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A theorem on ROD-hypersmooth equivalence relations in the Solovay model

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It is known that every Borel hypersmooth but non-smooth equivalence relation is Borel bi-reducible to E_1. We prove a ROD version of this result in the Solovay model.

1 Introduction

It is known since [5] that classical theorems on Borel and analytic sets tend to generalize to all projective, generally, all real-ordinal definable (ROD) sets in the Solovay model. In particular, as one of the authors demonstrated in [2], the fundamental theorem of Glimm-Effros classification for Borel equivalence relations admits such a generalization (although not straightforward). In this note we prove the following theorem:

Theorem 1 (Main Theorem) *In the Solovay model, if* E *is a ROD-hypersmooth equivalence relation, then either* $\mathsf{E} \le_{\rm ROD} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\rm ROD} \mathsf{E}_1$. *The two cases are incompatible.*

This is a partial generalization of a fundamental result on the Borel reducibility, saying that any Borel hypersmooth equivalence relation E satisfies either $\mathsf{E} \le_{\rm B} \mathsf{E}_0$ or $\mathsf{E} \sim_{\rm B} \mathsf{E}_1$ (Theorem 2.1 in [4], also known as "the third dichotomy theorem"). The generalization is not complete: due to a simple counterexample, we cannot claim that E is ROD-hyperfinite in the "or" case.

2 Notation

ROD means: *real-ordinal-definable*. $\mathrm{OD}(p)$ means: *ordinal-definable in a real* p, i. e, definable with p and any ordinals as parameters.

We consider ROD equivalence relations on (also ROD) sets. If E, F are ROD equivalence relations on sets X, Y, respectively, then, by analogy with the Borel reducibility, $\mathsf{E} \le_{\rm ROD} \mathsf{F}$ means that there exists a ROD map $\vartheta : X \longrightarrow Y$ such that $x \mathsf{E} x'$ iff $\vartheta(x) \mathsf{F} \vartheta(x')$. (In principle, it is not assumed here that X, Y carry any topological or other structure.) As usual, $\mathsf{E} \sim_{\rm ROD} \mathsf{F}$ iff $\mathsf{E} \le_{\rm ROD} \mathsf{F}$ and $\mathsf{F} \le_{\rm ROD} \mathsf{E}$ (ROD bi-reducibility), while $\mathsf{E} <_{\rm ROD} \mathsf{F}$ iff $\mathsf{E} \le_{\rm ROD} \mathsf{F}$ but $\mathsf{F} \not\le_{\rm ROD} \mathsf{E}$ (strict ROD-reducibility).

An equivalence relation E on X is *ROD-finite* iff it is ROD and every E-class $[x]_{\mathsf{E}} = \{y : x \mathsf{E} y\}$, $x \in X$, is finite. A *ROD-hyperfinite* equivalence relation is any one of the form $\bigcup_n \mathsf{E}_n$, where $\{\mathsf{E}_n\}_{n \in \mathbb{N}}$ is an increasing chain of ROD-finite equivalence relations.

An equivalence relation E on a set X is *ROD-smooth* iff $\mathsf{E} \le_{\rm ROD} \mathsf{D}(2^{\mathbb{N}})$, i. e., there is a ROD map $\vartheta : X \longrightarrow 2^{\mathbb{N}}$ such that $x \mathsf{E} y$ iff $\vartheta(x) = \vartheta(y)$. A *ROD-hypersmooth* equivalence relation is an increasing union of ROD-smooth equivalence relations. Obviously all ROD-hyperfinite and all ROD-hypersmooth equivalence relations are ROD.

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Recall that E_0 is an equivalence relation on $2^{\mathbb{N}}$ defined as follows: $x\mathsf{E}_0 y$ iff $x_n = y_n$ for almost all n: here we assume that $x = \{x_n\}_{n\in\mathbb{N}}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ belong to $2^{\mathbb{N}}$. This is a ROD-hyperfinite, moreover, Borel-hyperfinite equivalence relation. Further, E_1 is an equivalence relation on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ defined similarly, i. e, $x\mathsf{E}_1 y$ iff $x_n = y_n$ for almost all n. E_1 is a typical example of a ROD-hypersmooth equivalence relation, indeed, even Borel-hypersmooth equivalence relation.

Lemma 2 *An equivalence relation* E *is ROD-hypersmooth iff* $\mathsf{E} \le_{\mathrm{ROD}} \mathsf{E}_1$.

