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N. N. LUZIN'S PROBLEMS ON THE EXISTENCE OF CA-SETS WITHOUT PERFECT SUBSETS

V. G. Kanovei

After P. S. Aleksandrov, Hausdorff, and M. Ya. Suslin proved in 1916-1917 that any uncountable Borel set, and even any uncountable A-set on the real line, contains a perfect subset, and thus has the cardinality of the continuum $c = 2^{\aleph_0}$, one of the main problems in the descriptive set theory at that time was that of investigating the cardinality of CA-sets (i.e., sets complementary to A-sets). The essence of this problem is the following:

PROBLEM I. Is it true that (I) there exists a CA-set of cardinality strictly between the countable cardinality $\aleph_0$ and the cardinality of the continuum, c?

As possible approximations to the solution of this problem, two more questions can be posed on CA-sets:

PROBLEM II. Is it true that (II) there exists an everywhere defined function y = f(x), whose graph is a CA-set without any perfect subsets?

PROBLEM III. Is it true that (III) there exists an uncountable CA-set without any perfect subsets?

These three problems occur in many of Luzin's works on descriptive set theory. In particular, Problems I and II are formulated and discussed in the fifth chapter of [1] (see also pp. 240 and 246 of [2]), and Problem III arises in [3]. However, in classical descriptive set theory they remain unsolved.

In later years, P. S. Novikov, Solovay, and others (for details, see below) established that Problems I, II, and III are insoluble within the confines of the Zermelo—Frankel axiometric system ZFC, i.e., using this theory it is not possible to obtain a definite "yes" or "no" to any of the three questions posed.

The aim of this article is to explain the relation of the three problems to each other, and also their relation to the continuum hypothesis CH (which can be expressed by the equation $c = \aleph_1$, where $\aleph_1$ is the first uncountable cardinality). In order to justify the formulation of the fundamental result, we make several remarks connecting the conjectures (I), (II), and (III).

Remark 1. The conjecture (II) implies (III). Conjecture (I) also implies (III), since a set with a perfect subset must have continual cardinality. Moreover, the conjecture (I) implies the negation of CH.

Remark 2. (Luzin, Chap. v of [1]). An uncountable CA-set with no perfect subsets has cardinality equal to $\aleph_1$. Moreover, the graph of an everywhere defined real function has cardinality c. Therefore, conjecture (II) implies CH and is inconsistent with (II).

Remark 3. It follows from Remarks 1 and 2 that conjecture (I) is equivalent to the conjunction (III) $\wedge \neg$CH (the sign $\wedge$ denotes the connective "and," and $\neg$ is the symbol of negation).

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These remarks only leave possible the following five combinations of our conjectures and their negations with CH and its negation:

$\neg$(III) $\wedge$ CH [and then $\neg$ (I) and $\neg$ (II)];

$\neg$(III) $\wedge\neg$ CH [and then analogously $\neg$(I) and $\neg$(II)];

(II) [and then (III), $\neg$(I) and CH];

(III) $\wedge\neg$ CH [and then (I) and $\neg$(II)];

(III) $\wedge\neg$ (II) $\wedge$ CH [and then $\neg$(I)].

It turns out that each of these five combinations of hypotheses is consistent with the axioms of ZFC, i.e., is noncontradictory. The consistency of the combinations of $\neg$(III) with CH and its negation gives Theorems 2 and 3 of [4] (where, incidentally, a far stronger conjecture than $\neg$(III) is considered, implying that not only sets in the class CA, but also uncountable projective sets, have no perfect subsets).

The consistency of conjecture (II) was established by P. S. Novikov, using the inference of (II) from the constructivity axiom V = L. In fact, it was shown in Theorem 2 of [5] that the axiom V = L implies the existence of an everywhere defined function, whose graph is the set A_2 and has no perfect subsets. But, as was noted in [6] (Remark 32), using the Novikov—Condō uniformization theorem we can then derive an everywhere defined function with graph in the class CA and no perfect subsets. Finally, each corollary of the constructivity axiom is consistent, since this axiom itself does not contradict the axioms of ZFC (K. Gödel).

