ON INITIAL SEGMENTS OF DEGREES OF CONSTRUCTIBILITY

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Let $\mathfrak{M}$ be a fixed countable standard transitive model of ZF + V = L. We consider the structure Mod of degrees of constructibility of real numbers x with respect to $\mathfrak{M}$ such that $\mathfrak{M}(x)$ is a model. An initial segment Q ⊆ Mod is called realizable if some extension of $\mathfrak{M}$ with the same ordinals contains exclusively the degrees of constructibility of real numbers from Q (and is a model of ZFC). We prove the following: if Q is a realizable initial segment, then

$$[y \in Q \to y < x]] \& \forall z \exists y [z < x \to y \in Q \& \sim [y < z]]].$$

Introduction. Let L be a countable standard transitive (c.s.t.) model of ZFC, $\mathfrak{M} \models V = L$. For $x \subseteq \omega_0$ we define L(x) as the constructive closure of $L \cup \{x\}$ with respect to the ordinals of L (see [1]). Let $\mathrm{Mod}^0 = \{x \mid L(x)$ be a model of $ZFC \;\&\; x \subseteq \omega_0\}$.

Let us introduce a partial order on Mod^0 by $x \preccurlyeq y \equiv x \in L(y)$, and an equivalence:

$$x \approx y \equiv x \preccurlyeq y \;\&\; y \preccurlyeq x.$$

Let Mod be the factorization; $[x] = \{y \mid y \approx x\}$; $[x] \leqslant [y] \equiv x \preccurlyeq y$; $\mathrm{Mod} = \{[x] \mid x \in \mathrm{Mod}^0\}$.

Let Q be an initial segment of Mod. We call Q a realizable segment if $\exists \mathfrak{M} [L \subseteq \mathfrak{M} \;\&\; \mathfrak{M}$ is a c.s.t. model of $ZFC \;\&\; \mathrm{On}^{\mathfrak{M}} = \mathrm{On}^L \;\&\; \forall x\, [x \in \mathfrak{M} \;\&\; x \subseteq \omega_0 \to [x] \in Q] \;\&\; \forall x\, [[x] \in Q \to x \in \mathfrak{M}]]$.

It is trivial to prove that if Q is a realizable initial segment of Mod, then it is bound by a [x] ∈ Mod. In this paper we investigate the question of the existence of the smallest bound and in particular prove the following theorem.

THEOREM A. Let Q be a realizable initial segment of Mod. Then there exists [x] ∈ Mod such that

$$\forall y\, [[y] \in Q \to [y] \preccurlyeq [x]] \;\&\; \forall z \exists y\, [[z] \preccurlyeq [x] \;\&\; [z] \neq [x] \to \\ \to [y] \in Q \;\&\; \sim [[y] \preccurlyeq [z]]].$$

To prove this theorem we prove the following auxiliary theorem.

THEOREM B. Let $\mathfrak{M}$ be a c.s.t. model of ZFC of the form L(X), where $X \in \mathfrak{M}$, $X \subseteq \omega_1^{\mathfrak{M}}$. Then we find $x \subseteq \omega_0$ such that $\mathfrak{M}(x)$ is a model of ZFC,

$$L(x) = \mathfrak{M}(x) \text{ and } \forall y\, [y \in (\mathfrak{M}(x) - \mathfrak{M}) \;\&\; y \subseteq \omega_0 \to x \in \mathfrak{M}(y)].$$

The last theorem is obviously an extension of a result of Sacks [2] on minimal degrees (the extension concerns $L(x) = \mathfrak{M}(x)$).

We outline how Theorem B implies Theorem A. Let Q be a realizable initial segment of Mod. This means that there is a model $\mathfrak{N}^0$,

$$\mathfrak{N}^0 \models ZFC,\; \mathrm{On}^L = \mathrm{On}^{\mathfrak{N}^0},\; \forall x\, [x \in \mathfrak{N}^0 \;\&\; x \subseteq \omega_0 \to [x] \in Q], \\ \forall x\, [[x] \in Q \to x \in \mathfrak{N}^0].$$

