NEW RADON–NIKODYM IDEALS

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Abstract. Farah recently proved that many Borel P-ideals $\mathscr{I}$ on $\mathbb{N}$ satisfy the following requirement: any measurable homomorphism $F: \mathscr{P}(\mathbb{N})/ \to \mathscr{P}(\mathbb{N})/\mathscr{I}$ has a continuous lifting $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which is a homomorphism itself. Ideals having such a property were called *Radon–Nikodym* (RN) ideals. Answering some Farah's questions, it is proved that many non-P ideals, including, for instance, $\mathtt{Fin} \otimes \mathtt{Fin}$, are Radon–Nikodym. To prove this result, another property of ideals called *the Fubini property*, is introduced, which implies RN and is stable under some important transformations of ideals.

§1. *Introduction.* Below, *homomorphism* will mean a Boolean algebra homomorphism $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ or $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$, where $\mathscr{I}$ is an ideal on $\mathbb{N}$. (We shall also consider $\mathscr{P}(\mathbb{N})$ as a group, with the symmetric difference as the group operation, and call the related homomorphisms Δ-*homomorphisms*.)

We shall mainly deal with *Lebesgue measurable* (LM) (in the sense of the uniform Lebesgue probability measure on $\mathscr{P}(\mathbb{N}) = 2^{\mathbb{N}}$) and *Baire measurable* (BM) homomorphisms and other maps $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$. A map $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$ will be called *Lebesgue* or *Baire measurable* if it admits a resp. LM or BM *lifting*, *i.e.*, a map $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ such that $f(x) \in F(x)$ for any x. (Note that a lifting is not required to be necessarily a homomorphism.)

DEFINITION 1 (Farah [**1, 2, 3**]). An ideal $\mathscr{I}$ on $\mathbb{N}$ is *Radon–Nikodym* (RN, for brevity), if any BM or LM homomorphism $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$ admits a continuous lifting $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which is a homomorphism itself.

Note that the RN property is formulated in [**1, 2, 3**] for $\mathtt{Fin}$-invariant and only BM maps F, which does not appear to reflect anything essential in the problem. The requirement that g is a continuous Boolean algebra homomorphism is equivalent to the complete additivity of g, in Farah's definition of the RN property.

Velickovic [**10**] proved that the ideal $\mathtt{Fin}$ of all finite sets $x \subseteq \mathbb{N}$ is RN. The authors obtained a similar theorem in [**5**] for quotients $\mathbb{R}/G$, G being a countable subgroup of the additive group of $\mathbb{R}$. The general problem of liftings which are homomorphisms was first explicitly formulated in this context, perhaps, by Todorcevic [**9**]. Studying Borel P-ideals $\mathscr{I}$ (*i.e.*, those satisfying the property that, if $x_n \in \mathscr{I}$ for any n, then there is $x \in \mathscr{I}$ with $x_n \setminus x$ finite for any x), Farah [**1, 2, 3**] demonstrated that a big family of them, called *non-pathological*, are RN. This family contains practically all interesting Borel P-ideals (like summable ideals, density ideals, EU ideals); however, there are rather weird non-RN Borel P-ideals.

This note proves the RN property for another interesting, but quite different family of Borel ideals (*not* P-ideals).

DEFINITION 2. Let $1 \leqslant \xi < \omega_1$. Then $\mathtt{Fin}_\xi$ is the ideal of all sets $x \subseteq \omega^\xi$ having the order type $\mathtt{otp}\, x$ strictly smaller than ω^ξ.

Note that $\mathtt{Fin}_\xi$ are really ideals because the ordinals $\alpha = \omega^\xi$ are *indecomposable, i.e.,* α is not a sum of two smaller ordinals. The notation $\mathtt{Fin}_\xi$ is motivated by the fact that any $\mathtt{Fin}_{\xi+1}$ is isomorphic to the Fubina product $\mathtt{Fin} \otimes \mathtt{Fin}_\xi$: for example, $\mathtt{Fin}_2 \cong \mathtt{Fin} \otimes \mathtt{Fin}$. (Yet the recursion on limit steps is not so easy.) The following theorem answers a question of Farah (see Theorem 1.14.6 in [3]).