The consistency of the combination (III) $\wedge\neg$CH was obtained by Solovay [7] as an elementary corollary of the principal lemma showing that uncountable CA-sets without perfect subsets can be obtained from weaker premises than the axiom V = L. This simple argument will be set out below, after the proof of the following result.

THEOREM. The hypothesis (III)$\wedge\neg$(II) $\wedge$ CH does not contradict the axioms of ZFC.

To prove this theorem (which covers the problem of the relations between problems I, II, and III, and their relations to the continuous hypothesis within the axioms of ZFC), we shall use a model known as the ω_1-Cohen generic extension of the constructive model. We describe its construction.

Fix a countable transitive model M of the axioms ZFC + (V = L). As a set of forcing conditions (s.f.c.) for the generic extension of M, we take the $\omega_1{}^M$-Cohen set P, consisting (see p. 119 of [8]) of all possible functions p such that the domain of definition dom p is contained in $\omega_1{}^M \times \omega$ and is finite, and the range of values ran p is contained in the two-element set {0, 1}. As usual, we denote by $\omega_1{}^M$ in this definition the first uncountable cardinal in the model M.

The set P is ordered by inverse inclusion: $p \le q$ if the function p is an extension of the function q: in this case p, as a forcing condition, is more informative than q.

We now fix a set $G \subseteq P$ which is P-generic over M, and consider the generic model N = M[G]. The properties we require of the model N are contained in the following three lemmas:

LEMMA 1. The continuum hypothesis CH is true in N.

LEMMA 2. In N, $\omega_1{}^L = \omega_1$ is true.

LEMMA 3. The following statement is true in N: there is no point δ in the Baire space I such that $I \subseteq L[\delta]$.

Notation. $\omega_1{}^L$ is the first uncountable cardinal in the class L of all constructive sets. The Baire space $I = \omega^\omega$ consists of all possible ω-sequences of natural numbers: ω = {0, 1, 2, ...} is the natural series. For $\delta \in I$, we denote by $L[\delta]$ the class of all sets which are constructive with respect to δ.

We first show how Lemmas 1, 2, and 3 imply the truth in N of the combined hypothesis (III) $\wedge\neg$(II)$\wedge$ CH; by the same token, we shall have proved the theorem. We shall then give the proofs of the lemmas.

With the continuum hypothesis CH, this is obvious: Lemma 1.

The truth in N of the conjecture (III) is a corollary of Lemma 2 and another lemma, proved by Solovay [7] and Lyubetskii [9]:

LEMMA 4. If $\omega_1^L = \omega_1$, then there exists an uncountable CA-set with no perfect subsets.

We give an outline of the proof of Lemma 4. It is known that the set $I \cap L$ of all the constructive points in I belongs to the class A_2, and admits a canonical complete ordering $\prec$ of length ω_1^L, and also having (as its set of pairs) the class A_2 (in fact, the class Σ_2^1 as well as $I \cap L$). See, for example, the supplement of [8, Sec. 2].

In order to construct an uncountable CA-set with no perfect subsets in the space I (and thus also on the real line, as a result of the homeomorphism between I and the space of irrational points), it is quite sufficient to construct such a set in the class A_2: the step to the smaller class CA is easily realized using the Novikov–Condo uniformization theorem (see [8, p. 258]) and its corollary 4.22 for the class $\Pi_1^1 = CA$.

Thus it is sufficient to deduce from the equation $\omega_1^L = \omega_1$ that there exist uncountable A_2-sets in I with no perfect subsets. Suppose the contrary: there are no such sets. In particular, the uncountable (as $\omega_1^L = \omega_1$) set $I \cap L$ in the class A_2 has a perfect subset X_0. The set X_0 may be assumed to be compact. Set $F_0(\alpha) = \alpha$ for $\alpha \in X_0$; the function F_0: $X_0 \to I$ is continuous and one-to-one on X_0.

