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Pyramidal Structure of Constructibility Degrees

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Introduction

We consider a model whose structure of degrees of constructibility includes

(1) degrees $\mathbf{0} < \boldsymbol{a}_0 < \boldsymbol{a}_1 < \cdots$, where $\mathbf{0}$ is the constructive degree;
(2) a degree $\boldsymbol{b} > \mathbf{0}$ incomparable with any of the $\boldsymbol{a}_n$;
(3) "concatenations" $\boldsymbol{a}_0\boldsymbol{b} < \boldsymbol{a}_1\boldsymbol{b} < \boldsymbol{a}_2\boldsymbol{b} < \cdots$;
(4) the greatest degree $\boldsymbol{a}_\omega\boldsymbol{b}$.

The degrees $\boldsymbol{a}_0, \boldsymbol{a}_1, \dots,$ and $\boldsymbol{b}$ could be obtained by means of the $\mathrm{Sacks}^\omega \times \mathrm{Sacks}$ forcing. (Here Sacks^ω is the iteration of the Sacks forcing Sacks of length ω with countable support; see [1]. Accordingly, Sacks^m is the iteration of Sacks of length m.) However, in this process, another degree, the upper bound $\boldsymbol{a}_\omega$ of the degrees $\boldsymbol{a}_n$, $n \in \omega$, would emerge; this degree is incomparable with $\boldsymbol{b}$, and so it is distinct from $\boldsymbol{a}_\omega\boldsymbol{b}$. Therefore, we need another form of iteration.

Elements of the set ω^ω will be called (real) *numbers*.

Theorem. *Let ω_1^L be countable. Then there exists a generic extension $\mathbf{M} = \mathbf{L}[\langle a_n : n \in \omega\rangle, b]$ generated by the real numbers a_n and b such that*

(i) *for any n, the sequence $\langle\langle a_0, \dots, a_n\rangle, b\rangle$ is $(\mathrm{Sacks}^{n+1} \times \mathrm{Sacks})$-generic over $\mathbf{L}$;*
(ii) *each real number $x \in \mathbf{M}$ either belongs to $\mathbf{L}[a_0, \dots, a_n, b]$ for a certain n or satisfies the property $\mathbf{L}[x] = \mathbf{L}[\langle a_n : n \in \omega\rangle, b]$.*

By the known properties of ordinary iterated Sacks models (see [1, 2]), the constructibility degrees of numbers in such a model $\mathbf{M}$ have the structure described by (1)–(4).

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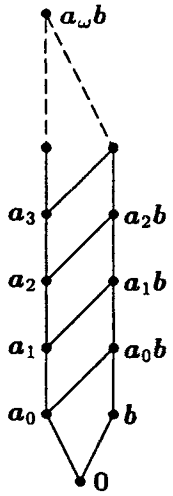

FIG. 1. The structure of L-degrees in the model under study

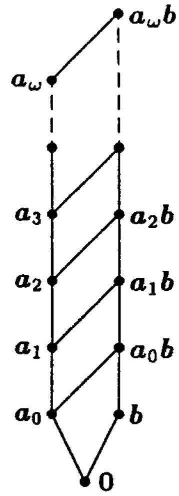

FIG. 2. The structure of L-degrees in the ordinary iterated Sacks model with the same system of generators a_n and b

§1. Forcing

The arguments below are performed in the model $\mathbf{L}$.

Let $\mathbf{S} = \text{Sacks}$ be the Sacks forcing. For any n, we denote by $\mathbf{S}^n$ its iteration of length n with the associated forcing relation $\Vdash_n$. Each $\tau \in \mathbf{S}^n$ is a function defined on the set $n = \{0, \dots, n-1\}$ so that $\tau \restriction k \Vdash_k$ "$\tau(k) \in \check{\mathbf{S}}$" for all $k < n$.

If $u \in 2^n$ and f is a function such that each restriction of the form $u \restriction k$, $k < n$, belongs to $\operatorname{dom} p$, then we denote by $f|_u$ the function defined on n by the formula $(f|_u)(k) = f(u \restriction k)$ for all $k < n$.

