BOREL IRREDUCIBILITY BETWEEN TWO LARGE FAMILIES OF BOREL EQUIVALENCE RELATIONS

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Abstract. We prove that if $\mathcal{I}$ is a Borel ideal, which includes a dense summable ideal, and E is a Borel equivalence relation that can be obtained from $\mathtt{Fin}$ using certain elementary operations like the Fubini product and countable power relation, then $\mathsf{E}_{\mathcal{I}}$ is not Borel reducible to E. The ideals $\mathcal{I}$ in the scope of this theorem include, for instance, all Borel P-ideals except for $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ and (trivial variations of) $\mathtt{Fin}$.

§1. The result. Let $\mathfrak{E}$ be the smallest class of Borel equivalence relations (or ERs, for brevity) E on Polish spaces, containing the equality relations on finite and countable sets and closed under the the following transformations:

(1) *countable union* (if it results in a ER) and *countable intersection* of ERs on one and the same space;

(2) *countable disjoint union* $\mathsf{E} = \bigvee_k \mathsf{E}_k$ of ERs E_k on pairwise disjoint spaces $\mathbb{S}_k$, that is, a ER on the union $\bigcup_k \mathbb{S}_k$ defined by: $x \mathrel{\mathsf{E}} y$ iff x, y belong to the same $\mathbb{S}_k$ and $x \mathrel{\mathsf{E}_k} y$;

(3) the *Fubini product* $\mathtt{Fin} \bigotimes_k \mathsf{E}_k$ of ERs E_k on spaces $\mathbb{S}_k$, over the ideal $\mathtt{Fin}$ of all finite subsets of $\mathbb{N}$, that is, the ER on the product space $\prod_k \mathbb{S}_k$ defined as follows: $x \mathrel{\mathsf{E}} y$ iff $x(k) \mathrel{\mathsf{E}_k} y(k)$ for all but finite k;

(4) *product* $\mathsf{E} = \prod_k \mathsf{E}_k$ of ERs E_k on spaces $\mathbb{S}_k$, that is, the ER on the product space $\prod_k \mathbb{S}_k$ defined by: $x \mathrel{\mathsf{E}} y$ iff $x(k) \mathrel{\mathsf{E}_k} y(k)$ for all k;

(5) *countable power* ER E^∞ of a ER E on a space $\mathbb{S}$, that is, a ER on $\mathbb{S}^{\mathbb{N}}$ defined as follows: $x \mathrel{\mathsf{E}^\infty} y$ iff $\{[x(k)]_{\mathsf{E}} : k \in \mathbb{N}\} = \{[y(k)]_{\mathsf{E}} : k \in \mathbb{N}\}$, where $[z]_{\mathsf{E}}$ is the E-class of $z \in \mathbb{S}$: thus, it is required that for any k there is l with $x(k) \mathrel{\mathsf{E}} y(l)$ and for any l there is k with $x(k) \mathrel{\mathsf{E}} y(l)$.

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THEOREM 1 (The main result). *If $\mathcal{Z}$ is a nontrivial*[1] *Borel P-ideal on $\mathbb{N}$ and E is a ER in $\mathfrak{E}$ then $\mathsf{E}_{\mathcal{Z}}$ is not Baire-measurable reducible to E unless $\mathcal{Z}$ is isomorphic*[2] *to $\mathtt{Fin}$, a trivial variation of $\mathtt{Fin}$, or $\mathcal{I}_3 = 0 \times \mathtt{Fin}$.*

Special notions involved in this theorem (all of them known in this direction of descriptive set theory, see, *e.g.*, [1, 4]) are explained in the next section.

§2. Notation and comments. Recall that any ideal $\mathcal{I}$ on a set A induces an ER $\mathsf{E}_{\mathcal{I}}$ on 2^A: $x \mathrel{\mathsf{E}_{\mathcal{I}}} y$ iff the set $x \,\Delta\, y = \{i \in A : x(i) \neq y(i)\}$ belongs to $\mathcal{I}$.

- An equivalence relation E on $\mathbb{S}$ is *Borel* or *Baire measurable* (BM, for brevity) *reducible* to a ER E' on $\mathbb{S}'$ if there is a resp. Borel or BM *reduction* E to E', that is, a resp. Borel or BM map $F : \mathbb{S} \to \mathbb{S}'$ such that we have $x \mathrel{\mathsf{E}} y \iff F(x) \mathrel{\mathsf{E}'} F(y)$ for all $x, y \in \mathbb{S}$.
- *P-ideals* are those ideals $\mathcal{Z}$ which satisfy the requirement that for any sequence of sets $x_0, x_1, x_2, \ldots \in \mathcal{Z}$ there is $x \in \mathcal{Z}$ with $x_n \subseteq^* x$ for all n, where $y \subseteq^* x$ means that $y \setminus x$ is finite.

Borel P-ideals (in fact all of them belong to Borel class $\boldsymbol{\Pi}^0_3$) admit different characterizations (see, *e.g.*, the proof of Proposition 2 below) and form an important and widely studied class, which includes, for instance,

(i) the ideal $\mathtt{Fin}$ of all finite subsets of $\mathbb{N}$,

(ii) the ideal $\mathcal{I}_3 = 0 \times \mathtt{Fin}$ of all sets $x \subseteq \mathbb{N} \times \mathbb{N}$ such that every cross-section $(x)_n = \{k : \langle n, k\rangle \in x\}$ is finite,

(iii) *trivial variations of* $\mathtt{Fin}$, *i.e.*, by Kechris [5], ideals of the form $\mathcal{I} = \{x \subseteq \mathbb{N} : x \cap W \in \mathtt{Fin}\}$, where $W \subseteq \mathbb{N}$ is infinite and coinfinite (all of them are isomorphic to each other).