Our aim now is to construct a decreasing sequence of perfect (and thus compact) sets $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ and a sequence of continuous and one-to-one functions F_i: $X_i \to I$ ($i = 0, 1, 2, \ldots$) on the corresponding X_i, such that we have the condition $F_{i+1}(\alpha) \prec F_i(\alpha)$ for each point $\alpha \in X_{i+1}$. The existence of such a sequence immediately leads to a contradiction and to the proof of Lemma 4; take a point α in the (obviously nonempty) intersection of all the X_i, and we obtain $F_{i+1}(\alpha) \prec F_i(\alpha)$ for all indices i, which contradicts the completeness of the ordering $\prec$.

The set X_0 and the function F_0 have already been constructed. As it is perfect and compact, X_0 is homeomorphic to the Cantor discontinuum $C = 2^\omega$. The latter admits a homeomorphism h: C onto C such that $\gamma \neq h(\gamma)$ and $\gamma = h(h(\gamma))$ for all $\gamma \in C$ (if $\gamma = \langle j_0, j_1, j_2, \ldots\rangle \in C$, then $h(\gamma) = \langle 1 - j_0, j_1, j_2, \ldots\rangle$). Therefore there is also a homeomorphism H: X_0 to X_0 such that $\alpha \neq H(\alpha)$ and $\alpha = H(H(\alpha))$ for each point $\alpha \in X_0$.

By the choice of H, and as F_0 is one-to-one, the sets

$$Y = \{\alpha \in X_0 \colon F_0(H(\alpha)) \prec F_0(\alpha)\}$$

and

$$Z = \{\alpha \in X_0 \colon F_0(\alpha) \prec F_0(H(\alpha))\}$$

give X_0 as their union and, moreover, Y is the H-image of the set Z. Thus Y is not countable. Moreover, Y is a set in the class A_2, since the relation $\prec$ has this class. Therefore, in view of the above assumption of the contrary, Y contains a perfect subset X_1. It remains to define $F_1(\alpha) = F_0(H(\alpha))$ for $\alpha \in X_1$.

In exactly the same way, we construct for the pair X_1, F_1 another pair X_2, F_2, etc.

The proof of Lemma 4 and the verification of the truth of the conjecture (III) in the model N are complete.

Finally, we verify that the conjecture (II) is false in N; thus we shall complete the deduction of the theorem from Lemmas 1, 2, and 3. We obtain the required result as a corollary of Lemma 3 and the following statement.

LEMMA 5. If the conjecture (II) is satisfied, then there exists a point δ in the space I such that $I \subseteq L[\delta]$.

Proof. Let the function F: $I \to I$ be such that its graph $X = \{\langle\alpha, F(\alpha)\rangle\colon \alpha \in I\}$ is a CA-set with no perfect subsets [the case of real functions of a real argument in conjecture (II) reduces easily to the case of functions from I to I]. By a theorem of Solovay [7] and Mansfield [10] (see also [8, p. 327, result 16.7], there exists a point $\delta \in I$ such that $X \subseteq L[\delta]$; it is sufficient to take δ such that X is a set in the class $\Pi_1^{1,\delta}$. But then also $I \subseteq L[\delta]$. In fact, let $\alpha \in I$ and $\beta = F(\alpha)$. We have $\langle\alpha, \beta\rangle \in X$, and hence $\langle\alpha, \beta\rangle \in L[\delta]$ and $\alpha \in L[\delta]$.

Therefore, in order to prove the theorem it remains to prove Lemmas 1, 2, and 3.

Proof of Lemma 1. Our ω_1^M-Cohen s.f.c. P satisfies the chain countability condition c.c.c. (also known as the antichain ω_1-condition) in the original model M (see [8, p. 121]), and has cardinality ω_1^M in M. Moreover, the continuum hypothesis CH is true in M (as a corollary of the constructivity axiom in M), and by the same token the equation $\omega_1 = \omega_1^{\omega\times\omega}$ is satisfied. The truth of CH in the model N = M[G] is given by Theorem 3.15 of [8, p. 125] for $\kappa = \nu = \omega_1^M$ and $\lambda = \mu = \omega$.