THEOREM 3 (The main theorem). *All ideals* $\mathtt{Fin}_\xi$ *are Radon–Nikodym.*

To prove this theorem, we show that all ideals $\mathtt{Fin}_\xi$ belong to a much bigger family of ideals satisfying a kind of Fubini theorem for the product of the submeasure associated with the ideal, and the uniform Lebesgue probability measure on $\mathscr{P}(\mathbb{N})$—this is why we call them *Fubini* ideals. The family of Borel Fubini ideals is closed under several important transformations of ideals (like pointwise limits or Fubini products) and contains ideals very different from $\mathtt{Fin}_\xi$, for instance, all non-pathological Borel P-ideals of Farah. The other part of the proof demonstrates that every Borel Fubini ideal has the RN property.

§2. *Fubini ideals.*

Let λ be the uniform Lebesgue probability measure on $\mathscr{P}(\mathbb{N})$.

DEFINITION 4. An ideal $\mathscr{I}$ on a countable set I is a *Fubini ideal* if, whenever $p > 0$, $B \subseteq I$, $B \notin \mathscr{I}$, and, for any $i \in B$, $W(i) \subseteq \mathscr{P}(\mathbb{N})$ is a LM set satisfying $\lambda(W(i)) \geqslant p$, the set X of all points $x \in \mathscr{P}(\mathbb{N})$, such that $W_x = \{i: x \in W(i)\} \notin \mathscr{I}$, is LM and satisfies $\lambda(X) \geqslant p$.

To explain this definition, let us associate with the ideal $\mathscr{I}$ a submeasure $\varphi_{\mathscr{I}}$ on $\mathscr{P}(I)$ defined so that $\varphi_{\mathscr{I}}(x) = 0$ whenever $x \in \mathscr{I}$ and $\varphi_{\mathscr{I}}(x) = 1$ otherwise. In these terms, Definition 4 can be viewed as a form of the Fubini theorem (which, generally speaking, fails for submeasures, of course). (After this note was submitted, the authors learned from a preprint of S. Solecki that the property, which defines Fubini ideals here, is equivalent to a form of Fatou's lemma, so that the ideals may equally be called *Fatou ideals*.)

We shall show that the family of Borel Fubini ideals on countable and finite sets is closed w.r.t. the following operations 1° through 5°:

1°. Bijections and extensions of underlying sets: for instance, if $\mathscr{I}$ is an ideal on a set $I \subseteq J$, and J is countable, then we consider the following ideals: (a) $\mathscr{I}$ as an ideal on the set J, and (b) the ideal $\{Y \subseteq J: Y \cap I \in \mathscr{I}\}$.

2°. Convergent pointwise limit, applied to sequences of ideals on one and the same set. In particular, both increasing unions and decreasing intersections.

An ideal $\mathcal{I}$ is a pointwise limit of a sequence of ideals $\mathcal{I}_n$ if any $y \in \mathcal{I}$ belongs to all but finite $\mathcal{I}_n$ and any $y \notin \mathcal{I}$ satisfies $y \notin \mathcal{I}_n$ for all but finite n.

3°. Finite and countable intersection of ideals on one and the same set.

4°. *Fubini products*: if $\mathcal{I}$, $\mathcal{J}$ are ideals on countable sets I, J then we consider the ideal $\mathcal{I} \otimes \mathcal{J}$ of all sets $Z \subseteq I \times J$ such that $\{i \in I: (Z)_i \notin \mathcal{J}\} \in \mathcal{I}$.

Here $(Z)_i = \{j: \langle i, j\rangle \in Z\}$ is the *cross-section*. This can be generalized.

5°. If $\mathcal{I}$ is an ideal on a countable set I and $\mathcal{J}_i$ is an ideal on a countable set J_i for any $i \in I$, then we consider the ideal $\mathcal{J} = \mathcal{I} \otimes_{i \in I} \mathcal{J}_i$ of all sets $Z \subseteq J = \{\langle i, j\rangle: i \in I \wedge j \in J_i\}$ such that $\{i \in I: (Z)_i \notin \mathcal{J}_i\} \in \mathcal{I}$. Clearly $\mathcal{I} \otimes_{i \in I} \mathcal{J}_i = \mathcal{I} \otimes \mathcal{J}$ provided that $J_i = J$ and $\mathcal{J}_i = \mathcal{J}$ for all i.

