PROOF OF A THEOREM OF LUSIN

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In [1, 2], Lusin published a theorem (with proof) asserting that a very simple set constructed by him is not Borel. Lunina [3] discovered an error in Lusin's proof. It is proved that Lusin's theorem is nonetheless valid.

In 1926, Lusin described a set using the "laws of arithmetic" and proposed proving that this set is not Borel (it followed in particular from the method of proof that the set under consideration turned out to be analytic). The example constructed and the corresponding proof form the content of the note [1] of Lusin as well as Sec. 61 of his memoirs [2].

As was discovered by Lunina [3], the proof proposed by Lusin is invalid. Nevertheless it can be shown that the "arithmetic example" of Lusin has the desired properties, i.e., is an example of an analytic set which is not a Borel set. We give here a proof of this fact.

Lusin considered subsets of the set of all irrational numbers of the interval [0, 1]. It is sometimes more convenient to consider (and this is what we will do) subsets of the Baire space of [4, p. 227] which consists of all sequences of natural numbers and is homeomorphic to the set of all irrational numbers in [0, 1] by virtue of the representation of an irrational number x by a continuous fraction:

$$x=\cfrac{1}{n_1+\cfrac{1}{n_2+\ddots+\cfrac{1}{n_k+\ddots}}}$$

In the notational system [5, 6], associated with the projective hierarchy, the set studied in [1] is defined as follows. Introduce the Borel space $\mathbf{N}^{\mathbf{N}}$ of all functions of the natural numbers $\mathbf{N}=\{1, 2,\ldots\}$ into N (i.e., all sequences of natural numbers). A function (sequence) $\alpha=(\alpha_n,\ n\in\mathbf{N})\in\mathbf{N}^{\mathbf{N}}$ is called composite [1] if "there exists among its members an infinite set of numbers which divide one another." The set E of all composite sequences constitutes Lusin's example.

<u>Assertion [1].</u> The set E is analytic but not Borel.

There is an ambiguity in the definition of a composite sequence, i.e., do we assume that among the elements of the "infinite set of numbers" identical natural numbers can occur which are distinct elements (i.e., have distinct indices) of the sequence $(\alpha_n, n\in\mathbf{N})$? For example, is the sequence defined by $\alpha_n = 1$ for all $n\in\mathbf{N}$ composite?

Taking the definition of a composite sequence in accordance with the above two alternatives, we obtain, respectively, two "realizations" of the set E discussed by Lusin.

$E_1=\{\alpha\in\mathbf{N}^{\mathbf{N}}$: there exists a function

$$\gamma\in\mathbf{N}^{\mathbf{N}}, \text{such that } \gamma(k)\neq\gamma(n) \text{ for all } k\neq n$$

and $\alpha(\gamma(k+1))/(\alpha(\gamma(k)))$ is an integer for every $k\in\mathbf{N}\}$;

$E_2=\{\alpha\in\mathbf{N}^{\mathbf{N}}$: there exists a function

$$\gamma\in\mathbf{N}^{\mathbf{N}},\ \text{such that } \alpha(\gamma(k+1))\,/\,\alpha(\gamma(k))$$

is an integer greater than or equal to 2 for every $k\in\mathbf{N}\}$. For one variant of the definition of composite sequence we have E = E_1, for the other $E=E_2$; $E_2\subseteq E_1$.

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We prove the following theorem.

THEOREM. The sets E_1 and E_2 are analytic but not Borel.

Proof. The proof of this theorem consists of two lemmas.

LEMMA 1. The sets E_1 and E_2 are analytic.

LEMMA 2. The sets E_1 and E_2 are not Borel.