Borel P-ideals also include summable ideals, density ideals, and many more (see Farah [1], Solecki [7, 8]).

The class $\mathfrak{E}$ of ERs contains, for instance, the equality $\mathsf{D}(2^{\mathbb{N}})$ on $2^{\mathbb{N}}$,[3] it also contains the ERs $\mathsf{E}_{\mathtt{Fin}}$ and $\mathsf{E}_{0\times\mathtt{Fin}}$ (usually denoted by resp. E_0 and E_3), associated with the ideals $\mathtt{Fin}$ and $\mathcal{I}_3 = 0 \times \mathtt{Fin}$,[4] as well as those associated with trivial variations of $\mathtt{Fin}$. Thus the exclusion of $\mathtt{Fin}$, $0 \times \mathtt{Fin}$, and trivial variations of $\mathtt{Fin}$ in Theorem 1 is necessary and fully motivated.

Furthermore $\mathfrak{E}$ contains all ERs associated with the *iterated Frechet ideals*, *i.e.*, the smallest family of ERs containing equality relations on finite and countable sets and closed under (3). Class $\mathfrak{E}$ also contains all ERs associated with the *indecomposable ideals* (Farah [1]) $\mathcal{I}_\xi = \{x \subseteq \omega^\xi : \mathtt{otp}\, x < \omega^\xi\}$, $\xi < \omega_1$ ($\mathtt{otp}\, x$ is the order type of a set $x \subseteq \mathrm{Ord}$), yet in this case it takes

[1] An ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is nontrivial if it is not equal to $\mathcal{P}(X)$ for some $X \subseteq \mathbb{N}$.

[2] By isomorphism we mean isomorphism via a bijection between the underlying sets.

[3] To see that $\mathsf{D}(2^{\mathbb{N}})$ belongs to $\mathfrak{E}$ let each E_k be the equality on a 2-element set in (4).

[4] To see that $\mathsf{E}_{\mathtt{Fin}}$ belongs to $\mathfrak{E}$ let each E_k be the equality on a 2-element set in (3). To see that $\mathsf{E}_{0\times\mathtt{Fin}}$ belongs to $\mathfrak{E}$ take each E_k to be $\mathsf{E}_{\mathtt{Fin}}$ in (4).

some effort to find a recursive construction of $\mathcal{I}_\xi$ in terms of the transformations (1)–(4). Class $\mathfrak{C}$ also contains all ERs T_α of Friedman and Stanley [3, 2], obtained from the equality on $\mathbb{N}$ via operations (2) and (5): they can be seen as all (modulo Borel reduction) ERs which, in some broad sense, admit classification by countable structures.

Two earlier related results must be mentioned. Friedman and Stanley announced in [3] (Friedman gives a full proof in [2]) that $\mathsf{E}_{\mathcal{Z}_0}$, the ER associated with the density-0 ideal $\mathcal{Z}_0$, is not BM reducible to any ER of the form T_α (see above). Kechris [5] proved (as a part of the proof of another result on ideals) the particular case $\mathsf{E} = \mathsf{E}_3 = \mathsf{E}_{0\times\mathtt{Fin}}$ of Theorem 1.

The arguments of Kechris and (implicitly) Friedman were based on ideas of Hjorth's *turbulence theory*.[5] So is our proof, its scheme is to show that a certain stronger form of irreducibility (called: the generic ergodicity) is preserved under the transformations of equivalence relations (1)–(5). Our proof is self-contained and rather elementary, in particular, it makes no use of model theory or facts related to topological group theory, yet it makes use of forcing.

§3. "Special" ideals. Before the main part of the proof of Theorem 1 begins, we are going to simplify the task. This section reduces the problem to ideals which include some kind of summable ideals. Recall that any sequence of reals $r_n \geq 0$ produces an ideal

$$\mathcal{S}_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} \colon \sum_{n\in x} r_n < +\infty \right\}.$$

Ideals of this kind are called *summable*, and $\mathcal{S}_{\{r_n\}}$ is *nontrivial dense* if $\{r_n\} \to 0$ and $\sum_n r_n = +\infty$ (then $\mathcal{S}_{\{r_n\}} \neq \mathcal{P}(\mathbb{N})$). Say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is *"special"* if there is a nontrivial dense summable ideal $\mathcal{S}_{\{r_n\}}$ with $\mathcal{S}_{\{r_n\}} \subseteq \mathcal{I} \subsetneqq \mathcal{P}(\mathbb{N})$.

PROPOSITION 2 (Essentially, Kechris [5]). *Let $\mathcal{Z}$ be a nontrivial Borel P-ideal on $\mathbb{N}$, not isomorphic to one of* $\mathtt{Fin}$, $\mathcal{I}_3 = 0 \times \mathtt{Fin}$, *or a trivial variation of* $\mathtt{Fin}$. *Then there is a set $W \notin \mathcal{Z}$ such that $\mathcal{Z} \restriction W = \{x \cap W : x \in \mathcal{Z}\}$ is isomorphic* (via a bijection W onto $\mathbb{N}$) *to a "special" Borel ideal.*

PROOF. Recall that a lower semicontinuous (l.s.c.) submeasure on $\mathbb{N}$ is any map $\varphi \colon \mathcal{P}(\mathbb{N}) \to [0, +\infty]$ satisfying $\varphi(x) \leq \varphi(x \cup y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{P}(\mathbb{N})$, $\varphi(\emptyset) = 0$, and $\varphi(x) = \mathtt{sup}_{n\in\mathbb{N}}\, \varphi(x \cap [0, n))$ for all $x \in \mathcal{P}(\mathbb{N})$. By Solecki [7, 8], as $\mathcal{Z}$ is a Borel P-ideal, there is an l.s.c. submeasure $\varphi \colon \mathcal{P}(\mathbb{N}) \to [0, +\infty]$ such that $\mathcal{Z} = \{x \in \mathcal{P}(\mathbb{N}) \colon \varphi_\infty(x) = 0\}$, where $\varphi_\infty(x) = \mathtt{lim}_{n\to\infty}\, \varphi(x \cap [n, \infty))$.

