SINGULAR CARDINALS

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The problem is studied of the existence of nonconstructive subsets of cardinals belonging to an original countable standard transitive model of ZF theory of sets that do not generate new subsets of smaller cardinals of this same model. It is found that a fairly extensive class of properties of the extended model is closely related to the corresponding properties of the original model.

I. An ordinal is any set x such that $y \in x \to y \subseteq x$, and a cardinal is the least ordinal or given power. On is the class of all ordinals.

If τ and ν are cardinals, we shall write $\tau = \nu^+$, if $\tau > \nu$ and there are no other cardinals between ν and τ. We shall write $\tau = 2^\nu$ if τ is the power of the set of all subsets of ν.

All cardinals of the form ν^+ are said to be nonlimit cardinals, whereas the others are limit cardinals. A cardinal τ is said to be singular if it can be represented in the form $\tau = \Sigma_{\alpha\in\lambda}\nu_\alpha$, where $\lambda < \tau$ and for any $\alpha \in \lambda$ we have $\nu_\alpha < \tau$.

Nonsingular cardinals are said to be regular.

Now let us examine without proof a number of results belonging to the Zermelo–Fränkel theory of sets (ZF).

The axiom of choice (AC) can be expressed as follows: If X is a set and F a function defined on X that assumes nonempty values, then there exists a function f defined on X and such that $f(x) \in F(x)$ for any $x \in X$.

The generalized continuum hypothesis (GCH) is the assertion $2^\nu = \nu^+$ for any cardinal ν.

Cohen [2] has defined a singular Gödel function $F(\alpha)$ with a domain of definition On and a domain of values L. The class L is called the class of constructive sets; this class is specified by a well-defined formula of ZF.

An axiom of constructivity is the following sequence: All sets belong to L. In brief this axiom can be written as V = L.

It is possible to define a function $F(\alpha, x)$ that differs from $F(\alpha)$ only by the fact that $F(o, x)$ is assumed equal to x, and not to the empty set. The corresponding class of values of $F(\alpha, x)$ for fixed x is denoted by L(x). It is a model of ZF and $On \subseteq L(x)$.

Now let $\mathfrak{M}$ be a model of ZF. We shall say that $\mathfrak{M}$ is a standard model if for $x \in \mathfrak{M}$, $y \in \mathfrak{M}$ the expressions "$x \in y$" and "$\mathfrak{M} \models$ "$x \in y$" " are equivalent; i.e., the relation of membership in $\mathfrak{M}$ is a restriction of this relation to the set $\mathfrak{M}$.

We shall say that $\mathfrak{M}$ is transitive if $y \in \mathfrak{M}$ & $x \in y \to x \in \mathfrak{M}$.

In the following we shall consider only countable standard transitive models.

Now let $\mathfrak{M}$ be a model of ZF + V = L, let x be a set (possibly not belonging to $\mathfrak{M}$), and let $On_{\mathfrak{M}}$ be the set of ordinals of $\mathfrak{M}$.

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Let us define

$$\mathfrak{M}(x) = \{y \mid \exists\alpha\, [\alpha \in On_{\mathfrak{M}} \;\&\; y = F(\alpha, x)]\}.$$

It was shown by Cohen [2] that not for every x will $\mathfrak{M}(x)$ be a model. There exists only one method of obtaining sets x such that $\mathfrak{M}(x)$ is a model, namely, the forcing method.

Now let us consider the following problem.

Let $\mathfrak{M}$ be a model of ZF + V = L, and λ a cardinal in $\mathfrak{M}$. It is required to find a set $a \subseteq \lambda$ such that:

1°. $\mathfrak{M}(a)$ is a model of ZF.

2°. $a \notin \mathfrak{M}$.

3°. If $\nu < \lambda$ is a cardinal in $\mathfrak{M}$ and $x \in \mathfrak{M}(a)$, $x \subseteq \nu$, then $x \in \mathfrak{M}$.

