf{B}} y \iff f(x) \mathrel{\mathsf{E}} f(y) \iff f(x) \mathrel{\mathsf{B}} f(y)\,.$$

Assume that $x \in Y$. Put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [f(x)]_{\mathsf{E}} = [y]_{\mathsf{E}} \cup [f(y)]_{\mathsf{E}}$ for any other $y \in Y \cap [x]_{\mathbb{E}_0}$, and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup [f(x)]_{\mathsf{B}} = [y]_{\mathsf{B}} \cup [f(y)]_{\mathsf{B}}$ for any other $y \in Y \cap [x]_{\mathbb{E}_0}$. Define the other B'-class within $[x]_{\mathsf{E}'}$ as $[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'}$.

A routine verification shows that in either case the relations E', B' are Borel (as in the proof of Lemma 5.3), and the pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a DBP which extends $\langle \mathsf{B}, \mathsf{E} \rangle$ and corralls $f \restriction Y$ (because we have $f(x) \in [x]_{\mathsf{E}'}$ for all $x \in Y$ simply by construction).□

Lemma 6.2. *Let $\langle \mathsf{B}, \mathsf{E} \rangle$ be a DBP with $\mathbb{E}_0 \subseteq \mathsf{E}$, and $R, X \subseteq 2^\omega$ be canonical $\mathbb{E}_0$-large sets. There exist: a canonical $\mathbb{E}_0$-large set $Y \subseteq X$, canonical $\mathbb{E}_0$-large maps $f, g : Y \to R$, and a DBP $\langle \mathsf{B}', \mathsf{E}' \rangle$ that extends $\langle \mathsf{B}, \mathsf{E} \rangle$, positively corralls $f \restriction Y$, and negatively corralls $g \restriction Y$.*

Proof. By Theorem 4.4, we w.l.o.g. assume that $\mathsf{E} = \mathsf{B} = \mathbb{E}_0$ on R. By definition, $R = \mathbb{X}_{\mathbf{r}}$ for a $\mathbb{E}_0$-matrix $\mathbf{r} = \langle r_n^i \rangle_{n<\omega, i=0,1}$. Now let $\mathbf{p} = \langle p_n^i \rangle_{n<\omega, i=0,1}$, $\mathbf{q} = \langle q_n^i \rangle_{n<\omega, i=0,1}$, where $p_n^i = r_{2n}^0 {}^\frown r_{2n+1}^i$ and $q_n^i = r_{2n}^1 {}^\frown r_{2n+1}^i$. Thus $\mathbf{p}, \mathbf{q}$ are $\mathbb{E}_0$-matrices, and the sets $\mathbb{X}_{\mathbf{p}}$, $\mathbb{X}_{\mathbf{q}}$ satisfy $\mathbb{X}_{\mathbf{p}} \cup \mathbb{X}_{\mathbf{q}} \subseteq \mathbb{X}_{\mathbf{r}} = R$ and $[\mathbb{X}_{\mathbf{p}}]_{\mathbb{E}_0} \cap [\mathbb{X}_{\mathbf{q}}]_{\mathbb{E}_0} = \varnothing$, hence, $[\mathbb{X}_{\mathbf{p}}]_{\mathsf{E}} \cap [\mathbb{X}_{\mathbf{q}}]_{\mathsf{E}} = \varnothing$ by the assumption above. It follows by Corollary 4.5

that there is a canonical $\mathbb{E}_0$-large set $X_0 \subseteq X$ satisfying $[X_0]_{\mathsf{E}} \cap [\mathbb{X}_{\mathbf{p}}]_{\mathsf{E}} = \varnothing$ or $[X_0]_{\mathsf{E}} \cap [\mathbb{X}_{\mathbf{q}}]_{\mathsf{E}} = \varnothing$. Assume, for instance, that $[X_0]_{\mathsf{E}} \cap [\mathbb{X}_{\mathbf{p}}]_{\mathsf{E}} = \varnothing$. As just above, there exist $\mathbb{E}_0$-matrices $\mathbf{p}', \mathbf{p}''$ such that the canonical $\mathbb{E}_0$-large sets $R' = \mathbb{X}_{\mathbf{p}'}$, $R'' = \mathbb{X}_{\mathbf{p}''}$ satisfy $R' \cup R'' \subseteq \mathbb{X}_{\mathbf{p}}$ and $[R']_{\mathsf{E}} \cap [R'']_{\mathsf{E}} = \varnothing$.

To conclude, we have canonical $\mathbb{E}_0$-large sets $X_0 \subseteq X$ and $R', R'' \subseteq R$ satisfying (*) $[R']_{\mathsf{E}} \cap [R'']_{\mathsf{E}} = [X_0]_{\mathsf{E}} \cap [R']_{\mathsf{E}} = [X_0]_{\mathsf{E}} \cap [R'']_{\mathsf{E}} = \varnothing$. As E, B are countable equivalence relations and $\mathbb{E}_0 \subseteq \mathsf{E}$, Theorem 4.4 yields a canonical $\mathbb{E}_0$-large set $Y = \mathbb{X}_{\mathbf{u}} \subseteq X_0$ such that $\mathsf{E} = \mathbb{E}_0$ on Y while B coincides with either $\mathbb{E}_0$ or the equality on Y. Then since B splits E-classes into two pieces, we conclude that B on Y also agrees with $\mathbb{E}_0$, so that $\mathsf{E} = \mathsf{B} = \mathbb{E}_0$ on Y.

Consider the canonical $\mathbb{E}_0$-large maps $f = h_{\mathbf{up}'} : Y \xrightarrow{\text{onto}} R'$ and $g = h_{\mathbf{up}''} : Y \xrightarrow{\text{onto}} R''$. We define the equivalence relations E', B' as follows.

If $x \in 2^\omega$ does **not** belong to *the critical domain* $\Delta = [Y \cup R' \cup R'']_{\mathsf{E}}$, then put $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$ and $[x]_{\mathsf{B}'} = [x]_{\mathsf{B}}$, so such a E-class and its B-subclasses are not changed. But within Δ, if $x \in Y$ then we have to merge $[x]_{\mathsf{E}}$ with $[f(x)]_{\mathsf{E}}$ and $[g(x)]_{\mathsf{E}}$, and $[x]_{\mathsf{B}}$ with $[f(x)]_{\mathsf{B}}$ but not with $[g(x)]_{\mathsf{B}}$, hence we put

$$[x]_{\mathsf{E}'} = [x]_{\mathsf{E}} \cup [f(x)]_{\mathsf{E}} \cup [g(x)]_{\mathsf{E}}\,, \quad [x]_{\mathsf{B}'} = [x]_{\mathsf{B}} \cup [f(x)]_{\mathsf{B}} \cup ([g(x)]_{\mathsf{E}} \smallsetminus [g(x)]_{\mathsf{B}})\,,$$

and define the other B'-class within $[x]_{\mathsf{E}'}$ as

$$[x]_{\mathsf{E}'} \smallsetminus [x]_{\mathsf{B}'} = ([x]_{\mathsf{E}} \smallsetminus [x]_{\mathsf{B}}) \cup ([f(x)]_{\mathsf{E}} \smallsetminus [f(x)]_{\mathsf{B}}) \cup [g(x)]_{\mathsf{B}}\,.$$

