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On effective σ-boundedness and σ-compactness

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We prove several dichotomy theorems which extend some known results on σ-bounded and σ-compact-pointsets. In particular we show that, given a finite number of Δ^1_1 equivalence relations $\mathsf{F}_1, \dots, \mathsf{F}_n$, any Σ^1_1 set A of the Baire space either is covered by compact Δ^1_1 sets and lightface Δ^1_1 equivalence classes of the relations F_i, or A contains a superperfect subset which is pairwise F_i-inequivalent for all $i = 1, \dots, n$. Further generalizations to Σ^1_2 sets A are obtained.

1 Introduction

Effective descriptive set theory appeared in the 1950s as a useful technique of simplification and clarification of constructions of classical descriptive set theory (cf., e.g., [5] or [18]). Yet it has become clear that development of effective descriptive set theory leads to results having no direct analogies in classical descriptive set theory. As an example we recall the following *basis theorem*: any countable Δ^1_1 set A of the Baire space $\mathscr{N} = \omega^\omega$ consists of Δ^1_1 points. Its remote predecessor in classical descriptive set theory is the Luzin-Novikov theorem on Borel sets with countable sections.

We shall focus on effectivity aspects of the properties of σ-compactness and σ-boundedness of pointsets in this paper. Our starting point will be a pair of classical dichotomy theorems on pointsets, together with their effective versions obtained in the end of 1970s.

The first of them deals with the property of σ-boundedness. Recall that a pointset is σ-*bounded* iff it is a subset of a σ-compact set. For subsets of the Baire space $\mathscr{N} = \omega^\omega$, the property of σ-boundedness is equivalent to being bounded in $\mathscr{N}$ with the *eventual domination* order. Saint Raymond [15] proved that if X is a $\boldsymbol{\Sigma}^1_1$ set then one and only one of the following two (obviously incompatible) conditions holds:

(I) the set X is σ-bounded;

(II) there is a superperfect set $Y \subseteq X$ (i.e., a closed set homeomorphic to $\mathscr{N}$).

This result can be compared with an older theorem by Hurewicz [3], which deals with the property of σ-compactness instead of σ-boundedness. It says that if X is a $\boldsymbol{\Sigma}^1_1$ set then again one and only one of the following two (incompatible) conditions (I′), (II′) holds:

(I′) the set X is σ-compact;

(II′) there is a set $Y \subseteq X$ homeomorphic to $\mathscr{N}$ and relatively closed in X.

There is an effective version of the first result: Theorem 3.1 below, by Kechris. It says that if X is a Σ^1_1 set then condition 1 can be strengthened to Δ^1_1-effective σ-boundedness, so that a given set X is covered by a Δ^1_1 sequence of compact sets. Accordingly, an effective version of the second result, Theorem 3.2 below by Louveau,

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asserts that if X is a lightface Δ^1_1 set then condition (I$'$) can be strengthened to Δ^1_1-effective σ-compactness, so that a given set is equal to the union of a Δ^1_1 sequence of compact sets.

It occurs that Theorem 3.2 fails for Σ^1_1 sets, but we prove a similar more complicated dichotomy theorem on Σ^1_1 sets in Section 4. Several counterexamples with sets outside of Σ^1_1 will be outlined in Section 5.

Section 6–8 contain a generalization of Theorem 3.1 (Theorem 8.1) which replaces σ-bounded sets by $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded sets, where $\mathsf{F}_1,\dots,\mathsf{F}_n$ are given Δ^1_1 equivalence relations and being $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded means being covered by the union of a σ-bounded set and countably many equivalence classes of $\mathsf{F}_1,\dots,\mathsf{F}_n$. Accordingly the condition of existence of a superperfect set strengthens by the requirement that the superperfect set is pairwise F_i-inequivalent for $i=1,\dots,n$. Section 6 develops a necessary technique while the proof of the generalized dichotomy is presented in Section 8. In the classical form, the case of a single equivalence relation F in this dichotomy was earlier obtained by Zapletal, cf. [7].

In parallel to this, we prove in Section 7 that a σ-bounded set and a countable union of equivalence classes as above can be defined so that they depend only on a given set X (and the collection of equivalence relations F_j), but are independent of the choice of a parameter p such that X is $\Sigma^1_1(p)$ and the relations are $\Delta^1_1(p)$.

In the remaining parts of the paper, we prove a generalization of another Kechris's result of [8], related to Σ^1_2 sets, which by necessity involves uncountable unions of equivalence classes and σ-bounded sets.

2 Preliminaries

We use standard notation Σ^1_1, Π^1_1, Δ^1_1 for effective classes of points and pointsets in $\mathscr{N}$, as well as $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$ for corresponding projective classes.

Let $\omega^{<\omega}$ be the set of all finite strings of natural numbers, including *the empty string* Λ. If $u,v\in\omega^{<\omega}$ then $\operatorname{lh} u$ is the *length* of u, and $u\subset v$ means that v is a *proper extension* of u. If $u\in\omega^{<\omega}$ and $n\in\omega$ then $u^\frown n$ is the string obtained by adding n to u as the rightmost term. Let, for $u\in\omega^{<\omega}$,

$$\mathscr{N}_u=\{x\in\mathscr{N}: u\subset x\}\quad (\textit{a Baire interval} \text{ in } \mathscr{N}).$$

If a set $X\subseteq\mathscr{N}$ contains at least two elements then there is a longest string $u=\operatorname{stem}(X)$ such that $X\subseteq\mathscr{N}_u$. We put $\operatorname{diam}(X)=\frac{1}{1+\operatorname{stem}(X)}$ in this case, and additionally $\operatorname{diam}(X)=0$ whenever X has at most one element.

A set $T\subseteq\omega^{<\omega}$ is a *tree* if $u\in T$ holds whenever $u^\frown n\in T$ for at least one n, and a *pruned* tree iff $u\in T$ implies $u^\frown n\in T$ for at least one n. Any non-empty tree contains Λ. A string $u\in T$ is a *branching point* of T if there are $k\ne n$ such that $u^\frown k\in T$ and $u^\frown n\in T$; let $\mathbf{bran}(T)$ be the set of all branching points of T. The *branching height* $\mathbf{BH}_T(u)$ of a string $u\in T$ in a tree T is equal to the number of strings $v\in\mathbf{bran}(T)$, $v\subset u$. For instance, if $T=\omega^{<\omega}$ then $\mathbf{BH}_{\omega^{<\omega}}(u)=\operatorname{lh} u$ for any string u.

A tree $T\subseteq\omega^{<\omega}$ is *compact*, if it is pruned and has *finite branchings*, i.e., if $u\in\mathbf{bran}(T)$ then $u^\frown n\in T$ holds for finitely many n. Then

$$[T]=\{x\in\mathscr{N}:\forall m\,(x{\restriction}m\in T)\},$$

the *body* of T, is a compact set. Conversely, if $X\subseteq\mathscr{N}$ is compact then

$$\operatorname{tree}(X)=\{x{\restriction}n: x\in X\wedge n\in\omega\}$$

is a compact tree. Let **CT** be the Δ^1_1 set of all non-empty compact trees.

A pruned tree $T\subseteq\omega^{<\omega}$ is *perfect*, if for each $u\in T$ there is a string $v\in\mathbf{bran}(T)$ with $u\subset v$. Then $[T]$ is a perfect set. A perfect tree T is *superperfect*, if for each $u\in\mathbf{bran}(T)$ there are infinitely many numbers n such that $u^\frown n\in T$. Then $[T]$ is a superperfect set. Conversely, if $X\subseteq\mathscr{N}$ is a perfect set then $\operatorname{tree}(X)$ is a perfect tree, while for any superperfect set $X\subseteq\mathscr{N}$ there is a superperfect tree $T\subseteq\operatorname{tree}(X)$.

If $\mathbb{X}$, $\mathbb{Y}$ are any sets and $P\subseteq\mathbb{X}\times\mathbb{Y}$ then

$$\operatorname{proj}P=\{x\in\mathbb{X}:\exists y\,(\langle x,y\rangle\in P)\}\text{ and }(P)_x=\{y\in\mathbb{Y}:\langle x,y\rangle\in P\}$$

are, respectively, the *projection* of P to $\mathbb{X}$, and the ("vertical") *section* of P corresponding to $x\in\mathbb{X}$. A set $P\subseteq\mathbb{X}\times\mathbb{Y}$ is *uniform* if every section $(P)_x$ $(x\in\mathbb{X})$ contains at most one element. Let a *product space* be any finite product of factors ω, $\omega^{<\omega}$, $\mathscr{N}$, $\mathscr{P}(\omega^{<\omega})$. A *discrete* product space is a finite product of ω, $\omega^{<\omega}$.

We'll make use of several known results of effective descriptive set theory. They are listed below, with a few proofs (of claims which are not in common use in this area) attached to make the text self-contained.

Fact 2.1 (Kreisel selection) *If $\mathbb{X}$ is a discrete product space, $P \subseteq \mathscr{N} \times \mathbb{X}$ is a Π^1_1 set, and $A \subseteq \operatorname{proj} P$ is a Σ^1_1 set, then there is a Δ^1_1 map $f : \mathscr{N} \to \mathbb{X}$ such that $\langle x, f(x)\rangle \in P$ for all $x \in A$* [14, 4B.5].

Fact 2.2 *If $P(x, y, z, \dots)$ is a Π^1_1 relation on a product space then the following derived relations belong to Π^1_1 as well:*

$$\exists\, x \in \Delta^1_1\, P(x, y, z, \dots) \quad \textit{and} \quad \exists\, x \in \Delta^1_1(y)\, P(x, y, z, \dots)$$

[14, 4D.3].

Fact 2.3 (enumeration of Δ^1_1 sets) *Let $\mathbb{X}$ be a product space. There exist Π^1_1 sets $E \subseteq \omega$ and $W \subseteq \omega \times \mathbb{X}$, and a Σ^1_1 set $W' \subseteq \omega \times \mathbb{X}$ such that*

(i) *if $e \in E$ then $(W)_e = (W')_e$* (where $(W)_e = \{x \in \mathbb{X} : \langle e, x\rangle \in W\}$)*;*

(ii) *a set $X \subseteq \mathbb{X}$ is Δ^1_1 iff there is $e \in E$ such that $X = (W)_e$*

[14, 4D.2].

There is a useful uniform version of Fact 2.3.

Fact 2.4 *Let $\mathbb{X}$ be a product space. There exist Π^1_1 sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W} \subseteq \mathscr{N} \times \omega \times \mathbb{X}$, and a Σ^1_1 set $\mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathbb{X}$ such that*

(i) *if $\langle p, e\rangle \in \mathbf{E}$ then $(\mathbf{W})_{pe} = (\mathbf{W}')_{pe}$* (where, as above, $(\mathbf{W})_{pe} = \{x \in \mathbb{X} : \langle p, e, x\rangle \in \mathbf{W}\}$)*;*

(ii) *if $p \in \mathscr{N}$ then a set $X \subseteq \mathbb{X}$ is $\Delta^1_1(p)$ iff there is a number $e \in E$ such that $T = (\mathbf{W})_{pe} = (\mathbf{W}')_{pe}$.*

This result implies the following stronger version of Fact 2.1.

Fact 2.5 *Suppose that $\mathbb{X}$ is a product space, $Q \subseteq \mathscr{N} \times \mathbb{X}$ is Π^1_1, $A \subseteq \operatorname{proj} Q$ is Σ^1_1, and for each $a \in A$ there is a point $x \in \Delta^1_1(a)$ such that $\langle a, x\rangle \in Q$. Then there is a Δ^1_1 map $f : \mathscr{N} \to \mathbb{X}$ such that $\langle a, f(a)\rangle \in Q$ for all $a \in A$* [14, 4D.6].

Proof. Assume that $\mathbb{X} = \mathscr{N}$, for the sake of brevity. Then any $x \in \mathbb{X}$ satisfies $x \subseteq \mathbb{Y} = \omega \times \omega$. Making use of the sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W}, \mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathbb{Y}$ as in Fact 2.4, we let

$$P = \big\{\langle a, e\rangle \in \mathbf{E} : (\mathbf{W})_{ae} \in \mathscr{N} \wedge \langle a, (\mathbf{W})_{ae}\rangle \in Q\big\}.$$

Easily the set P and its projection $\operatorname{proj} P$ both are Π^1_1, and $A \subseteq \operatorname{proj} P$. By Fact 2.1, there is a Δ^1_1 map $f : \mathscr{N} \to \omega$ such that $\langle a, f(a)\rangle \in P$ for all $a \in A$. It remains to define $f(a) = (\mathbf{W})_{a, f(a)}$ for $a \in A$; to prove that f is Δ^1_1 use both sets $\mathbf{W}$ and $\mathbf{W}'$. □

Fact 2.6 *If $X \neq \varnothing$ is a countable Δ^1_1 set then there exists a Δ^1_1 map defined on ω such that $X = \big\{f(n) : n < \omega\big\}$* [14, 4F.17].

In addition, Facts 2.1, 2.2, 2.3, and 2.5, remain true for relativized classes $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$, where $p \in \mathscr{N}$ is any fixed parameter.

3 Two effective dichotomy theorems

The following two theorems were briefly discussed in the introduction.

Theorem 3.1 *If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims* (I), (II) *holds*:

(I) *A is Δ^1_1-effectively σ-bounded, in the sense that there is a Δ^1_1 sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ satisfying $A \subseteq \bigcup_n [T_n]$;*

(II) *there is a superperfect set $Y \subseteq A$*

[8, p. 198].

Theorem 3.2 *If $A \subseteq \mathscr{N}$ is a Δ^1_1 set then one and only one of the next two claims holds*:

- (I′) *A is Δ^1_1-effectively σ-compact, in the sense that there is a Δ^1_1 sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ satisfying $A = \bigcup_n [T_n]$*;
- (II′) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and relatively closed in A*

(*cf.* [10] *and* [14, 4F.18]).

Corollary 3.3 *If $A \subseteq \mathscr{N}$ is a σ-bounded Σ^1_1 set then it is Δ^1_1-effectively σ-bounded in the sense of condition* 3.1 *of Theorem* 3.1. *Accordingly, if $A \subseteq \mathscr{N}$ is a σ-compact Δ^1_1 set then it is Δ^1_1-effectively σ-compact in the sense of condition* (I) *of Theorem* 3.2.