The third property can be formulated more roughly as follows: a does not generate new subsets consisting of cardinals smaller than λ.

This problem has been completely solved for cardinals λ that are regular in $\mathfrak{M}$.

The method of its solution, presented in the second section of this paper, is a simplification of the method of [3] (where Easton introduces not one, but many subsets λ such that property 3° holds and in the obtained model we have $2^{\lambda} > \lambda^{+}$).

For singular λ the search for a set a with the above properties is in general an unsolved problem.

In the third section we present one of the particular results obtained for this problem. For understanding the proofs, it is necessary to be acquainted with the theory of sets as formulated, for example, in [2].

II. Thus, let $\mathfrak{M}$ be a standard transitive model of ZF + V = L, and let $\lambda \in \mathfrak{M}$ be a regular cardinal in $\mathfrak{M}$. All the subsequent constructions belong to $\mathfrak{M}$.

Let us construct a parametric space S: $S_0 = \{\,|\alpha|\mid \alpha < \lambda\}$ being a set of symbols for ordinals smaller than λ;

$S_1 = \{a\}$ is a set consisting of the symbol a alone; S_β, $\beta \geq 2$, is defined in accordance with the general rules as a set of formulas of one free variable and of constants belonging to $\bigcup_{\gamma<\beta} S_\gamma$, that are relativized to $\bigcup_{\gamma<\beta} S_\gamma$.

The forcing condition will be any pair $p = \langle u, v\rangle$, where $u \subseteq \lambda$, $v \subseteq \lambda$, $u \cap v = \phi$ and card $(u \cup v) < \lambda$. As usual, we shall define $\langle u, v\rangle \leq \langle u', v'\rangle$, if $u \subseteq u'$, $v \subseteq v'$. The set of forcing conditions with such an order will be denoted by P.

Now let us define the forcing of elementary statements (such as the statements "$|\alpha| \in a$," where $\alpha < \lambda$) by $\langle u, v\rangle$ Forc "$|\alpha| \in a$" if $\alpha \in u$.

The definition of a predicate can be extended to more complex statements by a well-known induction method [2].

<u>LEMMA 1.1.</u> Let $p = \langle u, v\rangle \in P$, $c \in S$. Then $p \Vdash$ "$c \cap |\nu|$ constructively" for any $\nu < \lambda$. ($\Vdash$ — is the symbol of weak forcing, i.e., $p \Vdash \equiv p\,\text{Forc} \sim\sim A$.)

<u>Proof.</u> As is easy to see, by assuming the contrary we find that there exists a $q \geq p$ such that $q \Vdash$ "$c \cap |\nu|$ nonconstructively."

For obtaining a contradiction, we shall construct a system of forcing conditions $\{p_\alpha \mid \alpha < \nu\}$ such that

1) $\alpha \leqslant \beta \to p_\alpha \leqslant p_\beta$,

2) if $\alpha < \beta$, then $p_\beta \Vdash$ «$|\alpha| \in c \cap |\nu|$» or $p_\beta \Vdash$ «$|\alpha| \notin c \cap |\nu|$» . The construction will be carried out by induction on α. Let us write $p_0 = q$.

Suppose we have constructed all p_β, $\beta < \alpha$, and let α be a limit ordinal. Let $p_\beta = \langle u_\beta, v_\beta\rangle$. Then we shall write $u_\alpha = \bigcup_{\beta<\alpha} u_\beta$, $v_\alpha = \bigcup_{\beta<\alpha} v_\beta$.

By virtue of the induction hypothesis the condition 1) holds for $\{p_\beta \mid \beta < \alpha\}$; therefore $u_\alpha \cap v_\alpha = \phi$.

Moreover, card $(u_\alpha \cup v_\alpha) = \text{card}\,(\bigcup_{\beta<\alpha} (u_\beta \cup v_\beta)) < \lambda$, since λ is a regular cardinal by assumption, i.e., it is not reachable.