The definition of $[x]_{\mathsf{E}'}$ and $[x]_{\mathsf{B}'}$ is suitably invariant, so that if $x, y \in Y$ and $x \mathrel{\mathsf{E}} y$ then $[f(x)]_{\mathsf{E}'} = [f(y)]_{\mathsf{E}'}$ and $[f(x)]_{\mathsf{B}'} = [f(y)]_{\mathsf{B}'}$, because 1) $x \mathrel{\mathbb{E}_0} y$ and $x \mathrel{\mathsf{B}} y$ since $\mathsf{E} = \mathsf{B} = \mathbb{E}_0$ on Y, therefore 2) $f(x) \mathrel{\mathbb{E}_0} f(y)$ and $g(x) \mathrel{\mathbb{E}_0} g(y)$ since f, g are $\mathbb{E}_0$-isomorphisms, and further 3) $[f(x)]_{\mathsf{E}} = [f(y)]_{\mathsf{E}}$, $[g(x)]_{\mathsf{E}} = [g(y)]_{\mathsf{E}}$, $[f(x)]_{\mathsf{B}} = [f(y)]_{\mathsf{B}}$, $[g(x)]_{\mathsf{B}} = [g(y)]_{\mathsf{B}}$, since $\mathsf{E} = \mathsf{B} = \mathbb{E}_0$ on $R' \cup R'' \subseteq R$.

Note also that if $x, y \in Y$ and $x \mathrel{\not\mathsf{E}} y$ then $x \mathrel{\not\mathbb{E}_0} y$, and it follows by approximately the same arguments (plus (*) above) that the sets $[x]_{\mathsf{E}}$, $[f(x)]_{\mathsf{E}}$, $[g(x)]_{\mathsf{E}}$, $[y]_{\mathsf{E}}$, $[f(y)]_{\mathsf{E}}$, $[g(y)]_{\mathsf{E}}$ are pairwise disjoint. Using this observation, a routine verification shows that the relations E', B' are Borel (as in the proof of Lemma 5.3), and the pair $\langle \mathsf{B}', \mathsf{E}' \rangle$ is a DBP that extends $\langle \mathsf{B}, \mathsf{E} \rangle$, corralls $f \restriction Y$ positively, and corralls $g \restriction Y$ negatively. □

7. Definability aspects

The following Theorem 8.1 asserts the existence of a certain transfinite sequence of Borel equivalence relations. We have to explain how definability properties of such sequences can be considered. In particular, we consider $\in$-definability in the set HC of all hereditarily countable sets. We'll make use of the following well-known fact:

Proposition 7.1 (see, *e.g.*, Lemma 25.25 in [9]). *Let $n \geq 1$. A set of reals is Σ^1_{n+1} iff it is Δ^{HC}_n. The same for Π and Δ.* □

Note that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ as long as $\mathbf{V} = \mathbf{L}$ is assumed. Let $<_{\mathbf{L}}$ be Gödel's Δ^{HC}_1 wellordering of HC. The next result exploits the "goodness" of $<_{\mathbf{L}}$.

Proposition 7.2 ($\mathbf{V} = \mathbf{L}$). *Let $W \subseteq \mathrm{HC} \times \mathrm{HC}$ be a Δ^{HC}_n relation, $n \geq 1$. For* *any* $x \in X = \operatorname{dom} W$ *let* y_x *be the* $<_{\mathbf{L}}$*-least* y *satisfying* $W(x,y)$*. Then the set* $W' = \{\langle x, y_x\rangle : x \in X\}$ *is* Δ^{HC}_n *as well.*

Proof. Make use of the well-known fact, called the "goodness" of $<_{\mathbf{L}}$ that bounded quantifiers $\exists\, x <_{\mathbf{L}} y$, $\forall\, x <_{\mathbf{L}} y$ preserve the property of being Δ^{HC}_1. This property of bounded quantifiers is similar and essentially equivalent, via Proposition 7.1, to the fact that $\exists\, a <_{\mathbf{L}} b$, $\forall\, a <_{\mathbf{L}} b$ (where a, b vary over 2^ω) preserve the property of being Δ^1_n. See, *e.g.*, 5A.2 and 8F.7 in [20], or the proof of Lemma 25.27 in [9] on this descriptive set theoretic form of the "goodness" result. □

Borel coding and effective descriptive set theory

The following are useful technicalities. See, *e.g.*, Section II.1 in [23], Section 25 in [9], or Section 2.9 in [11] on Borel coding.

(I) Let $\mathcal{E} \subseteq 2^\omega$ be a Π^1_1 set of all Borel codes for Borel subsets of $2^\omega \times 2^\omega$, and if $\varepsilon \in \mathcal{E}$ then let $\mathbb{E}_\varepsilon \subseteq 2^\omega \times 2^\omega$ be the Borel set coded by ε. There exist two ternary Σ^1_1 relations $\mathfrak{E}$, $\mathfrak{E}'$ on 2^ω such that if $\varepsilon \in \mathcal{E}$ and $x, y \in 2^\omega$ then $\langle x, y\rangle \in \mathbb{E}_\varepsilon$ iff $\mathfrak{E}(\varepsilon, x, y)$ iff $\neg\, \mathfrak{E}'(\varepsilon, x, y)$.

(II) Let $\mathcal{F} \subseteq 2^\omega$ be a Π^1_1 set of all Borel codes for Borel maps $2^\omega \to 2^\omega$, and if $\varphi \in \mathcal{F}$ then let $\mathbb{f}_\varphi : 2^\omega \to 2^\omega$ be the Borel map coded by φ. There exist two ternary Σ^1_1 relations $\mathfrak{F}$, $\mathfrak{F}'$ on 2^ω such that if $\varphi \in \mathcal{E}$ and $x, y \in 2^\omega$ then $\mathbb{f}_\varphi(x) = y$ iff $\mathfrak{F}(\varepsilon, x, y)$ iff $\neg\, \mathfrak{F}'(\varepsilon, x, y)$.

(III) Let $\mathcal{T} \subseteq \mathscr{P}(2^{<\omega})$ be the set of all perfect trees in $2^{<\omega}$, and if $\tau \in \mathcal{T}$ then let $[\tau] \subseteq 2^\omega$ be the corresponding perfect set.