In spite of certain differences between the theorems, both of them easily follow from the next much more general result (which was actually extended by Louveau and Saint Raymond to all levels of the Borel hierarchy).

Theorem 3.4 (Louveau, Saint Raymond [11, 12]) *If $A, B \subseteq \mathscr{N}$ are disjoint Σ^1_1 sets then one and only one of the next two claims holds*:

- (I) *there exists a Δ^1_1 real p such that A is separated from B by a $\Sigma^0_2(p)$ set S—then S is Δ^1_1, and moreover, there is a Δ^1_1 sequence $\{T_n\}_{n\in\omega}$ of trees $T_n \subseteq \omega^{<\omega}$ such that $S = \bigcup_n [T_n]$*;
- (II) *there is a set $C \subseteq A \cup B$ homeomorphic to 2^ω* (*hence by necessity closed*) *and such that $C \cap B$ is a countable set dense in C.*

Let's show how this result implies Theorems 3.1 and 3.2.

P r o o f o f T h e o r e m 3.1. Recall that the Baire space $\mathscr{N}$ is homeomorphic to the Π^0_2 set $\mathscr{N}'$ of all points $x \in 2^\omega$ with infinitely many terms $x(k)$ equal to 1, via the map $H : \mathscr{N} \xrightarrow{\text{onto}} \mathscr{N}'$ sending each $a \in \mathscr{N}$ to

$$H(a) = 1, \underbrace{0,\dots,0}_{a(0)\text{ zeros}}, 1, \underbrace{0,\dots,0}_{a(1)\text{ zeros}}, 1, \underbrace{0,\dots,0}_{a(2)\text{ zeros}}, \dots .$$

Let $A' = H[A] = \{H(a) : a \in A\} \subseteq \mathscr{N}'$ and $B' = 2^\omega \setminus \mathscr{N}'$.

Assume that (I) of Theorem 3.4 holds, via a Δ^1_1 sequence of trees T'_n. We can assume that $T'_n \subseteq 2^{<\omega}$, of course. Then $[T'_n] \subseteq \mathscr{N}'$ by the choice of B', so that the sets $X_n = H^{-1}([T'_n]) \subseteq \mathscr{N}$ are compact, the trees $T_n = \mathrm{tree}(X_n)$ are compact, too, which leads us to (I) of Theorem 3.1.

Assume that (II) of Theorem 3.4 holds, via a (closed) set $C \subseteq A' \cup B'$ homeomorphic to 2^ω. Then $C' = C \setminus B' = C \cap A' = C \cap \mathscr{N}'$ is a relatively closed subset of A' homeomorphic to $\mathscr{N}$. We may note in passing by that (I) of Theorem 3.4 fails, and moreover A is not even $\boldsymbol{\Sigma}^0_2$-separated from B—as otherwise C' would be a relative $\boldsymbol{\Sigma}^0_2$ subset of C, which is impossible.

Further, $C = H^{-1}(C) \subseteq \mathscr{N}$ is a relatively closed subset of A and a Σ^1_1 set, of course. It remains to prove that C is not σ-bounded—then it contains a superperfect subset by a Saint Raymond's theorem mentioned in the introduction. Suppose, to the contrary, that $C \subseteq F$, where $F \subseteq \mathscr{N}$ is σ-compact. The set $F' = H[F] \subseteq \mathscr{N}'$ is then σ-compact, too, and hence $\boldsymbol{\Sigma}^0_2$, thus A is $\boldsymbol{\Sigma}^0_2$-separated from B, contrary to the above. □

P r o o f o f T h e o r e m 3.2. Let $A' = H[A] \subseteq \mathscr{N}'$, as above, and now $B' = 2^\omega \setminus A'$. If 3.4 of Theorem 3.4 holds, via a Δ^1_1 sequence of trees $T'_n \subseteq 2^{<\omega}$, then just $A' = \bigcup_n [T'_n]$, so that, pulling this back to $\mathscr{N}$ via H^{-1}, we easily get (I′) of Theorem 3.2. If (II) of Theorem 3.4 holds, then the set $C' = C \cap A$ is a relatively closed subset of A' homeomorphic to $\mathscr{N}$, thus pulling it back to $\mathscr{N}$ via H^{-1}, we get (II′) of Theorem 3.2. □

The original proof of Theorem 3.4 in [11] was based on determinacy ideas and technique. A proof by methods of effective descriptive set theory is also known to those working in this field. It combines two rather independent results and techniques. One of them is the famous effective separation theorem by Louveau [10]. The other one is (essentially) Hurewicz's [3] result cited in the introduction—in a more advanced form of Theorem 21.22 (by Kechris, Louveau, Woodin) in [9], given there with a proof involving some game. The original Hurewicz proof was purely topological, and a more transparent version of this proof is given in [16, Lemma 7].

4 Effective σ-compactness dichotomy for Σ^1_1 sets

There is a difference between Theorem 3.1 and Theorem 3.2: the first theorem deals with Σ^1_1 sets while the other one—with Δ^1_1 sets. The proof of Theorem 3.2 in Section 3 does not work in the case when A is a Σ^1_1 set, and in fact Theorem 3.2 fails for Σ^1_1 sets A, as the next counterexample shows.

Example 4.1 Let $\{y\}$ be a Π^1_1 singleton such that $y \in 2^\omega$ is not Δ^1_1. The set $A = 2^\omega \setminus \{y\}$ is then Σ^1_1 and an open subset of 2^ω, hence, σ-compact. Suppose towards the contrary that Theorem 3.2 holds for A. Then (I$'$) of Theorem 3.2 must be true. Let $\{T_n\}_{n\in\omega}$ be a Δ^1_1 sequence of compact trees such that $A = \bigcup_n [T_n]$. Therefore y is Δ^1_1, as the only point in 2^ω which does not belong to $\bigcup_n [T_n]$, a contradiction.

The next theorem is our best result so far, in the direction of Theorem 3.2 for Σ^1_1 sets, with still some amount of effectivity in condition (I$'$).

Theorem 4.2 *If $A \subseteq \mathscr{N}$ is a Σ^1_1 set then one and only one of the following two claims holds*:

(I) *A is Δ^1_3-effectively σ-compact, so that there exists a Δ^1_3 sequence $\{T^n\}_{n<\omega}$ of compact Δ^1_3 trees $T^n \subseteq \omega^{<\omega}$ such that $A = \bigcup_{n<\omega} [T^n]$*;

(II) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and relatively closed in A.*

The following proof is essentially the classical proof of the Hurewicz theorem, at least as presented in [16] (while in his original proof in [3], Hurewicz deletes in one step all open sets whose images are contained in some σ-compact subset of the given set). We only add the computation of the complexity of this classical construction.

Proof. Given a tree $S \subseteq (\omega \times \omega)^{<\omega}$, define a *derived tree* $S' \subseteq S$ so that

($*$) S' consists of all nodes $\langle u, v\rangle \in S$ such that $\overline{\mathrm{proj}\,[S\restriction\langle u, v\rangle]} \not\subseteq A$, where $S\restriction\langle u, v\rangle = \{\langle u', v'\rangle \in S : (u \subset u' \wedge v \subset v') \vee (u' \subseteq u \wedge v' \subseteq v)\}$.

Note that S' can contain maximal nodes even if S contains no maximal nodes. Yet if $\langle u, v\rangle$ is a maximal node in S, or generally a note in the well-founded part of S (so $[S\restriction\langle u, v\rangle] = \varnothing$), then definitely $\langle u, v\rangle \notin S'$.

Lemma 4.3 *The set $\{\langle S, u, v\rangle : \langle u, v\rangle \in S'\}$ is Σ^1_2. In addition, $S' \subseteq S$, and if $S \subseteq T$ then $S' \subseteq T'$. Moreover, if $\mathfrak{M}$ is a countable transitive model of a large enough fragment of* ZFC *and $S \in \mathfrak{M}$ then $(S')^{\mathfrak{M}} \subseteq S'$.*

Proof. As A is Σ^1_1, the key condition $\overline{\mathrm{proj}\,[S\restriction\langle u, v\rangle]} \not\subseteq A$ is Σ^1_2. □

Beginning the proof of Theorem 4.2, we w.l.o.g. assume, by Theorem 3.1, that A, the given set, is σ-bounded, and hence if $F \subseteq A$ is an arbitrary closed set then F is σ-compact. Let $P \subseteq \mathscr{N} \times \mathscr{N}$ be a Π^0_1 set such that $A = \mathrm{proj}\, P$. We define

$$S = \{\langle x\restriction n, y\restriction n\rangle : n \in \omega \wedge \langle x, y\rangle \in P\} \subseteq \omega^{<\omega} \times \omega^{<\omega},$$

so that $P = [S]$. A decreasing sequence of derived trees $S^{(\alpha)}$, $\alpha \in \mathrm{Ord}$, is defined by transfinite induction so that $S^{(0)} = S$, if λ is a limit ordinal then naturally $S^{(\lambda)} = \bigcap_{\alpha<\lambda} S^{(\alpha)}$, and $S^{(\alpha+1)} = \big(S^{(\alpha)}\big)'$ for any α.

Obviously there is a countable ordinal λ such that $S^{(\lambda+1)} = S^{(\lambda)}$.

Case 1: $S^{(\lambda)} = \varnothing$. Then, if $x \in A = \mathrm{proj}\, P$ then by construction there exist an ordinal $\alpha < \lambda$ and a node $\langle u, v\rangle \in S^{(\alpha)}$ such that

$$x \in A^{(\alpha)}_{uv} \subseteq \overline{A^{(\alpha)}_{uv}} \subseteq A, \quad \text{where} \quad A^{(\alpha)}_{uv} = \mathrm{proj}\,\big[S^{(\alpha)}\restriction\langle u, v\rangle\big],$$

and hence A is a countable union of sets $F \subseteq A$ of the form $\overline{A^{(\alpha)}_{uv}}$, where $\alpha < \lambda$ and $\langle u, v\rangle \in S^{(\alpha)}$, closed, therefore σ-compact by the above.

Let us show how this leads to (I) of the theorem.

It easily follows from Lemma 4.3 that both the ordinal λ, and each ordinal $\alpha < \lambda$, and the sequence $\{S^{(\alpha)}\}_{\alpha<\lambda}$ itself, are Δ^1_3. Therefore there is a Δ^1_3 sequence $\{U^{(n)}\}n < \omega$ of the same trees, i.e.,

$$\{S^{(\alpha)} : \alpha < \lambda\} = \{U^{(n)} : n < \omega\}.$$

Each tree $U^{(n)}$, $n < \omega$, is Δ^1_3 either, as well as all restricted subtrees of the form $U^{(n)} \upharpoonright \langle u, v\rangle$ (where $\langle u, v\rangle \in U^{(n)}$) and their "projections"

$$T^{(n)}_{uv} = \{u : \exists\, v\, (\langle u, v\rangle \in U^{(n)} \upharpoonright \langle u, v\rangle)\} \subseteq \omega^{<\omega}.$$

On the other hand, if $\alpha < \lambda$ and $\langle u, v\rangle \in S^{(\alpha)}$ then we have $\overline{A^{(\alpha)}_{uv}} = [T^{(n)}_{uv}]$ for some $n = n(\alpha)$ by construction.

To conclude, if $x \in A$ then there is a Δ^1_3 tree $T^{(n)}_{uv} \subseteq \omega^{<\omega}$ such that $x \in [T^{(n)}_{uv}] \subseteq A$—and $[T^{(n)}_{uv}]$ is σ-compact in this case. Then by Theorem 3.2 (relativized version) there is a $\Delta^1_1(T^{(n)}_{uv})$ sequence of compact trees $T^{(n)}_{uv}(k)$ such that $[T^{(n)}_{uv}] = \bigcup_k [T^{(n)}_{uv}(k)]$. This easily leads to (I) of the theorem.[1]

Case 2: $S^{(\lambda)} \neq \varnothing$, and then $S^{(\lambda)} \subseteq S$ is a pruned tree and $\langle \Lambda, \Lambda\rangle \in S^{(\lambda)}$.

Lemma 4.4 *If $\langle u, v\rangle \in S^{(\lambda)}$, $u' \in \omega^{<\omega}$, $u \subset u'$, and $A^{(\lambda)}_{uv} \cap \mathscr{N}_{u'} \neq \varnothing$ then there is a string $v' \in \omega^{<\omega}$ such that $v \subset v'$ and $\langle u', v'\rangle \in S^{(\lambda)}$.*

We'll define a pair $\langle u(t), v(t)\rangle \in S^{(\lambda)}$ for each $t \in \omega^{<\omega}$, such that

(1) if $t \in \omega^{<\omega}$ then $t \subseteq u(t)$;
(2) if $s, t \in \omega^{<\omega}$ and $s \subseteq t$ then $u(s) \subseteq u(t)$ and $v(s) \subseteq v(t)$;
(3) if $t \in \omega^{<\omega}$ and $k \neq n$ then $u(t^\frown k)$ and $u(t^\frown n)$ are $\subseteq$-incomparable;
(4) if $s \in \omega^{<\omega}$ then there exists a point $y_s \in \overline{A^{(\lambda)}_{u(s)v(s)}} \setminus A$ such that any sequence of points $x_k \in A^{(\lambda)}_{u(s^\frown k)v(s^\frown k)}$ converges to y_s.

Suppose that such a system of pairs is defined. Then the associated map $f(a) = \bigcup_n u(a\upharpoonright n) : \mathscr{N} \to A$ is 1-1 and is a homeomorphism from $\mathscr{N}$ onto its full image $Y = \operatorname{ran} f = \{f(a) : a \in \mathscr{N}\} \subseteq A$.

Let's prove that Y is relatively closed in A. Consider a sequence of points $a_n \in \mathscr{N}$ such that the corresponding sequence of points $y_n = f(a_n) \in Y$ converges to a point $y \in \mathscr{N}$; we have to prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence convergent to $b \in \mathscr{N}$ then $\{y_n\}$ converges to $f(b) \in Y$. So suppose that the sequence $\{a_n\}$ has no convergent subsequences. Then there exist a string $s \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and for each $k \in K$—a number $n(k)$, such that $s^\frown k \subset a_{n(k)}$. Then $y_{n(k)} \in A^{(\lambda)}_{u(s^\frown k)v(s^\frown k)}$ by construction. Therefore the subsequence $\{y_{n(k)}\}_{k\in\omega}$ converges to a point $y_s \notin A$ by (4), as required.