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Since $\mathfrak{N}^0 \models ZFC$, in $\mathfrak{N}^0$ there exists a total ordering of $S(\omega_0) \cap \mathfrak{N}^0$ of the type $\exp^{\mathfrak{N}^0}(\omega_0)$ (notation: $S(u) = \{x \mid x \subseteq u\}$, $\exp(u) = \mathrm{card}(S(u))$, assuming the axiom of choice). Let $X = \{\langle x_\alpha, \alpha\rangle \mid \alpha \in \exp^{\mathfrak{N}^0}(\omega_0)\}$ be such an ordering. We consider $\mathfrak{N} = L(X)$. It is easy to see that $X \in \mathfrak{N}$, $\mathfrak{N} \models ZFC$, $\forall x\,[x \in \mathfrak{N} \cap S(\omega_0) \to [x] \in Q]$, $\forall x\,[[x] \in Q \to x \in \mathfrak{N}]$. It is also easy to see that if we take a generic extension $\mathfrak{M}$ of $\mathfrak{N}$ by collapsing $\exp^{\mathfrak{N}}(\omega_0)$ onto $\omega_1^{\mathfrak{N}}$ (taking as forcing conditions the functions $f\colon D \to \exp^{\mathfrak{N}}(\omega_0)$, where D is an arbitrary countable subset of $\omega_1^{\mathfrak{N}}$ and, naturally, $f \in \mathfrak{N}$) then $\mathfrak{M}$ will have the same properties as $\mathfrak{N}$, and, moreover, it will satisfy the assumption of Theorem B. Obviously, if $x \subseteq \omega_0$ is as in Theorem B, then [x] satisfies the condition of Theorem A. Let us therefore consider B.

If $\mathfrak{M}$ satisfies the conditions of Theorem B, then we can easily find in $\mathfrak{M}$ a set $X = \{\langle\alpha, x_\alpha\rangle \mid \alpha \in \omega_1^{\mathfrak{M}}\}$ such that $\forall\alpha\,[\alpha \in \omega_1^{\mathfrak{M}} \to x_\alpha \subseteq \omega_0]$, $\mathfrak{M} = L(X)$, $\forall\alpha\,[\alpha \in \omega_1^{\mathfrak{M}} \to \alpha$ is countable in $L(x_\alpha)]$ and $\forall\alpha\,[\alpha \in \omega_1^{\mathfrak{M}} \to \{\langle\beta, x_\beta\rangle \mid \beta \in \alpha\} \in L(x_\alpha)]$. We also assume that $\forall x\,[x \in \mathfrak{M} \cap S(\omega_0) \to \mathfrak{M} \neq L(x)]$, for otherwise the theorem could be proved using the method of [2] by choosing a generic $a \subseteq \omega_0$ (with respect to the perfect forcing of $\mathfrak{M} = L(x)$) such that $\forall n\,[n \in x \equiv 2n \in a]$. In this case we can obviously assume that

$$\forall\alpha\,[\alpha \in \omega_1^{\mathfrak{M}} \to x_\alpha \notin L(\{\langle\beta, x_\beta\rangle \mid \beta \in \alpha\})].$$

Throughout §§ 1-5 it is supposed that $\mathfrak{M} = L(X)$ satisfies the above mentioned conditions

§ 1. Basic Notation

1.1. Let for $\lambda \in \omega_1^{\mathfrak{M}}$ $\mathfrak{M}_\lambda = \mathfrak{M}(\{\langle\alpha, x_\alpha\rangle \mid \alpha \in \lambda\})$, and let $\leq(\lambda)$ be a canonical total order on $\mathfrak{M}_\lambda$. The collection $\{\leqslant(\lambda) \mid \lambda \in \omega_1^{\mathfrak{M}}\}$ can be chosen in such a way that $\lambda \leqslant \mu \to \leqslant(\lambda)$ agrees on $\mathfrak{M}_\lambda$ with the induced $\leq(\mu)$. Let $\theta_\lambda = \exp^{\mathfrak{M}_\lambda}(\omega_0)$. We assume that $\leq(\lambda)$ orders $\mathfrak{M}_\lambda \cap S(\omega_0)$ into the type θ_λ. For $x \in \mathfrak{M}_\lambda$, we denote by $N_\lambda(x)$ the index of x in the sense $\leq(\lambda)$ and for $x \in \mathfrak{M}_{\omega_1}$ $\lambda(x) = \inf\{\lambda \mid x \in \mathfrak{M}_\lambda\}$.