[5] For instance, Kechris observed, that any nontrivial Borel P-ideal $\mathcal{Z}$, with the same exceptions, induces a *turbulent* Δ-action on $\mathcal{P}(\mathbb{N})$, while $\mathsf{E}_{0\times\mathtt{Fin}}$ is induced by a continuous action of S_∞, the group of all permutations of $\mathbb{N}$, which is enough, by the first turbulence theorem, for $\mathsf{E}_{\mathcal{Z}}$ to be Borel irreducible to $\mathsf{E}_{0\times\mathtt{Fin}}$.

Put $U_n = \{k : \varphi(\{k\}) \le \frac{1}{n}\}$, separately $U_0 = \mathbb{N}$, thus, $U_{n+1} \subseteq U_n$ for all n. We claim that $\inf_{m\in\mathbb{N}} \varphi(U_m) > 0$. Suppose otherwise. Then a set $x \subseteq \mathbb{N}$ belongs to $\mathcal{Z}$ iff $x \setminus U_n$ is finite for any n. If the set $N = \{n : U_n \setminus U_{n+1}$ is infinite$\}$ is empty then easily $\mathcal{Z} = \mathcal{P}(\mathbb{N})$. If N is finite and nonempty then $\mathcal{Z}$ is isomorphic to either $\mathtt{Fin}$ (if eventually $U_n = \emptyset$) or a trivial variation of $\mathtt{Fin}$ (if U_n is nonempty for all n). If finally N is infinite then easily $\mathcal{Z}$ is isomorphic to $0 \times \mathtt{Fin}$ (for instance, if all sets $D_n = U_n \setminus U_{n+1}$ are infinite then $x \in \mathcal{Z}$ iff $x \cap D_n$ is finite for all n). Thus we always have a contradiction to the assumptions of the Proposition.

Thus there is $\varepsilon > 0$ such that $\varphi(U_m) > \varepsilon$ for all m. As φ is l.s.c., we can define an increasing sequence of numbers $n_1 < n_2 < n_3 < \dots$ and for any l a finite set $w_l \subseteq U_{n_l} \setminus U_{n_{l+1}}$ with $\varphi(w_l) > \varepsilon$. Then $W = \bigcup_l w_l \notin \mathcal{Z}$ and obviously $\{r_k\}_{k\in W} \to 0$ and $\sum_{k\in W} r_k \ge \sum_l \varphi(w_l) = \infty$, where $r_k = \varphi(\{k\})$. Finally, if a set $x \subseteq W$ satisfies $\sum_{k\in x} r_k < +\infty$ then $x \in \mathcal{Z}$: indeed, we have $\varphi_\infty(x) \le \sum_{k\in x} r_k$ because φ is a l.s.c. submeasure. ⊣

It follows (indeed, $\mathsf{E}_{\mathcal{Z}\restriction W} \le_{\mathrm{B}} \mathsf{E}_{\mathcal{Z}}$) that the next theorem implies Theorem 1:

THEOREM 3. *If $\mathcal{I}$ is a "special" Borel ideal then $\mathsf{E}_{\mathcal{I}}$ is not BM reducible to any* ER E *in* $\mathfrak{E}$.

§4. Ergodicity and dense summable ideals.

The next preliminary step is to further reduce the task to summable ideals. This involves the following special form of irreducibility.

DEFINITION 4. Let E, F be ERs on Polish spaces, resp., $\mathbb{X}$, $\mathbb{Y}$. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is

- *a.e.* E, F-*invariant* if there is a co-meager set $D \subseteq \mathbb{X}$ such that $x \mathrel{\mathsf{E}} x' \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$ for all $x, x' \in D$, and E, F-*invariant* if this holds for $D = \mathbb{X}$.
- *a.e.* F-*constant* if there is a co-meager set $D \subseteq \mathbb{S}$ such that $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$ holds for all $x, x' \in D$.

Finally, following Kechris [6, 12.1] and Hjorth [4, 3.6], we say that E is *generically* F-*ergodic* if every BM E, F-invariant map ϑ is a.e. F-constant. ⊣

REMARK 5. To see that E is generically F-ergodic, it suffices to demonstrate that every <u>Borel</u> and <u>a.e.</u> E, F-invariant map ϑ is a.e. F-constant: indeed, any BM map is continuous, hence, Borel, on a comeager set. ⊣

LEMMA 6. *Suppose that $\mathcal{I}$ is a nontrivial Borel ideal on $\mathbb{N}$ and E is a Borel* ER. *If $\mathsf{E}_{\mathcal{I}}$ is generically E-ergodic then $\mathsf{E}_{\mathcal{I}}$ is not* BM *reducible to* E.