Finally, on the construction of pairs $\langle u(t), v(t)\rangle$. Put $\langle u(\Lambda), v(\Lambda)\rangle = \langle \Lambda, \Lambda\rangle$. Suppose that a pair $\langle u(t), v(t)\rangle \in S^{(\lambda)}$ is defined. Then $\overline{A^{(\lambda)}_{u(t)v(t)}} \not\subseteq A$ by the choice of λ. There is a sequence of pairwise different points $x_n \in A^{(\lambda)}_{u(t)v(t)}$ which converges to a point $y_s \in \overline{A^{(\lambda)}_{u(t)v(t)}} \setminus A$. We can associate a string $u_n \in \omega^{<\omega}$ with each x_n such that $u(t) \subset u_n \subset x_n$, the strings u_n are pairwise $\subseteq$-incompatible, and $\operatorname{lh} u_n \to \infty$. Then, by Lemma 4.4, for each n there is a matching string v_n such that $v(t) \subset v_n$ and $\langle u_n, v_n\rangle \in S^{(\lambda)}$. Put $u(t^\frown n) = u_n$ and $v(t^\frown n) = v_n$ for all n. □

5 Counterexamples above Σ^1_1

Here we outline several counterexamples to Theorems 3.1 and 3.2 with sets A more complicated than Σ^1_1.

Example 5.1 Suppose that the universe is a Cohen real extension $\mathbf{L}[a]$ of the constructible universe $\mathbf{L}$. The set $A = \mathscr{N} \cap \mathbf{L}$ is Σ^1_2 and it is not σ-bounded in $\mathbf{L}[a]$. On the other hand, it is known from [2] that A has no perfect

[1] Class Δ^1_3 in (I) of the theorem looks too bad. One may want to improve it to Δ^1_2 at least. This would be the case if the ordinal λ in the argument of Case 1 could be shown to be Δ^1_2. Yet by Martin [13] closure ordinals of inductive constructions of this sort may exceed the domain of Δ^1_2 ordinals.

subsets, let alone superperfect ones. Thus A is a Σ^1_2 counterexample to both Theorem 3.1 and Theorem 3.2 in $\mathbf{L}[a]$. We then immediately obtain a similar Π^1_1 counterexample, using the Π^1_1 uniformization theorem.

Example 5.2 Suppose that the universe is a dominating real extension $\mathbf{L}[d]$ of $\mathbf{L}$. The set $A = \mathscr{N} \cap \mathbf{L}$ is then σ-bounded in $\mathbf{L}[d]$. The dominating forcing is homogeneous enough for any OD (ordinal-definable) real in $\mathbf{L}[d]$ to be constructible, and hence it is true in $\mathbf{L}[d]$ that A cannot be covered by a countable union of OD compact sets in $\mathbf{L}[d]$. Thus A is a Σ^1_2 counterexample to Corollary 3.3.

Yet we don't know whether there exists a similar definable counterexample to Corollary 3.3.

Example 5.3 Let $A = \{y\}$ be a Π^1_1 singleton such that y is not a Δ^1_1 real. Then conditions (I), (II) of Theorem 3.1 obviously fail for A. The same for Theorem 3.2. Moreover, A is a Π^1_1 counterexample to Corollary 3.3 as well, although not in the same strong sense as in Example 5.2.

It is known that there is a countable Π^1_1 set $A \subseteq \mathscr{N}$ containing at least one non-Δ^1_2 element. Can it serve as a more profound Π^1_1 counterexample than the singleton A of Example 5.3 ?

6 Generalization of the σ-bounded dichotomy: preliminaries

Below, in Section 8, we establish a generalization of Theorem 3.1 for a certain system of pointset ideals which include the ideal of σ-bounded sets along with equivalence classes of a given finite or countable family of equivalence relations. The next definition introduces a necessary framework.

Definition 6.1 Let $\mathscr{F}$ be a family of equivalence relations on a set $X_0 \subseteq \mathscr{N}$. A set $X \subseteq X_0$ is $\mathscr{F}$-σ-*bounded*, iff it is covered by a union of the form $B \cup \bigcup_{n\in\omega} Y_n$, where B is a σ-bounded set and each Y_n is an F-equivalence class for an equivalence relation $\mathsf{F} = \mathsf{F}(n) \in \mathscr{F}$ which depends on n.

A set $X \subseteq X_0$ is $\mathscr{F}$-*superperfect*, if it is a superperfect pairwise F-inequivalent set (*i.e.*, a partial F-transversal) for every $\mathsf{F} \in \mathscr{F}$.

Clearly $\mathscr{F}$-σ-bounded sets form a σ-ideal containing all σ-bounded sets, and no $\mathscr{F}$-σ-bounded set can be $\mathscr{F}$-superperfect. What are properties of these ideals? Do they have some semblance of the superperfect ideal itself? We begin with a lemma and a corollary afterwards, which show that this is indeed the case w.r.t. the property of being $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The lemma is a generalization of Corollary 3.3, of course.

Lemma 6.2 *Suppose that $\{\mathsf{F}_n\}_{n<\omega}$ is a Δ^1_1 sequence of equivalence relations on $\mathscr{N}$, and a Σ^1_1 set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded. Then X is Δ^1_1-effectively $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded, in the sense that there exist*:

(1) *a Δ^1_1 sequence of compact trees T_k,*
(2) *a Δ^1_1 sequence of numbers n_k, and*
(3) *a Δ^1_1 set $H \subseteq \omega \times \mathscr{N}$*

such that, for every $k < \omega$ the section $(H)_k = \{a : \langle k, a\rangle \in H\}$ is an F_{n_k}-equivalence class and $X \subseteq \bigcup_k [T_k] \cup \bigcup_k (H)_k$.

In particular, if a Σ^1_1 set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then X is covered by the union of all Δ^1_1 F_0-classes, all Δ^1_1 F_1-classes, all Δ^1_1 F_2-classes, et cetera, and all Δ^1_1 compact sets.

P r o o f. The set $C = \mathbf{CT} \cap \Delta^1_1$ of all Δ^1_1 compact trees is Π^1_1, and hence so is $K = \bigcup_{T\in C}[T]$. If $n < \omega$ then let U_n be the union of all Δ^1_1 F_n-classes. Let's show that $U = \bigcup_n U_n$ is Π^1_1 either. We make use of sets $E \subseteq \omega$ and $W, W' \subseteq \omega \times \mathscr{N}$ as in Fact 2.3. The Π^1_1 formula

$$\varphi(e,n) := e \in E \;\wedge\; \forall y, z \in (W')_e\; (y \;\mathsf{F}_n\; z) \;\wedge\; \forall y \in (W')_e\, \forall z\, (y \;\mathsf{F}_n\; z \implies z \in (W)_e)$$

says that $e \in E$ and $(W')_e = (W)_e$ is a F_n-equivalence class. Moreover

$$x \in U \iff \exists n\, \exists e\, (\varphi(e,n) \wedge x \in (W)_e).$$

Case 1: $X \subseteq K \cup U$. Then the set S of all pairs $\langle x, h\rangle$ such that

- either $h = T \in C$ and $x \in [T]$,
- or $h = \langle e, n\rangle \in \Phi = \{\langle e, n\rangle \in E \times \omega : \varphi(e,n)\}$ and $x \in (W)_e$,

is a Π^1_1 set satisfying $X \subseteq \operatorname{proj} S$. By Fact 2.5 there is a Δ^1_1 map f defined on $\mathscr{N}$ and such that $\langle a, f(a)\rangle \in S$ for each $a \in X$. The sets

$$X' = \{x \in X : f(x) \in \mathbf{CT}\} \quad \text{and} \quad X'' = \{x \in X : f(x) \in \Phi\}$$

are Σ^1_1 as well as their images

$$R' = \big\{f(x) : x \in X'\big\} \subseteq C \text{ and } R'' = \big\{f(x) : x \in X''\big\} \subseteq \Phi,$$

and $X' \cup X'' = X$, $R' \cup R'' = \{f(x) : x \in X\}$. By the Σ^1_1 Separation theorem there is a Δ^1_1 set τ such that $R' \subseteq \tau \subseteq C$, and by Fact 2.6 we have $\tau = \{T_k : k < \omega\}$, where $k \longmapsto T_k$ is a Δ^1_1 map. By similar reasons, there is a Δ^1_1 map $k \longmapsto \langle e_k, n_k\rangle$ such that $R'' \subseteq \rho = \{\langle e_k, n_k\rangle : k < \omega\} \subseteq \Phi$. To finish the proof in Case 1, it remains to define

$$H = \big\{\langle k, x\rangle \in \omega \times \mathscr{N} : x \in (W)_{e_k}\big\} = \big\{\langle k, x\rangle \in \omega \times \mathscr{N} : x \in (W')_{e_k}\big\}.$$

Case 2: $A = X \setminus (K \cup U) \neq \varnothing$. Then A is a non-empty Σ^1_1 set. We are going to derive a contradiction. By definition, we have $X \setminus A \subseteq \bigcup_k C_k \cup \bigcup_n \bigcup_k E_{nk}$, where each C_k is compact and each E_{nk} is an F_n-class. Let M be a countable elementary substructure of a sufficiently large structure, containing, in particular, the whole sequence of covering sets C_k and E_{nk}. Below "generic" will mean Gandy-Harrington generic over M.

As $A \neq \varnothing$ is Σ^1_1, there is a perfect set $P \subseteq A$ of points both generic and pairwise generic. It is known that then P is a pairwise F_n-inequivalent set for every n, hence, definitely a set not covered by a countable union of F_n-classes for all $n < \omega$. Thus to get a contradiction it suffices to prove that $P \cap C_k = \varnothing$ for all k. In other words, we have to prove that if $k < \omega$ and $x \in A$ is any generic real then $x \notin C_k$.

Suppose towards the contrary that a non-empty Σ^1_1 condition $Y \subseteq A$ forces that $\mathbf{a} \in C_k$, where $\mathbf{a}$ is a canonical name for the Gandy-Harrington generic real. We claim that Y is not σ-bounded. Indeed otherwise we have $Y \subseteq \bigcup_n [T_n]$ by Theorem 3.1, where all trees $T_n \subseteq \omega^{<\omega}$ are Δ^1_1 and compact, which contradicts the fact that A does not intersect any compact Δ^1_1 set.

Therefore $Y \not\subseteq C_k$. Then there is a point $x \in Y$ and a number m such that the set $I = \{y \in \mathscr{N} : y\restriction m = x\restriction m\}$ does not intersect C_k. But then the Σ^1_1 condition $Y' = Y \cap I$ forces that $\mathbf{a} \notin C_k$, a contradiction. □

Corollary 6.3 *If $\{\mathsf{F}_n\}_{n<\omega}$ is a Δ^1_1 sequence of equivalence relations on $\mathscr{N}$ then the ideal of $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded sets is Π^1_1 on Σ^1_1 and $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$.*

Cf. [23, Section 3.8] on Π^1_1 on Σ^1_1 and $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$ ideals.

Proof. Consider a Σ^1_1 set $P \subseteq \mathscr{N} \times \mathscr{N}$. We have to prove that

$$X = \big\{x \in \mathscr{N} : (P)_x = \{y : \langle x, y\rangle \in P\} \text{ is } \{\mathsf{F}_n\}_{n<\omega}\text{-}\sigma\text{-bounded}\big\}$$

is a Π^1_1 set. By the relativized version of Lemma 6.2, $x \in X$ iff

(∗) there exist $\Delta^1_1(x)$ sequences $\{T_k\}_{k<\omega}$ (of compact trees) and $\{n_k\}_{k<\omega}$ and a $\Delta^1_1(x)$ set $H \subseteq \omega \times \mathscr{N}$ such that, for every $k < \omega$ the section $(H)_k$ is an F_{n_k}-equivalence class and $(P)_x \subseteq \bigcup_k [T_k] \cup \bigcup_k (H)_k$.

A routine analysis (as in the proof of Lemma 6.2) shows that this is a Π^1_1 description of the set X. □

7 Digression: another look on the effectivity

As usual, Lemma 6.2 and Corollary 6.3 remain true for relativized classes. In particular, if $p \in \mathscr{N}$, F_n are $\Delta^1_1(p)$ equivalence relations, and a $\Sigma^1_1(p)$ set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then X is covered by the union of all $\Delta^1_1(p)$ F_n-classes, $n = 0, 1, 2, \dots$, and all $\Delta^1_1(p)$ compact sets. If now $p \neq q \in \mathscr{N}$ is a different parameter, but still F_n are $\Delta^1_1(q)$ and X is $\Sigma^1_1(q)$ and $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded then accordingly X is covered by the union of all $\Delta^1_1(q)$ F_n-classes, $n = 0, 1, 2, \dots$, and all $\Delta^1_1(q)$ compact sets. Those two countable coverings of the same set X can be different, of course. This leads to the question: is there a covering of X of the type indicated, which depends on X and F_n themselves, but not on the choice of a parameter p such that X is $\Sigma^1_1(p)$ and F_n are $\Delta^1_1(p)$. We are able to answer this question in the positive at least in the case of finitely many equivalence relations. The next theorem will be instrumental in the proof of a theorem in Section 10.

Theorem 7.1 *Suppose that $n \geq 1$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ are Borel equivalence relations on $\mathscr{N}$, and a $\mathbf{\Sigma}^1_1$ set $X \subseteq \mathscr{N}$ is $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Then there exist Borel sets $Y_1, \dots, Y_n, X_{n+1} \subseteq \mathscr{N}$ such that*

(i) $X \subseteq Y_1 \cup \dots \cup Y_n \cup X_{n+1}$,

(ii) *each set Y_j is a countable union of F_j-equivalence classes while the set X_{n+1} is σ-bounded,*

(iii) *if $w \in \mathscr{N}$, X is $\Sigma^1_1(w)$, and all relations F_m are $\Delta^1_1(w)$, then there is a parameter $\bar{w} \in \mathscr{N}$ in $\Delta^1_2(w)$ such that both X_{n+1} and all sets Y_j are $\Delta^1_1(\bar{w})$, hence $\Delta^1_2(w)$.*

Thus, under the assumptions of the theorem, there is a Borel covering of the set X satisfying (i) and (ii), and effective as soon as X and F_j are granted some effectivity. Note that the covering (i.e., the sets $Y_1, \dots, Y_n, X_{n+1}$) depends only of X and $\mathsf{F}_1, \dots, \mathsf{F}_n$, but does not depend on w in the context of (iii). It is a challenging problem to get rid of $\bar{w}$ in (iii) (so that the sets X_{n+1} and Y_j are just $\Delta^1_1(w)$ with the same parameter w), but this remains open.