P r o o f. Similar to the Borel case, see [4, 1.3] for the nontrivial direction. □

By *the Solovay model* we mean a $\mathbb{P}^{\Omega}$-generic extension of L, the constructible universe[1)], where Ω is an inaccessible cardinal in L. $\mathbb{P}^{\Omega} = \prod_{\gamma<\Omega}\mathbb{P}_\gamma$ (the product with finite support), and $\mathbb{P}_\gamma = \gamma^{<\omega} = \bigcup_n \gamma^n$ for every $\gamma < \Omega$.

Assume that $\gamma < \Omega$. Let $\mathbb{T}_\gamma[p]$ be the set of all *terms* $t = \langle \gamma, \{t_n\}_{n\in\mathbb{N}}\rangle \in \mathrm{L}[p]$, where $t_n \subseteq \mathbb{P}_\gamma$ for all n. If $f \in \gamma^{\mathbb{N}}$ (an infinite sequence), then let $t[f] = \{n : \exists m\,(f \restriction m \in t_n)\}$.

Let $\mathbb{F}_\gamma[p]$ be the set of all over $\mathrm{L}[p]$ $\mathbb{P}_\gamma$-generic functions $f \in \gamma^{\mathbb{N}}$. Put $t[w] = \{t[f] : w \subset f \in \mathbb{F}_\gamma[p]\}$ for any $w \in \mathbb{P}_\gamma$ and $t \in \mathbb{T}_\gamma$. The following result is established, e. g, in [2, Proposition 5].

Proposition 3 (In the Solovay model) *Let* p *be a real. Then*

(i) *If* $\emptyset \ne X \subseteq \mathcal{P}(\mathbb{N})$ *is* $\mathrm{OD}(p)$*, then there exist* $\gamma < \Omega$*,* $w \in \mathbb{P}_\gamma$*, and* $t \in \mathbb{T}_\gamma[p]$ *such that* $t[w] \subseteq X$.

(ii) *If* $\gamma < \Omega$*,* $w \in \mathbb{P}_\gamma$*, and* $\emptyset \ne X \subseteq t[w]$ *is* $\mathrm{OD}(p)$*, then there exists* $w' \in \mathbb{P}_\gamma$ *such that* $w \subset w'$ *and* $t[w'] \subseteq X$. □

3 Incompatibility in the main theorem

It suffices to show that $\mathsf{E}_1 \not\le_{\mathrm{ROD}} \mathsf{E}_0$ in the Solovay model. The proof that $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{E}_0$, moreover, $\mathsf{E}_1 \not\le_{\mathrm{B}} \mathsf{F}$ for any countable Borel equivalence relation F in [4, 1.4 and 1.5] actually gives non-reducibility even via *Baire measurable* functions, i. e, those continuous on a dense $\mathbf{G}_\delta$ set. However it is known (see [5]) that in the Solovay model any ROD function is Baire measurable.

4 The partition into cases

This section begins the essential part of the proof of Theorem 1.

We argue in the Solovay model.

Let E be a ROD equivalence relation on a set X. Suppose that E is ROD-hypersmooth. We have $\mathsf{E} \le_{\mathrm{ROD}} \mathsf{E}_1$ by Lemma 2. Let this be witnessed by a ROD map $\vartheta : X \longrightarrow \mathcal{P}(\mathbb{N})^{\mathbb{N}}$. We put $P = \operatorname{ran}\vartheta$, the full image of ϑ. This is still a ROD set, hence, there is a real p such that P is $\mathrm{OD}(p)$.

The real p *is fixed until the end of the proof.*

To define the partition into two cases, we need the following notation. If $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$, then $x\restriction_{\ge n}$ is the restriction of x (a function defined on $\mathbb{N}$) to the domain $[n, \infty)$. If $X \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$, then let $X\restriction_{\ge n} = \{x\restriction_{\ge n} : x \in X\}$. Define $x\restriction_{>n}$ and $X\restriction_{>n}$ similarly. In particular, $\mathcal{P}(\mathbb{N})^{\mathbb{N}}\restriction_{\ge n} = \mathcal{P}(\mathbb{N})^{\ge n} = \mathcal{P}(\mathbb{N})^{[n,\infty)}$. For a sequence $x \in \mathcal{P}(\mathbb{N})^{\ge n}$ let $\operatorname{dep} x$ (the *depth of* x) be the number (finite or ∞) of elements of the set

$$J(x) = \{j \ge n : x(j) \notin \mathrm{OD}(p, x\restriction_{>j})\}.$$

Recall that, in the Solovay model, $x \in \mathrm{OD}(y)$ iff $x \in \mathrm{L}[y]$ for any two reals x, y.

C a s e 1. All $x \in P = \operatorname{ran}\rho$ satisfy $\operatorname{dep} x < \infty$.