Note that the operations preserve the Borelness.

THEOREM 5. *The family of all Borel Fubini ideals on countable sets is closed under operations* 1°–5°.

Proof. 1° is elementary.

2°. Let $\mathcal{I}$, an ideal on $\mathbb{N}$, be a pointwise limit of a sequence of Borel Fubini ideals $\mathcal{I}_n$. To prove that $\mathcal{I}$ is Fubini, let $B, p, W(i), X$ be as in Definition 4. Define $X^{(n)}$ as X in Definition 4, but using the ideal $\mathcal{I}_n$ instead of $\mathcal{I}$. We may assume that the sets $W(i)$ are Borel. Then the sets X and $X^{(n)}$ are Borel, too. Moreover, for any $\varepsilon > 0$ there is an n such that $\lambda(X \Delta X^{(n)}) \leqslant \varepsilon$. We can assume that $B \notin \mathcal{I}_n$. Then $\lambda(X^{(n)}) \geqslant p$ because $\mathcal{I}_n$ is a Fubini ideal, hence, $\lambda(X) \geqslant p - \varepsilon$, but $\varepsilon > 0$ is arbitrary.

3°. As 2° has been proved, it suffices to consider only intersections of two ideals, where the result is obtained by a routine verification.

5°. Let I, $\mathcal{I}$, J_i, $\mathcal{J}_i$, J, $\mathcal{J}$ be as in 5°. Suppose that $\mathcal{I}$ and $\mathcal{J}_i$ are Borel Fubini ideals; we prove that the ideal $\mathcal{J} = \mathcal{I} \otimes_{i \in I} \mathcal{J}_i$ is Fubini, too. Fix a real $0 < p \leqslant 1$ and a family of LM sets $W(i, j) \subseteq \mathcal{P}(\mathbb{N})$ satisfying $\lambda(W(i, j)) \geqslant p$ for all pairs $\langle i, j\rangle \in B$, where $B \subseteq J$, $B \notin \mathcal{J}$. We prove that the set $X = \{x \in \mathcal{P}(\mathbb{N}): W_x \notin \mathcal{J}\}$ satisfies $\lambda(X) \geqslant p$, where

$$W_x = \{\langle i, j\rangle \in B: x \in W(i, j)\} \quad \text{for any} \quad x \in \mathcal{P}(\mathbb{N}).$$

By definition $U = \{i \in I: (B)_i \notin \mathcal{J}_i\} \notin \mathcal{I}$, so that, as all ideals $\mathcal{J}_i$ are Fubini, for any $i \in U$ there is a set $X(i) \subseteq \mathcal{P}(\mathbb{N})$ such that $\lambda(X)i)) \geqslant p$ and $(W_x)_i \notin \mathcal{J}_i$ for all $x \in X(i)$. Since $\mathcal{I}$ itself is Fubini, there is a set $X' \subseteq \mathcal{P}(\mathbb{N})$ such that $\lambda(X') \geqslant p$ and $W'_x \notin \mathcal{I}$ for all $x \in X'$, where $W'_x = \{i \in U: x \in X(i)\}$. It remains to show that $X' \subseteq X$. Let $x \in X'$. By definition $W'_x \notin \mathcal{I}$. Moreover if $i \in W'_x$ then $x \in X(i)$ and $(W_x)_i \notin \mathcal{J}_i$. It follows that the set

$$W' = \{\langle i, j\rangle: i \in W'_x \wedge j \in (W_x)_i\} \subseteq W_x$$

does not belong to $\mathcal{J}$, therefore $x \in X$, as required.

§3. *Fubini property of the "indecomposability" ideals.* We are going to prove that all ideals $\mathtt{Fin}_\xi$ are Fubini ideals, using induction on ξ and Theorem 5. Unfortunately, it is not so easy to pass limit steps. We have to consider a bigger family of ideals.

- For all $1 \leqslant \eta \leqslant \xi < \omega_1$, let $\mathscr{I}^\eta_\xi$ be the ideal of all sets $x \subseteq \omega^\xi$ having the order type $\mathtt{otp}\, x$ strictly smaller than ω^η. Thus, $\mathtt{Fin}_\xi = \mathscr{I}^\xi_\xi$.