Proof. We define sets $X = X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots \supseteq X_n \supseteq X_{n+1}$ so that $X_{j+1} = X_j \setminus Y_j$, where

(1) $$Y_j = \{x \in \mathscr{N} : \text{the set } X_j \cap [x]_{\mathsf{F}_j} \text{ is not } \{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\}$$

by induction. In particular,

$$\begin{aligned}
Y_1 &= \{x \in \mathscr{N} : \text{the set } X_1 \cap [x]_{\mathsf{F}_1} \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\},\\
Y_2 &= \{x \in \mathscr{N} : \text{the set } X_2 \cap [x]_{\mathsf{F}_2} \text{ is not } \{\mathsf{F}_3, \dots, \mathsf{F}_n\}\text{-}\sigma\text{-bounded}\},\\
&\dots\\
Y_{n-1} &= \{x \in \mathscr{N} : \text{the set } X_{n-1} \cap [x]_{\mathsf{F}_{n-1}} \text{ is not } \{\mathsf{F}_n\}\text{-}\sigma\text{-bounded}\},\\
Y_n &= \{x \in \mathscr{N} : \text{the set } X_n \cap [x]_{\mathsf{F}_n} \text{ is not } \varnothing\text{-}\sigma\text{-bounded}\},
\end{aligned}$$

where $\varnothing$-σ-bounded is the same as just σ-bounded.

Lemma 7.2 *If $1 \leq j \leq n$ then Y_j is a countable union of F_j-equivalence classes and the set $X_{j+1} = X_j \setminus Y_j$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded.*

Proof. Let $\mathscr{Y}_j$ be the family of all sets Y such that Y is a union of at most countably many F_j-classes and $X_j \setminus Y$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Note that $\mathscr{Y}_j$ is a non-empty (since X_j is $\{\mathsf{F}_j, \dots, \mathsf{F}_n\}$-$\sigma$-bounded by induction) σ-filter (since the collection of all $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded sets is a σ-ideal). Therefore $Y'_j = \bigcap \mathscr{Y}_j$ is a set in $\mathscr{Y}_j$, in fact, the $\subseteq$-least set in $\mathscr{Y}_j$.

It remains to show that $Y_j = Y'_j$. We claim that if C is an F_j-class then $C \subseteq Y'_j$ iff $C \subseteq Y'_j$. Indeed if $C \cap Y_j = \varnothing$ then $X_j \cap C$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, thus $Y'_j \setminus C$ is still a set in $\mathscr{Y}_j$, therefore $C \cap Y'_j = \varnothing$. Conversely if $C \cap Y'_j = \varnothing$ then $(X_j \cap C) \subseteq (X_j \setminus Y'_j)$, and hence $X_j \cap C$ is $\{\mathsf{F}_{j+1}, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, so $C \cap Y_j = \varnothing$, as required. □

Thus by the lemma the sets Y_j and X_{n+1} satisfy (i) and (ii). To verify (iii), assume that $w \in \mathscr{N}$, X is $\Sigma^1_1(w)$, and all F_m are $\Delta^1_1(w)$. The main issue is that the sets Y_j, albeit Borel (as countable unions of Borel equivalence classes) do not seem to be $\Delta^1_1(w)$, at least straightforwardly. For instance, Y_1 is $\Sigma^1_1(w)$ by Corollary 6.3 (relativized), and accordingly X_2 is $\Pi^1_1(w)$ (instead of $\Delta^1_1(w)$), which makes it very difficult to directly estimate the class of Y_2 at the next step. This is where a new parameter appears.

We precede the last part of the proof of the theorem with the following auxiliary fact on equivalence relations, perhaps, already known.

Lemma 7.3 *Let E be a Δ^1_1 equivalence relation on $\mathscr{N}$, and $X \subseteq \mathscr{N}$ be a Σ^1_1 set which intersects only countably many E-classes. Then all E-classes $[x]_{\mathsf{E}}$, $x \in X$, are Δ^1_1 sets, and there is an E-invariant Δ^1_1 set $Y \subseteq \mathscr{N}$ such that $X \subseteq Y$ and all E-classes $[y]_{\mathsf{E}}$, $y \in Y$, are Δ^1_1 sets (therefore Y still contains only countably many E-classes).*

Proof. The union C of all Δ^1_1 E-classes is an E-invariant Π^1_1 set. (Cf., e.g., [6, Claim 10.1.2].) Thus, if $X \not\subseteq C$ then $H = X \setminus C$ is a non-empty Σ^1_1 set which does not intersect Δ^1_1 E-classes. Then (see, *e.g.*, Case 2 in the proof of [6, Theorem 10.1.1]) the set H contains a perfect pairwise E-inequivalent set, which contradicts our assumptions. Therefore $X \subseteq C$, so indeed all E-classes $[x]_{\mathsf{E}}$, $x \in X$, are Δ^1_1. To prove the second claim apply the invariant Σ^1_1 separation theorem (cf., e.g., [6, 10.4.2]), which yields an E-invariant Δ^1_1 set Y satisfying $X \subseteq [X]_{\mathsf{E}} \subseteq Y \subseteq C$. □

We continue the proof of Theorem 7.1. The next goal is to find a parameter $q_1 \in \mathscr{N}$ in $\Delta^1_2(w)$ such that the $\Sigma^1_1(w)$ set Y_1 is $\Delta^1_1(q_1)$. Let Π^1_1 sets $\mathbf{E} \subseteq \mathscr{N} \times \omega$ and $\mathbf{W} \subseteq \mathscr{N} \times \omega \times \mathscr{N}$, and a Σ^1_1 set $\mathbf{W}' \subseteq \mathscr{N} \times \omega \times \mathscr{N}$ be as in Fact 2.4. If $w \in \mathscr{N}$ then let $E(w) = \{e : \langle w, e\rangle \in \mathbf{E}\}$ and, for $e < \omega$,

$$W_e(w) = \{x : \langle w, e, x\rangle \in \mathbf{W}\} \quad \text{and} \quad W'_e(w) = \big\{x : \langle w, e, x\rangle \in \mathbf{W}'\big\},$$

so that $W_e(w) = W'_e(w)$ for all $e \in E(w)$.

Assume that $w \in \mathscr{N}$ and X is $\Sigma^1_1(w)$, as in (iii) of the theorem. Let $Q_1(w)$ contain all numbers $e \in E(w)$ such that the set $W_e(w)$ is an F_1-class, and the set $W'_e(w) \cap X_1$ is not $\{\mathsf{F}_2, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Then

$$x \in Y_1 \iff \exists\, e \in Q_1(w)\, (x \in W_e(w)) \iff \exists\, e \in Q_1(w)\, \big(x \in W'_e(w)\big)$$

holds for all $x \in \mathscr{N}$ by Lemma 7.3 (relativized). Thus the set Y_1 is $\Delta^1_1(q_1)$, where $q_1 = Q_1(w)$, and accordingly the set $X_2 = X_1 \setminus Y_1$ is $\Sigma^1_1(w, q_1)$.

If $e \in E(w)$ $\big($and this is a Π^1_1 formula$\big)$, then using $W_e(w)$ and $W'_e(w)$ interchangeably, we express "$W_e(w)$ is an F_1-class" as a Π^1_1 property

$$\forall x, y \big(x \,\mathsf{F}_1\, y \Longrightarrow \big(x \in W'_e(w) \Longrightarrow y \in W_e(w)\big) \wedge \big(y \in W'_e(w) \Longrightarrow x \in W_e(w)\big)\big). \tag{2}$$

Finally, "$W'_e(w) \cap X_1$ is not $\{\mathsf{F}_2, \dots, \mathsf{F}_n\}$-$\sigma$-bounded" is a Σ^1_1 property by Corollary 6.3. It follows that $e \in Q_1(w)$ is a Δ^1_2 relation $\big($more precisely, a conjunction of Π^1_1 and $\Sigma^1_1\big)$.

Arguing the same way, we let $Q_2(w, q_1)$ contain all $e \in E(w)$ such that $W_e(w)$ is an F_2-class and $W'_e(w) \cap X_2$ is not $\{\mathsf{F}_3, \dots, \mathsf{F}_n\}$-$\sigma$-bounded. Then, by the same reasons, $e \in Q_2(w, q_1)$ is a Δ^1_2 relation, Y_2 is $\Delta^1_1(q_2)$, where $q_2 = Q_2(w, q_1)$, and accordingly $X_3 = X_2 \setminus Y_2$ is $\Sigma^1_1(w, q_1, q_2)$.

Iterating this construction, we define parameters $q_1, q_2, \dots, q_n$ such that each Y_j is $\Delta^1_1(q_j)$ and each q_{j+1} is $\Delta^1_2(w, q_1, q_2, \dots, q_j)$, and hence $\Delta^1_2(w)$ by induction. The concatenation $Q'(w) \in \mathscr{N}$ of the reals $w, q_1, q_2, \dots, q_n$ is then $\Delta^1_2(w)$, therefore $\bar{w} = Q'(w)$ implies (iii). □

We don't know whether the theorem still holds for countably infinite sequences of equivalence relations. Yet the proof miserably fails in this case. Indeed, let, for any n, F_n be an equivalence relation on $\mathscr{N}$ whose classes are $I_k = \{x \in \mathscr{N} : x(0) = k\}$, $k = 0, 1, \dots, n$, and all singletons outside of these large classes. The whole space $\mathscr{N} = \bigcup_n I_n$ is $\{\mathsf{F}_0, \mathsf{F}_1, \mathsf{F}_2, \dots\}$-$\sigma$-bounded, of course. But running the construction as above, we'll obviously have $Y_0 = Y_1 = Y_2 = \cdots = \varnothing$ (as each F_n-class is covered by an appropriate F_{n+1}-class), which results in nonsense.

There is another interesting problem. Under the assumptions of the theorem, the covering of X by sets $Y_1, \dots, Y_n, X_{n+1} \subseteq \mathscr{N}$ depends on X but is independent of the choice of a parameter p as in (iii). On the other hand, if such a parameter p, and accordingly $\bar{p}$ as in (iii), is given then not only each Y_j but also a representation of $Y_j = \bigcup_m Y_{jm}$ as a countable union of F_j-classes Y_{jm}, can be obtained in $\Delta^1_1(\bar{p})$ by Lemma 6.2. One may ask whether such a decomposition of each Y_j is available in a way independent of the choice of p (as the sets Y_j themselves). The answer in the negative is expected, but it may likely take a lot of work.

8 Generalization of the σ-bounded dichotomy: the theorem

Coming back to the content of Section 6, we'll prove the following theorem in this section.

Theorem 8.1 *Suppose that $n < \omega$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ are Δ^1_1 equivalence relations on $\mathscr{N}$, and $A \subseteq \mathscr{N}$ is a Σ^1_1 set. Then one and only one of the following two claims holds*:

(I) *the set A is $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded—and therefore Δ^1_1-effectively $\{\mathsf{F}_n\}_{n<\omega}$-$\sigma$-bounded as in Lemma 6.2*;

(II) *there exists an $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-superperfect set $P \subseteq A$.*

If $n = 0$ then this theorem is equivalent to Theorem 3.1: indeed, if $\mathscr{F} = \varnothing$ then $\varnothing$-σ-bounded sets are just σ-bounded, while $\varnothing$-superperfect sets are just superperfect.

The following key result of Solecki and Spinas [20, Corollary 2.2] will be an essential pre-requisite in the proof of Theorem 8.1.

Proposition 8.2 *Let $E \subseteq \mathscr{N} \times \mathscr{N}$ be a $\mathbf{\Sigma}^1_1$ set such that every section $(E)_x$, $x \in \mathscr{N}$, is σ-bounded. Then there is a superperfect set $P \subseteq \mathscr{N}$* ***free for*** *E in the sense that if $x \neq y$ belong to P then $\langle x, y\rangle \notin E$.*

Note that if E is an equivalence relation then a set free for E is the same as a pairwise E-inequivalent set.

Proof of Theorem 8.1. We argue by induction on n. The case $n = 0$ $(\text{then } \{\mathsf{F}_1, \dots, \mathsf{F}_n\} = \varnothing)$ is covered by Theorem 3.1. Now the step $n \to n+1$.

Let $\mathsf{F}_1, \dots, \mathsf{F}_n, \mathsf{F}_{n+1}$ be Δ^1_1 equivalence relations on $\mathscr{N}$, and $A \subseteq \mathscr{N}$ be a Σ^1_1 set. The set

$$\boldsymbol{U} = \{x \in A : [x]_{\mathsf{F}_1} \cap A \text{ is non-}\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}\text{-}\sigma\text{-bounded}\}$$

is Σ^1_1 by Corollary 6.3.

Case 1: the Σ^1_1 set $\boldsymbol{U}$ has only countably many F_1-classes. Then by Lemma 7.3, there is an F_1-invariant Δ^1_1 set D such that $\boldsymbol{U} \subseteq D$, D contains only countably many F_1-classes, and all of them are Δ^1_1.

Subcase 1.1: the complementary Σ^1_1 set $B = A \setminus D$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. Then the whole domain $A = D \cup B$ is $\{\mathsf{F}_1, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded, hence we have (I) for $\mathsf{F}_1, \dots, \mathsf{F}_n, \mathsf{F}_{n+1}$.