Let us define a forcing $\mathbf{P}$ as the family of all $p = \langle T_p, f_p \rangle$ such that

(a) $T_p \subseteq 2^{<\omega}$ is a perfect tree;
(b) f_p is a function defined on T_p so that $f_p|_u \in \mathbf{S}^n$ for all $u \in T_p \cap 2^n$.

We shall say that q is *stronger* than p (and write $q \leq p$), if $T_q \subseteq T_p$ and $f_q|_u \leq f_p|_u$ in $\mathbf{S}^n$ for each $u \in T_q \cap 2^n$. The relation of $\mathbf{P}$-forcing will be denoted by $\Vdash$.

Recall that $u \in T$ is called a *splitting node* of the tree $T \subseteq 2^{<\omega}$ if the nodes $u^\wedge 0$ and $u^\wedge 1$ belong to T. A splitting node of *level* n has exactly n splitting nodes below it. A perfect tree T contains exactly 2^n splitting nodes at each level n.

Let S and T be perfect trees. We shall write $S \leq_n T$ if $S \subseteq T$ and the nth splitting levels in S and T coincide. It is known that if T_n are perfect trees and $T_0 \geq_0 T_1 \geq_1 T_2 \geq_2 \cdots$, then $T = \bigcap T_n$ are also perfect trees.

For the forcing $\mathbf{P}$, this construction takes the following form.

Let $p, q \in \mathbf{P}$. Set $q \leq_n p$ if $q \leq p$, $T_q \leq_n T_p$, and each $u \in T_q \cap 2^m$, $m < n$, satisfies the property $f_q|_u \Vdash_m$ "$f_q(u) \leq_n f_p(u)$." Then any decreasing chain $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \cdots$ of forcing conditions $p_n \in \mathbf{P}$ has a lower bound in $\mathbf{P}$.

§2. Extension

If a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic over $\mathbf{L}$, then $T = \bigcap_{p \in G} T_p$ is a generic chain in $2^{<\omega}$, so that $b = \bigcup T \in 2^\omega$ is the Sacks number over $\mathbf{L}$.

Let $n \in \omega$. Set $u = b \restriction n$, $u \in 2^n$. Then the set $G|_u = \{f_p|_u : p \in G\}$ is $\mathbf{S}^n$-generic over $\mathbf{L}$, i.e., there is a $\mathbf{S}^n$-generic sequence of numbers $\langle a_0, \dots, a_{n-1} \rangle$ defined over $\mathbf{L}$, in which each element $a_k \in 2^\omega$ is the Sacks number over $\mathbf{L}[a_0, \dots, a_{k-1}]$. (Of course, a_k here does not depend on the choice of $n > k$.) Moreover, the sequence $\langle \langle a_0, \dots, a_{n-1} \rangle, b \rangle$ is $(\mathbf{S}^n \times \mathbf{S})$-generic over $\mathbf{L}$.

Now it remains to prove statement (ii) of the theorem. It follows from the following lemma.

Lemma 1. *Suppose that a number $x \in \mathbf{L}[G]$ does not belong to the class $\mathbf{L}[a_0, \dots, a_n, b]$ for any n. Then $G \in \mathbf{L}[x]$.*

We shall begin the proof with a definition. A *roster* of size l is a finite sequence of the form $R = \langle u, w_0, \dots, w_{l-1} \rangle$ all the elements $u, w_0, \dots, w_{l-1}$ of which belong to 2^l. A roster $R = \langle u, w_0, \dots, w_{l-1} \rangle$ can be understood as a condition in $\mathbf{P}$ that forces $\breve{b}$ to extend u and each $\breve{a}_k$, $k < l$, to extend w_k.

A roster $R = \langle u, w_0, \dots, w_{l-1} \rangle$ *agrees* with the generic set G if $u \subset b$ and $w_k \subset a_k$ for all $k < l$. A roster R *agrees* with a condition $p \in \mathbf{P}$ if there exists a stronger condition $q \le p$ that forces R to agree with $\breve{G}$. In this case there exists a greatest (i.e., a weakest) condition q of this sort (the *restriction* of p to R), which is denoted by $q = p \restriction R$: q is obtained by appending to p the information that $\breve{b}$ extends u and each $\breve{a}_k$, $k < l$, extends w_k.

