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On the Uniform Projection and Covering Problems in Descriptive Set Theory Under the Axiom of Constructibility

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Abstract: The following two consequences of the axiom of constructibility $\mathbf{V} = \mathbf{L}$ will be established for every $n \geq 3$: 1. Every linear $\mathbf{\Sigma}^1_n$ set is the projection of a uniform planar $\mathbf{\Pi}^1_{n-1}$ set. 2. There is a planar $\mathbf{\Pi}^1_{n-1}$ set with countable cross-sections not covered by a union of countably many uniform $\mathbf{\Sigma}^1_n$ sets. If $n = 2$ then claims 1 and 2 hold in **ZFC** alone, without the assumption of $\mathbf{V} = \mathbf{L}$.

Keywords: constructibility; projective hierarchy; uniform sets; projections; covering

MSC: 03E15; 03E45

1. Introduction

The following theorem is the main result of this paper. It relates to the problems of *uniform projection* and *countable uniform covering* in descriptive set theory.

Theorem 1. *Assume that $n \geq 2$ and either* (I) *the axiom of constructibility $\mathbf{V} = \mathbf{L}$ holds or* (II) $n = 2$. *Then, we have the following:*

(a) (Uniform projection) *any $\mathbf{\Sigma}^1_n$ set $X \subseteq \omega^\omega$ is the projection of a uniform $\mathbf{\Pi}^1_{n-1}$ set $P \subseteq (\omega^\omega)^2$;*

(b) (Countable uniform non-covering) *there is a $\mathbf{\Pi}^1_{n-1}$ set $P \subseteq (\omega^\omega)^2$ with countable cross-sections* **not** *covered by a union of countably many uniform $\mathbf{\Sigma}^1_n$ sets.*

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For those not exactly versed in modern set theory, we recall that the axiom of constructibility was introduced by Gödel [1] as a statement saying that all sets are *constructible*, i.e., all sets admit a certain type of direct transfinite construction. The class of all sets is traditionally denoted by $\mathbf{V}$, the class of all constructible sets — by $\mathbf{L}$; hence, the equality $\mathbf{V} = \mathbf{L}$ symbolically expresses the content of this axiom.

It is customary in modern descriptive set theory to consider sets in the *Baire space* ω^ω, identified with the irrationals of the real line $\mathbb{R}$. Sets in the product spaces $(\omega^\omega)^m$ are also considered. Sets $X \subseteq \omega^\omega$, resp., $P \subseteq (\omega^\omega)^2$, are called *linear*, resp., *planar* for clear reasons.

As it is customary in texts on modern set theory, we use $\mathtt{dom}\, P$ for the *projection* $\mathtt{dom}\, P = \{x : \exists\, y\, P(x,y)\}$ of a planar set P to the first coordinate, and we use more compact *relational expressions* like $P(x,y)$, $Q(x,y,z)$, etc., instead of $\langle x,y\rangle \in P$, $\langle x,y,z\rangle \in Q$, etc.

The uniform projection problem. By definition [2,3], a set X in the Baire space ω^ω belongs to $\mathbf{\Sigma}^1_{n+1}$ iff it is equal to the *projection* $\mathtt{dom}\, P = \{x : \exists\, y\, P(x,y)\}$ of a planar $\mathbf{\Pi}^1_n$ set $P \subseteq (\omega^\omega)^2$; hence, in symbol, $\mathbf{\Sigma}^1_{n+1} = \mathbf{proj}\, \mathbf{\Pi}^1_n$. The picture drastically changes if we consider only *uniform* sets $P \subseteq (\omega^\omega)^2$, i.e., those satisfying $P(x,y) \land P(x,z) \implies y = z$.

Proposition 1 (Luzin [4,5], see also Section 2 below)**.** *The following three classes coincide:*

- *Class* $\mathbf{\Delta}^1_1$ *of all Borel sets in* ω^ω;
- *Class* $\mathbf{proj\,unif\,\Delta}^1_1$ *of projections of uniform* $\mathbf{\Delta}^1_1$ *(that is, Borel) sets in* $(\omega^\omega)^2$;
- *Class* $\mathbf{proj\,unif\,\Pi}^1_0$ *of projections of uniform* $\mathbf{\Pi}^1_0$ *(that is, closed) sets in* $(\omega^\omega)^2$.

Thus, symbolically, $\mathbf{proj\,unif\,\Pi}^1_0 = \mathbf{proj\,unif\,\Delta}^1_1 = \mathbf{\Delta}^1_1 \subsetneqq \mathbf{\Sigma}^1_1 = \mathbf{proj\,\Pi}^1_0$.

In Luzin's monograph [5], it is indicated that after constructing the projective hierarchy, "we immediately meet" with a number of questions, the general meaning of which is as follows: can some properties of the first level of the hierarchy be transferred to the following levels? Luzin raised several concrete problems of this kind in ([5], pp. 274–276, 285) related to different results on Borel ($\mathbf{\Delta}^1_1$), analytic ($\mathbf{\Sigma}^1_1$), and coanalytic ($\mathbf{\Pi}^1_1$) sets already known by that time. In particular, in connection with the results of Proposition 1, Luzin asked a few questions in [5], the common content of which can be formulated as follows.

Problem 1 (Luzin [5])**.** *For any given* $n \geq 2$, *figure out the relations between the classes* $\mathbf{\Delta}^1_n \subsetneqq \mathbf{\Sigma}^1_n = \mathbf{proj\,\Pi}^1_{n-1}$ *and* $\mathbf{proj\,unif\,\Pi}^1_{n-1} \subseteq \mathbf{proj\,unif\,\Delta}^1_n$.

Proposition 1 handles case $n = 1$ of the problem, of course.

Case $n = 2$ in Problem 1 was solved with *the Novikov–Kondo uniformization theorem* [6,7], which asserts that every $\mathbf{\Pi}^1_1$ set $P \subseteq (\omega^\omega)^2$ is *uniformizable* by a $\mathbf{\Pi}^1_1$ set Q; that is, $Q \subseteq P$ is uniform and $\mathtt{dom}\,Q = \mathtt{dom}\,P$, and hence,

$$\mathbf{proj\,unif\,\Pi}^1_1 = \mathbf{proj\,unif\,\Delta}^1_2 = \mathbf{\Sigma}^1_2 = \mathbf{proj\,\Pi}^1_1, \tag{1}$$

which, by the way, implies Theorem 1(a) in case $n = 2$.

Thus, we have a pretty different state of affairs in cases $n = 1$ and $n = 2$. In this context, the result of our Theorem 1(a) answers Luzin's problem under Gödel's axiom of constructibility in such a way that $\mathbf{V} = \mathbf{L}$ implies

$$\mathbf{proj\,unif\,\Pi}^1_{n-1} = \mathbf{proj\,unif\,\Delta}^1_n = \mathbf{\Sigma}^1_n = \mathbf{proj\,\Pi}^1_{n-1}. \tag{2}$$

for all $n \geq 3$, which is pretty similar to the solution in case $n = 2$ given by (1).

The countable uniform non-covering problem. Assertion (b) of Theorem 1 also has its origins in some results of classical descriptive set theory. It concerns the following important result.

Proposition 2 (Luzin [4,5], see also Section 2 below)**.** *Every planar* $\mathbf{\Delta}^1_1$ *set* $P \subseteq (\omega^\omega)^2$, *with all cross-sections* $P_x = \{y : \langle x, y\rangle \in P\}$ *(where* $x \in \omega^\omega$*) being at most countable, is covered by the union of a countable number of uniform* $\mathbf{\Delta}^1_1$ *sets.*

Luzin was also interested in knowing whether this result transfers to levels $n \geq 2$.

Problem 2 (Luzin [5])**.** *For any given* $n \geq 2$, *find out if it is true that every* $\mathbf{\Delta}^1_n$ *set* $P \subseteq (\omega^\omega)^2$ *with countable cross-sections* P_x *is covered by the union of countably many uniform* $\mathbf{\Delta}^1_n$ *sets.*

Our Theorem 1(b) solves this problem *in the negative*, outright for $n = 2$, and under the assumption of the axiom of constructibility for $n \geq 3$. We may note that this solution seems to be the strongest possible under assumption (I) $\vee$ (II) of Theorem 1, since this assumption implies that every planar $\mathbf{\Pi}^1_{n-1}$ set, and even $\mathbf{\Sigma}^1_n$ set, with countable cross-sections **can be** covered by a union of countably many uniform $\mathbf{\Delta}^1_{n+1}$ sets.

On the other hand, even much stronger non-covering results are known in generic models of **ZFC**. For instance, it is true in the Solovay model [8,9] that the Σ^1_2 set

$P = \{\langle x, y \rangle \in (\omega^\omega)^2 : y \in \mathbf{L}[x]\}$ is a set with countable cross-sections not covered by a countable union of uniform projective sets of any class, and even real-ordinal definable sets. Different models containing a $\mathit{\Pi}^1_2$ set with the same properties were defined in [10,11], and, unlike the Solovay model, without the assumption of the existence of an inaccessible cardinal.