Proof of Lemma 2. As a result of the c.c.c., the cardinals of M remain cardinals in N, i.e., in particular, $\omega_1^M = \omega_1^N$. But M is the "constructive part" of the model N.

Proof of Lemma 3. On the contrary, let $\delta \in I \cap N$ and let $I \subseteq L[\delta]$ be true in N. By the "existence and minimality lemma" (see [8, p. 111]), there exists a set $t \in M$, $t \subseteq P \times (\omega \times \omega)$, such that

$$\delta = i_G(t) = \{\langle k, l\rangle : \exists p \in G\ (\langle p, k, l\rangle \in t)\}$$

[bearing in mind that $\langle k, \ell\rangle \in \delta$, when $\delta(k) = \ell$]. Moreover, if the c.c.c. is satisfied in M for P, we may assume without loss of generality that t is no more than countable in M (see the argument in the proof of Theorem 3.15 of [8, p. 125]). But then the set

$$U_1 = \{\xi < \omega_1^M: \text{ there exist natural numbers } k, \ell, m \text{ and } p \in P \text{ such that } \langle p, k, \ell\rangle \in t \text{ and } \langle \xi, m\rangle \in \operatorname{dom} p\}$$

is no more than countable in M; really, it is only important that the complementary set $U_2 = \omega_1^M - U_1$ is nonempty.

Let $\theta = 1$ or 2. If $p \in P$, then we denote by p_θ the restriction of the function p to the set $(U_\theta \times \omega) \cap \operatorname{dom} p$. Set $P_\theta = \{p_\theta : p \in P\}$ and $G_\theta = \{p_\theta : p \in G\}$.

It is easily seen that the mapping $p \mapsto \langle p_1, p_2\rangle$ gives (in M) an order isomorphism from P to $P_1 \times P_2$, and the image of the set G under this isomorphism is $G_1 \times G_2$ (the second statement uses the fact that G is generic). Therefore, the set $G_1 \times G_2$ is $P_1 \times P_2$-generic over M. In this situation, by Theorem 2.5 of [6, p. 13], each point $\alpha \in I$ which belongs to both $M[G_1]$ and $M[G_2]$, also belongs to M. In particular, since the point $\delta = i_G(t)$ belongs to the model $M[G_1]$ (by the definition of U_1), and $I \subseteq L[\delta]$ in M[G] (by the choice of δ), we have $\alpha \in M$ each time that $\alpha \in I \cap M[G_2]$.

In order for this to obtain the required contradiction, we fix an ordinal $\xi \in U_2$ and define a point $\alpha \in I$ by the relation $\alpha(n) = j$, when $\exists p \in G\ (\langle \xi, n\rangle \in \operatorname{dom} p \wedge p(\xi, n) = j)$ (naturally, here j = 0 or 1). Clearly, $\alpha \in M[G_2]$, and thus $\alpha \in M$ by the above. Thus, the set

$$D = \{r \in P: \text{ there exists } n \in \omega \text{ such that } \langle \xi, n\rangle \in \operatorname{dom} r \text{ and } r(\xi, n) \neq \alpha(n)\}$$

belongs to the model M.

However, it is easily verified that the set D is dense in P: for any $q \in P$ there exists $r \in D$, $r \leq q$. Therefore, the intersection $D \cap G$ is nonempty. Let $r \in D \cap G$, and let $n \in \omega$ be such that $\langle \xi, n\rangle \in \operatorname{dom} r$ and $r(\xi, n) \neq \alpha(n)$. We now see that the conditions $p, r \in G$ cannot be consistent in P, i.e., there is no $q \in P$ such that $q \leq p$ and $q \leq r$; and this contradicts the genericity of the set G.

This contradiction completes the proof of Lemma 3 and of the theorem.