(IV) If $x \in 2^\omega$ and $Y \subseteq 2^\omega$ is a countable $\Sigma^1_1(x)$ set (*i.e.*, Σ^1_1-definable with x as a parameter) then $Y \subseteq \Delta^1_1(x)$, see, *e.g.*, 2.10.5 in [11].

(V) If $\Phi(x, y, \dots)$ is a Π^1_1 formula then so is $\exists\, y \in \Delta^1_1(x)\, \Phi(x, y, \dots)$, see, *e.g.*, 4D.3 in [20] or 2.8.6 in [11].

Lemma 7.3. *The following sets and relations are* Π^1_1*, hence* Δ^{HC}_1 *by Proposition 7.1:*

(i) *the set* $\mathcal{E}^{\mathit{cnt}} = \{\varepsilon \in \mathcal{E} : \mathbb{E}_\varepsilon$ *is a countable equivalence relation on* $2^\omega\}$;

(ii) *the sets* $\{\langle \delta, \varepsilon\rangle \in \mathcal{E} \times \mathcal{E} : \mathbb{E}_\delta \subseteq \mathbb{E}_\varepsilon\}$ *and* $\{\langle \delta, \varepsilon\rangle \in \mathcal{E} \times \mathcal{E} : \mathbb{E}_\delta = \mathbb{E}_\varepsilon\}$;

(iii) *the set* $\mathcal{E}^{\mathit{DBS}} = \{\langle \delta, \varepsilon\rangle : \delta, \varepsilon \in \mathcal{E}^{\mathit{cnt}} \wedge \langle \mathbb{E}_\delta, \mathbb{E}_\varepsilon\rangle$ *is a DBP*$\}$ *and the relation of extension of coded DBP, as in Definition 3.1;*

(iv) *the set* $\{\langle \tau, \varphi\rangle : \tau \in \mathcal{T} \wedge \varphi \in \mathcal{F} \wedge \mathbb{f}_\varphi \restriction [\tau]$ *is 1-1*$\}$;

(v) *the set* $\{\langle \tau, \rho, \varphi\rangle : \tau, \rho \in \mathcal{T} \wedge \varphi \in \mathcal{F} \wedge \mathbb{f}_\varphi\text{"}[\tau] \subseteq [\rho]\}$;

(vi) *the set* $\{\langle \delta, \varepsilon, \varphi, \tau\rangle : \tau \in \mathcal{T} \wedge \varphi \in \mathcal{F} \wedge \langle \delta, \varepsilon\rangle \in \mathcal{E}^{\mathit{DBS}} \wedge \langle \mathbb{E}_\delta, \mathbb{E}_\varepsilon\rangle$ *corralls* $\mathbb{f}_\varphi \restriction [\tau]\}$*, the same goes for positive and negative corralling.*

It follows that all those sets and relations belong to Δ^{HC}_1*.*

Proof. (i) Let $\varepsilon \in \mathcal{E}$. For $\mathbb{E}_\varepsilon$ to be an equivalence relation is equivalent to the following sentence:

$$\left.\begin{array}{r} \forall\, x \neg\, \mathfrak{E}'(\varepsilon, x, x) \wedge \forall\, x, y \left(\mathfrak{E}(\varepsilon, x, y) \implies \neg\, \mathfrak{E}'(\varepsilon, y, x)\right) \wedge \qquad\qquad \\ \wedge\, \forall\, x, y, z \left(\mathfrak{E}(\varepsilon, x, y) \wedge \mathfrak{E}(\varepsilon, y, z) \implies \neg\, \mathfrak{E}'(\varepsilon, x, z)\right), \end{array}\right\}$$

which is a Π^1_1 relation. Further, by (IV), $\mathbb{E}_\varepsilon$ is a **countable** equivalence relation iff

$$\forall\, x, y\, (\mathfrak{E}(\varepsilon, x, y) \implies y \in \Delta^1_1(\varepsilon, x)),$$

which is equivalent to

$$\forall\, x, y\, (\mathfrak{E}(\varepsilon, x, y) \implies \exists\, y' \in \Delta^1_1\,(\varepsilon, x)\,(y' = y)),$$

and this is a Π^1_1 relation by (V).

(iii) If $\delta, \varepsilon \in \mathcal{E}^{\mathtt{cnt}}$ and $\mathbb{E}_\delta \subseteq \mathbb{E}_\varepsilon$, then $\langle \mathbb{E}_\delta, \mathbb{E}_\varepsilon\rangle$ is a DBP iff, first, among any triple of $\mathbb{E}_\varepsilon$-equivalent elements there is a pair that is $\mathbb{E}_\delta$-equivalent, and second, $\forall\, x\, \exists\, y, y' \in \Delta^1_1(\varepsilon, x)\, (y \;\mathbb{E}_\varepsilon\; y' \wedge \neg\, y \;\mathbb{E}_\delta\; y')$, which is Π^1_1 by (V).

Claims (ii), (iv), (v), (vi) are established similarly. □

8. Increasing systems of equivalence relations

Theorem 8.1 (in L). *There is an $\preccurlyeq$-increasing sequence of DBPs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$, $\alpha < \omega_1$, beginning with $\langle \mathsf{B}_0, \mathsf{E}_0\rangle = \langle \mathbb{E}_0^{\mathtt{even}}, \mathbb{E}_0\rangle$ of Example 3.2 and such that*

(i) *if $X \subseteq 2^\omega$ is perfect and $f : X \to 2^\omega$ Borel and 1-1, then there exist: a perfect $X' \subseteq X$ and an ordinal $\alpha < \omega_1$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ corralls $f \restriction X'$;*

(ii) *if $X, R \subseteq 2^\omega$ are perfect sets, then there exist: a perfect set $Y \subseteq X$, an ordinal $\alpha < \omega_1$, and Borel 1-1 maps $f, g : Y \to R$, such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ corralls f positively and corralls g negatively;*

(iii) *the sequence of pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ is Δ^{HC}_1, in the sense that there exists a Δ^{HC}_1 sequence of codes for Borel sets B_α and E_α.*

Proof. We argue in L. The following is a straightforward inductive construction using Lemma 5.3 and Lemma 5.4 and taking the Gödel-least code amongst all possible extensions fitting the given inductive step. Let's carry out this plan in detail.

(∗) Fix an enumeration $\langle \widehat{X}_\alpha, \widehat{R}_\alpha, \widehat{f}_\alpha\rangle_{\alpha<\omega_1}$ of all triples $\langle X, R, f\rangle$, where $X, R \subseteq 2^\omega$ are perfect sets and $f : 2^\omega \to 2^\omega$ is a Borel map.

The base. Take $\mathbb{E}_0$ and $\mathsf{B}_0 = \mathbb{E}_0^{\mathtt{even}}$ as indicated.