PROOF. Assume, towards the contrary, that $\vartheta : \mathcal{P}(\mathbb{N}) \to \mathbb{Y}$ (where $\mathbb{Y}$ is the domain of E) is a BM reduction of $\mathsf{E}_{\mathcal{I}}$ to E. There is a co-meager set $D \subseteq \mathcal{P}(\mathbb{N})$ such that $\vartheta \restriction D$ is Borel (even continuous). Let $\vartheta' : \mathcal{P}(\mathbb{N}) \to \mathbb{Y}$ coincide with ϑ on D and be constant on $\mathcal{P}(\mathbb{N}) \setminus D$. Then ϑ' is a Borel a.e. $\mathsf{E}_{\mathcal{I}}, \mathsf{E}$-invariant map, therefore, it is a.e. E-constant, so that, by the ergodicity, we have a comeager $\mathsf{E}_{\mathcal{I}}$-equivalence class in $\mathcal{P}(\mathbb{N})$. It follows that $\mathcal{I}$ itself is comeager in $\mathcal{P}(\mathbb{N})$, which easily implies that $\mathcal{I} = \mathcal{P}(\mathbb{N})$, a contradiction to the nontriviality of $\mathcal{I}$. $\dashv$

The remainder of the note contains the proof of the following theorem:

THEOREM 7. *For any nontrivial dense summable ideal* $\mathcal{S}_{\{r_n\}}$ *the equivalence relation* $\mathsf{E}_{\{r_n\}} = \mathsf{E}_{\mathcal{S}_{\{r_n\}}}$ *is generically* E*-ergodic for any* ER E *in* $\mathfrak{E}$.

This implies Theorem 3, hence, Theorem 1. Indeed, first, if $\mathcal{I}$ is a "special" Borel ideal then, by Proposition 2 there is a nontrivial dense summable ideal $\mathcal{S}_{\{r_n\}} \subseteq \mathcal{I}$. The latter is generically E-ergodic for any ER E in $\mathfrak{E}$ by Theorem 7, hence, $\mathsf{E}_{\mathcal{I}}$ itself is generically E-ergodic because now any $\mathsf{E}_{\mathcal{I}}, \mathsf{E}$-invariant map is $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant. It follows that $\mathsf{E}_{\mathcal{I}}$ is BM irreducible to E by Lemma 6, as required.

§5. Preliminaries to the proof. In the proof of Theorem 7 we shall use the following notation and keep the following agreements.

- For the course of the proof of Theorem 7, we fix a sequence of nonnegative reals $\{r_n\} \to 0$ with $\sum_{n\in\mathbb{N}} r_n = +\infty$.
- $2^{\mathbb{N}}$ is the Cantor space of all infinite dyadic sequences, with the ordinary product topology.
- $2^{<\omega}$ is the set of all finite dyadic sequences.
- For $u \in 2^{<\omega}$ we define $I_u = \{a \in 2^{\mathbb{N}} : u \subset a\}$, a basic clopen set in $2^{\mathbb{N}}$.
- If X is a nonempty open set in a Polish space then the phrase "$P(x)$ holds for a.a. $x \in X$" will mean that $\{x \in X : P(x)\}$ is comeager in X.
- We shall systematically identify sets $X \subseteq \mathbb{N}$ with their characteristic functions, unless it becomes ambiguous. In particular, for $a, b \in 2^{\mathbb{N}}$ we define $a \,\Delta\, b = \{n : a(n) \neq b(n)\}$ (as a set) and $(a \,\Delta\, b)(n) = |a(n) - b(n)|$ (as an infinite dyadic sequence).
- $\mathsf{E}_{\{r_n\}}$ is $\mathsf{E}_{\mathcal{S}_{\{r_n\}}}$, thus, for $x, y \in 2^{\mathbb{N}}$, $x \;\mathsf{E}_{\{r_n\}}\; y$ iff $\sum_{x(n)\neq y(n)} r_n < +\infty$.
- In accordance with Remark 5, we shall consider only Borel maps ϑ.

DEFINITION 8. A ER E is $\{r_n\}$-*irreducing* if $\mathsf{E}_{\{r_n\}}$ is generically E-ergodic. $\dashv$

The method of the proof of Theorem 7 will be to show, by induction on the construction of ERs in $\mathfrak{E}$, that all ERs in $\mathfrak{E}$ are $\{r_n\}$-irreducing.

§6. Base of induction. To begin with, we prove

LEMMA 9. *If* $\mathbb{S}$ *is a Polish space then* $\mathsf{D}_{\mathbb{S}}$, *the equality on* $\mathbb{S}$, *is* $\{r_n\}$-*irreducing.*

PROOF. First of all, as all (uncountable) Polish spaces are Borel isomorphic, we may assume that $\mathbb{S} = 2^{\mathbb{N}}$. Furthermore, as any invariant $\vartheta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ splits into a family of invariant "coordinate" maps $\vartheta_k : 2^{\mathbb{N}} \to \{0,1\}$, we can suppose that $\mathbb{S} = \{0,1\}$, a two-element discrete set. Assume, towards the contrary, that an a.e. $\mathsf{E}_{\{r_n\}}, \mathsf{D}_{\mathbb{S}}$-invariant Borel map $\vartheta : 2^{\mathbb{N}} \to \{0,1\}$ is <u>not</u> a.e. constant.

By definition there is a dense $\boldsymbol{G}_\delta$ set $D \subseteq \mathcal{P}(\mathbb{N})$ with $a \mathsf{E}_{\{r_n\}} b \Longrightarrow \vartheta(a) = \vartheta(b)$ for all $a, b \in D$.

According to the contrary assumption, there exist sequences $u, v \in 2^{<\omega}$ such that $\vartheta(a) = 0$ and $\vartheta(b) = 1$ for a.a. (in the sense of category) $a \in I_u$ and a.a. $b \in I_v$. We can assume that $\vartheta(a) = 0$ and $\vartheta(b) = 1$ actually for all $a \in D \cap I_u$ and $b \in D \cap I_v$, and that $u, v \in 2^m$ for one and the same m. (Clearly $u \neq v$.)