THEOREM 6. *All ideals $\mathscr{I}^\eta_\xi$, in particular all ideals $\mathtt{Fin}_\xi$, are Fubini.*

Proof. To begin with note that the ideal $\mathrm{O} = \{\varnothing\}$ (on any countable set) is Fubini. The ideal $\mathtt{Fin}$ of all finite subsets of $\mathbb{N} = \omega$ is Fubini, too: this can be easily checked directly, but we can also use the equality $\mathtt{Fin} = \bigcup_n \mathscr{P}([0, n))$ which reduces the question to the entirely clear Fubini-correctness of ideals of the form $\mathscr{P}([0, n))$, by Theorem 5. It follows that all ideals $\mathscr{I}^1_\xi$ (namely, all finite subsets of ξ) are Fubini ideals.

To maintain the induction step, recall that two ideals $\mathscr{I}$ and $\mathscr{J}$, on sets resp. X and Y, are *isomorphic*, in symbols $\mathscr{I} \cong \mathscr{J}$, if there is a bijection $b: X \xrightarrow{\text{onto}} Y$ which transforms $\mathscr{I}$ onto $\mathscr{J}$.

LEMMA 7. *If* $1 \leqslant \eta < \xi$, *then* $\mathscr{I}^{\eta+1}_{\xi+1} \cong (\mathrm{O} \otimes \mathscr{I}^{\eta+1}_\xi) \cap (\mathtt{Fin} \otimes \mathscr{I}^\eta_\xi)$. *In addition,* $\mathscr{I}^{\xi+1}_{\xi+1} \cong \mathtt{Fin} \otimes \mathscr{I}^\xi_\xi$.

Proof. We consider only the case $\eta < \xi$; the other case is similar. Note that $\mathscr{I}^{\eta+1}_{\xi+1}$ is an ideal on $X = \omega^{\xi+1}$ while both $\mathrm{O} \otimes \mathscr{I}^{\eta+1}_\xi$ and $\mathtt{Fin} \otimes \mathscr{I}^\eta_\xi$ are ideals on the cartesian product $Y = \omega \times \omega^\xi$. There is a natural bijection $b(\omega^\xi \cdot n + \gamma) = \langle n, \gamma \rangle$: $\xrightarrow{\text{onto}} Y$, mapping, order preservingtly, each interval $[\omega^\xi \cdot n, \omega^\xi \cdot (n+1))$ in X onto the corresponding vertical cross-section $\{n\} \times \omega^\xi$ in Y. One easily proves that b transforms $\mathscr{I}^{\eta+1}_{\xi+1}$ onto $(\mathrm{O} \otimes \mathscr{I}^{\eta+1}_\xi) \cap (\mathtt{Fin} \otimes \mathscr{I}^\eta_\xi)$; the crucial fact is that a sum of the form $\sigma = \sum_{n \in \omega} \alpha_n$, where $\alpha_n < \omega^{\eta+1}$ for any n, satisfies $\sigma = \omega^{\eta+1}$ if and only if infinitely many of ordinals α_n satisfy $\alpha_n \geqslant \omega^\eta$.

LEMMA 8. *If $\lambda \leqslant \xi$ is a limit ordinal then* $\mathscr{I}^\lambda_\xi = \bigcup_{\eta < \lambda} \mathscr{I}^\eta_\xi$.

Unlike this obvious result, the case of a limit lower index needs some work. Assume that $\eta < \lambda < \omega_1$ and that λ is a limit ordinal. Fix an increasing sequence of ordinals ξ_n which converges to λ and starts with $\xi_0 = 0$ and $\xi_1 = \eta + 1$.

LEMMA 9. *In this case,* $\mathscr{I}^{\eta+1}_\lambda \cong \mathrm{O} \otimes_{n \in \omega} \mathscr{I}^{\eta+1}_{\xi_{n+1}} \cap \mathtt{Fin} \otimes_{n \in \omega} \mathscr{I}^\eta_{\xi_{n+1}}$.