Subcase 1.2: B is non-$\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. By the inductive hypothesis there is an $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect set $P \subseteq B$. Let $x \in P$. Then the class $[x]_{\mathsf{F}_1}$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. We claim that the set $P_x = [x]_{\mathsf{F}_1} \cap P$ is just σ-bounded. Indeed by definition $P_x \subseteq Y \cup \bigcup_k X_k$, where Y is σ-bounded while each X_k is an $\mathsf{F}_{n(k)}$-equivalence class for some $n(k) = 2, 3, \dots, n+1$. By construction P has at most one common point with each X_k. Therefore the set $P_x \setminus Y$ is at most countable, hence, σ-bounded, and we are done.

Thus all F_1-classes inside P are σ-bounded. By Proposition 8.2, there is a superperfect pairwise F_1-inequivalent set $Q \subseteq P$—then the set Q is $\{\mathsf{F}_1, \dots, \mathsf{F}_{n+1}\}$-superperfect by construction. Thus (II) holds.

Case 2: $\boldsymbol{U}$ has uncountably many F_1-classes. Then by Silver there exists a perfect pairwise F_1-inequivalent set $X \subseteq \boldsymbol{U}$. If $x \in X$ then by definition the set $[x]_{\mathsf{F}_1} \cap A$ is not $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. Therefore by the inductive hypothesis there exists an $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect set $Y \subseteq [x]_{\mathsf{F}_1} \cap A$, and hence a superperfect tree $T \subseteq \omega^{<\omega}$ such that $[T] = Y$. The next step is to get such a tree T by means of a Borel function.

Lemma 8.3 *In our assumptions, there is a perfect set $X' \subseteq X$ and a Borel map $x \longmapsto T_x$ defined on X', such that if $x \in X'$ then T_x is a superperfect tree, $[T_x] \subseteq [x]_{\mathsf{F}_1} \cap A$, and $[T_x]$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect.*

Proof. Let $p \in \mathscr{N}$ be a parameter such that X is $\Pi^0_1(p)$.

Let $\mathbf{V}$ be the set universe considered, and let $\mathbf{V}^+$ be a generic extension of $\mathbf{V}$ such that $\omega_1^{\mathbf{L}[p]}$ is countable in $\mathbf{V}^+$. Let X^+ be the $\mathbf{V}^+$-extension of X, so that X^+ is $\Pi^0_1(p)$ in $\mathbf{V}^+$ and $X = X^+ \cap \mathbf{V}$. Let A^+ and F^+_i be similar extensions of resp. A, F_i. It is true then in $\mathbf{V}^+$ by the Shoenfield absoluteness that each F^+_i is a Δ^1_1 equivalence relation on $\mathscr{N}$, and X^+ is a perfect set in $\Pi^0_1(p)$. Moreover, it is true in $\mathbf{V}^+$ by Shoenfield that

$(*)$ if $x \in X^+$ then the set $[x]_{\mathsf{F}^+_1} \cap A^+$ is not $\{\mathsf{F}^+_2, \dots, \mathsf{F}^+_{n+1}\}$-$\sigma$-bounded

— simply because the formula

$$\forall\, x \in X\ ([x]_{\mathsf{F}_1} \cap A \text{ is not } \{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}\text{-}\sigma\text{-bounded})$$

is essentially Π^1_2 by Corollary 6.3, and is true in $\mathbf{V}$. It follows by the inductive hypothesis (applied in $\mathbf{V}^+$) that, in $\mathbf{V}^+$, the $\Pi^1_1(p)$ set W^+ of all pairs $\langle x, T\rangle$ such that $x \in X^+$, $T \subseteq \omega^{<\omega}$ is a superperfect tree, and

$$[T] \subseteq [x]_{\mathsf{F}^+_1} \cap A^+ \ \wedge \text{ the set } [T] \text{ is } \{\mathsf{F}^+_2, \dots, \mathsf{F}^+_{n+1}\}\text{-superperfect},$$

— satisfies $\operatorname{proj} W^+ = X^+$. Therefore by the Shoenfield absoluteness theorem the set $W = W^+ \cap \mathbf{V}$ is $\Pi^1_1(p)$ and satisfies $\operatorname{proj} W = X$ in $\mathbf{V}$.

Applying the Kondô-Addison uniformization in $\mathbf{V}^+$, we get a $\Pi^1_1(p)$ set $U^+ \subseteq W^+$ which uniformizes W^+, in particular, $\operatorname{proj} U^+ = \operatorname{proj} W^+ = X^+$. The corresponding set $U = U^+ \cap \mathbf{V}$ of type $\Pi^1_1(p)$ in $\mathbf{V}$ then uniformizes W and satisfies $\operatorname{proj} U = \operatorname{proj} W = X$ still by Shoenfield.

Now, by the choice of the universe $\mathbf{V}^+$, the uncountable $\Pi^1_1(p)$ set U^+ must contain a perfect subset $P^+ \subseteq U^+$ of class $\Pi^0_1(q)$ for a parameter $q \in \mathbf{L}[p]$, hence, $q \in \mathbf{V}$. The according set $P = P^+ \cap \mathbf{V}$ is then a perfect subset of U in $\mathbf{V}$, and hence $X' = \operatorname{proj} P \subseteq X$ is a perfect set.

Finally, if $x \in X'$ then let T_x be the only element such that $\langle x, T_x \rangle \in P$. The map $x \longmapsto T_x$ is Borel. On the other hand, still by the Shoenfield theorem, if $x \in X'$ then $[T_x] \subseteq [x]_{\mathsf{F}_1} \cap A$, and the set $[T_x]$ is $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect. □

We continue the proof of Theorem 8.1. Let $X' \subseteq X$ and a Borel map $x \longmapsto T_x$ be as in the lemma. If $x \in X'$ and $i = 2, \dots, n+1$, then every F_i-class $[y]_{\mathsf{F}_i}$ has at most one point common with the set $Y_x = [T_x]$. Thus if C is a $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded set then the intersection $C \cap Y_x$ is σ-bounded and hence $C \cap Y_x$ is meager in Y_x.

There is a Borel set $W \subseteq X' \times \mathscr{N}$ such that the collection of all sections $(W)_x$, $x \in X'$, is equal to the family of all countable unions of F_i-classes, $i = 2, \dots, n+1$, plus a σ-bounded $\mathbf{F}_\sigma$ set. (Note that σ-bounded $\mathbf{F}_\sigma$ sets is the same as σ-compact sets, and that every σ-bounded set is a subset of a σ-bounded $\mathbf{F}_\sigma$ set.) Thus if $x \in X'$ then $(W)_x \cap Y_x$ is meager in Y_x by the above. Therefore, by a version of "comeager uniformization", there is a Borel map f defined on X' such that if $x \in X'$ then $f(x) \in Y_x \setminus (W)_x$. Clearly f is $1-1$, hence the set $R = \{f(x) : x \in X'\}$ is Borel.

Moreover R is pairwise F_1-inequivalent by construction. We assert that R is non-$\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded, in particular, not σ-bounded!

Indeed suppose otherwise. Then there is $x \in X'$ such that $R \subseteq (W)_x$. But then $f(x) \in (W)_x$, which contradicts the choice of f.

Thus indeed R is non-$\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-$\sigma$-bounded. It follows by the inductive hypothesis that there exists a $\{\mathsf{F}_2, \dots, \mathsf{F}_{n+1}\}$-superperfect set $P \subseteq R$. And P is pairwise F_1-inequivalent since so is R. We conclude that P is even $\{\mathsf{F}_1, \dots, \mathsf{F}_{n+1}\}$-superperfect, which leads to (II) of the theorem. □

It is an interesting problem to figure out whether Theorem 8.1 is true for a countable infinite family of equivalence relations (as in Lemma 6.2). The inductive proof presented above is of little help, of course. Another problem is to figure out whether the theorem still holds for $\mathbf{\Pi}^1_1$ equivalence relations, as the classical Silver dichotomy does. This is open even for the case of one $\mathbf{\Pi}^1_1$ equivalence relation, since the background result, Proposition 8.2, does not cover this case. And finally we don't know whether Theorem 8.1 can be strengthened to yield the existence of sets free (as in Proposition 8.2) for a given (finite or countable) collection of Borel sets.

It remains to note that Theorem 8.1 (in its relativized form) implies the following theorem, perhaps, not known previously in such a generality.

Theorem 8.4 *Suppose that $\mathsf{F}_1, \dots, \mathsf{F}_n$ are Borel equivalence relations on a Polish space $\mathbb{X}$, and $A \subseteq \mathbb{X}$ is a $\mathbf{\Sigma}^1_1$ set. Then either A is $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-bounded, or there exists an $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-superperfect set $P \subseteq A$.*

Yet the case $n = 1$ is known in the form of the following (not yet published) superperfect dichotomy theorem of Zapletal:

Theorem 8.5 *If E be a Borel equivalence relation on $\mathscr{N}$ and $A \subseteq \mathscr{N}$ is a $\mathbf{\Sigma}^1_1$ set then either A is covered by countably many E-classes and a σ-bounded set or there is a superperfect pairwise E-inequivalent set $P \subseteq A$.*

Theorem 8.5 can be considered as a "superperfect" version of Silver's dichotomy (cf. [19] or [6, 10.1]), saying that if E is a Borel equivalence relation then either the domain of E is a union of countably many E-classes or there is a perfect pairwise E-inequivalent set $Y \subseteq D$.

9 The case of $\mathbf{\Sigma}^1_2$ sets: preliminaries

In view of the counterexamples in Section 5, one can expect that positive results for Σ^1_2 sets similar to Theorems 3.1, 8.1, and 3.2 should be expected in terms of ω_1-unions of compact sets. And indeed using a determinacy-style argument, Kechris proved in [8] the following theorem, presented here in a somewhat abridged form.

Theorem 9.1 *If* $A \subseteq \mathscr{N}$ *is a* Σ^1_2 *set then one of the following two claims* (I), (II) *holds*:

(I) A *is* $\mathbf{L}$-σ*-bounded, in the sense that it is covered by the union of all sets* $[T]$*, where* $T \in \mathbf{L}$ *is a compact tree*[2] *(hence not necessarily a countable union)—or equivalently, for each* $x \in A$ *there is* $y \in \mathscr{N} \cap \mathbf{L}$ *with* $x \le^* y$*, where* $\le^*$ *is the eventual domination order on* $\mathscr{N}$*,*

(II) *there is a superperfect set* $P \subseteq A$. □

Our next goal is to generalise this result in the directions of Theorem 8.1. We are going to change superperfect sets in (II) by $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-superperfect sets, where $\mathsf{F}_1, \dots, \mathsf{F}_n$ is a given collection of Δ^1_1 equivalence relations. As for (I), one would naturally look for a condition like: for each $x \in A$, either there is $y \in \mathscr{N} \cap \mathbf{L}$ with $x \le^* y$, or there is $j = 1, \dots, n$ and an "$\mathbf{L}$-presented" F_j-equivalence class containing x, whatever being "$\mathbf{L}$-presented" would mean. The following example shows that the most elementary definition of "$\mathbf{L}$-presented" as "containing a constructible element" fails.

Example 9.2 Let F be the equivalence relation of equality of countable sets of reals. That is, its domain is the set $\mathscr{N}^\omega$ of all infinite sequences of reals, and for $x, y \in \mathscr{N}^\omega$, $x \;\mathsf{F}\; y$ iff $\operatorname{ran} x = \operatorname{ran} y$. Let us work in a $\mathrm{Coll}(\omega_1^{\mathbf{L}})$-generic extension $\mathbf{L}[f]$ of $\mathbf{L}$, where $f : \omega \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$ is a generic collapse map. Let A consist of all $x \in \mathscr{N}^\omega$ such that $\operatorname{ran} x$ (a set of reals) belongs to $\mathbf{L}$ (but x itself does not necessarily belong to $\mathbf{L}$). Then A is Σ^1_2 in $\mathbf{L}[f]$. Moreover if $x \in A$ then the F-class $[x]_{\mathsf{F}}$ is not σ-bounded, and the quotient A/F (the set of all F-classes inside A) is uncountable in $\mathbf{L}[f]$.

We believe that there is no perfect (let alone superperfect) pairwise F-inequivalentset $P \subseteq A$ in $\mathbf{L}[f]$, which is quite a safe conjecture in view of the results in [2]. Yet to make the example self-contained let us add to $\mathbf{L}[f]$ a set C of $\aleph_3^{\mathbf{L}} = \aleph_2^{\mathbf{L}[f]}$ Cohen reals. By a simple cardinality argument, there are no perfect pairwise F-inequivalent sets $P \subseteq A$ in $\mathbf{L}[f, C]$.

However, in $\mathbf{L}[f, C]$, the quotient A/F has uncountably many particular F-classes which are non-σ-bounded and even non-$\mathbf{L}$-σ-bounded in the sense of (I) above, but contain no constructible elements. Thus A neither contains an F-superperfect subset nor satisfies the condition that *for each* $x \in A$*, either there is* $y \in \mathscr{N} \cap \mathbf{L}$ *with* $x \le^* y$*, or there is an* F*-equivalence class containing* x *and containing a constructible element*.

Our model for "$\mathbf{L}$-presented" will be somewhat more complex than just "containing a constructible element". It will be based on a certain uniform version of Δ^1_1, with ordinals as background parameters.

Let $\mathrm{WO} \subseteq \mathscr{N}$ be the Π^1_1 set of all codes of countable (including finite) ordinals, and if $\xi < \omega_1$ then let $\mathrm{WO}_\xi = \{w \in \mathrm{WO} : w \text{ codes } \xi\}$.

Definition 9.3 A Σ^1_2 map $h : \mathscr{N} \to \mathscr{N}$ is *absolutely total* if it remains total in any set-generic extension of the universe. In other words, it is required that there is a Σ^1_2 formula $\sigma(\cdot,\cdot)$ such that $h = \{\langle x, y\rangle : \sigma(x, y)\}$ and the sentence $\forall\, x\, \exists\, y\, \sigma(x, y)$ is forced by any set forcing.

A total but not absolutely total map can be defined in $\mathbf{L}$ by letting $h(x)$ be the Gödel-least $w \in \mathrm{WO}$ such that x appears at the ξ-th step of the Gödel construction, where $\xi < \omega_1$ is the ordinal coded by w.