The axiom of constructibility and consistency. As for the axiom of constructibility in Theorem 1, it was proved by Gödel [1] that $\mathbf{V} = \mathbf{L}$ is consistent with $\mathbf{ZFC}$; therefore, all of its consequences, like (a) and (b) of Theorem 1, are consistent as well. We recently succeeded ([12], [Theorem 74.1]) in proving that the negations of (a), in the forms $\mathbf{\Sigma}^1_n \not\subseteq \textbf{proj unif}\,\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n \not\subseteq \textbf{proj unif}\,\mathbf{\Pi}^1_{n-1}$, for any given $n \geq 3$, hold in appropriate generic models of $\mathbf{ZFC}$.

Corollary 1. *If $n \geq 3$, then each of the following three statements is consistent with and independent of* $\mathbf{ZFC}$: $\mathbf{\Sigma}^1_n = \textbf{proj unif}\,\mathbf{\Pi}^1_{n-1}$, $\mathbf{\Sigma}^1_n \not\subseteq \textbf{proj unif}\,\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n \not\subseteq \textbf{proj unif}\,\mathbf{\Pi}^1_{n-1}$.

No consistency result related to a positive solution of Problem 2 is known so far; in particular, both $\mathbf{V} = \mathbf{L}$ and generic models tend to solve the problem in the negative. This raises *the problem of the consistency of the positive solution* (Problem 5 in the final section), which can definitely inspire further research.

Outline of the proof. We will use a wide range of methods related to constructibility and effective descriptive set theory. Section 3 contains a brief introduction to universal sets and constructibility and presents some known results used in the proof of Theorem 1; it is written for the convenience of the reader.

Section 4 contains a proof of Claim (a) of Theorem 1. To prove the result, we define the class Γ as the closure of $\mathbf{\Sigma}^1_{n-1} \cup \mathbf{\Pi}^1_{n-1}$ under finite intersections and countable pairwise disjoint unions. Then, we prove, under $\mathbf{V} = \mathbf{L}$, that every set in Γ is a uniform projection of a $\mathit{\Pi}^1_{n-1}$ set (Lemma 1, an easy result), and that every set in $\mathbf{\Sigma}^1_n$ is a uniform projection of a set in Γ. To prove the latter result (Lemma 2), we make use of such a consequence of $\mathbf{V} = \mathbf{L}$ as a $\mathit{\Delta}^1_2$ well-ordering $<_{\mathbf{L}}$ of the reals, combined with an elaborate technique of effective descriptive set theory due to Harrington [13].

Section 5 contains a proof of Claim (b) of Theorem 1. The proof revolves around the set $U = U[n]$ of all pairs $\langle x, f \rangle \in \omega^\omega \times 2^\omega$ such that f is the indicator function of a $\Sigma^1_n(x)$ set $u \subseteq \omega$. We prove that U is not covered by countably many uniform $\mathbf{\Sigma}^1_n$ sets (Lemma 3, rather elementary) and then prove that U is Σ^1_n (Lemma 4) using quite a complex argument. Finally, a $\mathit{\Pi}^1_{n-1}$ set with necessary properties is obtained from U by Claim (a) of Theorem 1.

Section 6 presents alternative, shorter, and more transparent proofs of Claims (a) and (b) of Theorem 1, suggested by an anonymous reviewer.

Section 7 contains some conclusions and offers several problems for further study.

2. Some References

This section is written to provide some exact references and comments related to the problems and results discussed in Section 1.

Problem 1. In the following quote from Luzin ([5], p. 274), (B_n) is the early set theoretic notation for $\mathbf{\Delta}^1_n$ (sometimes to the exclusion of lower classes), whereas “ensemble de classe $< n$” means a set in $\mathbf{\Sigma}^1_m \cup \mathbf{\Pi}^1_m$ for some $m < n$.

> *Si $\mathcal{E}$ est un ensemble (B_n) plan uniforme relativement à l'axe OX, la projection E de $\mathcal{E}$ sur cet axe est-elle nécessairement un (B_n), ou un ensemble de classe $< n$? Un ensemble uniforme plan de classe $n-1$ a-t-il pour projection un (B_n), ou un ensemble de classe $< n$?* [Our italic here and below]

Thus, in modern terms, Luzin asks (1) whether the projection of any uniform $\mathbf{\Delta}^1_n$ set is necessarily a $\mathbf{\Delta}^1_n$ set and (2) whether the projection of any uniform $\mathbf{\Pi}^1_{n-1}$ set is necessarily a $\mathbf{\Delta}^1_n$ set. The question of inverse relations between linear sets and uniform projections is formulated by Luzin ([5], p. 276) in somewhat different terms as follows:

Or, dès que cette analogie est constatée, il est naturel de se poser les questions suivantes: peut-on trouver pour chaque ensemble (B_n) *une représentation paramétrique régulière?*

Here, a regular parametric representation of a $\mathbf{\Delta}^1_n = \mathrm{B}_n$ set is its representation as a 1–1 continuous image of a set in $\mathbf{\Pi}^1_{n-1}$, which is easily seen to be equivalent to a uniform projection of a $\mathbf{\Pi}^1_{n-1}$ set. Thus, essentially, Luzin asks (3) whether any linear $\mathbf{\Delta}^1_n$ set is the projection of a uniform $\mathbf{\Pi}^1_{n-1}$ set. We combine Luzin's questions (1), (2), and (3) in the form of Problem 1. Note that Theorem 1(a) answers (1) and (2) in the negative (assuming $n \geq 3$ and $\mathbf{V} = \mathbf{L}$, or just $n = 2$) and answers (3) in the positive, even more, for $\mathbf{\Sigma}^1_n$ instead of $\mathbf{\Delta}^1_n$.

Problem 2. Here, we refer to the following excerpt from Luzin ([5], p. 274).

Nous avons vu que chaque ensemble analytique uniforme est contenu dans une courbe uniforme mesurable B *et que chaque ensemble* E *mesurable* B *qui est conpé par chaque parallèle à l'axe OY en un ensemble de points au plus dénombrables est composé d'une infinité dénombrable d'ensembles uniformes mesurables* B. *Il est très naturel de se poser des questions analogues relativement aux ensembles projectifs* (A_n) *et* (B_n).

Thus, in particular, Luzin cites the result of Proposition 2 and asks if it also holds for $\mathbf{\Delta}^1_n$ for any n, that is, (4) whether every $\mathbf{\Delta}^1_n$ set $P \subseteq (\omega^\omega)^2$ with countable cross-sections P_x is covered by the union of countably many uniform $\mathbf{\Delta}^1_n$ sets. We reformulate it as Problem 2. Theorem 1(a) answers (4) in the negative (assuming $n \geq 3$ and $\mathbf{V} = \mathbf{L}$, or just $n = 2$).

Proposition 1. Every linear $\mathbf{\Delta}^1_1$ *(i.e., Borel) set* $X \subseteq \omega^\omega$ *is equal to the projection of a uniform closed set* $P \subseteq (\omega^\omega)^2$ (see Luzin ([4], [§ 39]) and ([5], [p. 114])), and conversely, *every projection of a uniform closed or even* $\mathbf{\Delta}^1_1$ *set* $P \subseteq \omega^\omega \times \omega^\omega$ *is a* $\mathbf{\Delta}^1_1$ *set in* ω^ω (see Luzin ([4], [§ 47]) and ([5], [p. 166])). For a modern treatment, see Kechris ([2], [15.1, 15.3]) and Moschovakis ([3], [2E.7, 2E.8, 4A.7]).

Proposition 2. See Luzin ([4], [§ 54]) (with a reference to Novikov's research) and ([5], [P. 243]), Kechris ([2], [18.15]), and Moschovakis ([3], [4F.17]). By the way, Moschovakis ([3], [p. 195]) refers to Novikov [14] regarding Proposition 2, yet our inspection showed that there is no such statement there, at least not in explicit form. Novikov, in fact, proves that every Borel ($= \mathbf{\Delta}^1_1$) set with countable sections has a Borel projection (§ 7 in [14]) and admits Borel uniformization (§ 9).

3. Preliminaries

We make use of the modern notation [2,3,15] $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ for classes of the projective hierarchy (*boldface* classes), and Σ^1_n, Π^1_n, and Δ^1_n for the corresponding effective (or *lightface*) classes of sets in the spaces of the form $(\omega^\omega)^m \times \omega^k$, $m, k < \omega$, which we call *product spaces*. As usual, elements $a, b, \dots \in \omega^\omega$ will be called *reals*. If $a, b, \dots \in \omega^\omega$ is a finite list of reals, then $\Sigma^1_n(a, b, \dots)$, $\Pi^1_n(a, b, \dots)$, and $\Delta^1_n(a, b, \dots)$ are the effective classes *relative to* $a, b, \dots$. Every real $x \in \omega^\omega$ is formally a subset of ω^2; hence, it can belong to one of the effective classes, say Δ^1_n or $\Delta^1_n(a)$.

Proposition 3 (universal sets)**.** (i) *If* $n \geq 1$, $\mathscr{X}$ *is a product space, and* K *is a class of the form* Σ^1_n *or* $\Sigma^1_n(a)$, $a \in \omega^\omega$, *then there is a set* $U \subseteq \omega \times \mathscr{X}$ universal *in the sense that if* $X \subseteq \mathscr{X}$ *belongs to* K, *then there exists an* m *such that* $X = U_m = \{x : \langle m, x \rangle \in U\}$.