1.2. Let F_x be some effective coding of closed subsets $S(\omega_0)$ by real numbers so that ϕ is the code of ϕ and ω_0 is the code of $S(\omega_0)$. We will write $x \leqslant_F y \equiv F_y \subseteq F_x$, $x \leqslant_{FB} y \equiv F_y \subseteq F_x$ & F_y is nowhere dense in F_x; $x \wedge y$ is the code of $F_x \cap F_y$; $x \vee y$ is the code of $F_x \vee F_y$; $l(x)$ is the length of the smallest segment in $S(\omega_0)$ which completely contains F_x.

If $f\colon K \to S(\omega_0)$, then $\bigwedge_{i \in K} f(i)$ will denote the code of $\bigcap_{i \in K} F_{f(i)}$ and $\bigvee_{i \in K} f(i)$ will denote the code of $\bigcup_{i \in K} F_{f(i)}$ (provided the set is closed).

1.3. Let $Z \subseteq S(\omega_0) \cap \mathfrak{M}$. Z will be called λ-weakly homogeneous if $\forall x \forall \mu \exists y\,[x \in Z\ \&\ \mu \in \lambda \to \lambda(y) \geqslant \mu\ \&\ y \geqslant_{FB} x\ \&\ y \in Z\ \&\ F_x$ is perfect.

Let $Y \subseteq S(\omega_0) \cap \mathfrak{M}_\lambda$ be λ-weakly homogeneous. The collection $S = \{\langle i, m, S_m^i\rangle \mid m \in \omega_0\ \&\ i \in 2\}$ is called λY-collection if

(i) $\forall m \forall i\,[S_m^i \subseteq Y]$;

(ii) $\forall m \forall i \forall x \forall y\,[x \in S_m^i\ \&\ y \in Y\ \&\ y \geqslant_F x \to y \in S_m^i]$;

(iii) $\forall m \forall x \forall i \forall y\,[x \in S_m^i\ \&\ y \in S_m^i \to x \vee y \in S_m^i]$;

(iv) $\forall m \forall x \exists i \exists y\,[x \in Y \to y \geqslant_{FB} x\ \&\ y \in S_m^i]$;

(v) $\forall m\,[S_m^0 \cap S_m^1 = \phi]$;

(vi) $\forall x \forall y \exists m \exists u \exists v\,[x \in Y\ \&\ y \in Y \to u \geqslant_{FB} x\ \&\ v \geqslant_{FB} y\ \&\ u \in S_m^0\ \&\ v \in S_m^1]$.

§ 2. The Successor Case

2.1. Let $Y \subseteq S(\omega_0) \cap \mathfrak{M}_{\lambda+1}$ be $(\lambda+1)$-weakly homogeneous, $S \in \mathfrak{M}_{\lambda+1}$ be a $(\lambda+1)$Z-collection, and $z \in Y$. We define on $S(\omega_0)$ a function $H_{\lambda+1,S,z}(x) = y$ as follows:

(i) if $x \notin F_z$, we consider y undefined, otherwise we put $z = z_0$;

(ii) we put $\langle\bar t, \bar u\rangle = \min_{\leqslant(\lambda+1)} \{\langle t, u\rangle \mid t \wedge u = \phi\ \&\ t \vee u \geqslant_{FB} z_0\ \&\ l(u) + l(t) \leqslant (1/2)\, l(z_0)\ \&\ \exists m\,[t \in S_m^0\ \&\ u \in S_m^1]\}$,

$$\langle\bar{\bar t}, \bar{\bar u}\rangle = \min_{\leqslant(\lambda+1)} \{\langle t, u\rangle \mid t \wedge u = \phi\ \&\ t \vee u \geqslant_{FB} z_0\ \&\ l(u) + l(t) \leqslant$$

$$\leqslant (1/2)\, l(z_0)\ \&\ \exists m\,[t \in S_m^0\ \&\ u \in S_m^1]\ \&\ x \in F_u \cup F_t\};$$

(iii) if $\langle \bar{\bar{t}}, \bar{\bar{u}} \rangle$ or $\langle \bar{t}, \bar{u} \rangle$ is undefined, we consider y undefined. Otherwise, if $\langle \bar{t}, \bar{u} \rangle = \langle \bar{\bar{t}}, \bar{\bar{u}} \rangle$ it is assumed that $0 \in y$; but if $\langle \bar{t}, \bar{u} \rangle \neq \langle \bar{\bar{t}}, \bar{\bar{u}} \rangle$, then $0 \notin y$. We put $z_1 = \bar{\bar{u}}$ or $z_1 = \bar{\bar{t}}$ depending on whether $x \in F_{\bar{\bar{u}}}$ or $x \in F_{\bar{\bar{t}}}$.