Thus, $p_\alpha = \langle u_\alpha, v_\alpha\rangle$ will be a condition. The properties 1 and 2 are trivial to verify for the extended system.

Now let α be a nonlimit ordinal, $\alpha = \beta + 1$, and suppose that $p_\beta = \langle u_\beta, v_\beta\rangle$ has been constructed. If $p_\beta \Vdash$ «$|\beta| \notin c \cap |v|$», we shall write $p_\alpha = p_\beta$, and the properties are retained. Otherwise there exists a $q \geq p_\beta$ such that $q \Vdash$ «$|\beta| \in c \cap |v|$». Let us take such a q and write $p_\alpha = q$.

The system $\{p_\alpha \mid \alpha < v\}$ has been constructed. Now let us write $u' = \bigcup_{\alpha<v} u_\alpha$, $v' = \bigcup_{\alpha<v} v_\alpha$. As before, $p' = \langle u', v'\rangle$ is a condition. Moreover, it follows from the properties of $\{p_\alpha\}$ that if $\alpha < \nu$, then $p' \Vdash$ «$|\alpha| \in c \cap |v|$» or $p' \Vdash$ «$|\alpha| \notin c \cap |v|$».

Hence we can easily prove that $p \Vdash$ "$c \cap |v|$ constructively". (For example, let $y = \{\alpha \in v \mid p' \Vdash$ «$|\alpha| \in c \cap |v|$»$\}$; then evidently $p' \Vdash$ «$c \cap |v| = |y|$», where $|y| \in S$ is an element of parametric space whose filling is always equal to y.) But by construction we have $p' \geq q$. The obtained contradiction proves the lemma.

Now it is trivial to construct a set that satisfies 1°–3°.

In fact, let $\{p^n\}_{n\in\omega_0}$ be a complete sequence of conditions belonging to P, and let $\bar{a}$ be the corresponding set. Then 2° will follow from general forcing theorems, 1° is evident, and 3° easily follows from the lemma (for example, let $\mathfrak{M}(\bar{a}) \models$ "$\bar{c} \subseteq v$ and not constructively." Then a p^n belonging to the complete sequence will force the nonconstructivity of $c \cap |\nu|$, which contradicts Lemma 1.1). Thus we have proved.

THEOREM 1. If λ is a regular cardinal in $\mathfrak{M}$, then there exists a set a that satisfies 1°–3°.

It is easy to see that such a method of proof cannot be applied to singular cardinals.

In the next section we shall present a different method which, however, does not completely solve the problem for singular numbers.

III. We shall prove the following

THEOREM 2. Let $\mathfrak{M}$ be a model of ZF + V = L, let λ be a cardinal in $\mathfrak{M}$, and let $\Omega = \lambda^+$ (the cardinal of $\mathfrak{M}$ that follows λ). Then there exists a subset $a \subsetneq \lambda$ such that $a \notin \mathfrak{M}$, and $\mathfrak{M}(a)$ is a model of ZF that does not contain nonconstructive subsets of ordinals $\nu < \lambda$ up to the step Ω.

It is evident that for completely solving the problem of singular numbers, it is necessary to prove that in the model under consideration the ordinals λ and Ω do not have the same power.

At first let us make a stipulation. Let S be a parametric space with only one symbol g for a generic set.

For finding out the future properties of the model $\mathfrak{M}(\bar{g})$, it is appropriate to replace g by sets belonging to the model $\mathfrak{M}$. Let A be a statement concerning S. We shall take a set $x \in \mathfrak{M}$ such that transitivity is ensured. Then we shall fill the elements of the parametric space S by substituting this set x for g. Let us write $\mathfrak{M}(x) \models A$, if $\mathfrak{M} \models \bar{A}$, where $\bar{A}$ is a formula of ZF with constants belonging to $\mathfrak{M}$ that has been obtained from A by replacing each occurrence of $c \in S$ by the set $\bar{c}$ (filled according to x), and by restricting in the same way the bounded quantifiers belonging to A (i.e., the quantifier $\exists_\alpha y$ is transformed into $\exists y (y \in \bigcup_{\beta<\alpha} \bar{S}_\beta)$).