(ii) *If* $n \geq 1$, *then there is a* Σ^1_n *set* $W \subseteq \omega \times \omega^\omega \times \omega$ *such that if* $x \in \omega^\omega$ *and a set* $u \subseteq \omega$ *belongs to* $\Sigma^1_n(x)$, *then there is an* $m < \omega$ *satisfying* $u = W_{xm} = \{k : \langle m, x, k \rangle \in W\}$.

Proof (sketch). (i) is a well-known standard fact; see, e.g., ([3], [3F.6]) or ([16], [Theorem 4.9 in Chapter C.8]). To prove (ii), let $U \subseteq \omega \times (\omega^\omega \times \omega)$ be a universal Σ^1_n set as in (i) for $\mathscr{X} = \omega^\omega \times \omega$. Then, set $W = U$. □

Constructible sets were introduced by Gödel [1] as those that can be obtained by a certain transfinite construction. The axiom of constructibility claims that all sets are constructible, symbolically $\mathbf{V} = \mathbf{L}$, where $\mathbf{V}$ = all sets and $\mathbf{L}$ = all constructible sets. See [15,17] as modern references on the theory of constructibility. The analytical representation of Gödel's constructibility is well known since the 1950s; see, e.g., Addison [18,19] and Simpson's book ([20], [VII.4]). The next proposition gathers some facts on the Gödel well ordering of ω^ω.

Proposition 4 ($\mathbf{V} = \mathbf{L}$)**.** *There is a Δ^1_2 well-ordering $<_{\mathbf{L}}$ of ω^ω, of order type ω_1, such that we have the following:*

(i) *The binary relation $R(x,z)$, iff $\{(z)_m : m < \omega\} = \{y : y <_{\mathbf{L}} x\}$, on ω^ω belongs to Σ^1_2, where $(z)_m \in \omega^\omega$ is defined by $(z)_m(k) = z(2^m(2k+1)-1)$, $\forall k$;*

(ii) *If $n \geq 2$, K is a class of the form $\Sigma^1_n(b)$, $b \in \omega^\omega$, and $P \subseteq (\omega^\omega)^3$ is a set in K, then*

$$U = \{\langle x,a\rangle : \forall y <_{\mathbf{L}} x\, P(x,y,a)\} \quad \text{and} \quad V = \{\langle x,a\rangle : \exists y <_{\mathbf{L}} x\, P(x,y,a)\}$$

are still sets in K. The same is true for $K = \Pi^1_n(b)$ and $\Delta^1_n(b)$.

Proof (sketch). We let $<_{\mathbf{L}}$ be the restriction of the Gödel well ordering of $\mathbf{L}$, the constructible universe, to $\omega^\omega \cap \mathbf{L}$. When assuming $\mathbf{V} = \mathbf{L}$, $<_{\mathbf{L}}$ is known to be a well ordering of ω^ω, of length ω_1, and a relation of class Δ^1_2; see, e.g., ([15], [Thm 25.26]).

Lemma 25.27 in [15] proves (i) for $<_{\mathbf{L}}$. Then, a simple argument, like that in the proof of Corollary 25.29 in [15], yields (ii). Namely, if, say, P is Σ^1_3, then

$$U(x,a) \iff \exists z \big(R(x,z) \wedge \forall m\, P(x,(z)_m,a)\big),$$

which is easily reducible to Σ^1_3 since the numerical quantifier $\forall m$ can be eliminated by the standard quantifier contraction rules. □

Claim (ii) of Proposition 4 is known as the Σ^1_2-goodness of the order $<_{\mathbf{L}}$; see ([3], [Section 5A]). This property of $<_{\mathbf{L}}$ was essentially singled out by Addison ([19], [Theorem 1]). The next corollary gives several further consequences of $\mathbf{V} = \mathbf{L}$ related to projective hierarchy, also attributed to Addison [19] and rather well known in set theoretic studies; see, e.g., ([3], [Section 5A]) or ([16], [Section C.8.5]). Yet, we add proofs for the convenience of the reader.

Corollary 2 ($\mathbf{V} = \mathbf{L}$)**.** *Let $n \geq 2$ and $a \in \omega^\omega$. Then, we have the following:*

(i) *If K is a class of the form Δ^1_n, Σ^1_n, $\Delta^1_n(a)$, or $\Sigma^1_n(a)$, then every set $P \subseteq \omega^\omega \times \omega^\omega$ in K is uniformizable by a set $Q \subseteq P$ still in K;*

(ii) *Any $\mathbf{\Sigma}^1_n$ set $X \subseteq \omega^\omega$ is the projection of a uniform $\mathbf{\Delta}^1_n$ set;*

(iii) *Any non-empty Σ^1_n, resp., $\Sigma^1_n(a)$ set $X \subseteq \omega^\omega$ contains a Δ^1_n, resp., $\Delta^1_n(a)$ real $x \in X$;*

(iv) *If $x, y \in \omega^\omega$ and $x <_{\mathbf{L}} y$, then $x \in \Delta^1_2(y)$.*

Proof. (i) If $P \in \Delta^1_n(a)$, then the set $Q = \{\langle x,y\rangle \in P : \forall y' <_{\mathbf{L}} y \neg P(x,y')\}$ obviously uniformizes P, whereas $Q \in \Delta^1_n(a)$ follows from Proposition 4(ii). Now, suppose that $P \in \Sigma^1_n(a)$. There is a Π^1_{n-1} set $C \subseteq (\omega^\omega)^3$ satisfying $P = \{\langle x,y\rangle : \exists z\, C(x,y,z)\}$. Using a canonical homeomorphism $H : (\omega^\omega)^2 \xrightarrow{\text{onto}} \omega^\omega$ and the result for $\Delta^1_n(a)$ already established, we can uniformize C as a $\Delta^1_n(a)$ subset of $\omega^\omega \times (\omega^\omega)^2$ via a $\Delta^1_n(a)$ set $D \subseteq C$ so that for any $x \in \omega^\omega$, $\exists\, y,z\, C(x,y,z) \implies \exists\,!\, y,z\, D(x,y,z)$. It remains to define $Q = \{\langle x,y\rangle \in P : \exists\, z\, D(x,y,z)\}$.

(ii) If $X \in \mathbf{\Sigma}^1_n$, then $X \in \Sigma^1_n(a)$ for some $a \in \omega^\omega$. By definition, $X = \mathtt{dom}\, P$ for some $\Pi^1_{n-1}(a)$ set $P \subseteq \omega^\omega \times \omega^\omega$. Let $Q \subseteq P$ be a $\Delta^1_n(a)$ set that uniformizes P by (i).

(iii) Define $\mathbf{0} \in \omega^\omega$ by $\mathbf{0}(k) = 0$, $\forall\, k$. If $X \in \Sigma^1_n(a)$, then the set $P = \{\mathbf{0}\} \times X = \{\langle \mathbf{0}, x\rangle : x \in X\}$ is $\Sigma^1_n(a)$ as well, and hence, by (i), it can be uniformized by a $\Sigma^1_n(a)$ set $Q \subseteq P$. In fact, $Q = \{\langle \mathbf{0}, x_0\rangle\}$ for some $x_0 \in X$. To see that x_0 is $\Delta^1_n(a)$, use the equivalence

$$x_0(j) = k \iff \exists\, x\,\big(Q(\mathbf{0},x) \wedge x(j) = k\big) \iff \forall\, x\,\big(Q(\mathbf{0},x) \implies x(j) = k\big).$$

(iv) If $f \in \omega^\omega$ and $m < \omega$, then define $(f)_m \in \omega^\omega$ as in Proposition 4(i). The set $X = \{f \in \omega^\omega : \forall\, x' <_{\mathbf{L}} y\, \exists\, m\, (x' = (f)_m)\}$ belongs to $\Delta^1_2(y)$ by Proposition 4(ii). Thus, X contains a $\Delta^1_2(a)$ element $f \in X$ by (iii). Then, $x = (f)_m \in \Delta^1_2(y)$ for some m. □

4. Proof of the Uniform Projection Theorem

Here, we prove Theorem 1(a). We may note that Case (II) ($n = 2$) of this statement is covered by the Novikov–Kondo uniformization theorem, and hence, we can assume that $n \geq 3$ and Case (I), the axiom of constructibility $\mathbf{V} = \mathbf{L}$, hold.

Thus, we fix a number $n \geq 3$ and assume $\mathbf{V} = \mathbf{L}$ in the course of the proof.

Note that the result will be achieved **not** by a reference to the $\mathbf{\Pi}^1_{n-1}$ uniformization claim, which actually fails for $n \geq 3$ under $\mathbf{V} = \mathbf{L}$.

Definition 1. *Let Γ be the closure of the union $\mathbf{\Sigma}^1_{n-1} \cup \mathbf{\Pi}^1_{n-1}$ under the operations (1) of finite intersections and (2) of countable unions of pairwise disjoint sets — both operations for sets in one and the same space, of course.*

The proof of Theorem 1(a) consists of two lemmas related to this intermediate class.