We shall say that a condition $p \in \mathbf{P}$ *fully n-splits below l* if the nth splitting level of T_p lies entirely below l and for any $u \in T_p \cap 2^m$, $m \le n$, we have

$$f_p|_u \Vdash_n \text{“the } n\text{th splitting level of } f_p(u) \text{ lies entirely below } l\text{”}.$$

Lemma 2. *Suppose that a roster $R = \langle u, w_0, \dots, w_{l-1} \rangle$ agrees with a condition $p \in \mathbf{P}$ which fully n-splits below $l \ge n$, and a condition $r \in \mathbf{P}$ is stronger than $p \restriction R$. Then there exists a condition $q \le_n p$ such that $q \restriction R$ coincides with r.*

Proof. Let us define T_q as the set of all $v \in T_p$ such that either $u \not\subseteq v$ or $v \in T_r$. (Then each $v \in T_r$ is $\subseteq$-comparable with u by the choice of r.)

Let us define $f_q(v)$ for $v \in T_q$. For $u \subseteq v$, we set $f_q(v) = f_r(v)$; for $\subseteq$-incomparable u and v, we set $f_q(v) = f_p(v)$. It remains to consider the case of the strict inclusion $v \subset u$. Set $m = \operatorname{dom} v$, $m < l$. Let $f_q(v)$ be the $\mathbf{S}_m$-name of

$$\text{“if } \exists j < m\ (w_j \not\subseteq \breve{a}_j), \text{ then I am } f_p(v);$$
$$\text{otherwise, I am } \{a \in f_p(v) : w_m \subset a \implies a \in f_r(v)\}.\text{”}$$

From the second part of the definition it follows that $q \restriction R = r$. Let $m < n$ and $v \in T_q \cap 2^m$. We shall show that $f_q|_v \Vdash_m$ “$f_q(v) \le_n f_p(v)$.” By definition, the only nontrivial case is $v = u \restriction m \subset u$. We proceed by arguing in the $\mathbf{S}^m$-generic extension of the universe. By definition, all distinctions between $f_p(v)$ and $f_q(v)$ are concentrated in the domain $D = \{a \in 2^\omega : w_m \subset a\}$, where $w_m \in 2^l$. On the other hand, the nth splitting level of $f_p(v)$ is defined *below* l, so these distinctions do not violate the property $f_q(v) \le_n f_p(v)$. Hence, $q \le_n p$. □

§3. Proof of Lemma 1

Let $\breve{x}$ be the name of our number x. By the assumption of the lemma, a certain $p \in G$ forces “$\breve{x} \notin \mathbf{L}[\breve{a}_0, \dots, \breve{a}_n, \breve{b}]$” for any n. By induction on n, we shall define

(a) a sequence $p = p_0 \ge_0 p_1 \ge_1 p_2 \ge_2 \cdots$ of conditions $p_n \in \mathbf{P}$;
(b) a sequence of positive integers $l_0 < l_1 < l_2 < \cdots$; and
(c) a function g that maps rosters of size l_n into $\Sigma \cup \{\bot\}$, where Σ is the set of all functions σ such that $\operatorname{dom} \sigma \subseteq \omega$ is finite and $\operatorname{ran} \sigma \subseteq \{0, 1\}$ and $\bot$ is a formal symbol for separating unessential cases

so that for each n and any roster R of size l_n,

$$p_{n+1} \Vdash \text{“} R \text{ agrees with } \breve{G} \iff g(R) \ne \bot \text{ and } g(R) \subset \breve{x}\text{”}. \tag{1}$$

Then any lower bound $q \in \mathbf{P}$ of the sequence of conditions p_n will force $\breve{G}$ to be the only generic set that agrees with all the rosters R satisfying the property $g(R) \subset \breve{x}$. Therefore, q will force “$\breve{G} \in \mathbf{L}[\breve{x}]$,” as required.

Suppose that p_n has already been constructed. Let us show how to define l_n, the action of g on rosters of size l_n, and the condition p_{n+1}.