Let $D = \bigcap_n D_n$, where each D_n is open dense. Put $r = \sum_{i<m, u(i) \neq v(i)} r_i$. Using the assumption $\{r_n\} \to 0$, we can easily define an increasing sequence $m = m_0 < m_1 < m_2 < \dots$ of natural numbers, and $u = u_0 \subset u_1 \subset u_2 \subset \dots$ and $v = v_0 \subset v_1 \subset v_2 \subset \dots$ of tuples $u_n, v_n \in 2^{m_i}$ with $\sum_{i<m_n, u_n(i) \neq v_n(i)} r_i < r+1$ and $I_{u_n} \cup I_{v_n} \subseteq D_n$ for all n. Then $a = \bigcup_n u_n \in D \cap I_u$, $b = \bigcup_n v_n \in D \cap I_v$, and $a \mathsf{E}_{\{r_n\}} b$, but $\vartheta(a) = x \neq y = \vartheta(b)$, which is a contradiction. ⊣

It remains to demonstrate that the property of $\{r_n\}$-irreducibility is preserved under the transformations of ERs which produce the class $\mathfrak{E}$.

§7. Inductive step of the Fubini product.

In this section, we show that the Fubini product preserves $\{r_n\}$-irreducibility.

Lemma 10. *Suppose that E_k, $k \in \mathbb{N}$, are Borel $\{r_n\}$-irreducing* ER*s on Polish spaces $\mathbb{S}_k$. Then $\mathsf{E} = \mathtt{Fin} \bigotimes_{k \in \mathbb{N}} \mathsf{E}_k$ is $\{r_n\}$-irreducing as well.*

Proof. Let $\mathbb{S} = \prod_k \mathbb{S}_k$, so that E is a ER on $\mathbb{S}$. Let $\vartheta \colon 2^{\mathbb{N}} \to \mathbb{S}$ be a Borel (according to Remark 5) function. It splits in the sequence of Borel functions

$$\vartheta_k(x) = \vartheta(x)(k) \colon 2^{\mathbb{N}} \to \mathbb{S}_k.$$

Suppose that ϑ is $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant on a dense $\boldsymbol{G}_\delta$ set $D \subseteq 2^{\mathbb{N}}$, so that

$$a \mathsf{E}_{\{r_n\}} b \Longrightarrow \exists k_0 \forall k \geq k_0 \big(\vartheta_k(a) \mathsf{E}_k \vartheta_k(b)\big)$$

for all $a, b \in D$. Our plan is to show that almost all ϑ_k are a.e. $\mathsf{E}_{\{r_n\}}, \mathsf{E}_k$-invariant.

In that we'll make use of two topologies. The first one is the ordinary product topology on $2^{\mathbb{N}}$. The other one is the topology on the set $\mathcal{S}_{\{r_n\}} \subseteq 2^{\mathbb{N}}$, generated by the metric $d_{\{r_n\}}(a, b) = \varphi_{\{r_n\}}(a \,\Delta\, b)$ on $\mathcal{S}_{\{r_n\}}$, where

$$\varphi_{\{r_n\}}(X) = \sum_{n \in X} r_n \text{ for } X \subseteq \mathbb{N}, \text{ so that } \mathcal{S}_{\{r_n\}} = \big\{X \colon \varphi_{\{r_n\}}(X) < +\infty\big\};$$

It is easy to verify (even in a much more general case, see [7]) that $d_{\{r_n\}} \restriction \mathcal{S}_{\{r_n\}}$ is a Polish (*i.e.*, complete separable) metric on $\mathcal{S}_{\{r_n\}}$. The $d_{\{r_n\}}$-topology is stronger than the product topology of $2^{\mathbb{N}}$ on $\mathcal{S}_{\{r_n\}}$, yet it yields the same Borel subsets of $\mathcal{S}_{\{r_n\}}$ as the product topology. Sets of the form

$$\mathcal{U}_\varepsilon(t) = \big\{z \in \mathcal{S}_{\{r_n\}} \colon d_{\{r_n\}}(z, t^*) < \varepsilon\big\}, \quad t \in 2^{<\omega},$$

where $t^* \in 2^{\mathbb{N}}$ denotes the extension of $t \in 2^{<\omega}$ (a finite sequence) by infinitely many zeros, and ε is a positive rational, provide a base of the $d_{\{r_n\}}$-topology on $\mathcal{S}_{\{r_n\}}$, and the countable set $\{t^* : t \in 2^{<\omega}\}$ is $d_{\{r_n\}}$-dense in $\mathcal{S}_{\{r_n\}}$.

Below, let $\mathbb{P} = 2^{<\omega}$ be the ordinary Cohen forcing for $2^{\mathbb{N}}$, and let $\mathbb{P}_{\{r_n\}}$ be the Cohen forcing for $\langle \mathcal{S}_{\{r_n\}}; d_{\{r_n\}} \rangle$, which consists of all sets of the form $\mathcal{U}_\varepsilon(t)$, where $t \in 2^{<\omega}$ and $\varepsilon > 0$ is rational. (Smaller sets are stronger conditions.) Let us fix a countable transitive model $\mathfrak{M}$ of a big enough fragment of ***ZFC***,[6] which contains all relevant objects or their codes, in particular, the sequence $\{r_n\}$ and a code of the Borel map ϑ.