C a s e 2. There exist $x \in P$ with $\operatorname{dep} x = \infty$.

The content of the remainder will be to prove $\mathsf{E} \le_{\mathrm{ROD}} \mathsf{E}_0$ in Case 1 and $\mathsf{E}_1 \le_{\mathrm{ROD}} \mathsf{E}$ in Case 2.

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1) Theorem 1 is true, with some rather clear adjustments of the proof, for the Solovay extensions not necessarily of the constructible universe.

4.1 Case 1

As obviously $\mathsf{E} \le_{\rm ROD} \mathsf{E}_1 \restriction P$, it suffices to show that $\mathsf{E}_1 \restriction P \le_{\rm ROD} \mathsf{E}_0$.

Suppose that $x \in P$. If $\operatorname{dep} x = \emptyset$, then let $f(x) = x$. If $\operatorname{dep} x \neq \emptyset$, then (as $\operatorname{dep} x$ is finite) let n_x be the largest $n \in \operatorname{dep} x$. Define $f(x) = y \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ so that $x\restriction_{>n_x} = y\restriction_{>n_x}$ while $y(j) = \emptyset$ for all $j \le n_x$. Easily f is a ROD reduction of $\mathsf{E}_1 \restriction P$ to $\mathsf{E}_1 \restriction Q$, where $Q = \operatorname{ran} f$, thus, it suffices to show that $\mathsf{E}_1 \restriction Q \le_{\rm ROD} \mathsf{E}_0$. The set Q belongs to $\mathrm{OD}(p)$ together with P.

Note that by definition any point $x \in Q$ satisfies $\operatorname{dep} x = \emptyset$, so that $x(n) \in \mathrm{OD}(p, x\restriction_{>n})$ for any $n \in \mathbb{N}$ and $x \in Q$. It follows that $x(n) \in \mathrm{L}[p, x\restriction_{>n}]$ for any $n \in \mathbb{N}$ and $x \in Q$, by known properties of the Solovay model. In other words, $Q \subseteq T = \{x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}} : \forall n\, (x(n) \in \mathrm{L}[p, x\restriction_{>n}])\}$, hence, it suffices to prove that $\mathsf{E}_1 \restriction T \le_{\rm ROD} \mathsf{E}_0$. Note that T is $\mathrm{OD}(p)$.

Fix $x \in T$. For any $n \in \mathbb{N}$ let $\xi_n(x)$ be the order of $x(n)$ in the sense of the canonical well-ordering of $\mathcal{P}(\mathbb{N}) \cap \mathrm{L}[p, x\restriction_{>n}]$; then $\xi_n(x) < \omega_1^{\mathrm{L}[p,x\restriction_{>n}]}$. Note that still $\xi(x) = \sup_n \xi_n(x) < \omega_1^{\mathrm{L}[p,x]}$, because the map $n \longmapsto \xi_n(x)$ is $\mathrm{OD}(p,x)$. Now define $\mu(x) = \inf\{\xi(y) : y \in T \wedge y\mathsf{E}_1 x\}$. This is E_1-invariant, i. e, $\mu(x) = \mu(y)$ whenever $x, y \in T$ and $x\mathsf{E}_1 y$, moreover, $\mu(x) < \omega_1^{\mathrm{L}[p,x]}$.

Let $W = \{x \in T : \xi(x) = \mu(x)\}$. This is an $\mathrm{OD}(p)$ subset of T, and there is a ROD reduction of $\mathsf{E}_1 \restriction T$ to $\mathsf{E}_1 \restriction W$. (Indeed: Let $x \in T$. By definition there is m such that $\xi_j(x) \le \mu(x)$ for all $j \ge m$; let m_x be the least of such numbers m. Define $y = g(x) \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ so that $x\restriction_{>m_x} = y\restriction_{>m_x}$ while $y(j) = \emptyset$ for all $j > m_x$. Then $y \in W$, under the natural assumption that $\emptyset$ has order 0 in any relevant well-ordering, and $y\mathsf{E}_1 x$. Thus, g is a ROD reduction of $\mathsf{E}_1 \restriction T$ to $\mathsf{E}_1 \restriction W$.) It suffices to prove that $\mathsf{E}_1 \restriction W \le_{\rm ROD} \mathsf{E}_0$.