Strong (syntactically defined) forcing will be denoted by Forc, weak forcing by $\Vdash$, and the truth symbol by $\models$.

Let S be a parametric space of the following form: $S_0 = \{|\alpha| \mid \alpha < \lambda\}$, with tags for all the ordinals smaller than λ; S_0 is necessary for transitive construction of $S_1 = \{a\}$. S_α, $\alpha \geq 2$, is defined in accordance with general rules as a set of formulas with one free variable and constants belonging to $\bigcup_{\beta<\alpha} S_\beta$ that are relativized to this same set.

As the set P of forcing conditions we shall take the set of all subsets 2^λ of power Ω that is ordered by inclusion. The forcing of elementary statements is also defined in the usual way: p Forc "$|\alpha| \in a$" (for $\alpha < \lambda$) if $\forall x \in p\,[\alpha \in x]$.

LEMMA 2.1. Let A be a limited statement concerning S of rank $< \Omega$ [i.e., all the constants of A are elements of $\bigcup_{\gamma<\Omega} S_\gamma$ and all the quantifers of A are bounded by ordinals $\beta < \Omega$], and let p be a forcing condition.

Hence if $\forall x \in p\,[\mathfrak{M}(x) \models A]$, then $p \Vdash A$ and if $p \Vdash A$, then $\{x \in p \mid \mathfrak{M}(x) \Vdash \sim A\}$ will have a power smaller than Ω.

Proof. The lemma can be proved by induction on the rank of formula A (defined as in [2]) by examining several cases; among the quantifiers and connectives we shall consider only $\sim$, & and $\exists$.

1. Let A be an elementary statement, A = "$|\alpha| \in a$". It then follows from the definition of forcing of elementary statements that

$$\forall x \in p\,[\mathfrak{M}(x) = «|\alpha| \in a»] \to \forall x \in p\,[|\alpha| \in x] \to p \Vdash «|\alpha| \in a».$$

Conversely, let $p \Vdash$ "$|\alpha| \in a$". Let $q = \{x \in p \mid \cdot\, |\alpha \in x\}$. It is evident that the power of q is smaller than Ω (since otherwise we would have $q \Vdash «|\alpha| \notin a»$), i.e., $\{\exp \mid \mathfrak{M}(x) \Vdash «|\alpha| \notin a»\}$ has a power smaller than Ω.

2. Let A be a negation, $A = \sim B$, and let $\forall x \in p\,[\mathfrak{M} \models (x) \models \sim B]$. Let us show that $p \Vdash \sim B$. In fact, otherwise there would exist a $q \geq p$ such that $q \Vdash B$. It now follows from the induction hypothesis that there exists at least one $x \in q$ such that $\mathfrak{M}(x) \models B$. But this contradicts $q \subseteq p$ and $\forall x \in p\,[\mathfrak{M}(x) \models \sim B]$.

Conversely, if $p \Vdash \sim B$ and there exists a $q \in P$, $q \geq p$ such that $\forall x \in q\ [\mathfrak{M}(x) \models B]$, then it follows from the induction hypothesis that $q \Vdash B$, which contradicts $q \geq p$ and $p \Vdash \sim B$.

3. Let $A = B \,\&\, C$, $\forall x \in p\ [\mathfrak{M}(x) \models B \,\&\, C]$. But in this case $\forall x \in p\ [\mathfrak{M}(x) \models B]$, whence follows from the induction hypothesis that $p \Vdash B$. Similarly, $p \Vdash C$. Hence $p \Vdash B \,\&\, C$.

Conversely, let $p \Vdash B \,\&\, C$. Then the sets $q_1 = \{x \in p \mid \mathfrak{M}(x) \models \sim B\}$ and $q_2 = \{x \in p \mid \mathfrak{M}(x) \models \sim C\}$ will have a power smaller than Ω, i.e., $\{x \in p \mid \mathfrak{M}(x) \models [B \,\&\, C]\}$ has a power smaller than Ω.