Lemma 1. *Every Γ set $X \subseteq \omega^\omega$ is the projection of a uniform $\mathbf{\Pi}^1_{n-1}$ set.*

Proof. The proof continues by induction on the construction of sets in Γ from the initial sets in $\mathbf{\Sigma}^1_{n-1} \cup \mathbf{\Pi}^1_{n-1}$. The result for $\mathbf{\Pi}^1_{n-1}$ sets is obvious, and for $\mathbf{\Sigma}^1_{n-1}$ sets, it follows from Corollary 2(ii). Now, the induction step follows.

Assume that sets $X_0, X_1, X_2, X_3, \ldots \subseteq \omega^\omega$ are pairwise disjoint, and, by the inductive hypothesis, $X_k = \mathtt{dom}\, P_k$ and $P_k \in \mathbf{\Pi}^1_{n-1}$, $P_k \subseteq \omega^\omega \times \omega^\omega$ is uniform for each $k < \omega$. Then, the set $X = \bigcup_k X_k$ satisfies $X = \mathtt{dom}\, P$, where $P = \bigcup P_k$ is uniform and belongs to $\mathbf{\Pi}^1_{n-1}$ (since the class $\mathbf{\Pi}^1_{n-1}$ is closed under countable operations $\bigcup$ and $\bigcap$.)

Now, assume that $X_0, X_1 \subseteq \omega^\omega$ and, by the inductive hypothesis, $X_k = \mathtt{dom}\, P_k$ and $P_k \in \mathbf{\Pi}^1_{n-1}$, $P_k \subseteq \omega^\omega \times \omega^\omega$ is uniform for each $k = 0, 1$. We set

$$P = \{\langle x,y,z\rangle : \langle x,y\rangle \in P_0 \wedge \langle x,z\rangle \in P_1\} \quad \text{and} \quad Q = \{\langle x, G(y,z)\rangle : \langle x,y,z\rangle \in P\},$$

where $G : \omega^\omega \times \omega^\omega \xrightarrow{\text{onto}} \omega^\omega$ is a homeomorphism. Then, the set $X = X_0 \cap X_1$ satisfies $X = \mathtt{dom}\, Q$, where Q is uniform and belongs to $\mathbf{\Pi}^1_{n-1}$. □

Lemma 2. *Every $\mathbf{\Sigma}^1_n$ set $X \subseteq \omega^\omega$ is the projection of a uniform Γ set.*

Proof. This is a much more involved argument. Consider a $\mathbf{\Sigma}^1_n$ set $X \subseteq \omega^\omega$ such that $X = \mathtt{dom}\, P$, where $P \subseteq (\omega^\omega)^2$ is $\mathbf{\Pi}^1_{n-1}$. We can w.l.o.g. assume that $P \subseteq \omega^\omega \times 2^\omega$, where $2^\omega \subseteq \omega^\omega$ (all infinite dyadic sequences) is the Cantor discontinuum. (If this is not the case, then replace P with $P' = \{\langle x, F(y)\rangle : P(x,y)\}$, where $F : \omega^\omega \to 2^\omega$ is the injection defined by $F(y) = 1^\frown 0^{y(0)\frown} 1^\frown 0^{y(1)\frown} 1^\frown 0^{y(2)\frown} \ldots$.)

Note that P belongs to $\Pi^1_{n-1}(a)$ for some $a \in \omega^\omega$. Assume that P is in fact lightface Π^1_{n-1}, and hence, X is Σ^1_n; the general case does not differ. Then, there is a Σ^1_{n-2} set $C \subseteq \omega^\omega \times 2^\omega \times 2^\omega$ satisfying $P = \{\langle x,y\rangle \in \omega^\omega \times 2^\omega : \forall z \in 2^\omega\, C(x,y,z)\}$.

From now on, we assume that $y, z, w, w' \in 2^\omega$ in all quantifiers and other occurrences in the course of the proof of Lemma 2.

Note that $x \in X \iff \exists y\, \forall z\, C(x,y,z)$. Consider the set

$$W = \{\langle x,w\rangle \in \omega^\omega \times 2^\omega : \forall y <_{\mathbf{L}} w\, \exists z <_{\mathbf{L}} w\, \neg C(x,y,z)\}.$$

Quite obviously, if $x \in \omega^\omega$, then the cross-section $W_x = \{w : \langle x,w\rangle \in W\} \subseteq 2^\omega$ is non-empty (contains the $<_{\mathbf{L}}$-least element of 2^ω), is closed in 2^ω in the sense of the order $<_{\mathbf{L}}$ (that is, in the sense of the topology induced on 2^ω by the order $<_{\mathbf{L}}$), and satisfies $\langle x,y\rangle \in P \wedge w \in W_x \implies w \leqslant_{\mathbf{L}} y$. We conclude that if $x \in X$, then there exists a $<_{\mathbf{L}}$-largest element $w_x \in W_x$. The following follow from the above:

(A) If $\langle x,y\rangle \in P$, then $w_x \in 2^\omega$ exists and $w_x \leqslant_{\mathbf{L}} y$.

Now, define the relation $B(x,y,w) := w \in W_x \wedge \forall w' \leqslant_{\mathbf{L}} y\, (w <_{\mathbf{L}} w' \implies w' \notin W_x)$. We conclude the following from (A):

(B) If $\langle x,y\rangle \in P$, then $B(x,y,w) \iff w = w_x$.

The next claim makes use of an idea presented in Harrington's paper [13]:

(C) If $x \in X$, then there is a $y \in \Delta^1_{n-1}(x, w_x)$ such that $\langle x,y\rangle \in P$.

To prove this crucial claim, *we fix* $x \in X$ and let $f \in 2^\omega$ be the $<_{\mathbf{L}}$-least element of the difference $2^\omega \smallsetminus \Delta^1_{n-1}(x, w_x)$. We assert the following:

(D) If $z \in 2^\omega$, then the equivalence $z <_{\mathbf{L}} f \iff z \in \Delta^1_{n-1}(x, w_x)$ holds.

Indeed, in the nontrivial direction, suppose that the left-hand side fails, i.e., $f \leqslant_{\mathbf{L}} z$. Then, $f \in \Delta^1_2(z)$ by Corollary 2(iv). We conclude that $z \notin \Delta^1_{n-1}(x, w_x)$. (Indeed, otherwise, $f \in \Delta^1_{n-1}(x, w_x)$, contrary to the choice of f). This completes the proof of (D).

Taking $z = w_x$ in (D), we obtain $w_x <_{\mathbf{L}} f$, and hence, $f \notin W_x$. By definition, there exists $y \in 2^\omega$, $y <_{\mathbf{L}} f$ satisfying the following:

(E) $\forall z <_{\mathbf{L}} f\, C(x,y,z)$.

Fix such a real y. We assert that $\langle x,y\rangle \in P$. Suppose otherwise. Then, the $\Pi^1_{n-2}(x,y)$ set $Z = \{z \in 2^\omega : \langle x,y,z\rangle \notin C\}$ is non-empty, and hence, there is a $\Delta^1_{n-1}(x,y)$ real $z \in Z$ by Corollary 2(iii). However, $y <_{\mathbf{L}} f$ by construction. We conclude by (D) that $y \in \Delta^1_{n-1}(x, w_x)$. This implies that $z \in \Delta^1_{n-1}(x, w_x)$, which contradicts (D), (E), and the choice of z. The contradiction ends the proof of $\langle x,y\rangle \in P$ and thereby completes the proof of (C) as well since $y \in \Delta^1_{n-1}(x, w_x)$ is already established.

Now, recall the following technical notation.

Definition 2. *The indicator function $\chi_u \in 2^\omega$ of a set $u \subseteq \omega$ is defined by $\chi_u(k) = 1$ in case $k \in u$ and $\chi_u(k) = 0$ in case $k \notin u$.*

If $h \in \omega^\omega$, $m < \omega$, then define $(h)_m \in \omega^\omega$ by $(h)_m(j) = h(2^m(2j+1) - 1)$, $\forall j$.

In continuation of the proof of Lemma 2, we note that Proposition 3(ii) yields a Σ^1_{n-1} set $D \subseteq (\omega^\omega)^2 \times \omega$ that is universal in the sense of the following:

(F) If $x \in \omega^\omega$, $w \in 2^\omega$, and a real $y \in 2^\omega$ belongs to $\Sigma^1_{n-1}(x,w)$, then there is an $m < \omega$ such that $y = (f[x,w])_m$, where $f[x,w] = \chi_{D[x,w]}$ and $D[x,w] = \chi_{\{k\,:\,D(x,w,k)\}}$.

The set $Q = \{\langle x, f[x, w_x]\rangle : x \in X\}$ is obviously uniform, and $\mathtt{dom}\, Q = X$ by (A). Thus, it remains to prove that $Q \in \Gamma$. This is the last step in the proof of Lemma 2. We claim the following:

(G) $$Q = \{\langle x, f\rangle \in \omega^\omega \times 2^\omega : \exists\, m\, P(x, (f)_m) \wedge \\ \wedge \forall\, j\, \big(f(j) = 1 \iff \exists\, w\, (B(x, (f)_m, w) \wedge D(x, w, j))\big)\}.$$

Direction $\subseteq$ in (G). Suppose that $x \in X$ and $f = f[x, w_x]$. By (C), take $y \in \Delta^1_{n-1}(x, w_x)$ such that $\langle x, y\rangle \in P$. Note that $y \in 2^\omega$ as $P \subseteq \omega^\omega \times 2^\omega$ was assumed in the beginning of the proof. Then, by (F), we have $y = (f)_m$ for some m.