Proof of Lemma 1. The proof is carried out by the syntactic method. Let α and γ be variables running over $\mathbf{N}^{\mathbf{N}}$; let n and k be variables running over N; let $\times$ be multiplication of natural numbers. It is clear from the definition of E_1 that $E_1 = \{\alpha : \varphi(\alpha)\}$, where $\varphi(\alpha)$ is the following formula:

$$\exists\gamma\,[\forall k\exists n\,[\alpha(\gamma(k+1)) = n \times \alpha(\gamma(k))]\;\&\;\forall k\forall n\,[k \neq n \rightarrow \gamma(k) \neq \gamma(n)]].$$

The formula $\varphi(\alpha)$ is evidently a Σ_1^1-formula, and therefore $E_1 = \{\alpha : \varphi(\alpha)\}$ is a Σ_1^1-set, i.e., it is in particular analytic (for the definition of Σ_1^1-formulas and their relation to analytic sets, see [6]).

In completely analogous fashion, one also proves the analyticity of the set E_2 (it is necessary to make use of the formula

$$\exists\gamma\forall k\exists n\,[\alpha(\gamma(k+1)) = n \times \alpha(\gamma(k))\;\&\;n \geqslant 2]).$$

Proof of Lemma 2. Fix some analytic set $Q \subseteq \mathbf{N}^{\mathbf{N}}$ which is not Borel. The idea of the proof consists in constructing a Borel function F from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}^{\mathbf{N}}$ such that $\alpha \in Q \equiv F(\alpha) \in E_1 \equiv F(\alpha) \in E_2$ for every $\alpha \in \mathbf{N}^{\mathbf{N}}$.

We turn to the details. Let J denote the set of all positive sequences of natural numbers of finite nonzero length. By the definition of analytic set [5], there exists a system $H = \{H_e : e \in J\}$ of closed subsets of $\mathbf{N}^{\mathbf{N}}$ such that $Q = \bigcup_{f\in\mathbf{N}^{\mathbf{N}}} \bigcap_{n\in\mathbf{N}} H_{f|n}$ ($f|n$ is the restriction of the function f to the first n natural numbers, i.e., the first n terms of the sequence f). Moreover, H can be assumed to be a regular system, i.e., if e_1 is an extension of e_2, then $H_{e_1} \subseteq H_{e_2}$ [5]. For every $\alpha \in \mathbf{N}^{\mathbf{N}}$ we define $t(\alpha) = \{e \in J : \alpha \in H_e\}$.

Let $q \subseteq J$. We will say that q has an infinite path if there exists an infinite sequence $e_1, e_2, \ldots, e_k, \ldots$ of elements of q such that e_{k+1} is a proper (i.e., $e_{k+1} \neq e_k$) extension of e_k for every k.

LEMMA 3. Let $\alpha \in \mathbf{N}^{\mathbf{N}}$. Then $\alpha \in Q$, if and only if $t(\alpha)$ has an infinite path.

Proof. Let $\alpha \in Q$. Then by the choice of H we have $\alpha \in \bigcap_{n\in\mathbf{N}} H_{f|n}$ for some $f \in \mathbf{N}^{\mathbf{N}}$. Thus $f|1, f|2, \ldots, f|k, \ldots$ is the desired infinite path in $t(\alpha)$.

Conversely, let $e_1, e_2, \ldots, e_k, \ldots$ be an infinite path in $t(\alpha)$, m_k the length of the sequence e_k. It is clear that there exists an $f \in \mathbf{N}^{\mathbf{N}}$, such that $e_k = f|m_k$ for every k. By the definition of $t(\alpha)$, this means that $\alpha \in H_{f|m_k}$ for every k. Now by the regularity of the system H and the obvious inequality $m_{k+1} > m_k$ for every k we obtain: if $m \in \mathbf{N}$, then $\alpha \in H_{f|m}$. But by the choice of the system H, the latter means precisely that $\alpha \in Q$. The lemma is proved.

Now let $P = \{p_{km} : k,m \in \mathbf{N}\} \cup \{p_n : n \in N\}$ be some enumeration of the set P of all prime numbers without repetition, i.e., $p_n = p_{km}$ is impossible for every m, k, and n; $p_{n_1} = p_{n_2}$ implies $n_1 = n_2$; $p_{k_1m_1} = p_{k_2m_2}$ implies $k_1 = k_2$ and $m_1 = m_2$.