4. Let $A = \exists_\gamma x B(x)$, $\gamma < \Omega$, and let $\forall y \in p\ [\mathfrak{M}(y) \models A]$. Let us assume the contrary. Then there exists a condition $q \geq p$ such that $q \Vdash \forall_\gamma x \sim B(x)$, i.e., for any $c \in \bigcup_{\beta<\gamma} S_\beta$ we have $q \Vdash \sim B(c)$.

For any such c let us write $q_c = \{x \in q \mid \mathfrak{M}(x) \models B(c)\}$. Let us note that $q = \bigcup_c q_c$. (This follows from the fact that for similarly defined sets p_c the assumption $\mathfrak{M}(x) \models A$ implies that $p = \bigcup_c p_c$, and $q_c = q \cap p_c$.) Hence at least one q_c will have the power Ω. Let r be this q_c. It follows from the definition of q_c $\forall x \in r\,[\mathfrak{M}(x) \models B(c)]$. Hence $r \Vdash B(c)$, which contradicts $r \geq q$ and $q \Vdash \sim B(c)$.

Conversely, let $p \Vdash \exists_\gamma xB(x)$, but suppose that a $q \geq p$ is such that $\forall x \in q\,[\mathfrak{M}(x) \models \sim \exists_\gamma xB(x)]$; i.e., for any $c \in \bigcup_{\beta<\gamma} S_\beta$ and $x \in q$ we have $\mathfrak{M}(x) \models \sim B$. But $q \geq p$ and $q \Vdash \exists_\gamma \cdot xB(x)$. Hence if $r \geq q$, then r Forc $\exists_\gamma xB(x)$, whence we can see that there exists a $c \in \bigcup_{\beta<\gamma} S_\beta$ such that $r \Vdash B(c)$. Now it follows from the induction hypothesis that there exists at least one $x \in r$ such that $\mathfrak{M}(x) \models B(c)$; this contradicts the above-mentioned property of q and $r \geq q$.

5. Let A be $c_1 \in c_2$, where $c_1 \in S_\alpha$, $c_2 \in S_\beta$, $\beta \leq \alpha < \Omega$ and $\forall x \in p\,[\mathfrak{M}(x) \models «c_1 \in c_2»]$. By virtue of the transitivity of the construction of $\mathfrak{M}(x)$ this signifies that $\forall x \in p \exists c \in \bigcup_{\gamma<\beta} S_\gamma\,[\mathfrak{M}(x) \models «c_1 = c \,\&\, c \in c_2»]$. For any $c \in \bigcup_{\gamma<\beta} S_\gamma$ let us denote $p_c = \{x \in p \mid \mathfrak{M}(x) \models «c_1 = c \,\&\, c \in c_2»\}$.

Then one of the p_c will have the power Ω. Let r be this p_c. Then $\forall x \in r\ [\mathfrak{M}(x) \models «c_1 = c \,\&\, c \in c_2»]$. It now follows from the induction hypothesis that

$$r \Vdash «c_1 = c \,\&\, c \in c_2», \text{ т. е. } r \Vdash «c_1 \in c_2».$$

Conversely, let $p \Vdash$ "$c_1 \in c_2$", but there exists a $q \geq p$ such that for any $c \in \bigcup_{\gamma<\beta} S_\gamma$, and $x \in q$ we have $\mathfrak{M}(x) \models «c_1 \neq c \vee c \notin c_2»$. But in this case $q \Vdash «c_1 \neq c \vee c \notin c_2»$ for any $c \in \bigcup_{\gamma<\beta} S_\gamma$, i.e., $q \Vdash$ "$c_1 \notin c_2$", which contradicts $q \geq p$.