(iv) is the same as (ii) but for the substitution of z_0 by z_1, and we recognize $1 \in y$, etc.

LEMMA 2.2. Let λ, Z, Y, and S be as in 2.1. Then $\mathfrak{M}_{\lambda+2} \models \exists x \forall y\, [x \subseteq \omega_0\ \&\ F_x$ is perfect $\&\ [y \in F_x \to H_{\lambda+1, S, z}(y) = x_{\lambda+1}]]$.

The proof is accomplished in $\mathfrak{M}_{\lambda+2}$. Let $E = 2^{\langle \omega_0 \rangle}$, and for $t \in E$ let $h(t) = D(t)$ be the domain of definition of t; $h(t) \in \omega_0$. Let $\phi \in E$,

$$h(\phi) = 0; \quad \langle 0 \rangle \in E, \quad \langle 1 \rangle \in E, \quad h(\langle 0 \rangle) = h(\langle 1 \rangle) = 1.$$

For u, $t \in E$, we let $ut \in E$ be such that

$$h(ut) = h(u) + h(t); \quad k < h(u) \to ut(k) = u(k);$$
$$k < h(t) \to ut(h(u) + k) = t(k).$$

We define $u \leq t$ if $\exists v\, [v \in E\ \&\ uv = t]$.

Let $f: E \to Y$ be a function such that

(i) $s \leqslant t \to f(s) \leqslant_{FB} f(t), \quad l(f(s)) \leqslant 1/2^{h(s)}, \quad f(\phi) = z$;

(ii) $f(s\langle 0 \rangle) \wedge f(s\langle 1 \rangle) = \phi$;

(iii) if $h(s) \in x_{\lambda+1}$, then $\langle f(s\langle 0 \rangle), f(s\langle 1 \rangle) \rangle = \min\limits_{\leq(\lambda+1)} \{\langle t, u \rangle \mid F_t$ and F_u's are perfect $\&\ (t \vee u) \geqslant_{FB} f(s)\ \&\ \exists m\, [t \in S_m^0\ \&\ u \in S_m^1]\ \&\ l(u) + l(t) \leqslant (1/2) l(f(s))\} = \langle \bar{t}, \bar{u} \rangle$;

(iv) if $h(s) \notin x_{\lambda+1}$, then $\langle f(s\langle 0 \rangle), f(s\langle 1 \rangle) \rangle = \min\limits_{\leqslant(\lambda+1)} \{\langle t, u \rangle \mid t \vee u \geqslant_{FB} f(s)\ \&\ \exists m\, [t \in S_m^0\ \&\ u \in S_m^1]\ \&\ F_t$ and F_u's are perfect $\&\ (t \vee u) \wedge (\bar{t} \vee \bar{u}) = \phi\ \&\ l(u) + l(t) \leqslant (1/2)\, l(f(s))\}$.

We consider $x = \bigwedge\limits_{n \in \omega_0} \bigvee\limits_{h(s)=n} f(s)$. It is easy to see that F_x is perfect (the proof is analogous to [3]) and the equality $H_{\lambda+1, S, z}(y) = x_{\lambda+1}$ for every $y \in F_x$ follows from the definitions of F_x and the function H.

A function f of such a kind can be easily constructed by taking into account the weak homogeneity of Y and the properties of S.

The lemma is proved.

We note that because of the properties of $X = \{\langle \alpha, x_\alpha \rangle \mid \alpha \in \omega_1^{\mathfrak{M}}\}$ the constructed x may not be in $\mathfrak{M}_{\lambda+1}$, for if we took in $\mathfrak{M}_{\lambda+1}$ a $y \in F_x$, then we could construct $x_{\lambda+1} = H_{\lambda+1, S, z}(y)$ in $\mathfrak{M}_{\lambda+1}$, and that is not possible since $x_{\lambda+1} \notin \mathfrak{M}_{\lambda+1}$.

The smallest x, in the sense $\leq(\lambda + 2)$, constructed as in 2.2, will be denoted by $x = W(\lambda + 1, S, z)$.