By definition, $\xi_n(x) \le \mu(x) < \omega_1^{\mathrm{L}[p,x]}$ for all $x \in W$ and $n \in \mathbb{N}$, hence, if $a \in W\restriction_{>n}$, then the set $S_W(a) = \{x(n) : x \in W \wedge a = x\restriction_{>n}\} \subseteq \mathrm{L}[p,x]$ is countable in $\mathrm{L}[p,x] = \mathrm{L}[p,a]$. Thus there exists an $\mathrm{OD}(p)$ map F with $S_W(a) = \{F(a,k) : k \in \mathbb{N}\}$ whenever $a \in A = \bigcup_{n\in\mathbb{N}} W\restriction_{>n}$. Assuming w. l. o. g. that $\mu(x) \ge \omega$ for any x. All sets $S_W(a)$, $a \in A$, are strictly countable, hence, we can assume that for any $a \in A$ the partial map $F_a(k) = F(a,k)$ is a bijection of $\mathbb{N}$ onto $S_W(a)$. Then for any $x \in W$ and n there is a unique $k = \kappa_n(x)$ such that $x(n) = F(x\restriction_{>n}, k)$. Let $\kappa(x) = \{\kappa_n(x)\}_{n\in\mathbb{N}}$. Note that if $x \neq y \in W$ and $x\mathsf{E}_1 y$, then $\kappa(x) \neq \kappa(y)$.

The next step is to uniformly define an ordering of any set of the form $[x]_{\mathsf{E}_1} \cap W$, $x \in W$, similar to $\mathbb{Z}$. Define $\sigma_n(x) = \max\{n, \max_{j\le n} \kappa_j(x)\}$ for all $x \in W$ and n. Define the infinite sequence

$$\sigma(x) = \langle \kappa_0(x), \sigma_0(x), \kappa_1(x), \sigma_1(x), \ldots, \kappa_n(x), \sigma_n(x), \ldots \rangle$$

of natural numbers. Easily if $x, y \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ satisfy $x\mathsf{E}_1 y$, i. e., $x\restriction_{>n} = y\restriction_{>n}$ for some n, then still $\sigma(x)\mathsf{E}_0\sigma(y)$, i. e., $\sigma(x)\restriction_{>k} = \sigma(y)\restriction_{>k}$ for some $k \ge n$. Define, for $x, y \in W$, $x <_0 y$ iff $\sigma(x) <_{\rm alex} \sigma(y)$ (the *anti-lexicographical ordering*), meaning that $\sigma_k(x) < \sigma_k(y)$, where k is the least number such that $\sigma(x)\restriction_{>k} = \sigma(y)\restriction_{>k}$. Easily $<_{\rm alex}$ orders any E_0-class of an element of $\mathbb{N} \times \mathbb{N}$ similarly to $\mathbb{Z}$, with the only exception of the E_0-class of the constant 0 which is ordered similarly to $\mathbb{N}$. It follows that any E_1-class $[x]_{\mathsf{E}_1} \cap W$ of $x \in W$ is ordered by $<_0$ similarly to either $\mathbb{Z}$ or $\mathbb{N}$. As a matter of fact, any class ordered similarly to $\mathbb{N}$ can be rearranged, in some trivial manner, to that its order is now $\mathbb{Z}$ instead of $\mathbb{N}$. This way we obtain an $\mathrm{OD}(p)$ binary relation $<_0$ which orders every set of the form $[x]_{\mathsf{E}_1} \cap W$, $x \in W$, similarly to $\mathbb{Z}$. In other words, we have defined an $\mathrm{OD}(p)$ action of $\mathbb{Z}$ on W whose orbits are exactly E_1-classes $[x]_{\mathsf{E}_1} \cap W$, $x \in W$.

The rest of the argument involves a construction given in [1]. For any $x \in W$ define $\zeta(x) \in W^{\mathbb{Z}}$ so that $\zeta(x)(0) = x$ and, for any $c \in \mathbb{Z}$, $\zeta(x)(c+1)$ is the $<_0$-next element of $[x]_{\mathsf{E}_1} \cap W$ after $\zeta(x)(c)$. Thus, ζ is an $\mathrm{OD}(p)$ map $W \longrightarrow Z = (\mathcal{P}(\mathbb{N})^{\mathbb{N}})^{\mathbb{Z}}$. For $\zeta, \eta \in Z$ define $\zeta\mathsf{F}\eta$ iff there is an integer $z \in \mathbb{Z}$ such that $\zeta(c) = \eta(c+z)$ for all $c \in \mathbb{Z}$. Thus, F is the equivalence relation $\mathsf{E}(\mathbb{Z}, \mathcal{P}(\mathbb{N})^{\mathbb{N}})$ on $Z = (\mathcal{P}(\mathbb{N})^{\mathbb{N}})^{\mathbb{Z}}$, in the sense of [1].