Finally, to check the equivalence $\forall\, j\, (\dots)$ in (G), let $j < \omega$. Assume that $f(j) = 1$ (*direction* $\Longrightarrow$). Take $w = w_x$. Then, $j \in D[x, w_x]$; that is, $D(x, w_x, j)$ holds, whereas $B(x, (f)_m, w)$ holds by (B) in the presence of $P(x, (f)_m)$. Now, assume that some w witnesses $B(x, (f)_m, w) \wedge D(x, w, j)$ (*direction* $\Longleftarrow$). Then, $w = w_x$ yet again by (B); hence, $j \in D[x, w_x]$ and $f(j) = 1$ by construction. This ends the proof $\forall\, j\, (\dots)$ and completes the direction $\subseteq$ in (G).

Direction $\supseteq$ in (G). Let $\langle x, f\rangle$ belong to the right-hand side of equality (G); we have to prove that $\langle x, f\rangle \in Q$, that is, that $f = f[x, w_x]$. As $P(x, (f)_m)$ holds for some m, (B) implies $B(x, (f)_m, w) \iff w = w_x$ once again, and hence, the second line in (G) takes the form $\forall\, j\, \big(f(j) = 1 \Leftrightarrow D(x, w_x, j)\big)$, obviously meaning that $f = f[x, w_x]$, as required.

The proof of (G) is accomplished. It remains to prove that Q is a set in Γ. We recall that C is Σ^1_{n-2}; hence, W is Π^1_{n-2} by Proposition 4(ii), and then B is Δ^1_{n-1} also by Proposition 4(ii). Finally, D is Σ^1_{n-1}. Therefore, we can rewrite the subformula $\forall\, j\, (\cdots \iff \dots)$ in (G) as $\forall\, j\, (\cdots \Longrightarrow \dots) \wedge \forall\, j\, (\cdots \Longleftarrow \dots)$, which yields the conjunction of a Σ^1_{n-1} formula and a Π^1_{n-1} formula. Finally, P is Π^1_{n-1}. Thus, Q can be represented in the form (*) $Q = \bigcup_{m<\omega}(S_m \cap T_m)$, where $S_m \in \Sigma^1_{n-1}$ and $T_m \in \Pi^1_{n-1}$, $\forall\, m$.

To obtain a representation in Γ, we let $S^-_m = \omega^\omega \smallsetminus S_m$ and $T^-_m = \omega^\omega \smallsetminus T_m$. Then, (*) implies that $Q = \bigcup_{m<\omega}\big((S_m \cap T_m) \cap [\bigcap_{j<m}(S^-_j \cup (S_j \cap T^-_j))]\big)$, where all unions on the right-hand side are pairwise disjoint unions. Thus, $Q \in \Gamma$, as required. $\square$

Proof of Theorem 1(a), Case (I). Immediately from Lemmas 1 and 2. $\square$

5. Proof of the Uniform Covering Theorem

Here, we prove Theorem 1(b). An essential part of the arguments will be common for both Case (I) and Case (II) of the theorem. Note that unlike Theorem 1(a), no classical theorem is known to immediately imply the result for $n = 2$.

Our plan is to first define a $\mathbf{\Sigma}^1_n$ (actually Σ^1_n) set $U \subseteq (\omega^\omega)^2$ with the required properties and then convert it into a $\mathbf{\Pi}^1_{n-1}$ set using claim (a) of Theorem 1, which is already proved.

Thus, we fix $n \geq 2$ and assume that either (I) $\mathbf{V} = \mathbf{L}$ holds or (II) $n = 2$.

Let $\vartheta(m, x, k)$ be a Σ^1_n formula that defines the universal set W as in Proposition 3(ii); hence, for any $x \in \omega^\omega$ and any Σ^1_n set $u \subseteq \omega$, there is an $m < \omega$ such that $u = \{k : \vartheta(m, x, k)\}$.

Let $f_{mx} \in 2^\omega$ be the indicator function (Definition 2) of the set $u_{mx} = \{k : \vartheta(m, x, k)\}$.

Definition 3. *We define $U = U[n] := \{\langle x, f_{mx}\rangle : x \in \omega^\omega \wedge m < \omega\}$. Thus,*

(*) $U = \{\langle x, a\rangle \in \omega^\omega \times 2^\omega : a = \chi_u\}$ *is the indicator function of a set $u \in \Sigma^1_n(x)$, $u \subseteq \omega\}$*

by the universality of ϑ.

Lemma 3. *$U \subseteq \omega^\omega \times 2^\omega$ is a set with countable cross-sections* **not** *covered by a union of countably many uniform $\mathbf{\Sigma}^1_n$ sets.*

Proof. Suppose the contrary that $U \subseteq \bigcup_m U_m$, where all sets $U_m \subseteq \omega^\omega \times 2^\omega$ are $\mathbf{\Sigma}^1_n$ and uniform. There is an $x \in \omega^\omega$ such that every U_m belongs to $\Sigma^1_n(x)$. Then, every non-empty cross-section $U_{mx} = \{a : \langle x, a \rangle \in U_m\}$ is a $\Sigma^1_n(x)$ singleton whose only element is $\Delta^1_n(x)$. Thus, the whole cross-section $U_x = \{a : \langle x, a \rangle \in U\}$ contains only $\Delta^1_n(x)$ elements. This contradicts (*) above because there exist sets $u \subseteq \omega$ in $\Sigma^1_n(x) \smallsetminus \Delta^1_n(x)$. □

Lemma 4. *U is a Σ^1_n set.*

Proof. This argument is somewhat different in the two cases considered.

Case (I): $\mathbf{V} = \mathbf{L}$. First of all, if φ is an analytic formula and $z \in \omega^\omega$, then let φ^z be the formal relativization of φ to $\{y \in \omega^\omega : y <_{\mathbf{L}} z\}$ so that all quantifiers $\exists y$, $\forall y$ over ω^ω are replaced with, resp. $\exists y <_{\mathbf{L}} z$, $\forall y <_{\mathbf{L}} z$.

Let $f^z_{mx} \in 2^\omega$ be the indicator function of $\{k : \vartheta^z(m, x, k)\}$. Proposition 4(ii) implies the following:

(1) The set $\{\langle m, x, z, f^z_{mx} \rangle : m < \omega \wedge x, z \in \omega^\omega\}$ is Δ^1_2.

Indeed, by definition, the relativized formula $\vartheta^z(m, x, k)$ has all its real number quantifiers of the form $\exists a <_{\mathbf{L}} b$, $\forall a <_{\mathbf{L}} b$. Therefore, $\{\langle m, x, k \rangle : \vartheta^z(m, x, k)\}$ is a Δ^1_2 set by Proposition 4(ii) applied enough times (equal to the number of quantifiers, $\exists a <_{\mathbf{L}} b$, $\forall a <_{\mathbf{L}} b$ in the prenex form). This immediately implies (1).

The Σ^1_n formula $\vartheta(m, x, k)$ has the form $\exists y\, \psi(y, m, x, k)$, where ψ is a Π^1_{n-1} formula. The following set E belongs to Δ^1_n by (1), the choice of ψ, and Proposition 4(ii):

$$E = \{z \in \omega^\omega : \forall m, k\, \forall x, y <_{\mathbf{L}} z\, (\psi^z(y, m, x, k) \iff \psi(y, m, x, k))\}.$$

Corollary 2(iii) implies the next claim:

(2) If $k < \omega$, $z \in E$, $x <_{\mathbf{L}} z$ and $\Delta^1_n(x) \cap \omega^\omega \subseteq C_z = \{c \in \omega^\omega : c <_{\mathbf{L}} z\}$, then $f^z_{mx} = f_{mx}$.

In addition, we have the following standard claim:

(3) If $C \subseteq \omega^\omega$ is countable, then there is a $z \in \omega^\omega$ with $C \subseteq C_z = \{c \in \omega^\omega : c <_{\mathbf{L}} z\}$.

We now prove that

(4) $U = \{\langle x, a \rangle : \exists m\, \exists z\, (z \in E \wedge x <_{\mathbf{L}} z \wedge a <_{\mathbf{L}} z \wedge a = f^z_{mx})\}$.

Indeed, suppose that $\langle x, a \rangle \in U$ so that $a = f_{mx}$ for some m. Let, by (3), $z \in \omega^\omega$ satisfy $\{a\} \cup (\Delta^1_n(x) \cap \omega^\omega) \subseteq C_z$. Then, $x, a <_{\mathbf{L}} z$, and hence, we have $a = f^z_{mx}$ by (2).

Conversely, suppose that $x, a <_{\mathbf{L}} z \in E$ and $a = f^z_{mx}$. We have two cases, A and B:

A: $\Delta^1_n(x) \cap \omega^\omega \subseteq C_z$. Then, $f^z_{mx} = f_{mx}$ by (2) as above; hence, $a = f_{mx}$ and $\langle x, a \rangle \in U$.