Proof. Note that both $\mathrm{O} \otimes_{n \in \omega} \mathscr{I}^{\eta+1}_{\xi_{n+1}}$ and $\mathtt{Fin} \otimes_{n \in \omega} \mathscr{I}^\eta_{\xi_{n+1}}$ are ideals on the set $J = \{\langle n, \gamma \rangle : n < \omega \wedge \gamma < \omega^{\xi_{n+1}}\}$, while $\mathscr{I}^{\eta+1}_\lambda$ is an ideal on $X = \omega^\lambda$. There is a bijection $b(\omega^{\xi_n} + \gamma) = \langle n, \gamma \rangle : X \xrightarrow{\text{onto}} J$, mapping, order preservingly, each interval $[\omega^{\xi_n}, \omega^{\xi_{n+1}})$ in X onto the cross-section $\{n\} \times \omega^{\xi_{n+1}}$ in J. One easily proves that b transforms $\mathscr{I}^{\eta+1}_\lambda$ onto $\mathrm{O} \otimes_{n \in \omega} \mathscr{I}^{\eta+1}_{\xi_{n+1}} \cap \mathtt{Fin} \otimes_{n \in \omega} \mathscr{I}^\eta_{\xi_{n+1}}$.

We now end the proof of Theorem 6. Lemmas 7, 8, 9 show that every ideal of the form $\mathscr{F}_\xi^\eta$ can be obtained from simple ideals like O and Fin, which clearly are Fubini, by a suitable iteration of operations $1° - 5°$ of 2.

§4. *Reduction to the group case.* Theorem 6 just proved is the first part of the proof of Theorem 3. To accomplish the latter, we now prove the following:

THEOREM 10. *Any Borel Fubini ideal is Radon–Nikodym.*

The scheme of the proof is as follows. We first prove the group version of the theorem. To explain this point, note that $\mathscr{P}(\mathbb{N})$ is a group with the symmetric difference Δ as the group operation, which is the same as $\mathbb{Z}_2^{\mathbb{N}}$. The group structure is weaker than the Boolean algebra one, of course. Clearly any Boolean algebra homomorphism is a Δ-homomorphism, but not conversely.

DEFINITION 11. An ideal $\mathscr{I}$ on $\mathbb{N}$ is *group-Radon–Nikodym* (GRN, for brevity) if any BM of LM Δ-homomorphism $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$ admits a continuous lifting $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which is a Δ-homomorphism itself. (Farah [2] attributes to A. S. Kechris the idea of study of the RN property for group homomorphisms. See Kechris [8] for some other properties of quotients of the form $\mathscr{P}(\mathbb{N})/\mathscr{I}$.)

Note that continuous Δ-homomorphisms $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ admit a complete description by the following proposition (see, *e.g.*, Farah [2]):

PROPOSITION 12. *Let* $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ *be a continuous* Δ*-homomorphism. Then, for any* $x \in \mathscr{P}(\mathbb{N})$ *and for any* n*, there is a finite set* $u_n \subseteq \mathbb{N}$ *such that* $n \in g(x)$ *if and only if* $\#(x \cap u_n)$ *is an odd number.*

THEOREM 13. *Any Borel Fubini ideal is group-Radon–Nikodym.*

Let us show how Theorem 13 implies Theorem 10.

Let $\mathscr{I}$ be a Borel Fubini ideal on $\mathbb{N}$. Consider a LM or BM Boolean algebra homomorphism $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$. Then F is a Δ-homomorphism, too. According to Theorem 13, F has a lifting $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which is a continuous Δ-homomorphism, to that, by Proposition 12, for any n there is a finite set $u_n \subseteq \mathbb{N}$ such that $n \in g(x)$ if and only if $\#(x \cap u_n)$ is an odd number—for any $x \in \mathscr{P}(\mathbb{N})$.

Note that the set $U_0 = \{n : u_n = \varnothing\}$ belongs to $\mathscr{I}$: indeed, otherwise one easily proves that f cannot lift the Boolean algebra homomorphism F, because clearly $U_0 \cap f(x) = \varnothing$ for any x. Thus it can be assumed that $U_0 = \varnothing$, *i.e.*, $u_n \neq \varnothing$ for every n, as, if this is not the case, we simply re-define $u_n = \{n\}$ for any $n \in U_0$.