The map ζ is obviously a reduction of $\mathsf{E}_1 \restriction W$ to F, hence, it suffices to show that $\mathsf{F} \le_{\rm ROD} \mathsf{E}_0$. But [1, 7.1] yields a stronger result: $\mathsf{F} \le_{\rm B} \mathsf{E}_0$. □ Case 1

Unlike the Borel case (see the implication (3) ⇒ (1) in [1, Theorem 5.1]) we cannot claim here that E is ROD-hyperfinite. Indeed, arguing in the Solovay model, consider the set T of all $x \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ such that $x(n) \in \mathrm{L}[x\restriction_{>n}]$ for every n. (See above.) Then $\mathsf{E}_1 \restriction T$ is a countable and ROD-hypersmooth equivalence relation. But $\mathsf{E}_1 \restriction T$ is not ROD-hyperfinite! Indeed: Otherwise, for some $p \in \mathcal{P}(\mathbb{N})$, $\mathsf{E}_1 \restriction T$ is the union of an increasing countable sequence of finite equivalence relations, which (i. e., the sequence) is $\mathrm{OD}(p)$. Then, for any n and $a \in T\restriction_{>n}$ the set $S_T(a)$ evidently is $\mathrm{OD}(p,a)$-countable. Taking a to be the constant p, we get a contradiction, because then $S_T(a) = \mathrm{L}[p] \cap \mathcal{P}(\mathbb{N})$, and this cannot be $\mathrm{OD}(p)$-countable.

4.2 Case 2

Thus, assume that the $\mathrm{OD}(p)$ set $R = \{x \in P : \operatorname{dep} x = \infty\}$ is non-empty. Our goal is to define an $\mathrm{OD}(p)$ subset $X \subseteq R$ with $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$.

We continue to argue in the Solovay model.

We begin with a reduction to the case when $J(x) = \{n : x(n) \notin \mathrm{L}[p, x\restriction_{>n}]\}$ is equal to $\mathbb{N}$ for any $x \in R$. Fix, for any k, a recursive bijection $b_k : \mathcal{P}(\mathbb{N})^{k+1} \times \mathbb{N}^2 \xrightarrow{\text{onto}} \mathcal{P}(\mathbb{N})$. Now let $x \in R$. Then $J(x) \subseteq \mathbb{N}$ is infinite; let $J(x) = \{j_0, j_1, j_2, \dots\}$ in the increasing order. For any m, put

$$y(m) = b_{j_m - j_{m-1} - 1}(x \restriction (j_{m-1}, j_m], j_m, j_m - j_{m-1})$$

(with $j_{-1} = -1$ for $m = 0$). The map $x \longmapsto y$ is $\mathrm{OD}(p)$, $x \mathsf{E}_1 x'$ iff $y \mathsf{E}_1 y'$, and also $J(y) = \mathbb{N}$. This observation justifies to assume w.l.o.g. $J(x) = \mathbb{N}$ for any $x \in R$, that is, $x(n) \notin \mathrm{OD}(p, x\restriction_{>n})$ for any $x \in R$ and n.

The following construction uses the basic idea of [4, Theorem 2.1], in the form of a splitting construction developed in [3] for the study of "ill"-founded Sacks iterations. Fix a recursive map $\varphi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$, which assumes each value $m \in \mathbb{N}$ infinitely many times so that $\{\varphi(k) : k < n\}$ is an initial segment of $\mathbb{N}$ for any n. For any n and finite sequences $u, v \in 2^n$, let $\nu_\varphi[u, v] = \max\{\varphi(k) : k < n \wedge u(k) \ne v(k)\}$. Separately, $\varphi[u, u] = -1$ for any $u \in 2^{<\omega}$. We are going to define for each $u \in 2^{<\omega}$ a non-empty $\mathrm{OD}(p)$ subset $X_u \subseteq R$, so that

(i) if $u, v \in 2^n$, then (a) $X_u\restriction_{>\nu_\varphi[u,v]} = X_v\restriction_{>\nu_\varphi[u,v]}$ and (b) $X_u\restriction_{\ge\nu_\varphi[u,v]} \cap X_v\restriction_{\ge\nu_\varphi[u,v]} = \emptyset$;

(ii) $X_{u^\frown i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(iii) $\max_{u\in 2^n} \operatorname{diam} X_u \to 0$ as $n \to \infty$ (a reasonable Polish metric on $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is assumed to be fixed);

(iv) $\bigcap_n X_{a\restriction n} \ne \emptyset$ for any $a \in 2^{\mathbb{N}}$.