B: There is a $\Delta^1_n(x)$ real y satisfying $z \leqslant_{\mathbf{L}} y$. Then, $a, x <_{\mathbf{L}} y$; hence, $a \in \Delta^1_2(y)$ by Corollary 2(iv). We conclude that $a \in \Delta^1_n(x)$ by the choice of y. Now, $\langle x, a \rangle \in U$ easily follows from (*). This ends the proof of (4).

We finally note that the right-hand side of (4) is definitely a Σ^1_n set because E is Δ^1_n, $<_{\mathbf{L}}$ is Σ^1_2, and the equality $a = f^z_{mx}$ is Δ^1_2 by (1). Thus, U is Σ^1_n, and we are finished with case $\mathbf{V} = \mathbf{L}$ in Lemma 4.

Case (II): $n = 2$, sketch. As the axiom of constructibility is not assumed any more in this case, we are going to use the technique of *relative constructibility*. For any real $w \in \omega^\omega$ (and in principle, for any set x, but we do not need such a generality here), the class $\mathbf{L}[w]$ is defined similarly to $\mathbf{L}$ itself; see ([15], [Chapter 12]). All major consequences of $\mathbf{V} = \mathbf{L}$ are preserved mutatis mutandis under the relative constructibility $\mathbf{V} = \mathbf{L}[w]$. In particular, we have the following:

1° There exists a Σ^1_2 formula $\zeta(w, x)$ such that for all $w, x \in \omega^\omega$: $x \in \mathbf{L}[w] \iff \zeta(w, x)$.

2° For any $w \in \omega^\omega$, there is a well-ordering $<_{\mathbf{L}[w]}$ of $\omega^\omega \cap \mathbf{L}[w]$ of order type $\omega_1^{\mathbf{L}[w]}$ such that the ternary relation $x, y \in \mathbf{L}[w] \wedge x <_{\mathbf{L}[w]} y$ on $(\omega^\omega)^3$ is Σ^1_2.

3° If $w, b \in \omega^\omega$, $\mathbf{V} = \mathbf{L}[w]$ holds, $m \geq 2$, K is a class of the form $\Sigma^1_m(w,b)$, and $P \subseteq (\omega^\omega)^3$ is a set in K, then similarly to Proposition 4(ii), the sets

$$U = \{\langle y,z\rangle : \forall\, x <_{\mathbf{L}[w]} y\, P(x,y,z)\} \quad \text{and} \quad V = \{\langle y,z\rangle : \exists\, x <_{\mathbf{L}[w]} y\, P(x,y,z)\}$$

are still sets in K. The same is true for $K = \Pi^1_m(w,b)$ and $K = \Delta^1_m(w,b)$.

After these remarks, let us prove that the set $U = U[2]$ (Definition 3) belongs to Σ^1_2 without any reference to the axiom of constructibility or anything beyond **ZFC**.

Indeed, the proof of Lemma 4 in Case (I): $\mathbf{V} = \mathbf{L}$ with $n = 2$ can be compressed to the existence of a Σ^1_2 formula $\mathbf{u}(x,f)$ such that $U = \{\langle x,a\rangle : \mathbf{u}(x,a)\}$ under $\mathbf{V} = \mathbf{L}$. The relativized version, essentially with nearly the same proof based on 2° and 3°, yields a Σ^1_2 formula $\mathbf{u}'(w,x,f)$ such that

4° If $w \in \omega^\omega$ and $\mathbf{V} = \mathbf{L}[w]$, then $U = \{\langle x,a\rangle : \mathbf{u}'(w,x,a)\}$.

Now, let $\mathbf{u}''(x,a)$ be the formula $x, a \in \omega^\omega \wedge f \in \mathbf{L}[x] \wedge \mathbf{u}'(x,x,a)$. Clearly $\mathbf{u}''$ is Σ^1_2 by 1° and the choice of $\mathbf{u}'$. Thus, it suffices to prove that $U = \{\langle x,a\rangle : \mathbf{u}''(x,a)\}$ (in **ZFC** with no extra assumptions).

Suppose that $\langle x,a\rangle \in U$. Then, $a \in \mathbf{L}[x]$ by the Shoenfield absoluteness theorem [21]. It follows from 4° (with $w = x$) that $\mathbf{u}'(x,x,f)$ holds in $\mathbf{L}[x]$ and hence holds in the universe by the same Shoenfield's absoluteness. Thus, we have $\mathbf{u}''(x,a)$, as required.

Conversely, assume $\mathbf{u}''(x,a)$ so that $a \in \mathbf{L}[x]$, and we have $\mathbf{u}'(x,x,a)$. Then, $\mathbf{u}'(x,x,a)$ holds in $\mathbf{L}[x]$ by Shoenfield, and hence, $\langle x,a\rangle \in U$ still by 4° (with $w = x$), as required. □

Proof of Theorem 1(b). As U is Σ^1_n by Lemma 4, Theorem 1(a) implies that there exists a Π^1_{n-1} set $Q \subseteq (\omega^\omega)^3$ such that $U = \mathtt{dom}_2\, Q := \{\langle x,y\rangle : \exists\, z\, Q(x,y,z)\}$ (the projection on $(\omega^\omega)^2$), and Q is uniform in $(\omega^\omega)^2 \times \omega^\omega$, i.e., $Q(x,y,z) \wedge Q(x,y,z') \implies z = z'$. Then, each cross-section $Q_x = \{\langle y,z\rangle : Q(x,y,z)\}$ is at most countable by the choice of U and Q.

We claim that Q is not covered by a countable union of $\mathbf{\Sigma}^1_n$ sets uniform in $\omega^\omega \times (\omega^\omega)^2$. Indeed, assume to the contrary that $Q \subseteq \bigcup_m Q_m$, where each Q_m is $\mathbf{\Sigma}^1_n$ and uniform in $\omega^\omega \times (\omega^\omega)^2$, i.e., $Q(x,y,z) \wedge Q(x,y',z') \implies y = y' \wedge z = z'$. Then, each set $U_m = \mathtt{dom}_2\, Q_m$ is still $\mathbf{\Sigma}^1_n$ and is uniform in $\omega^\omega \times \omega^\omega$ by the uniformity of Q_m. On the other hand, $U \subseteq \bigcup_m U_m$ by construction, which contradicts Lemma 3.

Finally, let $P = \{\langle x, H(y,z)\rangle : Q(x,y,z)\}$, where $H : (\omega^\omega)^2 \xrightarrow{\text{onto}} \omega^\omega$ is an arbitrary homeomorphism. Then, P witnesses (b) of Theorem 1. □

6. Alternative Proofs of the Main Results

This section contains alternative, shorter, and more transparent proofs of Theorem 1, suggested by an anonymous reviewer and presented here with their recommendation. We may note that these proofs also imply somewhat stronger results than the original ones; see Remarks 1 and 2 below.

Alternative Proof of Theorem 1(a), case $n \geq 3$ and $\mathbf{V} = \mathbf{L}$. Let $I_z = \{y \in \omega^\omega : y <_{\mathbf{L}} z\}$ for $z \in \omega^\omega$. Consider a $\mathbf{\Sigma}^1_n$ set $X \subseteq \omega^\omega$. Then, X belongs to $\Sigma^1_n(a)$ for some $a \in \omega^\omega$. Assume that X is in fact Σ^1_n; the general case does not differ. Then,

$$X = \{x : \exists\, y\, \forall\, z\, C(x,y,z)\},$$

where $C \subseteq (\omega^\omega)^3$ is Σ^1_{n-2}. Now, let $\Phi(x,y,F)$ be the conjunction of the following:

(A) $y \in \omega^\omega$ and $F : I_y \to \omega^\omega$;
(B) $\forall\, z\, C(x,y,z)$;
(C) $\forall\, y' <_{\mathbf{L}} y\, \neg\, C(x,y',F(y'))$;
(D) $\forall\, y' <_{\mathbf{L}} y\, \forall\, z <_{\mathbf{L}} F(y')\, C(x,y',z)$.

Lemma 5. *If* $x \in \omega^\omega$*, then* $x \in X \iff \exists\, y\, \exists\, F\, \Phi(x, y, F)$*.*

Proof. Indeed, if $x \in X$, then let y_x be the $<_{\mathbf{L}}$-least y $\forall\, z\, C(x, y, z)$, and then if $y' <_{\mathbf{L}} y_x$, then $\neg \forall\, z\, C(x, y', z)$; hence, let $F_x(y')$ be the $<_{\mathbf{L}}$-least z with $\neg\, C(x, y', z)$. Thus, we have $\Phi(x, y_x, F_x)$. Conversely, if $\Phi(x, y, F)$, then $x \in X$ by (B). □

Lemma 6. *If* $x \in X$*, then* $\langle y_x, F_x \rangle$ *is a unique pair satisfying* $\Phi(x, y_x, F_x)$*.*

Proof. Assume that some $\langle y, F \rangle$ satisfies $\Phi(x, y, F)$. If $y <_{\mathbf{L}} y_x$, then (B) for y is outright impossible by the $<_{\mathbf{L}}$-minimality of y_x. If $y_x <_{\mathbf{L}} y$, then $z = F(y_x)$ satisfies $\neg\, C(x, y_x, z)$ by (C), contrary to (B) for y_x. Thus, $y = y_x$.