Our second claim is that $U = \{n : \#(u_n) \geqslant 2\}$ also belongs to $\mathscr{I}$. Indeed, for any $n \in U$ the set P_n of all pairs $\langle x, y \rangle \in \mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})$ such that both $\#(x \cap u_n)$

and $\#(y \cap u_n)$ are odd while $\#((x \cup y) \cap u_n)$ is even, is non-empty, and, moreover, has measure $\lambda^2(P_n) > 1/20$ (a rough estimate). It follows, by the Fubini property of $\mathscr{I}$, that, if $U \notin \mathscr{I}$, then there is an $\langle x, y \rangle \in \mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})$ such that the set $U_{xy} = \{n \in U: \langle x, y \rangle \in P_n\} \notin \mathscr{I}$. In other words, x and y show that f cannot lift F, which is a contradiction.

Thus $U \in \mathscr{I}$. It follows that we can assume that $U = \varnothing$: for if not just redefine $u_n = \{n\}$ for all $n \in U$. Then any u_n is a singleton, say $u_n = \{h(n)\}$, so that $f(x) = \{n: h(n) \in x\}$. Thus, f is a Boolean algebra homomorphism, as required.

§5. *All Fubini ideals are RN: the group case.* This section is devoted to the proof of Theorem 13.

We first prove the result for LM Maps. Consider a Borel Fubini ideal $\mathscr{I}$ on $\mathbb{N}$; we prove that any Lebesgue measurable Δ-homomorphism $F: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{I}$ admits a continuous lifting $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which is a Δ-homomorphism. By definition, for F to be LM means that F has an LM lifting $f: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$. This map f itself may be not a homomorphism, but, as it lifts F, we have

(*) f is an *$\mathscr{I}$-approximate homomorphism*, which means that the set $\mathsf{D}^f_{xy} = f(x) \Delta f(y) \Delta f(x \Delta y)$ belongs to $\mathscr{I}$ for all $x, y \in \mathscr{P}(\mathbb{N})$.

(Recall that Δ is the group operation on $\mathscr{P}(\mathbb{N})$.) This is our starting point. The goal is to find a continuous Δ-homomorphism $g: \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ which *$\mathscr{I}$-approximates* f in the sense that $f(x) \Delta g(x) \in \mathscr{I}$ for all x.

The proposed argument is a modification of the proof of Theorem 2.1 in [4].

Below, a, b, x, y will always denote elements of $\mathscr{P}(\mathbb{N})$.

We say that an index $i \in \mathbb{N}$ is "*good*" when

(i) $\lambda^2\{\langle x, y \rangle: f_i(x) \Delta f_i(y) \neq f_i(x \Delta y)\} < 1/6$, where $f_i(z) = f(z) \cap \{i\}$.

We claim that $B = \{i: i \text{ is "bad"}\} \in \mathscr{I}$. Indeed, otherwise, by the Fubini property, there is a pair $\langle x, y \rangle \in \mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})$ (there is even a set of λ^2-measure $1/6$ of such pairs) such that the set $\{i: f_i(x) \Delta f_i(y) \neq f_i(x \Delta y)\} = \mathsf{D}^f_{xy}$ does not belong to $\mathscr{I}$, which contradicts the choice of f. Thus $B \notin \mathscr{I}$.

LEMMA 14. *For any "good" i there is an LM Δ-homomorphism g_i: $\mathscr{P}(\mathbb{N}) \to \mathscr{P}(\{i\})$ such that*

(ii) $\lambda\{x: f_i(a \Delta x) \Delta f_i(x) = g_i(a)\} > 2/3$ *for any* $a \in \mathscr{P}(\mathbb{N})$.

Proof. For any $a \in \mathscr{P}(\mathbb{N})$ we have, by (i),

(iii) $\lambda^2\{\langle x, y \rangle: f_i(x \Delta a) \Delta f_i(x) \neq f_i(y \Delta a) \Delta f_i(y)\} < 1/3$.

(Indeed, if $\langle x, y \rangle$ does not belong to that set, then either $f_i(x) \Delta f_i(y) \neq f_i(x \Delta y)$ or $f_i(x \Delta a) \Delta f_i(y \Delta a) \neq f_i(x \Delta y)$. But any of the two corresponding sets of pairs $\langle x, y \rangle$ has λ-measure $< 1/6$ by (i); recall that i is "good".)