In conclusion, we note that Lemma 4 equally allows us to verify that an ω_2-Cohen (i.e., obtained by means of an ω_2^M-Cohen s.f.c.) generic extension N' of the original model M by the axiom ZFC + (V = L) satisfies (III) $\wedge \neg$ CH. In fact, a λ-Cohen s.f.c. satisfies the chain countability condition for any λ, so that Lemma 2, and also conjecture (III) (by Lemma 4) are satisfied for N'. At the same time, it is well known that the continuum hypothesis is invalid in N' (see, for example, [8, pp. 118-122]). This argument proves the inconsistency of the combination (III) $\wedge \neg$ CH.

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SYMMETRY PROPERTIES OF HIGHER SPECTRAL DENSITIES OF STATIONARY RANDOM PROCESSES

V. G. Alekseev

Let $\{\xi(k), k \in \mathbf{Z}\}$ be a real random process with mean $\mathbf{E}\,\xi(k) \equiv 0$, stationary up to the sixth order, i.e., such that all its moments up to the sixth order, inclusive, exist and do not depend on the choice of the initial point of reference. For $\nu = \overline{2, 6}$ we set $r_\nu(\ell_1, \ldots, \ell_{\nu-1}) = s_\nu(k, k + \ell_1, \ldots, k + \ell_{\nu-1})$, where we denote by $s_\nu(k_1, \ldots, k_\nu)$ the ν-th order simple semiinvariant corresponding to the random vector $\{\xi(k_1), \ldots, \xi(k_\nu)\}$. We shall be interested in the spectral densities $f^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1})$ of orders $\nu = \overline{2, 6}$, which in the cases $\nu = 2$ and 3 are defined by the formulas

$$f^{(2)}(\lambda) \equiv f(\lambda) = (2\pi)^{-1} \sum_{l \in \mathbf{Z}} \exp(-il\lambda)\, r_2(l)$$

and

$$f^{(3)}(\lambda_1, \lambda_2) = (2\pi)^{-2} \cdot \sum_{l_1 \in \mathbf{Z}} \sum_{l_2 \in \mathbf{Z}} \exp(-il_1\lambda_1 - il_2\lambda_2)\, r_3(l_1, l_2), \tag{1}$$

and for $\nu \geq 4$ they are analogously expressed in terms of the corresponding semiinvariants $r_\nu(\ell_1, \ldots, \ell_{\nu-1})$ (see, for example, [1, Sec. 2.6]). The semiinvariants $r_\nu(\ell_1, \ldots, \ell_{\nu-1})$ ($\nu = \overline{2, 6}$) are assumed to decrease sufficiently quickly (in absolute value) as $\ell_1^2 + \ldots + \ell_{\nu-1}^2 \to \infty$, so that the spectral densities which interest us do exist.

As the domain of definition of the spectral density $f^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1})$ we may take, for example, the hypercube (square, interval) $\Pi^{\nu-1}$, where $\Pi = (-\pi, \pi]$. We may also assume that the function $f^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1})$ is defined on the whole space $R^{\nu-1}$ and is periodic (with period 2π) in each of its arguments. However, it will be more convenient for us to assume that the domain of definition of the spectral density $f^{(\nu)}(\lambda_1, \ldots, \lambda_{\nu-1})$ is a polyhedron (polygon, interval) Q_ν, where $Q_2 = [-\pi, \pi]$ and the polygon Q_3 and the polyhedra Q_4, Q_5, and Q_6 are described by the systems of inequalities

$$\begin{cases} -2\pi \leqslant \lambda_1 - \lambda_2 \leqslant 2\pi, \\ -2\pi \leqslant \lambda_j + 2\lambda_k \leqslant 2\pi \quad (j, k = 1, 2, \quad j \neq k), \end{cases}$$

$$\begin{cases} -2\pi \leqslant \lambda_j - \lambda_k \leqslant 2\pi \quad (j, k = 1, 2, 3, \quad j < k), \\ -2\pi \leqslant \lambda_j + \lambda_k + 2\lambda_l \leqslant 2\pi \quad (j \neq k \neq l \neq j), \end{cases}$$

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