To prove $F = F_x$, let $y' <_{\mathbf{L}} y = y_x$; show that $F(y') = F_x(y')$. If $z = F_x(y') <_{\mathbf{L}} F(y')$, then $C(x, y', z)$ holds by (D), i.e., $C(x, y', F_x(y'))$, which contradicts (D) for y_x and F_x. The case $F(y') <_{\mathbf{L}} F_x(y')$ leads to a contradiction in a similar manner. □

It follows from the lemma that X is equal to the projection of a uniform set

$$B = \{ \langle x, \langle y, F \rangle \rangle : \Phi(x, y, F) \}.$$

To replace B with a Π^1_{n-1} set with the same projection, let Q be the set of all tuples $\langle x, y, f, h \rangle \in (\omega^\omega)^4$ satisfying the following five properties (I)–(V):

(I) $\forall\, z\, C(x, y, z)$;

(II) (a) If $1 \le k < j$ and $(f)_k = (f)_j$, then $(h)_k = (h)_j$;
(b) The set $S_f := \{ (f)_k : k \ge 1 \} \smallsetminus \{ (f)_0 \}$ is equal to I_y (see Definition 2 on $(f)_k$; we remove $(f)_0$ here to take care of the case when S_f has to be the empty set) ;

(III) $\forall\, k \ge 1\, \big((f)_k \ne (f)_0 \implies \neg\, C(x, (f)_k, (h)_k) \big)$—compared to (C)—class Π^1_{n-2};

(IV) $\forall\, k \ge 1\, \forall\, z <_{\mathbf{L}} (h)_k\, \big((f)_k \ne (f)_0 \implies C(x, (f)_k, z) \big)$—compared to (D)—class Π^1_{n-2};

(V) $\langle f, h \rangle$ is the $<_{\mathbf{L}}$-least pair satisfying (II), (III), and (IV) for given x, y.

Lemma 7. Q *is* Π^1_{n-1}*.*

Proof. (II)(b) is Δ^1_2 by Proposition 4(ii); hence, the whole conjunction (II)∧ (III)∧(IV) is Δ^1_{n-1}. Therefore, (V) is Δ^1_{n-1} as well also by Proposition 4(ii). We conclude that the whole conjunction of (I)–(V) is Π^1_{n-1}, and such is the set Q. □

Lemma 8. *If* $x \in \omega^\omega$*, then* $x \in X \iff \exists\, y\, \exists\, f\, \exists\, h\, Q(x, y, f, h)$*. Moreover, if* $x \in X$*, then there is a unique triple of* $y, f,$ *and* h *with* $Q(x, y, f, h)$*.*

Proof. Assume that $x \in X$. By Lemma 6, there is a unique pair of y and F satisfying $P(x, y, F)$. Take any $f \in \omega^\omega$ satisfying (II)(b). Define $h \in \omega^\omega$ such that $(h)_k = F((f)_k)$, $\forall\, k$. Then, (II)(b) holds, and (III) and (IV) follow from, resp. (C) and (D) so that $\langle f, h \rangle$ satisfies (II), (III), and (IV). We can assume that $\langle f, h \rangle$ is the $<_{\mathbf{L}}$-least such pair, which yields $Q(x, y, f, h)$.

Conversely, suppose $Q(x, y, f, h)$. If $k \ge 1$ and $y' = (f)_k \in I_y$ by (II)(b), then set $F(y') = (h)_k$; this is consistent with (II)(a). Items (C) and (D) follow from, resp. (III) and (IV); hence, we have $U(x, y, F)$, and furthermore, $y = y_x$ by Lemma 6. We complete the proof of the uniqueness claim by referring to (V). □

Thus, Q is a Π^1_{n-1} set by Lemma 7, uniform in the sense of $\omega^\omega \times (\omega^\omega)^3$ by Lemma 8, and its projection is equal to X by Lemma 8. It remains to obtain a set $P \subseteq (\omega^\omega)^2$ with the same properties via any recursive homeomorphism $H : (\omega^\omega)^3 \xrightarrow{\text{onto}} \omega^\omega$. □

Remark 1. We may note that the alternative proof gives a stronger effective result than Claim (a) of Theorem 1. Namely, *under the assumptions of the theorem, any lightface Σ^1_n set $X \subseteq \omega^\omega$ is the projection of a uniform lightface Π^1_{n-1} set $P \subseteq (\omega^\omega)^2$, and the same is true for the lightface classes $\Sigma^1_n(a)$ and $\Pi^1_{n-1}(a)$ for any parameter $a \in \omega^\omega$.*

Alternative Proof of Claim (b) of Theorem 1. This proof deviates from the proof given in Section 5 Lemma 4, which is established differently. The main ingredient of the proof is the following proposition. (We refer to ([3], [5A.3]) in case $\mathbf{V} = \mathbf{L}$, and to ([3], [4B.3]) in case $n = 2$.)

Proposition 5. *If $n = 2$ or $\mathbf{V} = \mathbf{L}$ and $n \geq 3$, then the pre-well-ordering property holds for Σ^1_n, meaning that for any Σ^1_1 set W, there is a map $\varphi : W \to \omega_1$ such that the relations*

$$a \leq^*_\varphi b \quad \textit{iff} \quad a \in W \wedge \big(b \in W \implies \varphi(a) \leq \varphi(b)\big);$$
$$a <^*_\varphi b \quad \textit{iff} \quad a \in W \wedge \big(b \in W \implies \varphi(a) < \varphi(b)\big)$$

are both Σ^1_n-definable.

Let $W = \{\langle m, x, k\rangle \in \omega \times \omega^\omega \times \omega : \vartheta(m, x, k)\}$ be a universal Σ^1_n set as in Proposition 3(ii), where $\vartheta(m, x, k)$ is a universal Σ^1_n formula, as in Section 5.

Consider the set $U = U[n] \subseteq \omega^\omega \times 2^\omega$ introduced by Definition 3.

Alternative Proof of Lemma 4. Let, by Proposition 5, $\varphi : W \to \omega_1$ be a map such that the relations $\langle m, x, k\rangle \leq^*_\varphi \langle m', x', k'\rangle$ and $\langle m, x, k\rangle <^*_\varphi \langle m', x', k'\rangle$ are Σ^1_n.

Let $\langle x, a\rangle \in \omega^\omega \times 2^\omega$. **We claim** that $\langle x, a\rangle \in U$ is equivalent to the following formula:

$$\exists\, m\, \forall\, k\, \big[a(k) = 1 \implies W(m, x, k) \wedge \forall\, \ell\, \big(a(\ell) = 0 \implies \langle m, x, k\rangle <^*_\varphi \langle m, x, \ell\rangle\big)\big]. \tag{3}$$

Indeed, assume that $\langle x, a\rangle \in U$. By definition, this means that $a \in 2^\omega$, and for some m_0, we have got $a(k) = 1 \iff W(m_0, x, k)$ for all k. Now, if $a(\ell) = 0$, then $\langle m_0, x, \ell\rangle \notin W$; hence, $\langle m_0, x, k\rangle <^*_\varphi \langle m_0, x, \ell\rangle$ by the definition of $<^*_\varphi$, so (3) holds for $m = m_0$.

To prove the converse, assume that (3) holds for some $m = m_0$. Let us show that $\langle x, a\rangle \in U$. It suffices to prove that $a(k) = 1 \iff W(m_0, x, k)$ for all k. Suppose to the contrary that this is not the case. Then, as $a(k) = 1 \implies W(m_0, x, k)$ by (3), there are numbers k such that $a(k) = 0$, but $W(m_0, x, k)$ holds—let us call such numbers k "bad". Let k_0 be such a "bad" k for which the value $\varphi(m_0, x, k)$ is the least possible. We assert that

$$\forall\, k\, \big(a(k) = 1 \iff \langle m_0, x, k\rangle <^*_\varphi \langle m_0, x, k_0\rangle\big). \tag{4}$$

Indeed, if $a(k) = 1$, then we have $\langle m_0, x, k\rangle <^*_\varphi \langle m_0, x, k_0\rangle$ by (3) with $\ell = k_0$. Conversely, assume that (**) $\langle m_0, x, k\rangle <^*_\varphi \langle m_0, x, k_0\rangle$. However, $W(m_0, x, k_0)$ holds by the choice of k_0. Therefore, we have $W(m_0, x, k)$ as well by the definition of $<^*_\varphi$. Then, $a(k) = 1$, since if $a(k) = 0$, then k is "bad", so $\langle m_0, x, k_0\rangle \leq^*_\varphi \langle m_0, x, k\rangle$ by the choice of k_0, contrary to assumption (**). This ends the proof of (4).