Now, if $a \in \mathscr{P}(\mathbb{N})$, then we let $g_i(a) = s$, where $s \subseteq \{i\}$ satisfies

$$\lambda\{x: f_i(a \Delta x) \Delta f_i(x) = s\} > 2/3.$$

This is well-defined—in particular, we have (ii)—for if no such s exists then $\lambda\{x: f_i(a\Delta x)\Delta f_i(x)\neq f_i(a\Delta y)\Delta f_i(y)\}\geqslant 1/3$ for any $y\in\mathscr{P}(\mathbb{N})$, so that

$$\lambda^2\{\langle x,y\rangle: f_i(a\Delta x)\Delta f_i(x)\neq f_i(a\Delta y)\Delta f_i(y)\}\geqslant 1/3$$

by the Fubini theorem, which is a contradiction to (iii).

We prove that g_i is a Δ-homomorphism. To show that $g_i(a\Delta b)=g_i(a)\Delta g_i(b)$, note that each of the four following sets has λ-measure greater than 2/3:

$$X_1=\{x: g_i(a)=f_i(x\Delta a)\Delta f_i(x)\},\qquad X_3=\{x: g_i(a\Delta b)=f_i(x\Delta a\Delta b)\Delta f_i(x)\},$$

$$X_2=\{x: g_i(b)=f_i(x\Delta b)\Delta f_i(x)\},\qquad X_4=\{x: g_i(a\Delta b)=f_i(x\Delta a)\Delta f_i(x\Delta b)\}.$$

Indeed, for X_1, X_2, X_3 this follows from (ii). In addition, $x\in X_3\Leftrightarrow x\Delta b\in X_4$, so $\lambda(X_4)>2/3$, too. Hence, there is $x\in X_1\cap X_2\cap X_4$. This ends the proof of Lemma 14.

For any "good" i, let $g_i:\mathscr{P}(\mathbb{N})\to\mathscr{P}(\{i\})$ be a Δ-homomorphism given by the lemma. For any "bad" i, let $g_i(x)=\varnothing$ for all $x\in\mathscr{P}(\mathbb{N})$. Define a Δ-homomorphism g by $g(x)=\bigcup_{i\in\mathbb{N}} g_i(x)$. We prove that $f(x)\Delta g(x)\in\mathscr{I}$ for any $x\in\mathscr{P}(\mathbb{N})$.

Assume on the contrary that there exists $a\in\mathscr{P}(\mathbb{N})$ such that $f(a)\Delta g(a)\notin\mathscr{I}$. Then, by the above, $A=(f(a)\Delta g(a))\setminus B\notin\mathscr{I}$. Now, for any $i\in A$ we have $f_i(a)\neq g_i(a)$, hence, $\lambda\{x: i\in D^f_{ax}\}>2/3$ by (ii). We conclude that, by the Fubini property, there is an $x\in\mathscr{P}(\mathbb{N})$ such that $D^f_{ax}\notin\mathscr{I}$, which contradicts the choice of f. Thus g $\mathscr{I}$-approximates f.

By the construction g is λ-measurable. Let us show that then g is in fact continuous. Note that g is Borel on a Borel set $X\subseteq\mathscr{P}(\mathbb{N})$ of full λ-measure. It is clear that for any $x\in\mathscr{P}(\mathbb{N})$ there exist $x',x''\in X$ such that $x=x'\Delta x''$. Then $g(x)=g(x')\Delta g(x'')$ as g is a Δ-homomorphism. It follows that

$$\begin{aligned} g(x)=y &\Leftrightarrow \exists x',x''\in X\,(x=x'\Delta x''\wedge y=g(x')\Delta g(x''))\\ &\Leftrightarrow \forall x',x''\in X\,(x=x'\Delta x''\Rightarrow y=g(x')\Delta g(x'')),\end{aligned}$$

so g is Borel, and hence continuous by the Pettis theorem [**7**, **9.10**].