The successor step. Suppose that $\alpha < \omega_1$ and a DBP $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ is defined. If $\widehat{f}_\alpha \restriction \widehat{X}_\alpha$ is not 1-1 then let $Y' = \widehat{X}_\alpha$ and $\langle \mathsf{B}', \mathsf{E}'\rangle = \langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$. Otherwise, by Lemma 5.3, there exist: a perfect set $Y' \subseteq \widehat{X}_\alpha$ and a DBP $\langle \mathsf{B}', \mathsf{E}'\rangle$, such that $\langle \mathsf{B}', \mathsf{E}'\rangle$ extends $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha\rangle$ and positively corralls $\widehat{f}_\alpha \restriction Y'$. By Lemma 5.4, there exist: a perfect set $Y'' \subseteq Y'$, Borel maps $g, h : 2^\omega \to 2^\omega$, and a DBP $\langle \mathsf{B}'', \mathsf{E}''\rangle$, such that

$g{\restriction}Y''$, $h{\restriction}Y''$ are 1-1 maps into $\widehat{R}_\alpha$, $\langle \mathsf{B}'', \mathsf{E}''\rangle$ extends $\langle \mathsf{B}', \mathsf{E}'\rangle$, positively corralls $g{\restriction}Y''$ and negatively corralls $h{\restriction}Y''$. Let $Y_{\alpha+1}$, $g_{\alpha+1}$, $h_{\alpha+1}$, $\langle \mathsf{B}_{\alpha+1}, \mathsf{E}_{\alpha+1}\rangle$ be any particular choice of such Y'', g, h, $\langle \mathsf{B}'', \mathsf{E}''\rangle$. Thus the relationships between the αth and $(\alpha+1)$th steps, in terms of $(*)$, are as follows:

(†) $Y'' \subseteq \widehat{X}_\alpha$, $\langle \mathsf{B}'', \mathsf{E}''\rangle$ is a DBP and extends $\langle \mathsf{B}, \mathsf{E}\rangle$,
either $\widehat{f}_\alpha{\restriction}\widehat{X}_\alpha$ is not 1-1 or $\langle \mathsf{B}'', \mathsf{E}''\rangle$ positively corralls $\widehat{f}_\alpha{\restriction}Y''$,
$g{\restriction}Y''$ and $h{\restriction}Y''$ are 1-1 Borel maps from Y'' to $\widehat{R}_\alpha$,
$\langle \mathsf{B}'', \mathsf{E}''\rangle$ corralls $g{\restriction}Y''$ positively and $h{\restriction}Y''$ negatively.

The limit step. If $\lambda < \omega_1$ is limit then put $\mathsf{E}_\lambda = \bigcup_{\alpha<\lambda} \mathsf{E}_\alpha$, $\mathsf{B}_\lambda = \bigcup_{\alpha<\lambda} \mathsf{B}_\alpha$.

Conclusion. Whatever way different choices are made during the course of the above construction, the result satisfies (i), (ii). It remains to fulfill (iii). This will be achieved by making the construction more precise.

Complexity evaluation. In the remainder of the proof of Theorem 8.1 in this section, we shall freely use special notation of Section 7, like *e.g.* $\mathcal{T}, \mathcal{F}, <_{\mathbf{L}}, \mathcal{E}^{\mathtt{DBS}}$ et cetera. Let $\langle \tau_\alpha, \rho_\alpha, \varphi_\alpha\rangle$ be the αth element of the set $\mathcal{T} \times \mathcal{T} \times \mathcal{F}$, in the sense of Gödel's wellordering $<_{\mathbf{L}}$. We define $\widehat{X}_\alpha = [\tau_\alpha]$, $\widehat{R}_\alpha = [\rho_\alpha]$, $\widehat{f}_\alpha = \mathbb{f}_{\varphi_\alpha}$.

Corollary 8.2 (by Proposition 7.2). *$\langle \tau_\alpha, \rho_\alpha, \varphi_\alpha\rangle_{\alpha<\omega_1}$ is a Δ^{HC}_1 sequence, and the sequence $\langle \widehat{X}_\alpha, \widehat{R}_\alpha, \widehat{f}_\alpha\rangle_{\alpha<\omega_1}$ satisfies $(*)$.* □

Now let $P(\langle \alpha, \delta, \varepsilon\rangle, \langle \delta', \varepsilon', \tau, \zeta, \eta\rangle)$ say:

(‡) $\alpha < \omega_1$; both $\langle \delta, \varepsilon\rangle$ and $\langle \delta', \varepsilon'\rangle$ belong to $\mathcal{E}^{\mathtt{DBS}}$; $\tau \in \mathcal{T}$; $\zeta, \eta \in \mathcal{F}$, and the pairs $\langle \mathsf{B}, \mathsf{E}\rangle = \langle \mathbb{E}_\delta, \mathbb{E}_\varepsilon\rangle$, $\langle \mathsf{B}'', \mathsf{E}''\rangle = \langle \mathbb{E}_{\delta'}, \mathbb{E}_{\varepsilon'}\rangle$ satisfy (†) with $Y'' = [\tau]$, the Borel maps $g = \mathbb{f}_\zeta$ and $h = \mathbb{f}_\eta$, and $\widehat{X}_\alpha$, $\widehat{R}_\alpha$, $\widehat{f}_\alpha$ provided by 8.2.

It follows from Corollary 8.2 and Lemma 7.3 that P has definability class Δ^{HC}_1. In addition $\mathtt{dom}\, P = \omega_1 \times \mathcal{E}^{\mathtt{DBS}}$ (see the successor step in the beginning of the proof of Theorem 8.1). If $\alpha < \omega_1$ and $\langle \delta, \varepsilon\rangle \in \mathcal{E}^{\mathtt{DBS}}$ then let

$$\pi_\alpha(\delta, \varepsilon) = \langle \delta'_\alpha(\delta, \varepsilon), \varepsilon'_\alpha(\delta, \varepsilon), \tau_\alpha(\delta, \varepsilon), \zeta_\alpha(\delta, \varepsilon), \eta_\alpha(\delta, \varepsilon)\rangle$$

be the $<_{\mathbf{L}}$-least tuple $\langle \delta', \varepsilon', \tau, \zeta, \eta\rangle$ satisfying $P(\langle \alpha, \delta, \varepsilon\rangle, \langle \delta', \varepsilon', \tau, \zeta, \eta\rangle)$. The map $\alpha, \delta, \varepsilon \mapsto \pi_\alpha(\delta, \varepsilon)$ is Δ^{HC}_1 by Lemma 7.2, and hence so are all five component maps, in particular, the mappings $\alpha, \delta, \varepsilon \mapsto \delta'_\alpha(\delta, \varepsilon)$ and $\alpha, \delta, \varepsilon \mapsto \varepsilon'_\alpha(\delta, \varepsilon)$, because the domain $\mathtt{dom}\, P = \omega_1 \times \mathcal{E}^{\mathtt{DBS}}$ is a Δ^{HC}_1 set.