Let us demonstrate how such a system of sets accomplish Case 2. According to (iii) and (iv), for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a\restriction n}$ contains a single point, let it be $F(a)$, so that $F : 2^{\mathbb{N}} \longrightarrow \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is continuous and one-to-one.

Define a parallel system of sets Y_u, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = \mathcal{P}(\mathbb{N})^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$, and $\varphi(n) = j$. Let ℓ be the number of all indices $k < n$ satisfying $\varphi(k) = j$, perhaps $\ell = 0$. Put $Y_{u^\frown i} = \{x \in Y_u : x(j)(\ell) = i\}$ for $i = 0, 1$. Each of Y_u is clearly a basic clopen set in $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$, and one easily verifies that conditions (i) – (iv) are satisfied for the sets Y_u (instead of X_u, in particular, for any $a \in 2^{\mathbb{N}}$, the intersection $\bigcap_n Y_{a\restriction n} = \{G(a)\}$ is a singleton, and the map G is continuous and one-to-one. (We can define G explicitly: $G(a)(j)(\ell) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\varphi(n) = j$ and there is exactly ℓ numbers $k < n$ with $\varphi(k) = j$.) Note finally that $\{G(a) : a \in 2^{\mathbb{N}}\} = \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ since by definition $Y_{u^\frown 1} \cup Y_{u^\frown 0} = Y_u$.

We conclude that the map $\vartheta(x) = F(G^{-1}(x))$ is a continuous bijection, hence, a homeomorphism by the compactness of the spaces considered, of $\mathcal{P}(\mathbb{N})^{\mathbb{N}}$ onto the set $X = \{F(a) : a \in 2^{<\omega}\} = \bigcap_n \bigcup_{u\in 2^n} X_u$. We further assert that ϑ satisfying the following: for each $y, y' \in \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ and m,

$$(*) \qquad y\restriction_{\ge m} = y'\restriction_{\ge m} \quad \text{iff} \quad \vartheta(y)\restriction_{\ge m} = \vartheta(y')\restriction_{\ge m}.$$

Indeed: Let $y = G(a)$ and $x = F(a) = \vartheta(y)$, and similarly $y' = G(a')$ and $x' = F(a') = \vartheta(y')$, where $a, a' \in 2^{\mathbb{N}}$. Suppose that $y\restriction_{\ge m} = y'\restriction_{\ge m}$. According to (i)(b) for ψ and the sets Y_u we then have $m > \nu_\varphi[a \restriction n, a' \restriction n]$ for any n, hence, $X_{a\restriction n}\restriction_{\ge m} = X_{a\restriction n}\restriction_{\ge m}$ for any n by (i)(a). Assuming now that Polish metrics on all spaces $\mathcal{P}(\mathbb{N})^{\ge j}$ are chosen so that $\operatorname{diam} Z \ge \operatorname{diam}(Z\restriction_{\ge j})$ for all $Z \subseteq \mathcal{P}(\mathbb{N})$ and j, we easily obtain that $x\restriction_{\ge m} = x'\restriction_{\ge m}$, i. e, the right-hand side of $(*)$. The inverse implication in $(*)$ is proved similarly.

Thus we have $(*)$, but this means that ϑ is a continuous reduction of E_1 to $\mathsf{E}_1 \restriction X$, thus, $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$, as required. □ Theorem 1 modulo the construction (i) – (iv)

5 The construction

We continue to argue in the Solovay model.

Recall that $R \subseteq \mathcal{P}(\mathbb{N})^{\mathbb{N}}$ is a fixed non-empty $\mathrm{OD}(p)$ set such that $J(x) = \mathbb{N}$ for each $x \in R$. According to Proposition 3(i), there is $\gamma < \Omega$, $w_0 \in \mathbb{P}_\gamma$, and $t \in \mathbb{T}_\gamma[p]$ such that $X_\Lambda = t[w_0] \subseteq R$. Let us fix an enumeration (*not* $\mathrm{OD}(p)$) $\{D_n\}_{n\in\mathbb{N}}$ of all dense subsets of $\mathbb{P}_\gamma$ which belong to $\mathrm{L}[p]$. We define, along with sets X_u, a system $\{w_u\}_{u\in 2^{<\omega}}$ of finite sequences $w_u \in 2^{<\omega}$ satisfying

(v) $w_u \in D_{\mathrm{dom}\,u}$, and, for any i, $w_u \subset w_{u^\frown i}$ and $t[w_{u^\frown i}] \subseteq X_u \subseteq t[w_u]$.