Let us finally prove Theorem 10 in its part related to BM maps. (The results for measure and Baire category often admit similar proofs, but rarely imply each other formally. In this case however the results for (BM) maps is a rather elementary consequence. The essential point is that both LM and BM cases are quite easily reducible to the Borel case.) It suffices to demonstrate that any BM $\mathscr{I}$-approximate Δ-homomorphism $f:\mathscr{P}(\mathbb{N})\to\mathscr{P}(\mathbb{N})$ is $\mathscr{I}$-approximable by a Borel map g.

To prove this, fix a comeager Borel set D such that $f\restriction D$ is Borel. For any x, the Borel set $D_x=D\cap(x\Delta D)\subseteq D$ is still comeager, and $y\in D_x\Rightarrow x\Delta y\in D_x$. Now, by a known uniformization theorem (see, *e.g.*, Kechris [**7**, **18.7**]), there is a Borel map $\bar{y}:\mathscr{P}(\mathbb{N})\to\mathscr{P}(\mathbb{N})$ such that $\bar{y}(x)\in D_x$ for any x. The Borel map $g(x)=f(\bar{y}(x))\Delta f(x\Delta\bar{y}(x))$ is an $\mathscr{I}$-approximation of f. This ends the proof of Theorem 13.

§6. *Remarks and problems.* The definition (*) of $\mathscr{I}$-approximate maps in Section 4 can be weakened to

(†) For any $x \in \mathscr{P}(\mathbb{N})$, the set $\{y \in \mathscr{P}(\mathbb{N}) : \mathsf{D}^f_{xy} \notin \mathscr{I}\}$ has λ-measure 0.

But it cannot be weakened to the requirement that

(‡) $\lambda^2\{\langle x, y\rangle : \mathsf{D}^f_{xy} \notin \mathscr{I}\} = 0$;

to see this let $f(x) = x$ for all $x \neq \varnothing$ and $f(\varnothing) = \mathbb{N}$. Yet (‡) implies the existence of a continuous Δ-homomorphism g which $\mathscr{I}$-approximates f on a set of full λ-measure.

It is known (see *e.g.*, Proposition 1.3 in [**1**]) that, if $\mathscr{I}$ and $\mathscr{J}$ are Borel RN ideals on countable sets I and J, then the quotients $\mathscr{P}(I)/\mathscr{I}$ and $\mathscr{P}(J)/\mathscr{J}$ are Baire BA-isomorphic (*i.e.*, there exists a Boolean algebra isomorphism with a Baire measurable lifting) if and only if $\mathscr{I}$ and $\mathscr{J}$ are "almost" isomorphic themselves, symbolically $\mathscr{I} \approx \mathscr{J}$, in the sense that there are sets $A \subseteq I$ and $B \subseteq J$ such that $I \setminus A \in \mathscr{I}$ and $J \setminus B \in \mathscr{J}$ and a bijection $f : A \xrightarrow{\text{onto}} B$ which transforms $\mathscr{I} \upharpoonright A$ onto $\mathscr{J} \upharpoonright B$. In view of this result, is it true that $\mathscr{I}^{\eta}_{\xi} \approx \mathscr{I}^{\eta'}_{\xi'}$ if and only if $\eta = \eta'$ and $\xi = \xi'$?

Are there Borel ideals $\mathscr{I}$, $\mathscr{J}$ on $\mathbb{N}$ such that there is a BM Boolean algebra or group <u>iso</u>morphism $F : \mathscr{P}(\mathbb{N})/\mathscr{I} \xrightarrow{\text{onto}} \mathscr{P}(\mathbb{N})/\mathscr{J}$ which does not admit a continuous lifting g which is a homomorphism? (A problem of Farah.) Note that all violations of the RN property known so far (*e.g.*, those derived from pathological submeasures, see [**1, 2**]) are related to homomorphisms F which are <u>not</u> isomorphisms.

Finally it would be interesting to prove lifting theorems for quotients of other algebraic structures, for instance, the additive group of the reals.

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Added to proof. After this note had been submitted to *Mathematika*, the authors proved, using similar methods, that *Weiss ideals* $\mathscr{W}_\xi = \{X \subseteq \omega^{\omega^\xi} : |X| < \omega^\xi\}$, where $|X|$ denotes the Cantor–Bendixson rank of a set of ordinals X, are Radon–Nikodym. This answers another of Farah's questions in [**3**] (close to the end of Section 1.14). The result appeared in [**6**].

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