To complete the proof of Theorem 8.1, define a sequence $\langle\langle \delta_\alpha, \varepsilon_\alpha\rangle\rangle_{\alpha<\omega_1}$ of pairs $\langle \delta_\alpha, \varepsilon_\alpha\rangle \in \mathcal{E}^{\mathtt{DBS}}$ by transfinite induction. Let $\delta_0, \varepsilon_0 \in \mathcal{E}^{\mathtt{cnt}}$ be any recursive codes for the equivalence relations, resp., $\mathbb{E}_0^{\mathtt{even}}$ and $\mathbb{E}_0$, so $\langle \delta_0, \varepsilon_0\rangle \in \mathcal{E}^{\mathtt{DBS}}$. On successor steps, if $\langle \delta_\alpha, \varepsilon_\alpha\rangle \in \mathcal{E}^{\mathtt{DBS}}$ is defined, then let $\delta_{\alpha+1} = \delta'_\alpha(\delta, \varepsilon)$ and $\varepsilon_{\alpha+1} = \varepsilon'_\alpha(\delta, \varepsilon)$. On limit steps, if $\lambda < \omega_1$ is limit and $\langle \delta_\alpha, \varepsilon_\alpha\rangle \in \mathcal{E}^{\mathtt{DBS}}$ is defined for all $\alpha < \lambda$, then let $\langle \delta_\lambda, \varepsilon_\lambda\rangle$ be the $<_{\mathbf{L}}$-least pair of codes in $\mathcal{E}^{\mathtt{cnt}}$ satisfying $\mathbb{E}_{\delta_\lambda} = \bigcup_{\alpha<\lambda} \mathbb{E}_{\delta_\alpha}$ and $\mathbb{E}_{\varepsilon_\lambda} = \bigcup_{\alpha<\lambda} \mathbb{E}_{\varepsilon_\alpha}$. It follows by Proposition 7.2 and the choice of the maps

$\alpha, \delta, \varepsilon \mapsto \delta'_\alpha(\delta, \varepsilon)$ and $\alpha, \delta, \varepsilon \mapsto \varepsilon'_\alpha(\delta, \varepsilon)$ via standard estimation, that the sequence $\langle\langle \delta_\alpha, \varepsilon_\alpha \rangle\rangle_{\alpha<\omega_1}$ is Δ^{HC}_1, so we have (iii) of Theorem 8.1.

(Theorem 8.1) □

Theorem 8.3 (in L). *There is an $\preccurlyeq$-increasing sequence of DBPs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, $\alpha < \omega_1$, beginning with $\langle \mathsf{B}_0, \mathsf{E}_0 \rangle = \langle \mathbb{E}_0^{even}, \mathbb{E}_0 \rangle$ of Example 3.2, and such that*

(i) *if $X \subseteq 2^\omega$ is a Borel $\mathbb{E}_0$-large set and $f : X \to 2^\omega$ Borel and 1-1, then there exist: a canonical $\mathbb{E}_0$-large set $Y \subseteq X$ and $\alpha < \omega_1$ such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ corralls $f \restriction Y$;*

(ii) *if $X, R \subseteq 2^\omega$ are $\mathbb{E}_0$-large sets, then there exist: a canonical $\mathbb{E}_0$-large set $Y \subseteq X$, an ordinal $\alpha < \omega_1$, and canonical $\mathbb{E}_0$-large maps $f, g : Y \to R$, such that $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ corralls f positively and g negatively;*

(iii) *the sequence of pairs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$ is Δ^{HC}_1, in the sense that there exists a Δ^{HC}_1 sequence of codes for Borel sets B_α and E_α.*

Proof. Similar to Theorem 8.1. □

9. Proof of the main theorem in the Sacks case

We prove Theorem 1.1 separately in the Sacks case (this Section) and in the $\mathbb{E}_0$-large case (the next Section).

To begin the Sacks case proof, let us fix, in **L**, an $\preccurlyeq$-increasing sequence of DBPs $\langle \mathsf{B}_\alpha, \mathsf{E}_\alpha \rangle$, $\alpha < \omega_1$, satisfying conditions (i), (ii), (iii) of Theorem 8.1.

Arguing in a Sacks-generic extension $\mathbf{L}[a_0]$, we define a relation $\mathsf{B} = \bigcup_{\alpha<\omega_1} \mathsf{B}_\alpha$ on 2^ω; thus $x \;\mathsf{B}\; y$ iff $x \;\mathsf{B}_\alpha\; y$ for some $\alpha < \omega_1$. (We identify Borel sets B_α, E_α, formally defined in **L**, with their extensions, Borel sets in $\mathbf{L}[a_0]$ with the same codes.) Define a relation $\mathsf{E} = \bigcup_{\alpha<\omega_1} \mathsf{E}_\alpha$ on 2^ω similarly. Define a subdomain $U = 2^\omega \smallsetminus \mathbf{L}$ of all new reals. Then $a_0 \in U$ and all reals in U have the same **L**-degree by the minimality property of Sacks reals, see, *e.g.*, Theorem 15.34 in [9].

Lemma 9.1. *It is true in $\mathbf{L}[a_0]$ that*

(i) *E and B are equivalence relations and B is a subrelation of E;*

(ii) *all reals $x, y \in U = 2^\omega \smallsetminus \mathbf{L}$ are E-equivalent;*

(iii) *there are exactly two B-classes intersecting U — call them M, N;*

(iv) *B is lighface Σ^1_2 but $\mathsf{B} \restriction U$ is lighface Π^1_2;*

(v) *the sets M, N are not OD*[i]*, and we have $M \cup N = U$.*

Proof. (i) To see that E is an equivalence relation, let $a, b, c \in U$ and suppose that $a \;\mathsf{E}\; b$ and $a \;\mathsf{E}\; c$. Then by definition we have $a \;\mathsf{E}_\alpha\; b$ and $a \;\mathsf{E}_\alpha\; c$ for some $\alpha <$

[i] Note that M, N are indiscernible in a stronger sense: if $R(M, N)$ holds for some OD relation R, then $R(N, M)$ holds. Indeed, otherwise M can be distinguished from N by the property: "$R(\cdot, A)$ holds but $R(A, \cdot)$ fails, where A is the other element of $\{M, N\}$".

ω_1. However being an equivalence relation is absolute by Shoenfield's absoluteness theorem (Theorem 25.20 in [9]). Therefore $b \mathsf{E}_\alpha c$ and hence $b \mathsf{E} c$ holds, as required. The proof for B is similar.