Prove that this implies (iv). Let $a \in 2^{\mathbb{N}}$. Then there is $f \in \gamma^{\mathbb{N}}$ such that $w_{a\restriction n} \subset f$ for any n. This map f is generic over $\mathrm{L}[p]$, because for all n, $w_{a\restriction n} \in D_n$, that is, $f \in \mathbb{F}_\gamma[p]$. It follows that $t[f] \in \bigcap_n t[w_{a\restriction n}] = \bigcap_n X_{a\restriction n}$, as required.

To begin with, let w_Λ be any extension of w_0 which belongs to D_0. Put $X_\Lambda = t[w_0]$. Now suppose that the sets $X_u \subseteq R$ and sequences w_u with $u \in 2^n$ have been defined and satisfy the applicable part of (i) – (iii) and (v).

Lemma 4 *If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is a non-empty $\mathrm{OD}(p)$ set, then there is a system of $\mathrm{OD}(p)$ sets $\emptyset \neq X'_u \subseteq X_u$ with $X'_{u_0} = X'$, still satisfying* (i).

Proof. For any $u \in 2^n$, let $X'_u = \{x \in X_u : x\restriction_{>n(u)} \in X'\restriction_{>n(u)}\}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$ because, $\nu_\varphi[u_0, u_0] = -1$. The sets X'_u are as required, via a routine verification. □ Lemma

Step 1. Put $j = \varphi(n)$ and $Y_u = X_u\restriction_{>j}$. Take any $u_1 \in 2^n$. Under our assumptions, any element $x \in X_{u_1}$ satisfies $j \in J(x)$, so that $x(j) \notin \mathrm{OD}(p, x\restriction_{>j})$. Since X_{u_1} is an $\mathrm{OD}(p)$ set, it follows that the set $S_{X_{u_1}}(x\restriction_{>j}) = \{x'(j) : x' \in X_{u_1} \wedge x'\restriction_{>j} = x\restriction_{>j}\}$ is not a singleton, in fact is uncountable. Then there is a number l_{u_1} having the property that the set

$$Y'_{u_1} = \{y \in Y_{u_1} : (\exists x, x' \in X_{u_1})\,(x'\restriction_{>j} = x\restriction_{>j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))\}$$

is non-empty. We now put $X' = \{x \in X_{u_1} : x\restriction_{\prec j} \in Y'_{u_1}\}$ and define $\mathrm{OD}(p)$ sets $\emptyset \neq X'_u \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1}\restriction_{>j} = Y'_{u_1}$, still (i) is satisfied, and in addition

$$(1)\qquad (\forall y \in X'_{u_1}\restriction_{>j})(\exists x, x' \in X'_{u_1})\,(x'\restriction_{>j} = x\restriction_{>j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))$$

Now take some other $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j > \nu$, then $X_{u_1}\restriction_{>j} = X_{u_2}\restriction_{>j}$, so that we already have, for $l_{u_2} = l_{u_1}$,

$$(2)\qquad (\forall y \in X'_{u_2}\restriction_{>j})(\exists x, x' \in X'_{u_2})\,(x'\restriction_{>j} = x\restriction_{>j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j)),$$

and can pass to some $u_3 \in 2^n$. Suppose that $\nu \geq j$. Now things are somewhat nastier. As above there is a number l_{u_2} such that

$$Y'_{u_2} = \{y \in Y_{u_2} : (\exists x, x' \in X_{u_2})\,(x'\restriction_{>j} = x\restriction_{>j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\}$$

is a non-empty $\mathrm{OD}(p)$ set, thus, we can define $X'' = \{x \in X_{u_1} : x\restriction_{>j} \in Y'_{u_1}\}$ and maintain the construction of Lemma 4, getting non-empty $\mathrm{OD}(p)$ sets $X''_u \subseteq X'_u$ still satisfying (i) and $X''_{u_2} = X''$, therefore, we still have (2) for the set X''_{u_2}.

Yet it is most important in this case that (1) is preserved, i. e., it still holds for the set X''_{u_1} instead of X'_{u_1}! Indeed: According to the construction in the proof of Lemma 4, we have $X''_{u_1} = \{x \in X'_{u_1} : x\restriction_{>\nu} \in X''\restriction_{>\nu}\}$. Thus, although, in principle, X''_{u_1} is smaller than X'_{u_1}, for any $y \in X''_{u_1}\restriction_{>j}$ we have

$$\{x \in X''_{u_1} : x\restriction_{>j} = y\} = \{x \in X'_{u_1} : x\restriction_{>j} = y\},$$

simply because now we assume $\nu \geq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty $\mathrm{OD}(p)$ sets, let us call them "new" X_u, but they are subsets of the "original" X_u, still satisfying (i), and, for any $u \in 2^n$ a number l_u such that $j > \nu_\varphi[u, v]$ implies $l_u = l_v$ and

$$(*)\qquad (\forall y \in X_u\restriction_{>j})(\exists x, x' \in X_u)\,(x'\restriction_{>j} = x\restriction_{>j} = y \wedge l_u \in x(j) \wedge l_u \notin x'(j)).$$

Step 2. We define the $(n+1)$th-level by $X_{u^\frown 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u^\frown 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from $(*)$ that all these $\mathrm{OD}(p)$ sets are non-empty.