CLAIM 11. *Suppose that $\langle a, z \rangle \in 2^{\mathbb{N}} \times \mathcal{S}_{\{r_n\}}$ is $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}$. Then $b = a \,\Delta\, z$ is $\mathbb{P}$-generic over $\mathfrak{M}$.*

PROOF OF THE CLAIM. Actually, b is $\mathbb{P}$-generic even over $\mathfrak{M}[z]$: indeed, a is such by the product forcing lemma, and, for any fixed z, the map $a \longmapsto a \,\Delta\, z$ is a homeomorphism. $\dashv$

Fix $u \in 2^{<\omega}$. Then by the invariance of ϑ and Claim 11 there is another sequence $v \in 2^{<\omega}$ with $u \subset v$, a number k_0, and a non-empty $d_{\{r_n\}}$-nbhd $\mathcal{U}_\varepsilon(t)$ in $\mathcal{S}_{\{r_n\}}$ (a condition in $\mathbb{P}_{\{r_n\}}$), where $\varepsilon > 0$ and $t \in 2^{<\omega}$, such that $\vartheta_k(a) \mathsf{E}_k \vartheta_k(a \,\Delta\, z)$ holds for any $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}$ pair $\langle a, z \rangle$ of $a \in I_v$ and $z \in \mathcal{U}_\varepsilon(t)$ and any $k \geq k_0$. We can assume that the length $\mathtt{lh}\, v$ is big enough for $i \geq \mathtt{lh}\, v \Longrightarrow r_i < \varepsilon$.

CLAIM 12. *If $a, b \in I_v$ are $\mathbb{P}$-generic over $\mathfrak{M}$ and $a \mathsf{E}_{\{r_n\}} b$ then $\vartheta_k(a) \mathsf{E}_k \vartheta_k(b)$ holds for all $k \geq k_0$.*

PROOF OF THE CLAIM. First consider the case when $\varphi_{\{r_n\}}(a \,\Delta\, b) < \varepsilon$. Take any $z \in Z = \mathcal{U}_\varepsilon(t)$ with $\varphi_{\{r_n\}}(z \,\Delta\, t) < \varepsilon - \varphi_{\{r_n\}}(a \,\Delta\, b)$, $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a, b]$.[7] (This is possible as $r_n \to 0$.) Then z is $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a]$, hence, $\langle a, z \rangle$ is $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}$ by the product forcing lemma, thus, $\vartheta_k(a) \mathsf{E}_k \vartheta_k(a \,\Delta\, z)$. Moreover, $z' = z \,\Delta\, (a \,\Delta\, b)$ still belongs to Z and is $\mathbb{P}_{\{r_n\}}$-generic over $\mathfrak{M}[a, b]$, so that $\vartheta_k(b) \mathsf{E}_k \vartheta_k(b \,\Delta\, z')$ by the same argument. Yet we have $a \,\Delta\, z = b \,\Delta\, z'$.

Now consider the general case. By definition $X = a \,\Delta\, b$ satisfies $\sum_{n \in X} r_n = \varphi_{\{r_n\}}(X) < +\infty$, moreover, $\mathtt{min}\, X \geq \mathtt{lh}\, v$, hence, by the choice of v, all r_j with $j \in X$ satisfy $r_j < \varepsilon$. In this case X has the form $X = \{j_1, \ldots, j_n\} \cup X'$, where $\varphi_{\{r_n\}}(X') < \varepsilon$, and $r_{j_m} < \varepsilon$ for all m. Define $a_m = a \,\Delta\, \{j_1, \ldots, j_m\}$ for $m = 1, \ldots, n$. Then $a = a_0, a_1, a_2, \ldots, a_m, a_{m+1} = b$ is a chain of $\mathbb{P}$-generic,

[6] For instance, the first one million of ***ZFC*** axioms plus the Replacement for Σ_{100} formulas.

[7] By $\mathfrak{M}[a, b]$, we understand a countable transitive model of the same fragment of ***ZFC*** as mentioned in Footnote 6, which contains a, b, and all sets in $\mathfrak{M}$. This model may contain more ordinals than $\mathfrak{M}$ because the pair $\langle a, b \rangle$ of two generic elements is not necessarily generic. On the contrary, the models $\mathfrak{M}[a]$ and $\mathfrak{M}[b]$ are ordinary generic extensions of $\mathfrak{M}$, containing the same ordinals as $\mathfrak{M}$.

over $\mathfrak{M}$, elements of I_v with $d_{\{r_n\}}(a_m, a_{m+1}) < \varepsilon$ for all m, so that we can apply the particular case considered above. ⊣

To summarize, we have shown that for any $u \in 2^{<\omega}$ there exist k_0 and $v \in 2^{<\omega}$ with $u \subset v$ such that $\vartheta_k(a)$ is $\mathsf{E}_{\{r_n\}}, \mathsf{E}_k$-invariant for all $k \geq k_0$ and all sufficiently $\mathbb{P}$-generic $a \in I_v$, so that ϑ_k is a.e. $\mathsf{E}_{\{r_n\}}, \mathsf{E}_k$-invariant on I_v. Therefore, as all E_k are $\{r_n\}$-irreducing, the map ϑ_k is a.e. E_k-constant on I_v, for each $k \geq k_0$. (Indeed, we can trivially extend $\vartheta_k \restriction I_v$ on the whole domain $2^{\mathbb{N}}$ so that the invariance is preserved.) Thus ϑ is a.e. E-constant on I_v as well.