Definition 9.4 (1) Suppose that $\xi < \omega_1$. A set $X \subseteq \mathscr{N}$ is *essential* $\Sigma^1_n(\xi)$ if there is a Σ^1_n formula $\varphi(x, w)$ such that $X = \{x \in \mathscr{N} : \varphi(x, w)\}$ for every $w \in \mathrm{WO}_\xi$. Essential $\Pi^1_n(\xi)$ sets are defined similarly, while an essential $\Delta^1_n(\xi)$ set is any set both essential $\Sigma^1_n(\xi)$ and essential $\Pi^1_n(\xi)$.

(2) A set X is *essential* $\big(\Delta^1_1/\Delta^1_2\big)(\xi)$ if there is an absolutely total Σ^1_2 map h, a Σ^1_1 formula $\chi(\cdot,\cdot)$, and a Π^1_1 formula $\chi'(\cdot,\cdot)$, such that if $w \in \mathrm{WO}_\xi$ then $X = \{x \in \mathscr{N} : \chi(x, h(w))\} = \{x \in \mathscr{N} : \chi'(x, h(w))\}$.

Cf. Section 11 for more on essential $\big(\Delta^1_1/\Delta^1_2\big)(\xi)$ sets. In particular we'll show that those sets admit a direct Borel coding with codes in $\mathbf{L}$.

10 The case of Σ^1_2 sets: the result

Here we prove a theorem which generalizes Theorem 9.1. If F is an equivalence relation on $\mathscr{N}$ then let a σ-F-*class* be any finite or countable union of F-equivalence classes.

Theorem 10.1 *Assume that* $n < \omega$, $\mathsf{F}_1, \dots, \mathsf{F}_n$ *are* Δ^1_1 *equivalence relations on* $\mathscr{N}$ *and* $A \subseteq \mathscr{N}$ *is a* Σ^1_2 *set. Then we have one of the following*:

[2] $\mathbf{L}$ is the constructible universe.

(I) *A is $\mathbf{L}$-$\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded, in the sense that for each $x \in A$:*
- *either there is $y \in \mathscr{N} \cap \mathbf{L}$ such that $x \le^* y$,*
- *or there is $j = 1,\dots,n$ and a σ-F_j-class C which contains x and is essential $\big(\Delta^1_1/\Delta^1_2\big)(\xi)$ for some $\xi < \omega_1$;*

(II) *there exists an $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-superperfect set $P \subseteq A$.*

The "or" option in (I) of Theorem 10.1 leaves a certain sense of dissatisfaction since one would rather look for coverage by F_j-classes themselves than σ-F_j-classes. Cf. Section 12 on resolution of σ-F_j-classes into appropriately definable F_j-classes, in the context of (I) of the theorem.

Proof. (Based on Theorems 7.1 and 8.1 in key arguments.) First of all, we can w.l.o.g. assume that A is a Π^1_1 set. Indeed, by Kondô's uniformization, A is the projection of a uniform Π^1_1 set $B \subseteq \mathscr{N} \times 2^\omega$. For $\langle x,a\rangle, \langle x',a'\rangle$ being pairs in $\mathscr{N} \times 2^\omega$, let $\langle x,a\rangle\ \mathsf{F}'_j\ \langle x',a'\rangle$ iff $x\ \mathsf{F}_j\ x'$. If Theorem 10.1 holds for B and $\mathsf{F}'_1,\dots,\mathsf{F}'_n$ (with $\langle x,a\rangle \le^* y$ iff $x \le^* y$ in (I)) then quite clearly it holds for A and $\mathsf{F}_1,\dots,\mathsf{F}_n$.

Thus let A be a Π^1_1 set, and let $A = \bigcup_{\xi<\omega_1} A^\xi$ be the ordinary decomposition of A into pairwise disjoint Borel sets A^ξ (called *constituents*). There is a Σ^1_1 formula $\beta(w,x)$ and a Π^1_1 formula $\beta'(w,x)$ such that

($*$) if $\xi < \omega_1$ and $w \in \mathrm{WO}_\xi$ then $A^\xi = \{x : \beta(w,x)\} = \{x : \beta'(w,x)\}$ is a Borel set, and in fact even set essential $\Delta^1_1(\xi)$.

Case A: There is an ordinal $\xi < \omega_1$ such that A_ξ is **not** $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded. Then we have (II) of the theorem by Theorem 8.1.

Case B: All sets A_ξ are $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded. Then, by Theorem 7.1, for each ξ there exist Borel sets $Y^\xi_1,\dots,Y^\xi_n, X^\xi_{n+1} \subseteq \mathscr{N}$ satisfying (i), (ii), (iii) of Theorem 7.1 for $X = A^\xi$—in particular, each set Y^ξ_j is a countable union of F_j-equivalence classes, each set X^ξ_{n+1} is σ-bounded, and $A^\xi \subseteq Y^\xi_1 \cup \dots \cup Y^\xi_n \cup X^\xi_{n+1}$.

Our initial plan was to prove that the sets Y^ξ_j and X^ξ_{n+1} are essential $(\Delta^1_1/\Delta^1_2)(\xi)$, and moreover, each set X^ξ_{n+1} (and in fact any set both essential $(\Delta^1_1/\Delta^1_2)(\xi)$ and σ-bounded) is $\mathbf{L}$-$\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-bounded by virtue of exclusively the "either" option in (I) of the theorem. The "moreover" claim was based on a metamathematical product forcing argument, similar to the one used in the proof of Lemma 8.3. The anonymous referee suggested another argument of more conventional sort, which we present here with thanks.

Coming back to the proof of Theorem 7.1 with the given set $X = X(w) = \{x\beta(w,x)\}$ and $w \in \mathscr{N}$ being an arbitrary parameter, we observe that the argument yields a Δ^1_2 function $Q' : \mathscr{N} \to \mathscr{N}$ such that

(i) Q' is absolutely total Σ^1_2—since it is defined in n steps, such that each step is governed by a combination of Σ^1_1 and Π^1_1 formulas;

(ii) if $\xi < \omega_1$ and $w \in \mathrm{WO}_\xi$, then $X_{n+1}(w) = X^\xi_{n+1}$;

(iii) if $\xi < \omega_1$ and $w \in \mathrm{WO}_\xi$ then $X_{n+1}(w)$ is $\Sigma^1_1(Q'(w))$;

(iv) moreover, there is a single Σ^1_1-set $U \subseteq \mathscr{N} \times \mathscr{N}$ such that $X_{n+1}(w) = \{x : \langle Q'(w), x\rangle \in U\}$ whenever $\xi < \omega_1$ and $w \in \mathrm{WO}_\xi$.

It immediately follows that $B = \bigcup_{\xi<\omega_1} X^\xi_{n+1} = \bigcup_{w\in\mathrm{WO}} X_{n+1}(w)$ is a Σ^1_2 set. Therefore, by Theorem 9.1, either B is $\mathbf{L}$-*σ-bounded* as in (I) of Theorem 9.1, or there is a superperfect set $P \subseteq B$. Thus to prove Theorem 10.1 it remains to check that the "or" option here definitely fails.

Suppose towards the contrary that $S \subseteq B$ is a superperfect set. Then $S \subseteq A$, and hence, by the known properties of constituents, there is an ordinal $\xi < \omega_1$ such that $S \subseteq A^{<\xi} = \bigcup_{\eta<\xi} A^\eta$. Then obviously $S \subseteq B^{<\xi} = \bigcup_{\eta<\xi} X^\eta_{n+1}$. However each set X^η_{n+1} is σ-bounded by the above, and hence the set $B^{<\xi}$ is σ-bounded as well, so it cannot contain a superperfect subset, as required. □

11 The case of Σ^1_2 sets: Borel coding and absoluteness

Each essential $\big(\Delta^1_1/\Delta^1_2\big)(\xi)$ set X is Borel, hence, it admits a Borel code. Moreover, if X is essential $\big(\Delta^1_1/\Delta^1_2\big)(\xi)$ via an absolutely total Σ^1_2 map h, and $w \in \mathrm{WO}_\xi$, then X admits a Borel code in $\mathbf{L}[w]$. Our next goal will be to show that such a set X admits a Borel code—in a certain generalized sense which allows uncountable Borel operations—even in $\mathbf{L}$.

Let $\mathrm{Ord}^{<\omega}$ be the class of all strings (finite sequences) of ordinals. If $s \in \mathrm{Ord}^{<\omega}$ and $\xi \in \mathrm{Ord}$ then $s^\frown\xi$ denotes the string s extended by ξ; if $s \in \mathrm{Ord}^{<\omega}$ then $\mathrm{lh}\, s$ is the length of s; if $m < \mathrm{lh}\, s$ then $s \restriction m$ is the restricted string. By Λ we denote the empty string; $\mathrm{lh}\, \Lambda = 0$ and $\Lambda = s \restriction 0$ for any $s \in \mathrm{Ord}^{<\omega}$.

A set $T \subseteq \mathrm{Ord}^{<\omega}$ is a *tree* if $T \neq \varnothing$, and for any $s \in T$ and $m < \mathrm{lh}\, s$ we have $s \restriction m \in T$. Then let $\sup T$ be the least ordinal λ such that $T \subseteq \lambda^{<\omega}$, and let $\mathrm{Max}\, T$ be the set of all $\subseteq$-maximal elements $s \in T$. Obviously $\Lambda \in T$ for any tree T.

A tree T is *well-founded* iff it contains no infinite branches. In this case, a *rank function* $s \longmapsto |s|_T \in \mathrm{Ord}$ can be associated with T so that $|t|_T = \sup_{t^\frown\xi \in T}(|t^\frown\xi|_T + 1)$ (the least ordinal strictly bigger than all ordinals of the form $|t^\frown\xi|_T$, where $\xi \in \mathrm{Ord}$ and $t^\frown\xi \in T$) for each $t \in T$. In particular $|s|_T = 0$ for any $s \in \mathrm{Max}\, T$. Let $|T| = |\Lambda|_T$ (the *rank* of T).

Definition 11.1 Let $\mathbb{K}$ be the class of all *generalized Borel codes* in $\mathbf{L}$, i.e., all pairs $c = \langle T, d\rangle = \langle T_c, d_c\rangle \in \mathbf{L}$, where $T \subseteq \mathrm{Ord}^{<\omega}$ is a well-founded tree and $d \subseteq T \times \omega^{<\omega}$. In this case, a set $[T, d, s] \subseteq \mathscr{N}$ can be defined for each $s \in T$ by induction on $|s|_T$ so that

if $s \in \mathrm{Max}\, T$ then $[T, d, s] = \mathscr{N} \setminus \bigcup_{\langle s,u\rangle \in d} \mathscr{N}_u$;

if $|s|_T > 0$ then $[T, d, s] = \mathscr{N} \setminus \bigcup_{s^\frown\xi \in T} [T, d, s^\frown\xi]$.

Recall that $\mathscr{N}_u = \{a \in \mathscr{N} : u \subset a\}$ is a Baire interval. Finally we put $[T, d] = [T, d, \Lambda]$.

If $\langle T, d\rangle \in \mathbb{K}$ and $\sup T < \omega_1$ then $[T, d]$ is a Borel set in $\mathbf{\Pi}^0_{1+|T|}$. We stress that only *constructible* codes are considered.

Definition 11.2 If $\rho < \omega_1$ then let $\mathbb{K}_\rho \in \mathbf{L}$ be the set of all codes $\langle T, d\rangle \in \mathbb{K}$ such that $|T| \leq \rho$ and $\sup T \leq \omega_\rho^{\mathbf{L}}$. (Not necessarily $\sup T < \omega_1$.) Accordingly let $[\mathbb{K}_\rho] = \{[T, d] : \langle T, d\rangle \in \mathbb{K}_\rho\}$.

Any essential $(\Delta^1_1/\Delta^1_2)(\xi)$ set is essential $\Delta^1_2(\xi)$, and hence $\Delta^{\mathrm{HC}}_1(\xi)$. (Recall that HC is the set of all hereditarily countable sets.) This simple fact will allow us to make use of the following result, explicitly proved in [4, Lemma 4] on the base of ideas and technique developed in [21, 22].

Proposition 11.3 *Let $X, Y \subseteq \mathscr{N}$ are two disjoint sets in $\Sigma^{\mathrm{HC}}_1(\omega_1)$, i.e., Σ^{HC}_1 with any finite number of parameters in ω_1. Suppose that $\rho < \omega_1^{\mathbf{L}}$ and X is $\mathbf{\Pi}^0_{1+\rho}$-separable from Y. Then there is a separating set in $[\mathbb{K}_\rho]$. In particular if $X \subseteq \mathscr{N}$ is a set in $\Delta^{\mathrm{HC}}_1 \cap \mathbf{\Pi}^0_{1+\rho}$ then $X \in [\mathbb{K}_\rho]$.*

For instance, if $\rho = 0$, so that $\mathbf{\Pi}^0_{1+\rho} =$ closed sets, then the result takes the form: any closed Δ^{HC}_1 set $X \subseteq \mathscr{N}$ has a code in

$$\mathbb{K}_0 = \{\langle T, d\rangle \in \mathbb{K} : |T| = 0 \text{ (hence just } T = \{\Lambda\}) \ \wedge\ \sup T \leq \omega\},$$

but this can be easily established directly.

Thus sets essential $(\Delta^1_1/\Delta^1_2)(\xi)$, $\xi < \omega_1$, even those essential $\Delta^1_2(\xi)$, admit a straight Borel coding by (not necessarily *countable*) codes in $\mathbf{L}$. We'll show now that such a coding can be chosen in a certain **absolute** way.