6. Let A be $c_1 = c_2$, $c_1 \in S_2$, $c_2 \in S_\beta$, $\alpha \leq \beta < \Omega$. Then $p \Vdash A \to \forall c \in \bigcup_{\gamma<\alpha} S_\gamma [p \Vdash «c \in c_1 \equiv c \in c_2»]$, whence follows from the induction hypothesis that for any $c \in \bigcup_{\gamma<\alpha} S_\gamma$ the quantity $\{x \in p \mid \mathfrak{M}(x) \models "\sim [c \in c_1 \equiv c \in c_2]"\}$ will have a power smaller than Ω; i.e., by taking into account the transivity of the construction of $\mathfrak{M}(x)$, we find that $\{x \in p \mid \mathfrak{M}(x) \models «\sim [c_1 = c_2]»\}$ will have a power smaller than Ω.

Conversely, let $\forall x \in p [\mathfrak{M}(x) \models «c_1 = c_2»]$. Then for any $c \in \bigcup_{\gamma<\alpha} S_\gamma$ we have $\mathfrak{M}(x) \models «c \in c_1 \equiv c \in c_2»$, i.e., $p \Vdash$ "$c \in c_1 \equiv c \in c_2$", whence follows directly that $p \Vdash$ "$c_1 = c_2$".

7. The last case, $A =$ "$c_1 \in c_2$", $c_1 \in S_\alpha$, $c_2 \in S_\beta$, $\alpha < \beta < \Omega$. Let $c_2 = A(x, c', \ldots, c^n)$. If $p \Vdash$ "$c_1 \in c_2$", then $p \Vdash A(c_1, c', \ldots, c^n)$ and $\forall x \in p [\mathfrak{M}(x) \models A(c_1, c^1, \ldots, c^n)]$, i.e., $\forall x \in p [\mathfrak{M}(x) \models «c_1 \in c_2»]$ (with the possible exception of x that are smaller than Ω).

Conversely, let $\forall x \in p [\mathfrak{M}(x) \models «c_1 \in c_2»]$. Then

$$\forall x \in p [\mathfrak{M}(x) \models A(c_1, c^1, \ldots, c^n)], \ p \Vdash A(c_1, c^1, \ldots, c^n)$$

and $p \Vdash$ "$c_1 \in c_2$". By examining these cases, we have completed the proof of the lemma.

Now the proof of Theorem 2 can be obtained by simple calculations.

Let $c \in \bigcup_{\alpha<\Omega} S_\alpha$ and suppose that the condition p is such that $p \Vdash$ "c is nonconstructive and $c \subseteq |\tau|$" for a $\tau < \lambda$.

For any subset $y \subseteq \tau$ let us write $p_y = \{x \in p \mid \mathfrak{M}(x) \models «c = |y|»\}$. Let us consider $q = p - \bigcup_{y \subseteq \tau} p_y$. It is evident that the power of q is smaller than Ω (since otherwise q would be a condition and $\forall x \in q [\mathfrak{M}(x) \models «c \not\equiv |\tau|»]$, which contradicts Lemma 2.1 and $q \geq p$). It is also easy to see that $\{y \mid y \subseteq \tau\}$ has a power smaller than Ω. Hence one of the p_y will have a power Ω. Let r be this p_y. Then $\forall x \in r [\mathfrak{M}(x) \models "c = |y|"]$, whence follows from Lemma 2.1 that $r \models «c = y»$ (where $|z|$ is an element of parametric space such that always $|\bar{z}| = z$; its existence is proved, for example, in [1]); i.e., $r \Vdash$ "c is constructive."

The obtained contradiction proves the theorem.

LITERATURE CITED

1. P. S. Aleksandrov, Introduction to General Theory of Sets and Functions [in Russian], Moscow (1948).
2. P. J. Cohen, Set Theory and the Continuum Hypothesis, Benjamin, New York (1966).
3. W. B. Easton, "Powers of regular cardinals," Ann. Math. Logic, 1, No. 2, 69-112 (1970).