In other words, for each $u \in 2^{<\omega}$ there is $v \in 2^{<\omega}$ with $u \subset v$ such that ϑ is a.e. E-constant on I_v. To complete the proof of Lemma 10, it remains to demonstrate that these E-constants cannot be different. Thus assume that $s, t \in 2^{<\omega}$, $\mathtt{lh}\, s = \mathtt{lh}\, t = m$, $x, y \in \mathbb{S}$, and $\vartheta(a) \mathrel{\mathsf{E}} x$ for a.a. $a \in I_s$ while $\vartheta(b) \mathrel{\mathsf{E}} y$ for a.a. $b \in I_t$. We have to show that $x \mathrel{\mathsf{E}} y$. Indeed, the same construction as in the proof of Lemma 9 yields $a \in D \cap I_s$ and $b \in D \cap I_t$ with $a \mathrel{\mathsf{E}_{\{r_n\}}} b$ such that $\vartheta(a) \mathrel{\mathsf{E}} x$ and $\vartheta(b) \mathrel{\mathsf{E}} y$. Then $\vartheta(a) \mathrel{\mathsf{E}} \vartheta(b)$ by the invariance of ϑ, hence, $x \mathrel{\mathsf{E}} y$, as required. ⊣

§**8. Other inductive steps.** In this section, we show that all other operations over ERs, defined in §1, also preserve $\{r_n\}$-irreducibility.

LEMMA 13. *Suppose that* $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \ldots$ *are Borel* $\{r_n\}$*-irreducing* ER*s on a Polish space* $\mathbb{S}$, *and* $\mathsf{E} = \bigcup_k \mathsf{E}_k$ *is a* ER. *Then* E *also is* $\{r_n\}$*-irreducing.*

PROOF. Let $\vartheta \colon 2^{\mathbb{N}} \to \mathbb{S}$ be a Borel $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant map. For each $u \in 2^{<\omega}$, by the invariance of ϑ, there exist: $v \in 2^{<\omega}$ with $u \subset v$, a number k, and a non-empty $d_{\{r_n\}}$-nbhd $Z = \mathcal{U}_\varepsilon(t)$ in $\mathcal{S}_{\{r_n\}}$ (a condition in $\mathbb{P}_{\{r_n\}}$), where $\varepsilon > 0$ and $t \in 2^{<\omega}$, such that $\vartheta(a) \mathrel{\mathsf{E}_k} \vartheta(a \,\Delta\, z)$ for any $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic, over $\mathfrak{M}$, pair $\langle a, z\rangle$ of $a \in I_v$ and $z \in Z$. We can assume that $\mathtt{lh}\, v$ is big enough for $i \geq \mathtt{lh}\, v \Longrightarrow r_i < \varepsilon$. Then, similarly to Claim 12, it is true that $\vartheta(a) \mathrel{\mathsf{E}_k} \vartheta(b)$ for any pair of $\mathbb{P}$-generic, over $\mathfrak{M}$, elements $a, b \in I_v$. It follows, as in the proof of Lemma 10, that ϑ is a.e. E_k-constant on I_v, hence, a.e. E-constant on I_v as well. It remains to show that these E-constants are equal to each other, which is demonstrated as in the end of the proof of Lemma 10. ⊣

COROLLARY 14. *Let* $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \ldots$ *be Borel* $\{r_n\}$*-irreducing* ER*s on disjoint Polish spaces* $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \ldots$. *Then* $\mathsf{E} = \bigvee_k \mathsf{E}_k$ *also is* $\{r_n\}$*-irreducing.*

PROOF. Apply Lemma 13 for the relations E'_k defined on $\mathbb{S} = \bigcup_k \mathbb{S}_k$ as follows: $x \mathrel{\mathsf{E}'_k} y$ iff either $x = y$ or $x, y \in \mathbb{S}_k$ and $x \mathrel{\mathsf{E}_k} y$. ⊣

LEMMA 15. *Let* $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \ldots$ *be Borel* $\{r_n\}$*-irreducing* ER*s on a Polish space* $\mathbb{S}$. *Then* $\mathsf{E} = \bigcap_k \mathsf{E}_k$ *also is* $\{r_n\}$*-irreducing.*

PROOF. Any $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant map is $\mathsf{E}_{\{r_n\}}, \mathsf{E}_k$-invariant for all k. ⊣

COROLLARY 16. *Let* $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \ldots$ *be Borel* $\{r_n\}$*-irreducing* ER*s on Polish spaces* $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \ldots$. *Then* $\mathsf{E} = \prod_k \mathsf{E}_k$ *also is* $\{r_n\}$*-irreducing.*

PROOF. Apply Lemma 15 for the relations $x \mathsf{E}'_k y$ iff $x(k) \mathsf{E}_k y(k)$ on $\prod_k \mathbb{S}_k$. ⊣

LEMMA 17. *If a Borel* ER E *on a Polish space* $\mathbb{S}$ *is* $\{r_n\}$*-irreducing then* E^∞ *is also* $\{r_n\}$*-irreducing.*

PROOF. Suppose that a Borel map $\vartheta \colon 2^{\mathbb{N}} \to \mathbb{S}^{\mathbb{N}}$ is a.e. $\mathsf{E}_{\{r_n\}}, \mathsf{E}^\infty$-invariant, that is, $a \mathsf{E}_{\{r_n\}} b \Longrightarrow \vartheta(a) \mathsf{E}^\infty \vartheta(b)$ for all $a, b \in D$, where $D \subseteq 2^{\mathbb{N}}$ is a dense $\boldsymbol{G}_\delta$ set. Let $\vartheta_k(a) = \vartheta(a)_k$. The invariance of ϑ can be reformulated as follows:

$$a \mathsf{E}_{\{r_n\}} b \Longrightarrow \forall k \exists l (\vartheta_k(a) \mathsf{E} \vartheta_l(b)) \wedge \forall l \exists k (\vartheta_k(a) \mathsf{E} \vartheta_l(b))$$

for all $a, b \in D$. As in the proof of Lemma 10, for any k and any $u \in 2^{<\omega}$ there are: a number l, a sequence $v \in 2^{<\omega}$ with $u \subset v$, and a $d_{\{r_n\}}$-nbhd $Z = \mathcal{U}_\varepsilon(t)$ in $\mathcal{S}_{\{r_n\}}$ such that $\vartheta_k(a) = \vartheta_l(a \Delta z)$ for any $\mathbb{P} \times \mathbb{P}_{\{r_n\}}$-generic, over $\mathfrak{M}$, pair $\langle a, z \rangle$ of $a \in I_v$ and $z \in Z$. The same argument (Claim 12) shows that $\vartheta_k(a) \mathsf{F} \vartheta_l(b)$ whenever $a, b \in I_v$ are $\mathbb{P}$-generic over $\mathfrak{M}$.