Yet, it follows from (4) and the Σ^1_n definability of $<^*_\varphi$ that the set $\{k\}a(k) = 1$ is Σ^1_n as well, and hence, $\langle x, a\rangle \in U$. This completes the proof of the **claim** above. In other words, U is defined by formula (3). However, (3) is Σ^1_n since so are both W and the relation $<^*_\varphi$. We conclude that U is Σ^1_n, and this completes the alternative proof of Lemma 4. □

Given Lemma 4, the rest of the alternative proof of Claim (b) of Theorem 1 is finalized exactly as in the end of Section 5. □

Remark 2. Similarly to Remark 1, the alternative proof gives a stronger effective result than Claim (b) of Theorem 1. Namely, *under the assumptions of the theorem, there is a lightface* $\mathit{\Pi}^1_{n-1}$ *set* $P \subseteq (\omega^\omega)^2$ *with countable cross-sections* **not** *covered by a union of countably many uniform* $\mathbf{\Sigma}^1_n$ *sets.*

7. Conclusions and Problems

In this study, methods of effective descriptive set theory and constructibility theory are employed to obtain the solution of two old problems of classical descriptive set theory raised by Luzin in 1930, under the assumption of the axiom of constructibility $\mathbf{V} = \mathbf{L}$ (Theorem 1). In addition, we established Corollary 1, an ensuing consistency and independence result. These are new results, and they make a significant contribution to descriptive set theory in the constructible universe. The technique developed in this paper may lead to further progress in studies on different aspects of the projective hierarchy under the axiom of constructibility.

The following problems arise from our study.

Problem 3. Find a "classical" proof of Theorem 1(b) in case $n = 2$ without any reference to "effective" descriptive set theory.

Problem 4. Instead of the set $U = U[n]$ as in Definition 3, one may want to consider a somewhat simpler set $U'[n] = \{\langle x, f\rangle \in (\omega^\omega)^2 : f \text{ is } \mathit{\Delta}^1_n(x)\}$. Does it prove Theorem 1(b)?

Problem 5. Find a model of **ZFC** in which Problem 2 in Section 1 is solved in the positive, at least in the following form: for a given $n \geq 3$, every $\mathbf{\Pi}^1_{n-1}$ set $P \subseteq (\omega^\omega)^2$ with countable cross-sections is covered by a union of countably many uniform $\mathbf{\Sigma}^1_n$ sets.

Accordingly, find a model of **ZFC** in which, for a given $n \geq 3$, there exists a $\mathbf{\Sigma}^1_{n-1}$ set $X \subseteq \omega^\omega$ not equal to the projection of a uniform $\mathbf{\Pi}^1_n$ set $P \subseteq (\omega^\omega)^2$.

As for Problem 5, we hope that it can be solved with the method of definable generic forcing notions introduced by Harrington [22,23]. This method has been recently applied for some definability problems in modern set theory, including the following applications:

- A generic model of **ZFC**, with a Groszek–Laver pair (see [24]) that consists of two OD-indistinguishable E_0 classes $X \neq Y$, whose union $X \cup Y$ is a $\mathit{\Pi}^1_2$ set, in [25];
- A generic model of **ZFC**, in which, for a given $n \geq 3$, there is a $\mathit{\Delta}^1_n$ real coding the collapse of $\omega_1^{\mathbf{L}}$, whereas all $\mathit{\Delta}^1_n$ reals are constructible, in [26];
- A generic model of **ZFC** that solves the Alfred Tarski [27] 'definability of definable' problem, in [28].

We hope that this study of generic models will contribute to the solution of the following well-known problem by S. D. Friedman (see ([29], [p. 209]) and ([30], [p. 602])): *find a model of* **ZFC***, for a given* n*, in which all* $\mathbf{\Sigma}^1_n$ *sets of reals are Lebesgue measurable and have the Baire and perfect set properties, and at the same time, there is a* $\mathbf{\Delta}^1_{n+1}$ *well-ordering of the reals*.

We also hope that this research can be useful in creating algorithms or computational algorithmic models that represent the evolution of cell types and are related to the storage and processing of genomic information.

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References

1. Gödel, K. *The Consistency of the Continuum Hypothesis*; Annals of Mathematics Studies, No. 3; Princeton University Press: Princeton, NJ, USA, 1940. [CrossRef]
2. Kechris, A.S. *Classical Descriptive Set Theory*; Springer: New York, NY, USA, 1995; pp. xviii+402.
3. Moschovakis, Y.N. Descriptive Set Theory. In *Studies in Logic and the Foundations of Mathematics*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1980; Volume 100, pp. xii+637. [CrossRef]
4. Lusin, N. Sur les ensembles analytiques. *Fund. Math.* **1927**, *10*, 1–95. [CrossRef]
5. Lusin, N. *Leçons Sur les Ensembles Analytiques et Leurs Applications*; Gauthier-Villars: Paris, France, 1930; pp. XVI+328.
6. Lusin, N.; Novikoff, P. Choix effectif d'un point dans un complémentaire analytique arbitraire, donné par un crible. *Fundam. Math.* **1935**, *25*, 559–560. [CrossRef]
7. Kondô, M. L'uniformisation des complémentaires analytiques. *Proc. Imp. Acad.* **1937**, *13*, 287–291. [CrossRef]
8. Solovay, R.M. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math.* **1970**, *92*, 1–56. [CrossRef]
9. Kanovei, V. An Ulm-type classification theorem for equivalence relations in Solovay model. *J. Symb. Log.* **1997**, *62*, 1333–1351. accessed on 31 July 2024. [CrossRef]
10. Kanovei, V.; Lyubetsky, V. Counterexamples to countable-section $\mathit{\Pi}^1_2$ uniformization and $\mathit{\Pi}^1_3$ separation. *Ann. Pure Appl. Log.* **2016**, *167*, 262–283. [CrossRef]
11. Kanovei, V.; Lyubetsky, V. Non-uniformizable sets of second projective level with countable cross-sections in the form of Vitali classes. *Izv. Math.* **2018**, *82*, 61–90. [CrossRef]
12. Kanovei, V.; Lyubetsky, V. On the significance of parameters and the projective level in the Choice and Comprehension axioms. *arXiv* **2024**, arXiv:2407.20098.
13. Harrington, L. Π^1_2 sets and Π^1_2 singletons. *Proc. Am. Math. Soc.* **1975**, *52*, 356–360. [CrossRef]
14. Novikoff, P. Sur les fonctions implicites mesurables B. *Fundam. Math.* **1931**, *17*, 8–25. [CrossRef]
15. Jech, T. *Set Theory*, The Third Millennium Revised and Expanded ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2003; pp. xiii+772. [CrossRef]
16. Barwise, J. (Ed.) *Handbook of Mathematical Logic*; Elsevier: Amsterdam, The Netherlands, 1978; Volume 90.
17. Devlin, K.J. *Constructibility*; Springer: Berlin/Heidelberg, Germany, 1984.
18. Addison, J.W. Separation principles in the hierarchies of classical and effective descriptive set theory. *Fundam. Math.* **1959**, *46*, 123–135. [CrossRef]
19. Addison, J.W. Some consequences of the axiom of constructibility. *Fundam. Math.* **1959**, *46*, 337–357. [CrossRef]
20. Simpson, S.G. *Subsystems of Second Order Arithmetic*, 2nd ed.; Cambridge University Press: Cambridge, UK; ASL: Urbana, IL, USA, 2009; pp. xvi+444.
21. Shoenfield, J.R. The problem of predicativity. In *Essays on the Foundation of Mathematics*; Bar-Hillel, Y., Poznanski, E.I.J., Rabin, M.O., Robinson A., Eds.; North-Holland: Amsterdam, The Netherlands, 1962; pp. 132–139.
22. Harrington, L. The Constructible Reals Can Be Anything. Preprint Dated May 1974 with Several Addenda Dated up to October 1975: (A1) Models Where Separation Principles Fail, May 74; (A2) Separation Without Reduction, April 75; (A3) The constructible Reals Can Be (Almost) Anything, Part II, May 75. Available online: http://iitp.ru/upload/userpage/247/74harr.pdf (accessed on 31 July 2024).
23. Harrington, L. Long projective wellorderings. *Ann. Math. Log.* **1977**, *12*, 1–24. [CrossRef]
24. Groszek, M.; Laver, R. Finite groups of OD-conjugates. *Period. Math. Hung.* **1987**, *18*, 87–97. [CrossRef]
25. Golshani, M.; Kanovei, V.; Lyubetsky, V. A Groszek—Laver pair of undistinguishable E_0 classes. *Math. Log. Q.* **2017**, *63*, 19–31. [CrossRef]
26. Kanovei, V.; Lyubetsky, V. Definable minimal collapse functions at arbitrary projective levels. *J. Symb. Log.* **2019**, *84*, 266–289. [CrossRef]
27. Tarski, A. A problem concerning the notion of definability. *J. Symb. Log.* **1948**, *13*, 107–111. [CrossRef]

28. Kanovei, V.; Lyubetsky, V. On the 'definability of definable' problem of Alfred Tarski. II. *Trans. Am. Math. Soc.* **2022**, *375*, 8651–8686. [CrossRef]
29. Friedman, S.D. Fine Structure and Class Forcing. In *De Gruyter Series in Logic and Its Applications*; de Gruyter: Berlin, Germany, 2000; Volume 3, pp. x+ 222. [CrossRef]
30. Friedman, S.D. Constructibility and class forcing. In *Handbook of Set Theory*; Springer: Dordrecht, The Netherlands, 2010; Volume 3, pp. 557–604. [CrossRef]