Remark 11.4 Suppose that $\xi < \omega_1$ and a set $X \subseteq \mathscr{N}$ is essential $(\Delta^1_1/\Delta^1_2)(\xi)$, via an absolutely total Σ^1_2 map h and formulas χ, χ' as in Definition 9.4(2). Then the following is true in the ground universe $\mathbf{V}$:

(a) if $v, w \in \mathrm{WO}_\xi$ and $x \in \mathscr{N}$ then

$$\chi(x, h(v)) \iff \chi(x, h(w)) \iff \chi'(x, h(v)) \iff \chi'(x, h(w)).$$

If we eliminate h by a formula σ as in Definition 9.3 then (a) becomes a Π^1_2 sentence. Therefore (a) is true in any generic extension $\mathbf{V}[G]$ of $\mathbf{V}$ by Shoenfield, and moreover, in any generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ such that $\xi < \omega_1^{\mathbf{L}[G]}$. This allows us to unambiguously define extensions $h^{\mathbf{V}[G]}$ of h (a total map) and $X^{\mathbf{V}[G]}$ of X to $\mathbf{V}[G]$, using the same formulas, so that $X^{\mathbf{V}[G]}$ is an essential $(\Delta^1_1/\Delta^1_2)(\xi)$ set in $\mathbf{V}[G]$ still via $h^{\mathbf{V}[G]}, \chi, \chi'$. Then, assuming $\xi < \omega_1^{\mathbf{L}[G]}$, we define associated restrictions $h^{\mathbf{L}[G]} = h^{\mathbf{V}[G]} \cap \mathbf{L}[G]$ and $X^{\mathbf{L}[G]} = X^{\mathbf{V}[G]} \cap \mathbf{L}[G]$ to $\mathbf{L}[G]$, so that $X^{\mathbf{L}[G]}$ is an essential $(\Delta^1_1/\Delta^1_2)(\xi)$ set in $\mathbf{L}[G]$ via $h^{\mathbf{L}[G]}, \chi, \chi'$ as well.

And if E is a Δ^1_1 equivalence relation in $\mathbf{V}$, then, even easier, we define an extension $\mathsf{E}^{\mathbf{V}[G]}$ of E to $\mathbf{V}[G]$, using the same formulas which define E, so that $\mathsf{E}^{\mathbf{V}[G]}$ is a Δ^1_1 equivalence relation in $\mathbf{V}[G]$ by Shoenfield, and then define $\mathsf{E}^{\mathbf{L}[G]}{=}\mathsf{E}^{\mathbf{V}[G]}\cap\mathbf{L}[G]$ (a Δ^1_1 equivalence relation in $\mathbf{L}[G]$).

Definition 11.5 Let $\xi,\rho<\omega_1$. An essential $\left(\Delta^1_1/\Delta^1_2\right)(\xi)$ set $X\subseteq\mathscr{N}$ *absolutely belongs to* $[\mathbb{K}_\rho]$ if there is a code $\langle T,d\rangle\in\mathbb{K}_\rho$ such that we have $X^{\mathbf{V}[G]}=[T,d]$ in any set generic extension $\mathbf{V}[G]$ of the universe $\mathbf{V}$. Note that then by Shoenfield the equality $X^{\mathbf{L}[G]}=[T,d]$ also holds in any generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ such that $\xi<\omega_1^{\mathbf{L}[G]}$.

Lemma 11.6 *Suppose that $\xi<\omega_1$, $\rho<\omega_1^{\mathbf{L}}$, and a set $X\subseteq\mathscr{N}$ is essential $\left(\Delta^1_1/\Delta^1_2\right)(\xi)$. Then X absolutely belongs to $[\mathbb{K}_\rho]$.*

Proof. Let a map $f:\omega\xrightarrow{\text{onto}}\omega^{\mathbf{L}}_{\rho+1}$ be collapse generic over $\mathbf{V}$, the ground set universe. Let $X^{\mathbf{V}[f]}\in\mathbf{V}[f]$ be the extension of X to $\mathbf{V}[f]$, as above. Then $X^{\mathbf{V}[f]}$ is essential $\left(\Delta^1_1/\Delta^1_2\right)(\xi)$ in $\mathbf{V}[f]$, and hence by Proposition 11.3 there is a code $\langle T,d\rangle\in\mathbb{K}_\rho$ such that $X^{\mathbf{V}[f]}=[T,d]$ in $\mathbf{V}[f]$. To prove, that this code witnesses that X absolutely belongs to $[\mathbb{K}_\rho]$, consider any generic extension $\mathbf{V}[G]$. It can be assumed that G is generic even over $\mathbf{V}[f]$.

Let $X^{\mathbf{V}[G]}$, $X^{\mathbf{V}[f,G]}$ be the extensions of X (a set in $\mathbf{V}$) to resp. $\mathbf{V}[G]$, $\mathbf{V}[f,G]$ (cf. Remark 11.4). Then $\langle T,d\rangle$ is a countable Borel code in $\mathbf{V}[f]$ and in $\mathbf{V}[f,G]$ by the choice of f. Therefore the equality $X^{\mathbf{V}[f]}=[T,d]$ can be expressed by a Shoenfield-absolute formula. We conclude that $X^{\mathbf{V}[f,G]}=[T,d]$ holds in $\mathbf{V}[f,G]$, too, and then $X^{\mathbf{V}[G]}=[T,d]$ is true in $\mathbf{V}[G]$ as well since easily $X^{\mathbf{V}[G]}=X^{\mathbf{V}[f,G]}\cap\mathbf{V}[G]$ and $[T,d]^{\mathbf{V}[G]}=[T,d]^{\mathbf{V}[f,G]}\cap\mathbf{V}[G]$. □

12 The case of $\mathbf{\Sigma}^1_2$ sets: resolution of σ-classes

Here our goal will be to resolve σ-classes, as in the "or" option of (I) of Theorem 10.1, into countable unions of single "$\mathbf{L}$-definable" equivalence classes. We are going to prove the next theorem in this section.

Theorem 12.1 *Assume that, in the ground set universe* $\mathbf{V}$,

(A) $\rho<\omega_1^{\mathbf{L}}$, $\xi<\omega_1$, E *is an equivalence relation on* $\mathscr{N}$ *in* $\Delta^1_1\cap\mathbf{\Pi}^0_{1+\rho}$, $\varnothing\ne C\subseteq\mathscr{N}$ *is a* σ-E-*class and a set essential* $\left(\Delta^1_1/\Delta^1_2\right)(\xi)$.

Then each E-*class* $X\subseteq C$ *is a set in* $[\mathbb{K}_\rho]$.

Corollary 12.2 *Suppose that, in Theorem* 10.1, *additionally,* $\rho<\omega_1^{\mathbf{L}}$ *and each relation* F_j *belongs to* $\mathbf{\Pi}^0_{1+\rho}$. *Then* (I) *of Theorem* 10.1 *can be reformulated as follows:*

(I) A *is* $\mathbf{L}$-$\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-*bounded, in the sense that for each* $x\in A$:
- *either there is* $y\in\mathscr{N}\cap\mathbf{L}$ *such that* $x\le^* y$,
- *or there is an index* $j=1,\dots,n$ *and a* F_j-*class* X *which contains* x *and belongs to* $[\mathbb{K}_\rho]$.

The ordinal $\omega^{\mathbf{L}}_\rho$ as the measure of borelness in Definition 11.2 and subsequently in (I) of Corollary 12.2, is a point of certain dissatisfaction. Can it be reduced to considerably narrower trees (of the same height)? Examples given in [17] and more recently in [1] allow to conjecture that the value $\omega^{\mathbf{L}}_\rho$ cannot be reduced in any essential way.

A similar question can be addressed to the inequality $\omega^{\mathbf{L}}_{\rho+1}<\omega_1$ in the next remark.

Remark 12.3 If $\omega^{\mathbf{L}}_{\rho+1}<\omega_1$ then both $\mathscr{N}\cap\mathbf{L}$ and $\mathbb{K}_\rho$ are countable sets, and hence the number of points y involved in (I) of Corollary 12.2 via the "either" option, and the number of classes X involved in (I) of Corollary 12.2 via the "or" option is countable, too—so that condition (I) of Theorem 10.1 can be replaced by just the $\{\mathsf{F}_1,\dots,\mathsf{F}_n\}$-$\sigma$-boundedness of the set A.

We now move to the proof of Theorem 12.1.

Proof. Assume that ρ,ξ,E,C are as in (A) above. Then C is $\mathbf{\Sigma}^0_{1+\rho+1}$, therefore by Lemma 11.6, we conclude that that

(B) there is a code $\langle T_0, d_0\rangle \in \mathbb{K}_{\rho+2}$ in $\mathbf{V}$ such that, in any set generic extension $\mathbf{V}[G]$ of $\mathbf{V}$, $[T_0, d_0] = C^{\mathbf{V}[G]}$, and hence by Shoenfield $[T_0, d_0]$ is a σ-$\mathsf{E}^{\mathbf{V}[G]}$-class in $\mathbf{V}[G]$ containing only those $\mathsf{E}^{\mathbf{V}[G]}$-classes already presented in $[T_0, d_0]\cap$.

We begin the proof with a few definitions. If $\langle T, d\rangle$ and $\langle T', d'\rangle$ are codes in $\mathbb{K}$ then let $\langle T, d\rangle \preccurlyeq \langle T', d'\rangle$ mean that $[T, d] \subseteq [T', d']$ holds in any set generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. Then, using appropriate collapse extensions, we conclude by Shoenfield, that $[T, d] \subseteq [T', d']$ also holds in any set generic extension $\mathbf{V}[G]$ of the ground universe $\mathbf{V}$, including $\mathbf{V}$ itself. A code $\langle T, d\rangle \in \mathbb{K}$ is *"essentially non-empty"* if $[T, d] \neq \varnothing$ in at least one set-generic extension of $\mathbf{L}$. By Shoenfield, this is equivalent to $[T, d] \neq \varnothing$ in some/any extension $\mathbf{L}[G]$ with $\sup T < \omega_1^{\mathbf{L}[G]}$.

Definition 12.4 Let $\mathbb{P} \in \mathbf{L}$ be the forcing notion which consists of all "essentially non-empty" codes $\langle T, d\rangle \in \mathbb{K}$ such that $\langle T, d\rangle \preccurlyeq \langle T_0, d_0\rangle$ and $\sup T \leq \omega_{\rho+2}^{\mathbf{L}}$. We order $\mathbb{P}$ by $\preccurlyeq$, and $\langle T, d\rangle \preccurlyeq \langle T', d'\rangle$ is understood as $\langle T, d\rangle$ being a stronger forcing condition.

In particular condition $\langle T_0, d_0\rangle$ itself (cf. (A)) belongs to $\mathbb{P}$.

Lemma 12.5 *$\mathbb{P}$ forces a real over $\mathbf{L}$, so that if a set $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$ then the intersection $\bigcap_{\langle T,d\rangle\in G}[T, d]$ contains a single real in $\mathbf{L}[G]$.*

Proof. If $u \in \omega^{<\omega}$ is a string of length $n = \operatorname{lh} u$ then let $T^u = \{\Lambda\}$ and let d^u consist of all pairs $\langle \Lambda, v\rangle$ such that $v \in \omega^{<\omega}$, $v \neq u$, $\operatorname{lh} v = n$. Then $\langle T^u, d^u\rangle \in \mathbb{P}$ and $[T^u, d^u] = \mathscr{N}_u = \{a \in \mathscr{N} : u \subset a\}$. By the genericity, for any n there is a inuque $u = u[n] \in \omega^{<\omega}$ such that $\operatorname{lh} u[n] = n$ and $\langle T^{u[n]}, d^{u[n]}\rangle \in G$, and in addition $u[n] \subset u[m]$ whenever $n < m$. It follows that there is a real $x_G = \bigcup_n u[n] \in \mathbf{L}[G]$ such that $x_G \restriction n = u[n]$, and hence $x_G \in [T^{u[n]}, d^{u[n]}]$, $\forall\, n$. We claim that *if* $\langle T, d\rangle \in \mathbb{P}$ *then* $\langle T, d\rangle \in G$ *iff* $x_G \in [T, d]$ *in* $\mathbf{L}[G]$; this obviously proves the lemma.

We prove the claim by induction on the rank $|T|$. Suppose that $|T| = 0$, so that $T = \{\Lambda\}$, $d \subseteq \{\Lambda\} \times \omega^{<\omega}$, and $[T, d] = \mathscr{N} \setminus \bigcup_{v\in U} \mathscr{N}_v$, where $U = \{v \in \omega^{<\omega} : \langle \Lambda, v\rangle \in d\}$. We assert that

(1) any $\langle T', d'\rangle \in \mathbb{P}$ is compatible, in $\mathbb{P}$, either with $\langle T, d\rangle$ or with one of the codes $\langle T^v, d^v\rangle$, where $v \in U$— therefore either $\langle T, d\rangle$ or one of the codes $\langle T^v, d^v\rangle$, $v \in U$, belongs to G.

Indeed we have $[T, d] = \mathscr{N} \setminus \bigcup_{v\in U}[T^v, d^v]$ in any universe.

With (1) in hands, if $v \in U$ and $\langle T^v, d^v\rangle \in G$ then on the one hand $\langle T, d\rangle \notin G$ by (1), and on the other hand, obviously $v = u[n]$, where $n = \operatorname{lh} v$, so that $x_G \in [T^v, d^v]$ and $x_G \notin [T, d]$. Conversely, if there is no $v \in U$ with $\langle T^v, d^v\rangle \in G$ then on the one hand $\langle T, d\rangle \in G$ by (1), and on the other hand, $x_G \notin \bigcup_{v\in U}[T^v, d^v]$, so that $x_G \in [T, d]$.

To carry out the step, suppose that $|T| > 0$. Let $\Xi = \{\xi : \langle\xi\rangle \in T\}$ (where $\langle\xi\rangle$ is a one-term string). If $\xi \in \Xi$ then let

$$T^\xi = \{s \in \mathrm{Ord}^{<\omega} : \xi^\frown s \in T\} \quad \text{and} \quad d^\xi = \{\langle s, v\rangle \langle \xi^\frown s, v\rangle \in d\}.$$

Thus each $\langle T^\xi, d^\xi\rangle$ is a code in $\mathbb{P}$, $|T^\xi| < |T|$, and $[T, d] = \mathscr{N} \setminus \bigcup_{\xi\in\Xi}[T^\xi, d^\xi]$ in any universe containing $\langle T, d\rangle$. Similarly to (1) above, we have

(2) any $\langle T', d'\rangle \in \mathbb{P}$ is compatible, in $\mathbb{P}$, either with $\langle T, d\rangle$ or with one of the codes $\langle T^\xi, d^\xi\rangle$, where $\xi \in \Xi$— therefore either $\langle T, d\rangle$ or one of the codes $\langle T^\xi, d^\xi\rangle$, $\xi \in \Xi$, belongs to G.