(ii) Let $b \in U$; prove that $a_0 \mathsf{E} b$. By Lemma 4.7, there is a continuous map $f : 2^\omega \to 2^\omega$ with a code in $\mathbf{L}$, such that $b = f(a_0)$. By the genericity of a_0 and Corollary 4.3, there is a perfect set $X \subseteq 2^\omega$, coded in $\mathbf{L}$, such that $a_0 \in X$ and the restricted map $f \restriction X$ is 1-1 or a constant. However if $f \restriction X$ is a constant, say $f(x) = z_0 \in 2^\omega$ for all $x \in X$, then $f(a_0) = b = z_0 \in \mathbf{L}$, which contradicts $b \notin \mathbf{L}$. (Since f and X are coded in $\mathbf{L}$, if $f \restriction X$ is a constant function taking value z_0, then z_0 must be a constructible real.) Thus $f \restriction X$ is 1-1. It follows then from Theorem 8.1(i) that there exists a perfect set $Y \subseteq 2^\omega$, coded in $\mathbf{L}$ and such that $a_0 \in Y$ and E_α corralls $f \restriction Y$ for some α. In particular, $\langle a_0, b\rangle \in \mathsf{E}_\alpha$, hence we have $a_0 \mathsf{E} b$ as required.

(iii) Let $a, b, c \in U$; prove that two of these reals are B-equivalent. Note that $a \mathsf{E} b \mathsf{E} c$ by (ii), and hence there is an ordinal $\alpha < \omega_1$ such that $a \mathsf{E}_\alpha b \mathsf{E}_\alpha c$. But containing exactly two B_α-classes in each E_α-class is absolute by Shoenfield. Therefore at least one pair among a, b, c is B_α-equivalent, and hence B-equivalent. Therefore there are **at most** two B-classes in U. due to the construction, if $x, y \in U$ are not $\mathbb{E}_0^{\mathsf{even}}$-equivalent, then x and y are not B-equivalent, so there are **exactly** two B-classes intersecting U, as required.

(iv) That B as a whole is Σ^1_2 holds by Theorem 8.1(iii). Next we prove that $\mathsf{B} \restriction U$ is Π^1_2. If $y \in 2^\omega$ then define $y^- \in 2^\omega$ by $y^-(0) = 1 - y(0)$ but $y^-(k) = y(k)$ for all $k \geq 1$. Then $y \mathbb{E}_0 y^-$ but $\neg(y \mathbb{E}_0^{\mathsf{even}} y^-)$. It follows by construction that $y \not\mathsf{B} y^-$ in $\mathbf{L}[a_0]$ for all y. Thus if $x, y \in U$ then $x \mathsf{B} y$ iff $x \not\mathsf{B} y^-$ by (iii).

(v) Suppose to the contrary that M is OD. Then M is Sacks-forced over $\mathbf{L}$, meaning that there is a perfect set $R \subseteq 2^\omega$, coded in $\mathbf{L}$ and such that $R \cap U \subseteq M$ in $\mathbf{L}[a_0]$. By Theorem 8.1(ii), there exist: a perfect set $Y \subseteq 2^\omega$ coded in $\mathbf{L}$ and containing a_0, an ordinal $\alpha < \omega_1$, and Borel 1-1 maps $f, g : Y \to R$, also coded in $\mathbf{L}$ and such that E_α corralls $f \restriction Y$ positively and $g \restriction Y$ negatively. In other words the reals $b = f(a_0)$ and $c = g(a_0)$ in $U \cap R$ satisfy $a_0 \mathsf{B}_\alpha b$, $a_0 \mathsf{E}_\alpha c$, but $a_0 \not\mathsf{B}_\alpha c$. It easily follows that $b \not\mathsf{B} c$, which contradicts the fact that b, c belong to one and the same B-class.

Thus M is not OD in $\mathbf{L}[a_0]$. We conclude that M cannot contain any constructible reals since constructible reals are ordinal definable. In other words $M \subseteq U$ and M is a B-class inside of U. Similarly for N. Since (iii) states there are only two B-classes touching U, we finally have $U = M \cup N$, as required. □

To conclude, it is true in the Sacks extension $\mathbf{L}[a_0]$ that the restricted relation $\mathsf{Q} := \mathsf{B} \restriction U$ is a Π^1_2 equivalence relation on the nonconstructible domain $U = 2^\omega \smallsetminus \mathbf{L}$ (a Π^1_2 set), and the quotient U/Q contains exactly two classes, both of which are non-OD.

(Theorem 1.1, Sacks case) □

10. Proof of the main theorem in the $\mathbb{E}_0$-large case

Rather similar to the proof of the Sacks case above. **Arguing in an $\mathbb{E}_0$-large-generic extension $\mathbf{L}[a_0]$,** we define relations $\mathsf{B}= \bigcup_{\alpha<\omega_1} \mathsf{B}_\alpha$, $\mathsf{E}= \bigcup_{\alpha<\omega_1} \mathsf{E}_\alpha$ on 2^ω, and the subdomain $U = 2^\omega \smallsetminus \mathbf{L}$; $a_0 \in U$.

Lemma 10.1. *It is true in $\mathbf{L}[a_0]$ that*

(i) E *and* B *are equivalence relations and* B *is a subrelation of* E ;
(ii) *all reals $x, y \in U$ are E-equivalent;*
(iii) *there are exactly two B-classes intersecting U — call them M, N ;*
(iv) B *is lighface Σ^1_2 but $\mathsf{B}\restriction U$ is lighface Π^1_2 ;*
(v) *the sets M, N are not OD, and we have $M \cup N = U$.*

Proof. The proof of claims (i), (iv), (ii), (iii) are similar to the proofs of the corresponding claims in Lemma 9.1, with some obvious changes *mutatis mutandis*, in particular, the reference to Corollary 4.3 has to be replaced by Corollary 4.6 in the proof of (ii), Proposition 8.1 has to be replaced by Proposition 8.3, and so on. But the last claim needs special attention because not all new reals in $\mathbf{L}[a_0]$ are $\mathbb{E}_0$-large-generic unlike the Sacks case.

(v) First of all let's prove that each of the classes M, N of (iii) contains a real $b \in 2^\omega$ $\mathbb{E}_0$-large-generic over $\mathbf{L}$. Indeed in view of (iii) it suffices to prove that (*) *there are $\mathbb{E}_0$-large-generic, but not B-equivalent, reals $b, c \in \mathbf{L}[a_0] \cap 2^\omega$.* Emulating the proof of Theorem 9.1(v), but using Theorem 8.3(ii) instead of Theorem 8.1(ii), we find a canonical $\mathbb{E}_0$-large set $Y \subseteq 2^\omega$, coded in $\mathbf{L}$ and containing a_0, an ordinal $\alpha < \omega_1$, and canonical $\mathbb{E}_0$-large maps $f, g : Y \to 2^\omega$, also coded in $\mathbf{L}$ and such that E_α corralls $f\restriction Y$ positively and $g\restriction Y$ negatively. We conclude that the reals $b = f(a_0)$ and $c = g(a_0)$ in U satisfy $a_0 \mathrel{\mathsf{B}_\alpha} b$, $a_0 \mathrel{\mathsf{E}_\alpha} c$, but $a_0 \not\mathrel{\mathsf{B}}_\alpha c$, so that $b \not\mathrel{\mathsf{B}} c$. And finally, it is clear that b, c are $\mathbb{E}_0$-large-generic along with a_0. (Basically any image of a $\mathbb{E}_0$-large-generic real $a \in 2^\omega$ via a canonical $\mathbb{E}_0$-large map h, coded in $\mathbf{L}$, with $a \in \mathtt{dom}\, h$, is $\mathbb{E}_0$-large-generic by an easy argument.)