First part. In view of familiar properties of Sacks forcing and its finite iterations, there exist a condition $q \leq_n p_n$ and a positive integer $l_n > l_{n-1}$ such that $T_q = T_{p_n}$ and q fully n-splits below l_n. Choose an enumeration $\langle R_k^0, R_k^1 \rangle$, $k < K$, of all pairs of different rosters of size l_n. Using induction on $k \leq K$, let us define

(a) conditions $q = q_0 \geq_n q_1 \geq_n q_2 \geq_n \cdots \geq_n q_K$ in $\mathbf{P}$;
(b) positive integers $m_k \in \omega$ and $i_k \in \{0, 1\}$

so as to ensure that

$$q_{k+1} \Vdash \begin{cases} \text{"if } R_k^0 \text{ agrees with } \check{G}, \text{ then } \check{x}(m_k) = i_k\text{"}, \\ \text{"if } R_k^1 \text{ agrees with } \check{G}, \text{ then } \check{x}(m_k) \neq i_k\text{"} \end{cases}$$

for each $k < K$. Then let us set $p_{n+1} = q_K$ and, for any roster R of size l_n,

$$g(R) = \begin{cases} \{\langle m_k, i_k \rangle : k < K,\ R = R_k^0\} & \text{if } R \text{ agrees with } q_K, \\ \bot & \text{otherwise.} \end{cases}$$

Such a choice obviously implies (1).

Second part. Now that we have defined q_k, let us define q_{k+1}, m_k, and i_k. The construction consists of two steps.

Step 1. We find a pair of intermediate conditions q^0 and q^1. If the roster R_k^0 does not agree with q_k, we set $q^0 = q^1 = q_k$ and proceed to Step 2. Suppose that R_k^0 *agrees with* q_k. Recall that R_k^0 is a roster of size l_n, i.e., $R_k^0 = \langle u, w_0, \ldots, w_{l_n - 1} \rangle$, where u and w_j belong to 2^{l_n}.

Since q_k forces "$\check{x} \notin \mathbf{L}[\check{a}_0, \ldots, \check{a}_{l_n}, \check{b}]$," there exist conditions $r^0, r^1 \in \mathbf{P}$ that are stronger than $q_k \upharpoonright R_k^0$ and satisfy the properties $T_{r^0} = T_{r^1}$ (which implies $u \in T_{r_0} = T_{r_1}$) and $f_{r^0}|_u = f_{r^1}|_u$; also, there exists a number $m_k \in \omega$ such that $r^0 \Vdash$ "$\check{x}(m_k) = 0$" and $r^1 \Vdash$ "$\check{x}(m_k) = 1$."

The existence of conditions q^0 and q^1 in $\mathbf{P}$ such that $q^i \leq_n q_k$ and $q^i \upharpoonright R_k^0 = r^i$, $i = 0, 1$, is ensured by Lemma 2. Moreover, a closer examination of the proof of Lemma 2 shows that, after we have chosen the conditions r^i, the conditions q^i can be chosen so that $T_{q^0} = T_{q^1}$ and $f_{q^0}(v) = f_{q^1}(v)$ for all $v \in T_{q^0} = T_{q^1}$ not satisfying the inclusion $u \subseteq v$. In particular, $q^0 \upharpoonright R = q^1 \upharpoonright R$ holds for any roster of size l_n.

Step 2. If R_k^1 does not agree with q^0 (and so with q^1 as well by the above), then we set $q_{k+1} = q^0$ and $i_k = 0$. Suppose that R_k^1 *agrees with* r.

The condition $r \leq q^0 \upharpoonright R_k^1$ defines the value of $\check{x}(m_k)$ to be equal, say, to 0. Set $i_k = 1$. As was proved above, $r \leq q^1 \upharpoonright R_k^1$. Using Lemma 2, we obtain a condition $q \leq_n q^1$ such that $q \upharpoonright R_k^1 = r$. Thus, $q \leq_n q_k$, $q \upharpoonright R_k^1 = r \Vdash$ "$\check{x}(m_k) = 0$," and if q agrees with R_k^0, then $q \upharpoonright R_k^0 \leq q^1 \upharpoonright R_k^0$. Hence,

$$q \upharpoonright R_k^0 \Vdash \quad \text{"}\check{x}(m_k) = i_k = 1.\text{"}$$

It follows that the condition $q_{k+1} = q$ has the desired properties.

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