Thus for any $u \in 2^{<\omega}$ there is $v \in 2^{<\omega}$ with $u \subset v$ such that $\vartheta_k(a)$ is $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant for all generic, over $\mathfrak{M}$, elements $a \in I_v$, in other words, ϑ_k is a.e. $\mathsf{E}_{\{r_n\}}, \mathsf{E}$-invariant on I_v. It follows, as above, that ϑ_k is a.e. E-constant on I_v. It follows that there is a dense $\boldsymbol{G}_\delta$ set $D' \subseteq D$ and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{S}$ such that, for all k and $a \in D'$, we have $\vartheta_k(a) \mathsf{E} y_j$ for some j. Let $\xi_k(a)$ be the least such an index j, thus, $\xi_k(a)$ is defined for all $a \in D'$ and all k, and each $\xi_k \colon D' \to \mathbb{N}$ is a Borel map. Now, by the invariance of ϑ,

$$a \mathsf{E}_{\{r_n\}} b \Longrightarrow \{\xi_k(a) : k \in \mathbb{N}\} = \{\xi_k(b) : k \in \mathbb{N}\}$$

for all $a, b \in D'$. Lemma 9 then implies that there is a set $\Xi \subseteq \mathbb{N}$ such that $\{\xi_k(a) : k \in \mathbb{N}\} = \Xi$ for a.a. $a \in D'$. We conclude that ϑ, the given function, is a.e. E^∞-constant on D', as required. ⊣

COROLLARY 18. *All* ER*s in* $\mathfrak{E}$ *are* $\{r_n\}$*-irreducing.* ⊣

PROOF. Apply Lemma 9 and the results of this Section. ⊣

This ends the proof of Theorems 7, 3, and 1. ⊣(*Main Theorem*)

§9. A corollary and a question. Recall that an ER E is *countable* iff all E-equivalence classes are countable. E is *essentially countable* iff E is Borel reducible to a countable Borel ER.

COROLLARY 19. *If* $\mathcal{Z}$ *is a nontrivial Borel P-ideal and* $\mathsf{E}_{\mathcal{Z}}$ *is an essentially countable* ER *then* $\mathcal{Z}$ *is* $\mathtt{Fin}$ *or a trivial variation of* $\mathtt{Fin}$.

This is true even for Borel ideals which are not P-ideals: in such a general form the result appears in [7, Corollary 4.2]. To derive this generalization from Corollary 19, we can use the following two results:

(1) If a Borel ideal $\mathcal{Z}$ is not a P-ideal then $\mathsf{E}_{\mathtt{Fin}\times 0}$ is Borel reducible to $\mathsf{E}_{\mathcal{Z}}$ (Solecki [7, 8]. The ideal $\mathtt{Fin} \times 0 = \mathcal{I}_1$, consists of all sets $x \subseteq \mathbb{N} \times \mathbb{N}$ such that $x \subseteq \{0, 1, \dots, m\} \times \mathbb{N}$ for some m.)

(2) $\mathsf{E}_{\mathtt{Fin}\times 0}$ is not essentially countable.

Proof of Corollary 19. We first prove that

(3) Any countable Borel ER E on a Polish space $\mathbb{S}$ is Borel reducible to $\mathsf{D}(\mathbb{S})^\infty$.

This is enough to prove the corollary. Indeed, as $\mathsf{D}(\mathbb{S})^\infty$ clearly belongs to $\mathfrak{E}$, the ideal $\mathcal{Z}$ is either $0 \times \mathtt{Fin}$ or $\mathtt{Fin}$ or a trivial modification of $\mathtt{Fin}$ by Theorem 1. Yet the first option is impossible as it is known that

(4) $\mathsf{E}_{0\times\mathtt{Fin}}$ is not essentially countable.

To prove (3) note that, by a classical theorem of descriptive set theory, $\mathsf{E} = \bigcup_n E_n$, where each $E_n \colon \mathbb{S} \to \mathbb{S}$ is a Borel map (identified with its graph). For any $x \in \mathbb{S}$ let $\vartheta(x) \in \mathbb{S}^{\mathbb{N}}$ be defined by $\vartheta(x)_n = E_n(x)$. Then $\{\vartheta(x)_n \colon n \in \mathbb{N}\} = [x]_{\mathsf{E}}$, so that ϑ is a Borel reduction E to $\mathsf{D}(\mathbb{S})^\infty$, as required. ⊣

Question 20. Which ideals except for P-ideals satisfy Theorem 1? There is an interesting Borel ideal whose relations in terms of Borel reducibility are not yet clear. The *Weyl ideal* $\mathcal{Z}_{\mathrm{W}}$ consists of those sets $x \subseteq \mathbb{N}$ which satisfy

$$\lim_{n\to+\infty} \mathtt{sup}_k \frac{\#(x \cap [k, k+n))}{n} = 0.$$

Despite a semblance of the density-0 ideal $\mathcal{Z}_0$, $\mathcal{Z}_{\mathrm{W}}$ has quite different properties, in particular, it is not a P-ideal. Most likely, $\mathsf{E}_{\mathcal{Z}_{\mathrm{W}}}$ is not Borel reducible to any ER in $\mathfrak{E}$, but how to prove this claim?

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