Now, if $\xi \in \Xi$ and $\langle T^\xi, d^\xi\rangle \in G$ then on the one hand $\langle T, d\rangle \notin G$ by (2), and on the other hand, $x_G \in [T^\xi, d^\xi]$ by the inductive hypothesis, and hence $x_G \notin [T, d]$. Conversely, if there is no $\xi \in \Xi$ with $\langle T^\xi, d^\xi\rangle \in G$ then on the one hand $\langle T, d\rangle \in G$ by (2), and on the other hand, $x_G \notin \bigcup_{\xi\in\Xi}[T^\xi, d^\xi]$, by the inductive hypothesis, so that $x_G \in [T, d]$. □

Reals of the form $x_G =$ the only element of $\bigcap_{\langle T,d\rangle\in G}[T, d]$ in $\mathbf{L}[G]$, where $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic, *e.g.*, over $\mathbf{V}$, will be called $\mathbb{P}$-*generic over* $\mathbf{V}$, too. Let $\mathbf{x}$ be a canonical $\mathbb{P}$-name for x_G. Let $\mathbf{x}_{\text{left}}$, $\mathbf{x}_{\text{right}}$ be canonical $(\mathbb{P} \times \mathbb{P})$-names for the left and the right copies of x_G.

Let $\underline{\mathsf{E}}$ be a canonical $\mathbb{P}$-name for the extension $\mathsf{E}^{\mathbf{V}[G]}$ or $\mathsf{E}^{\mathbf{L}[G]}$ of E to any class like $\mathbf{L}[G]$ or $\mathbf{V}[G]$, G being generic.

Definition 12.6 A code $\langle T,d\rangle \in \mathbb{P}$ is *stable* if condition $(\langle T,d\rangle ; \langle T,d\rangle)$ $(\mathbb{P}\times\mathbb{P})$-forces, over $\mathbf{L}$, that $\mathbf{x}_{\rm left} \mathrel{\underline{\mathsf{E}}} \mathbf{x}_{\rm right}$.

Lemma 12.7 *If $\langle T,d\rangle \in \mathbb{P}$ is stable then, in $\mathbf{V}$, there is an element $y \in C$ such that $\langle T,d\rangle$ $\mathbb{P}$-forces, over $\mathbf{V}$, that $\mathbf{x} \mathrel{\underline{\mathsf{E}}} y$.*

P r o o f. Recall that C contains countably many single E-classes in $\mathbf{V}$. It easily follows by Shoenfield that the extended set $C^{\mathbf{V}[G]}$ has no new $\mathsf{E}^{\mathbf{V}[G]}$-classes in any extension $\mathbf{V}[G]$ of $\mathbf{V}$. Thus the contrary assumption leads to a pair of conditions $\langle T',d'\rangle \preccurlyeq \langle T,d\rangle$ and $\langle T'',d''\rangle \preccurlyeq \langle T,d\rangle$ in $\mathbb{P}$ and elements $y',y'' \in C$ in $\mathbf{V}$ such that

$$\langle T',d'\rangle \ \mathbb{P}\text{-forces } \mathbf{x} \mathrel{\underline{\mathsf{E}}} y', \quad \text{and} \quad \langle T'',d''\rangle \ \mathbb{P}\text{-forces } \mathbf{x} \mathrel{\underline{\mathsf{E}}} y'' \text{ — over } \mathbf{V},$$

and $y' \mathrel{\not{\underline{\mathsf{E}}}} y''$. To get a contradiction consider a set $G'\times G''$, $(\mathbb{P}\times\mathbb{P})$-generic over $\mathbf{V}$, and containing condition $(\langle T',d'\rangle ; \langle T'',d''\rangle)$. Then, on the one hand, the generic reals $x_{G'}$ and $x_{G''}$ satisfy $x_{G'} \ \mathsf{E}^{\mathbf{V}[G']}\ y'$ and $x_{G''} \ \mathsf{E}^{\mathbf{V}[G']}\ y''$, but on the other hand, $x_{G'} \ \mathsf{E}^{\mathbf{V}[G',G'']}\ x_{G''}$ holds by stability. Therefore $y' \ \mathsf{E}\ y''$, which contradicts to the choice of these reals. □

Lemma 12.8 *The set of all stable conditions $\langle T,d\rangle \in \mathbb{P}$ is dense in $\mathbb{P}$.*

P r o o f. By definition $\operatorname{card}\mathbb{P} = \omega^{\mathbf{L}}_{\rho+3}$ and $\operatorname{card}\mathscr{P}(\mathbb{P}) = \omega^{\mathbf{L}}_{\rho+4}$ in $\mathbf{L}$. Consider an extension $\mathbf{V}[g]$ by a collapse-generic map $g : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+4}$. Then, in $\mathbf{V}[g]$, there is an enumeration $\{D_n\}_{n<\omega}$ of all dense sets $D \subseteq \mathbb{P}\times\mathbb{P}$, $D \in \mathbf{L}$.

Now suppose towards the contrary that $\langle T^*,d^*\rangle \in \mathbb{P}$ and there is no stable $\langle T,d\rangle \preccurlyeq \langle T^*,d^*\rangle$ in $\mathbb{P}$. Then for any condition $\langle T,d\rangle \preccurlyeq \langle T^*,d^*\rangle$ there are stronger conditions $\langle T',d'\rangle \preccurlyeq \langle T,d\rangle$ and $\langle T'',d''\rangle \preccurlyeq \langle T,d\rangle$ such that $(\langle T',d'\rangle ; \langle T'',d''\rangle)$ $(\mathbb{P}\times\mathbb{P})$-forces $\neg\ \mathbf{x}_{\rm left} \mathrel{\underline{\mathsf{E}}} \mathbf{x}_{\rm right}$ over $\mathbf{L}$. This allows to define, in $\mathbf{V}[g]$, a family $\{\langle T_u,d_u\rangle\}_{u\in 2^{<\omega}}$ of conditions in $\mathbb{P}$ satisfying

(i) $\langle T_u,d_u\rangle = \langle T^*,d^*\rangle$,
(ii) $\langle T_{u^\frown i},d_{u^\frown i}\rangle \preccurlyeq \langle T_u,d_u\rangle$ for each $i=0,1$ and $u\in\omega^{<\omega}$,
(iii) if $u\neq v$ in $2^{<\omega}$ are of length $n+1$ then $(\langle T_u,d_u\rangle ; \langle T_v,d_v\rangle) \in D_n$,
(iv) if $u\in 2^{<\omega}$ then condition $(\langle T_{u^\frown 0},d_{u^\frown 0}\rangle ; T_{u^\frown 1},d_{u^\frown 1})$ $(\mathbb{P}\times\mathbb{P})$-forces $\neg\ \mathbf{x}_{\rm left} \mathrel{\underline{\mathsf{E}}} \mathbf{x}_{\rm right}$ over $\mathbf{L}$.

Then, in $\mathbf{V}[g]$, if $a\in 2^\omega$ then the intersection $\bigcap_n [T_{a\restriction n},d_{a\restriction n}]$ contains a single point $x_a \in [T^*,d^*]$ by Lemma 12.5, and we have $\neg\ (x_a \ \mathsf{E}^{\mathbf{V}[g]}\ x_b)$ for all $a\neq b$. But by construction $[T^*,d^*]^{\mathbf{V}[g]} \subseteq [T_0,d_0]^{\mathbf{V}[g]} = C^{\mathbf{V}[g]}$, so that $C^{\mathbf{V}[g]}$ contains uncountably many $\mathsf{E}^{\mathbf{V}[g]}$-classes in $\mathbf{V}[g]$. Yet this contradicts the assumption that C contains countably many E-classes in $\mathbf{V}$ (cf. the list of our blanket assumptions (A) above), since by Shoenfield the property of being a σ-E-class is preserved under extensions. □

Let H be the set of all codes $\langle T,d\rangle \in \mathbb{K}_\rho$ such that the $\omega^{\mathbf{L}}_{\rho+4}$-collapse forcing notion $\operatorname{Coll}\big(\omega^{\mathbf{L}}_{\rho+4}\big) = \big(\omega^{\mathbf{L}}_{\rho+4}\big)^{<\omega}$ forces, over $\mathbf{L}$, that

$$[T,d] \subseteq [T_0,d_0] \text{ and } [T,d] \text{ is an } \underline{\mathsf{E}}\text{- equivalence class,}$$

where $\mathbf{g}$ is a canonical name for the $\operatorname{Coll}\big(\omega^{\mathbf{L}}_{\rho+4}\big)$-generic map $g : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_{\rho+4}$.

Lemma 12.9 *If $\langle T,d\rangle \in H$ then it is true in the ground set universe $\mathbf{V}$ that $[T,d] \subseteq [T_0,d_0]$ and $[T,d]$ is a E-class.*

P r o o f. By definition this is true for $\operatorname{Coll}\big(\omega^{\mathbf{L}}_{\rho+4}\big)$-generic extensions of $\mathbf{L}$—hence by Shoenfield also for all generic extensions $\mathbf{V}[G]$ in which $\omega^{\mathbf{L}}_{\rho+4}$ is countable, and then, by quite obvious downward absoluteness, for the universe $\mathbf{V}$ itself. □

Lemma 12.10 $H \neq \varnothing$.

P r o o f. By Lemma 12.10 there is a stable condition $\langle T',d'\rangle \in \mathbb{P}$. Using an $\omega^{\mathbf{L}}_{\rho+4}$-enumeration of all dense sets $D\subseteq\mathbb{P}$ in $\mathbf{L}$, we easily get a code $\langle T^*,d^*\rangle \in \mathbb{K}$ such that $\sup T^* \le \omega^{\mathbf{L}}_{\rho+4}$ and the equality

$$[T^*,d^*] = \big\{x \in [T',d'] : x \text{ is } \mathbb{P}\text{-generic over } \mathbf{L}\big\}$$

holds in any class $\mathbf{V}[G]$. Lemma 12.7 implies that all elements $x \in [T^*, d^*]$ in $\mathbf{V}[G]$ are $\mathsf{E}^{\mathbf{V}[G]}$-equivalent to each other and to some $y^* \in C$ (so $y^* \in \mathbf{V}$).

Let $g : \omega \xrightarrow{\text{onto}} \omega_{\rho+4}^{\mathbf{L}}$ be a collapse-generic map.

We argue in $\mathbf{V}[g]$. By a simple cardinality argument, $[T^*, d^*] \neq \varnothing$ in $\mathbf{V}[g]$, and $[T^*, d^*]$ consists of pairwise $\mathsf{E}^{\mathbf{V}[g]}$-equivalent elements by the above. This allows us to define

$$Z = \{z : \exists\, x \in [T^*, d^*]\ (x\ \mathsf{E}^{\mathbf{V}[g]}\ z)\} = \{z : \forall\, x \in [T^*, d^*]\ (x\ \mathsf{E}^{\mathbf{V}[g]}\ z)\}$$

in the universe $\mathbf{V}[g]$, so that it is true in $\mathbf{V}[g]$ that Z is an entire $\mathsf{E}^{\mathbf{V}[g]}$-equivalence class, which includes $[T^*, d^*]$, hence, has a non-empty intersection with $[T', d'] \subseteq [T_0, d_0]$, therefore $Z \subseteq [T_0, d_0]$ as $[T_0, d_0]$ is an σ-$\mathsf{E}^{\mathbf{V}[g]}$-class in $\mathbf{V}[g]$ by (A).

It follows that Z is $\mathbf{\Pi}^0_{1+\rho}$ in $\mathbf{V}[g]$. Moreover, by the choice of g it is true in $\mathbf{V}[g]$ that $\langle T^*, d^*\rangle \in \mathbf{L} \cap \mathrm{HC}$, and hence $\langle T^*, d^*\rangle$ is $\Delta_1^{\mathrm{HC}}(\eta)$ in $\mathbf{V}[g]$ for an ordinal $\eta < \omega_1^{\mathbf{V}[g]}$. (Indeed let η be the first ordinal such that $\langle T^*, d^*\rangle$ is the η-th set in the Gödel construction of $\mathbf{L}$.) Then Z is $\Delta_1^{\mathrm{HC}}(\eta)$ in $\mathbf{V}[g]$. Therefore by Proposition 11.3 that there is a code $\langle T, d\rangle \in \mathbb{K}_\rho$ such that $Z = [T, d]$ in $\mathbf{V}[g]$. Let us demonstrate that $\langle T, d\rangle \in H$.

Consider a collapse-generic map $g' : \omega \xrightarrow{\text{onto}} \omega_{\rho+4}^{\mathbf{L}}$; we can assume that g' is $\mathrm{Coll}(\omega_{\rho+4}^{\mathbf{L}})$-generic even over $\mathbf{V}[g]$. We have to prove that

(A) in $\mathbf{L}[g']$: $[T, d] \subseteq [T_0, d_0]$ and $[T, d]$ is an $\mathsf{E}^{\mathbf{L}[g']}$-equivalence class.

Recall that by construction $Z = [T, d] \subseteq [T_0, d_0]$ and $[T, d]$ is an $\mathsf{E}^{\mathbf{V}[g]}$-class in $\mathbf{V}[g]$. But the Borel codes involved are countable in both classes $\mathbf{V}[g]$ and $\mathbf{L}[g']$. This implies (A) by Shoenfield. □

Now we have gathered everything necessary to end the proof of the theorem in a few lines. It suffices to prove that $C = [T_0, d_0] \subseteq \bigcup_{\langle T, d\rangle \in H} [T, d]$ in $\mathbf{V}$. Suppose tovards the contrary that this is not the case.

The set $H \subseteq \mathbb{K}_\rho$ belongs to $\mathbf{L}$ and $\operatorname{card} H \leq \omega_{\rho+1}^{\mathbf{L}}$ in $\mathbf{L}$, of course. As $\langle T_0, d_0\rangle \in \mathbb{K}_{\rho+2}$, we cal easily define a code $\langle T_1, d_1\rangle \in \mathbb{K}_{\rho+2}$ such that absolutely $[T_1, d_1] = [T_0, d_0] \setminus \bigcup_{\langle T, d\rangle \in H} [T, d]$, and hence $[T_1, d_1] \neq \varnothing$ in $\mathbf{V}$, and still $[T_1, d_1]$ is a σ-E-class in $\mathbf{V}$ since so is $C = [T_0, d_0]$ while each $[T, d]$, $\langle T, d\rangle \in H$, is a E-class by Lemma 12.9. □

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