§ 3. The Limit Case

3.1. Let λ be a limit, $Y \subseteq S(\omega_0) \cap \mathfrak{M}_\lambda$ be λ-weakly homogeneous, $S \in \mathfrak{M}_\lambda$ be a λY-collection, and $z \in Y$.

We define $H_{\lambda, S, z}(x)$ analogously to 2.1 (except that $\min\limits_{\leqslant(\lambda+1)}$ changes to $\min\limits_{\leqslant(\lambda)}$).

LEMMA 3.2. Let $\lambda \in \omega_1^{\mathfrak{M}}$ be a limit, and Z, Y, and S be as in 3.1. Then $\mathfrak{M}_{\lambda+1} \models \exists x\, \forall y\, \forall \mu\, [x \subseteq \omega_0\ \&\ F_x$ is perfect $\&\ [y \in F_x \to H_{\lambda, S, z}(y) = x_\lambda]\ \&\ [\mu < \lambda \to \exists x' [x' \in \mathfrak{M}_\lambda\ \&\ \lambda(x') \geqslant \mu\ \&\ x \geqslant_{FB} x']]]$.

Proof (in $\mathfrak{M}_{\lambda+1}$). For the proof it is enough to construct a function $f: E \to Y$ satisfying 2.2, (i) to (iv) (changing $\min\limits_{\leqslant(\lambda+1)}$ to $\min\limits_{\leqslant(\lambda)}$), and adding one more condition as follows:

(v) There is an increasing function $\mu: \omega_0 \to \lambda$ such that $\sup\limits_{n \in \omega_0} \mu(n) = \lambda$ and $\forall s\, [s \in E \to \lambda(f(s)) \geqslant \mu(h(s))]$. The condition (v) is needed to secure the additional conditions on x.

We finish the proof as in 2.2.

Let x = W(λ, S, z) be the smallest $x \in \mathfrak{M}_{\lambda+1}$ in the sense $\leq(\lambda+1)$ which can be constructed as in Lemma 3.2.

Again we note $x \notin \mathfrak{M}_{\lambda+1}$.

3.3. We assume that if $S_1 \neq S_2$, $z_1 \neq z_2$, then $F_{W(\lambda, S_1, z_1)} \cap F_{W(\lambda, S_2, z_2)} = \phi$ (for an arbitrary $\lambda \in \omega_1$).

§ 4. Proof of Theorem B

In $\mathfrak{M}$ we construct a collection of sets $\{Z_\alpha \mid \alpha \in \omega_1^{\mathfrak{M}}\}$, satisfying the following conditions:

(i) $Z_\alpha \subseteq S(\omega_0)$, $Z_\alpha \in \mathfrak{M}_\alpha$ α-weakly homogeneous;

(ii) $Z_\alpha \subseteq Z_{\alpha+1}$, $Z_{\alpha+1} - Z_\alpha \neq \phi$;

(iii) $\alpha < \beta \;\&\; x \in Z_\beta \;\&\; \lambda(x) = \beta \rightarrow \exists y\ [y \in Z_\alpha \;\&\; x \geqslant_{FB} y \;\&\; \lambda(y) = \alpha]$;

(iv) if Z_λ is defined, we put $\bar{Z}_\lambda = \{W(\lambda, S, z) \mid \exists Y [Y \subseteq Z_\lambda^*$ λ-weakly homogeneous, $\& Y \in \mathfrak{M}_\lambda \& S$ is λY-collection $\& z \in Y]\}$, and $Z_{\lambda+1} = Z_\lambda \cup \{y \mid F_y$ is perfect $\& \exists x [x \in \bar{Z}_\lambda \;\&\; y \geqslant_F x] \;\&\; y \in \mathfrak{M}_{\lambda+1}\}$, where $Z_\lambda^* = Z_\lambda$ if λ is a limit and $Z_\lambda^* = Z_\lambda - Z_\beta$ for $\lambda = \beta + 1$.

(v) $\alpha < \beta < \omega_1^{\mathfrak{M}} \;\&\; x \in Z_\alpha \rightarrow \exists y\ [y \in Z_\beta \;\&\; \lambda(y) = \beta \;\&\; y \geqslant_{FB} x]$;

(vi) for limit λ's, $Z_\lambda = \bigcup_{\alpha \in \lambda} Z_\alpha$;

(vii) $Z_0 = \mathfrak{M}_0 \cap S(\omega_0) = L \cap S(\omega_0)$.