Now suppose to the contrary that M is OD. Let $\varphi(\cdot)$ be an $\in$-formula, with ordinals as parameters, such that $M = \{x : \varphi(x)\}$ in $\mathbf{L}[a_0]$. By (*), there is a real $b_0 \in M$ (in $\mathbf{L}[a_0]$), such that b_0 is $\mathbb{E}_0$-large-generic over $\mathbf{L}$. Then it is true in $\mathbf{L}[a_0] = \mathbf{L}[b_0]$ that $\varphi(b_0)$, and any real x satisfying $\varphi(x)$ also satisfies $x \mathrel{\mathsf{B}} b_0$. This is $\mathbb{E}_0$-large-forced over $\mathbf{L}$, meaning that there is a canonical $\mathbb{E}_0$-large set $R \subseteq 2^\omega$, coded in $\mathbf{L}$ and satisfying the following: (1) $b_0 \in R$, (2) if $b \in R \cap \mathbf{L}[b_0]$ and b is $\mathbb{E}_0$-large-generic over $\mathbf{L}$, then $\varphi(b)$ holds in the model $\mathbf{L}[b] = \mathbf{L}[b_0] = \mathbf{L}[a_0]$, and hence we have $b \mathrel{\mathsf{B}} b_0$.

However, emulating the proof of Theorem 9.1(v) as above, we find a canonical $\mathbb{E}_0$-large set $Y \subseteq 2^\omega$, coded in $\mathbf{L}$ and containing b_0, an ordinal $\alpha < \omega_1$, and canonical $\mathbb{E}_0$-large maps $f, g : Y \to R$, also coded in $\mathbf{L}$ and such that E_α corralls $f\restriction Y$ positively and $g\restriction Y$ negatively. Then the reals $b = f(b_0)$ and $c = g(b_0)$ are $\mathbb{E}_0$-large-generic over $\mathbf{L}$ and satisfy $b_0 \mathrel{\mathsf{B}_\alpha} b$ and $b_0 \mathrel{\mathsf{E}_\alpha} c$ but $b_0 \not\mathrel{\mathsf{B}}_\alpha c$, hence $b \not\mathrel{\mathsf{B}} c$, which contradicts (2) above. □

(Theorem 1.1, $\mathbb{E}_0$-large case) □

11. Final remarks and questions

We proceed with two questions, also containing some related remarks.

Question 11.1. It is interesting to figure out whether Theorem 1.1 holds in other extensions of $\mathbf{L}$ by a single generic real, *e.g.* in extensions by a single Cohen-generic [j], or a single Solovay-random, or a single Silver real, et cetera.

The Silver case is especially interesting as it is close to the Sacks case in some forcing details like the property of continuous reading of names or the minimality of generic reals. The major technical obstacle is that one is not able to prove an analog of the corralling lemmas (Lemma 5.3 and Lemma 6.1) for the Silver perfect sets due to the lack of an appropriate canonization result. Indeed, comparable to Theorem 4.4, the Silver-related canonization theorem (Theorem 8.6 in [15]) weakens case (II) to the following: "or (II) a Borel subrelation $\mathsf{E}' \subseteq \mathbb{E}_0$". One can try to circumvent this by beginning the construction of a sequence of DBPs as in Theorem 8.1 and Theorem 8.3 with the pair $\langle \mathbb{E}_0, \widehat{\mathbb{E}_0} \rangle$ rather than $\langle \mathbb{E}_0^{\mathtt{even}}, \mathbb{E}_0 \rangle$, where $\widehat{\mathbb{E}_0}$ extends $\mathbb{E}_0$ by connecting each pair of $x, \hat{x}$, where $\hat{x}$ is defined by $\hat{x}(k) = 1 - x(k)$ for all k. Then we still have $\mathsf{E} = \mathsf{B} = \mathsf{F} = \mathbb{E}_0$ on Y in the context of Case 2 of Lemma 6.1, but still there is no way to accordingly simplify $\widehat{\mathsf{E}}$ and $\widehat{\mathsf{B}}$ to maintain the merger of equivalence classes.

On the positive side, we have been able to verify that Theorem 1.1 also holds for forcing by perfect non-σ-compact sets in ω^ω (the Miller forcing). □

Question 11.2. In view of Theorem 1.1, one may ask whether there is a model of **ZFC** in which, to the contrary, the following holds:

(¶) every finite, or even countable [k], OD set $X \neq \varnothing$ contains an OD element, and, to avoid trivialities, there is no OD wellordering of the universe.

In particular, could the Lévy-collapse model $\mathbf{V}[G]$, obtained by adjoining a collapse-generic set G (up to an inaccessible cardinal) to a ground universe $\mathbf{V}$, as in [23], Section I.3, be such a model?

(A) Assuming that $\mathbf{V} = \mathbf{L}$, this question has a positive answer in $\mathbf{L}[G]$ for countable OD *sets of reals*, since it is known from [23] that any OD set of reals

[j] Since adding a single Cohen reals is equivalent to adding many Cohen reals, it is fairly easy to show that there are indiscernible sets of reals in Cohen extensions, *e.g.* $[a]_{\mathbf{L}}$ and $[b]_{\mathbf{L}}$ for any Cohen-generic pair of reals $\langle a, b \rangle$, as shown in Theorem 3.1 of [3]. On the other hand, such indiscernibles hardly form an OD pair, or, equivalently, arise as equivalence classes of an OD equivalence relation E with only two equivalence classes.

[k] The question is less interesting for <u>un</u>countable sets X because if $\mathbb{R} \not\subseteq$ OD then the uncountable OD set $X = \mathbb{R} \smallsetminus$ OD of all non-OD reals gives an obvious counterexample.

in $\mathbf{L}[G]$, containing at least one non-OD real, contains a perfect subset. A minor further step was achieved in [13]: it is true in $\mathbf{L}[G]$ that any non-empty countable OD set *of sets of reals* still contains an OD element.