For every sequence $e = (m_1, \ldots, m_k) \in J$ we define a natural number $p(e) = p_{1m_1} \times \ldots \times p_{km_k}$. We observe the following obvious fact.

Remark. Let $e_1, e_2, \in J$. Then e_2 is a proper extension of e_1 if and only if $p(e_2)$ is divisible by $p(e_1)$ and $p(e_2)/p(e_1) \geqslant 2$.

We now prove the key lemma.

LEMMA 4. Let $q \subseteq J$, $U = \{p(e) : e \in q\}$ and $\beta \in N^N$ be defined by the condition $\beta(n) = n$ for $n \in U$ and $\beta(n) = p_n$ for $n \notin U$. Then the following three assertions are mutually equivalent:

1) q has an infinite path;

2) $\beta \in E_2$;

3) $\beta \in E_1$.

Proof. We remark that it follows from the definition of β that $\beta \in E_1 \equiv \beta \in E_2$ (since for $k \neq n$ we have $\beta(k) \neq \beta(n)$). It is therefore sufficient to prove the equivalence of the first two assertions in the hypothesis of the lemma.

First let $e_1, e_2, \ldots, e_k, \ldots$ be an infinite path in q. Define the function $\gamma \in N^N$ by the condition $\gamma(k) = p(e_k)$. It then follows from the Remark that $\beta(\gamma(k+1))/\beta(\gamma(k)) = \gamma(k) = p(e_{k+1})/p(e_k)$ is an integer greater than or equal to 2 for every k. This by definition means that $\beta \in E_2$.

Conversely, assume that $\beta \in E_2$. Then there exists a function $\gamma \in N^N$ such that $\beta(\gamma(k+1))/(\beta(\gamma(k)))$ is an integer greater than or equal to 2 for every k. It is clear from the definition of β that we can choose a sequence $e_1, e_2, \ldots, e_k, \ldots$ of elements of q such that $p(e_k) = \gamma(k)$ for every k. It follows from the Remark that this sequence is an infinite path in q. The lemma is proved.

We now turn to the definition of the function F. Let $\alpha \in N^N$. Define $F(\alpha) \in N^N$ by the condition: $F(\alpha)(n) = n$ for $n \in \{p(e): e \in t(\alpha)\}$; $F(\alpha)(n) = p_n$ in the opposite case.

LEMMA 5. The function F is a Borel function from N^N to N^N, i.e., F is a Borel subset of $N^N \times N^N$.

Proof. It is clear that $F = \bigcap_{n \in N} U_n$, where each U_n is defined as follows: 1) if $n = p(e)$, $e \in J$, then $U_n = \{(\alpha,\beta) \in N^N \times N^N: [\alpha \in H_e \to \beta(n) = n] \& [\alpha \notin H_e \to \beta(n) = p_n]\}$; 2) if $n \notin \{p(e): e \in J\}$, then $U_n = \{(\alpha,\beta) \in N^N \times N^N: \beta(n) = p_n\}$. But every U_n is Borel (since each H_e is closed). From this the lemma is obvious.

LEMMA 6. Let $\alpha \in N^N$. Then $\alpha \in Q \equiv F(\alpha) \in E_1 \equiv F(\alpha) \in E_2$.

Proof. The proof is obtained from Lemmas 3 and 4.

We complete the proof of Lemma 2. Assume the contrary, i.e., let E_i be a Borel set; i = 1 or i = 2. Then it follows from Lemma 6 that $Q = \{\alpha \in N^N: F(\alpha) \in E_i\}$ is also a Borel set (since F is a Borel function by Lemma 5 and the inverse image of a Borel set under a Borel function is Borel, cf. [5, Proof of Corollary 5]), which contradicts the choice of Q. This contradiction completes the proof of Lemma 2.

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