Lemma 5 *The system of sets $\{X_s\}_{s\in 2^{n+1}}$ just defined satisfies* (i).

Proof. Let $s = u^\frown i$ and $t = v^\frown i'$ belong to 2^{n+1}, so that $u, v \in 2^n$ and $i, i' \in \{0, 1\}$. Let $\nu = \nu_\varphi[u, v]$ and $\nu' = \nu_\varphi[s, t]$.

Case 1. $\nu \geq j = \varphi(n)$. Then easily $\nu = \nu'$, so that (i)(b) immediately follows from (i)(b) at level n for X_u and X_v. As for (i)(a), we have $X_s\restriction_{>\nu} = X_u\restriction_{>\nu}$ (because by definition $X_s\restriction_{>j} = X_u\restriction_{>j}$), and similarly $X_t\restriction_{>\nu} = X_v\restriction_{>\nu}$, therefore, $X_t\restriction_{>\nu'} = X_s\restriction_{>\nu'}$ since $X_u\restriction_{>\nu} = X_v\restriction_{>\nu}$ by (i)(a) at level n.

Case 2. $j > \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have (i)(b). Further, $X_u\restriction_{>\nu} = X_v\restriction_{>\nu}$ by (i)(a) at level n, hence, $X_u\restriction_{\geq j} = X_v\restriction_{\geq j}$ and $l_u = l_v$ as above. Assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that $X_s\restriction_{>\nu'} = \{y \in X_u\restriction_{>\nu} : l \in y(j)\} = \{y \in X_v\restriction_{>\nu} : l \in y(j)\} = X_t\restriction_{>\nu'}$.

Case 3. $j > \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$, Now $\nu' = j$. Yet by definition $X_s\restriction_{>j} = X_u\restriction_{>j}$ and $X_t\restriction_{>j} = X_v\restriction_{>j}$, so it remains to apply (i)(a) for level n. As for (i)(b), note that by definition $l \notin x(j)$ for any $x \in X_s = X_{u^\frown 0}$ while $l \in x(j)$ for any $x \in X_t = X_{v^\frown 1}$, where $l = l_u = l_v$. □ Lemma

Step 3. In addition to (i), we already have (ii) at level $n+1$. To achieve the remaining properties (iii) and (v), consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep $s = u^\frown i$ ($u \in 2^n$ and $i = 0, 1$), a non-empty $\mathrm{OD}(p)$ subset of X_s, and also an extension $w_s \in 2^{<\omega}$ of w_u, consistent with (iii) and (v). As for (iii), just take a subset whose diameter is $\leq 2^{-n}$. As for (iv), choose, using Proposition 3(ii), $w_s \in \mathbb{P}_\gamma$ such that the following holds: $w_s \in D_{n+1}$, $w_u \subset w_s$, and the set $t[w_s]$ is a subset of the "current value" of X_s. Finally, define the "new" value of X_s to be $t[w_s]$. Then reduce all other sets X_t, $t \in 2^{n+1}$, as in Lemma 4 at level $n+1$. Thus ends the substep s. We have to pass to another $s' \in 2^{n+1}$ and carry out substep s'. And so on, with the consideration of all $s \in 2^{n+1}$ one by one.

□ Construction and Theorem 1

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References

[1] R. Dougherty, S. Jackson, and A. S. Kechris, The structure of hyperfinite Borel equivalence relations. Trans. Amer. Math. Soc., **341**, 193–225 (1994).
[2] V. Kanovei, An Ulm-type classification theorem for equivalence relations in Solovay model. J. Symbolic Logic **62**, 1333–1351 (1997).
[3] V. Kanovei, On non-wellfounded iterations of the perfect set forcing. J. Symbolic Logic **64**, 551–574 (1999).
[4] A. S. Kechris and A. Louveau, The classification of hypersmooth Borel equivalence relations. J. Amer. Math. Soc. **10**, 215–242 (1997).
[5] R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable. Ann. of Math. (2), **92**, 1–56 (1970).