(B) Let now $\mathfrak{N}_1 \subseteq \mathbf{L}[G]$ consist of all sets hereditarily definable in $\mathbf{L}[G]$ from an ω-sequence of ordinals, as in [23], Section III.2.4. (This is the Solovay model, in which all sets of reals are measurable.) Then $\mathfrak{N}_1 \models \mathbf{ZF} + \mathbf{DC}$, and it is true in $\mathfrak{N}_1$ that there exists an OD surjection of $\mathbf{Ord} \times \mathbb{R}$ onto the whole universe $\mathbf{L}[G]$. In addition the result mentioned in (A) above remains true in $\mathfrak{N}_1$. Therefore (¶) is true in $\mathfrak{N}_1$ in full generality. (This observation was kindly communicated by W. Hugh Woodin.) But the problem with (¶) remains open for $\mathbf{L}[G]$ itself.

The following notes (C), (D), (E) also were kindly communicated to the authors by W. Hugh Woodin; they contain references to his unpublished results.

(C) If one assumes $\mathbf{AD}^+$ and $\mathbf{V} = \mathbf{L}(\mathscr{P}(\mathbb{R}))$, then every countable OD set X contains only OD elements, and moreover, either X has an OD wellordering or there is 1-to-1 map from $\mathbb{R}$ to X. Note that $\mathbf{AD}^+$ holds in all known models of $\mathbf{AD}$, and (assuming $\mathbf{ZF} + \mathbf{DC}$) it is implied by $\mathbf{AD}_{\mathbb{R}}$, the axiom of real determinacy. See [24] or [1, 2] on $\mathbf{AD}^+$ and related issues.

(D) Even in case $\mathbf{V} \neq \mathbf{L}$, one can consider a Solovay-style submodel $\mathbf{L}(\mathbb{R}) \subseteq \mathbf{V}[G]$ in the Lévy extension $\mathbf{V}[G]$. If suitable large cardinals exist in $\mathbf{V}$, then $\mathbf{L}(\mathbb{R})$ models AD, and subsequently (¶) holds in $\mathbf{L}(\mathbb{R})$. But without the assumption of AD in $\mathbf{L}(\mathbb{R})$, or $\mathbf{V} \neq \mathbf{L}$, it's still probably open whether (¶) holds in $\mathbf{L}(\mathbb{R})$, even if one assumes that X is finite. The trouble of course is the current limited knowledge of the analysis of ordinal definability in the Solovay model $\mathbf{L}(\mathbb{R})$ obtained by collapsing over some arbitrary $\mathbf{V}$.

(E) A good candidate for a positive answer to Problem 11.2 is the $\mathbb{P}_{\max}$-extension of $\mathbf{L}(\mathbb{R})$. See [24] or elsewhere on $\mathbb{P}_{\max}$.[1] □

12. History of this result

The proof of Theorem 1.1 given above was written in January 2020, after a short discussion of the following excerpt from an old email message from R. M. Solovay to Ali Enayat, quoted here thanks to Solovay's generous permission.

> [**Solovay to Enayat 25.10.2002:**]
>
> Here's a freshly minted theorem.
>
> Consider the Sacks extension of a model of $\mathbf{V} = \mathbf{L} + \mathbf{ZFC}$. Then LA

[1]Upon the completion of the revision of the manuscript, we received another relevant note from W. Hugh Woodin. It follows verbatim. *Here is probably the best version of 11.2. Assume determinacy. Thus the theory of $\mathbf{L}[x]$ is constant on a cone. So what happens on a cone? Does every countable OD-$\mathbf{L}[x]$ set contain only $\mathbf{L}[x]$ members? Rephrased: in $\mathbf{L}[x]$ if E is an OD equivalence relation on R with only countably many equivalence classes, must each class be OD? Note by Kechris-Solovay: OD-determinacy holds in $\mathbf{L}[x]$ on a cone so probably the answer is yes.* The reference to Kechris-Solovay is Theorem 3.1 in [18].

does not hold. [m]

My proof is a bit involved. Here's a high level - view.

By a transfinite construction of length $\aleph_1$ I construct a P-name [n] E such that the following are forced: [o]

(1) E is an equivalence relation on the set of non-constructible reals.
(2) E has precisely two equivalence classes.
(3) In each perfect set with constructible code there are representatives of both equivalence classes.
(4) E is ordinal definable.

The two distinct but indiscernable members of the generic extension are the two equivalence classes of E.

The proof is a bit too involved to type in using a web-interface like yahoo. (Shades of Fermat's margin!) The proof uses one standard but relatively deep fact from descriptive set theory. If B is an uncountable Borel set, then B contains a perfect subset.

– Bob

P.S. I don't use much about **L**. Just that it satisfies $\mathbf{V} = \mathrm{OD}$ and is uniformly definable in any extension and that it satisfies CH. [p] **[End]**

The above proof of Theorem 1.1 in the Sacks case follows Solovay's outline from the point of view of the general flow of the argument. [q] In particular our lemmas 5.3 and 6.1 were designed to take care of Solovay's item (1), lemmas 5.4 and 6.2 — to take care of Solovay's item (3), and the key notion of DBP — to take care of Solovay's item (2). However, we were not able to work strictly within the descriptive set theoretic instrumentarium explicitly restricted by Solovay to the perfect subset property of Borel sets. In light of the key role of the canonization theorems (Corollary 4.2 and Theorem 4.4), presented here in the proof of Theorem 1.1, but not mentioned by Solovay, we don't know to what degree our proof really coincides with the original proof by Solovay in all important details.

Upon the completion of the proof, the co-authors contacted R. M. Solovay, with an invitation to join as a co-author of this note, but he unfortunately did not accept

[m] In the context of this exchange, LA is the Mycielski axiom, the axiom formulated by Mycielski and investigated in Enayat's paper [3], in which it is referred to as the Leibniz-Mycielski axiom LM. The axiom LM states that given any pair of distinct sets a and b, there is some ordinal α, and some first order formula $\phi(x)$, such that $\mathbf{V}_\alpha$ contains a and b, and $\mathbf{V}_\alpha$ satisfies $\phi(a)$ but does not satisfy $\phi(b)$, that is, in brief, that any sets $a \neq b$ are OD-discernible. The motivation for establishing Theorem 1.1 was the guess (privately communicated by Enayat to Solovay) that the consistency of **ZFC**+LM+"$\mathbf{V} \neq \mathbf{HOD}$" can be shown by verifying that LM holds in the extension of the constructible universe by a Sacks real. The question of consistency of **ZFC**+LM+"$\mathbf{V} \neq \mathbf{HOD}$" has proved to be more difficult than meets the eye, and remains open.

[n] In the context of this letter, P is the Sacks forcing.

[o] Enumeration (1)–(4) is ours. - AE and VK.

[p] This is equally true for our proof.

[q] All technical arrangements made in the proofs, beyond the literal content of Solovay's email message to Enayat cited above, are our own, of course.

our invitation.

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