It is easy to see that the points (vii), (vi), and (iv) define the construction of Z_α while all the other points will be preserved (this follows from Lemmas 2.2, 3.2, and the definition of W(λ, S, z)).

It is also obvious that $\{\langle Z_\alpha, \alpha\rangle \mid \alpha \in \lambda\} \in \mathfrak{M}_\lambda$. We put $P = \bigcup_{\alpha \in \omega_1} Z_\alpha$.

§ 5. Properties of the Forcing Conditions

5.1. Let $G \subseteq P$ be a $\mathfrak{M}$-generic filter on P. Obviously G defines a unique real number $a = a_G = \bigcap_{x \in G} F_x$ and is determined by it: $G = G_a = \{x \mid x \in P \;\&\; a \in F_x\}$. Let $G \subseteq P$ be an $\mathfrak{M}$-generic filter on P.

<u>LEMMA 5.2.</u>

$$\{\langle \alpha, x_\alpha \rangle \mid \alpha \in \omega_1^{\mathfrak{M}}\} \in L(a_G).$$

<u>Proof.</u> We show that $x_0 \in L(a_G)$. Indeed $Z_0 \in L$ and $a_G \in F_x$ for some $x \in \bar{Z}_0$ [this follows from 4.1 (iii), (iv), (v), and (vii)]. It means that $x_0 = H_{0Sz}(a_G)$ for some S, $z \in \mathfrak{M}_0$, i.e., S, $z \in L$. Therefore, H_{0Sz} is defined in L and $x_0 \in L(a_G)$.

Let us assume that

$$\{\langle \alpha, x_\alpha \rangle \mid \alpha \in \lambda\} \in L(a_G).$$

Then we can similarly construct $x_\lambda = H_{\lambda Sz}(a_G)$, and we have $x_\lambda \in L(a_G)$.

It is clear that all the x_α's can be effectively reconstructed from a_G and L (we know that the Z_α's were constructed effectively). The lemma is proved.

<u>LEMMA 5.3.</u> For some $\mathfrak{M}$-generic $G \subseteq P$, a_G is minimal over $\mathfrak{M}$.

<u>Proof.</u> Let (in $\mathfrak{M}$, $c \in V^{(P)}$ and $p \in P$, where

$$d \Vdash «c \in \check{\omega}_0 \;\&\; c \notin \mathfrak{M} \;\&\; a_G \notin \mathfrak{M}(c)».$$

Clearly, we can assume that $p = \omega_0$ (for $S(\omega_0)$). We define

$$S_m^0 = \{p \mid p \in P \;\&\; p \Vdash «\check{m} \notin c»\}$$

and

$$S_m^1 = \{p \mid p \in P \;\&\; p \Vdash «\check{m} \in c»\}.$$

It is easy to see that $S = \{\langle i, m, S^i_m\rangle \mid m \in \omega_0 \;\&\; i \in 2\}$ satisfies 1.3 (i) to (vi) by changing Y to P.

Because of the condition on $\mathfrak{M}$, we have $S \in \mathfrak{M}_{\omega_1} = \mathfrak{M}$, .

Therefore, we can construct by the Skolem–Löwenheim method a limit $\lambda \in \omega_1^{\mathfrak{M}}$ and $Y \subseteq Z_\lambda$ such that $Y \in \mathfrak{M}_\lambda$, and Y is λ-weakly homogeneous; $S^i_m(\lambda) = \{p \mid p \in S^i_m \cap \mathfrak{M}_\lambda\} \in \mathfrak{M}_\lambda$; $S(\lambda) = \{\langle i, m, S^i_m(\lambda)\rangle \mid m \in \omega_0 \;\&\; i \in z\}$ is a λY-collection. We consider $x = W(\lambda, S(\lambda), \omega_0) \in \mathfrak{M}_{\lambda+1} \cap P$. As in [3] or [2] it is not difficult to prove that $x \Vdash$ «$a_G \in L(c, \check{x}, S(\check{\lambda}))$»," i.e., $x \Vdash \langle\!\langle a_G \in \mathfrak{M}(c)\rangle\!\rangle$, which contradicts our assumption. The lemma is proved.

From Lemmas 5.2 and 5.3 Theorem B follows immediately.

Theorem B and similar theorems are given in [4].

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