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A Countable Definable Set Containing no Definable Elements

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Abstract—The consistency of the existence of a countable definable set of reals, containing no definable elements, is established. The model, where such a set exists, is obtained by means of a countable product of Jensen's forcing with finite support.

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1. INTRODUCTION. THE PROBLEM OF CHOICE OF DEFINABLE ELEMENTS

Questions related to the definability of mathematical objects, appeared in the focus of attention of discussions on mathematical foundations immediately after the publication of the Zermelo's famous paper on the axiom of choice and its application to the problem of wellorderability of an arbitrary set, in 1905, and also, to some extent, in connection with a simultaneous publication of the Richard paradox. For instance, Hadamard, Borel, Baire, and Lebesgue, participants of the discussion published in [1], in spite of significant differences in their positions regarding problems of mathematical foundations, emphasized that a proof of nonemptiness, i.e., a proof of pure existence of an element in a given set, and a direct definition (or an effective construction) of such an element are different mathematical results, and the second one of them does not follow from the first. In particular, Lebesgue, in his part of [1], pointed out the difficulties in the problem of effective choice, i.e., a selection of a definable element in a definable (nonempty) set[1].

For the sake of convenience of references, we represent Lebesgue's remark as follows.

Problem 1 (Lebesgue). *Is it true that every nonempty definable set has a definable element?*

In the beginning of the 20th century, the level of development of mathematical foundations was insufficient to even give an adequate mathematical formulation of the problem, let alone its solution. After Tarski's work [2] on the impossibility to mathematically define the notions of truth and definability themselves, it became clear that the formulation of the problem needed to be elaborated. Such an elaboration was obtained on the base of the notion of ordinal definability. A set x is *ordinal definable* [3] if it can be defined by means of a set-theoretic formula which contains one or several ordinals in the role of parameters of the definition. Note that the class Ord of all ordinals is an extension of the natural numbers, unique in itself and determined enough for not to insist on the definability of the ordinals themselves.

Unlike pure (parameter-free) definability, ordinal definability admits a set-theoretic formula $\mathrm{od}(x)$, which adequately expresses the property of a set x to be ordinal definable; see [4, Sec. 3.5]. This allows to concretely define the class $\mathrm{OD} = \{x : \mathrm{od}(x)\}$ of all ordinal definable sets, and then to re-formulate Problem 1 as follows.

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[1] "Ainsi je vois déjà une difficulté dans ceci dans un M' déterminé je puis choisir un m' déterminé", in the original. Thus, I already see a difficulty with the assertion that "in a determinate M' I can choose a determinate m'", in the English translation.

Problem 2. *Is it true that every nonempty ordinal definable set has an element also in the class* OD*?*

The problem is correctly formulated in this form, but it has no definite solution in the modern set theory **ZFC**. To be more exact, the answer can be both positive and negative, depending on which set-theoretic universes (models of **ZFC**) are considered.

For instance, in the universe **L** of Goedel's constructible sets, all sets are ordinal definable; therefore, the answer is trivially *positive*.

On the other hand, in many models obtained as extensions of the universe **L** by forcing (generic extensions), it is true that even not all reals[2] are ordinal definable, and hence the set

$$X = \mathbb{R} \setminus \mathrm{OD} = \{x \in \mathbb{R} : \neg\, \mathrm{od}(x)\},$$

of all reals which are not ordinal definable, is nonempty, belongs to OD (is even definable by the formula $\mathrm{od}(x)$), but does not contain any real in OD, which answers the question in the *negative*. For instance, this takes place in the well-know Solovay model [5], in which all projective sets are Lebesgue measurable.

A simple argument shows that if the set $X = \mathbb{R} \setminus \mathrm{OD}$ is nonempty (as, for instance, in the Solovay's model mentioned above), then it is fairly *large*, i.e., it definitely has the cardinality of the continuum if it is measurable, then it has full measure, and so on. Is there a similar example among *small* sets, say countable ones? This leads to the following problem, considered on such well-known discussion boards of modern international mathematics as *Mathoverflow* [6] and *Foundations of Mathematics* (FOM) [7].

Problem 3. *Prove the consistency of the statement of existence of a nonempty countable ordinal definable set of reals, containing no element in* OD.

Note that every *finite* OD set of reals definitely consists only of OD elements, and hence it cannot serve as the example required.

The next theorem, our main result, contains the solution.

Theorem 4. *There is a generic extension* $\mathbf{L}[\langle x_n \rangle_{n<\omega}]$ *of the constructible universe* **L** *by a sequence of reals* $x_n \in 2^{\mathbb{N}}$*, in which it is true that* $\{x_n : n < \omega\}$ *is a (countable)* Π^1_2 *set containing no* OD *elements.*

Thus, the hypothesis of the existence of a countable set $X \subseteq \mathbb{R}$ of class Π^1_2, hence OD as well, not containing OD elements, in fact does not contradict the axioms of **ZFC**. The class Π^1_2 is here the best possible. Indeed, even the dual class Σ^1_2 canot contain such examples, because any (not only countable) Σ^1_2 set necessarily contains an element of class Δ^1_2 by the Π^1_1-uniformization theorem [8, 8.4.1] or [9, 2.4.1]. Classes Σ^1_n, Π^1_n, Δ^1_n of the effective projective hierarchy (see [8, Chap. 6] or [9, Chap. 1] on them) are subsets of OD, of course.

Following Enayat's conjecture [7], [10], to prove Theorem 4, we make use of the product $\mathbb{P}^{<\omega}$ (with finite support) of countably many copies of a forcing $\mathbb{P}$ introduced by Jensen in [11] to define a model of set theory with a nonconstructible real $x \in 2^{\mathbb{N}}$ of class Δ^1_3. This forcing is defined, in the constructible universe **L**, in the form $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{U}_\alpha$, where each $\mathbb{U}_\alpha$ is a countable set consisting of perfect trees in $2^{<\omega}$. We carry out this construction in Sec. 7, on the basis of the material of the previous Secs. 2–4. The required properties of $\mathbb{P}^{<\omega}$-generic extensions are established in Sec. 8 with the help of the key Theorem 19.

We add a few words about the results obtained by this method shortly before the publication of this paper. Kanovei and Lyubetsky [12] defined a model in which there exists a projective set of effective class Π^1_2 in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with countable vertical cross-sections, which does not admit uniformization by any

[2] In modern works on set theory, by *reals* (real numbers) one usually understands not elements of the real line proper, but the points of the Baire space $\mathbb{N}^{\mathbb{N}}$ or the Cantor discontinuum $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$. It is in this sense that one has to understand $\mathbb{R}$ and reals in this section. The exact meaning depends on the context, but everything said here is equally related to $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, or the real line proper, because of the existence of definable 1-to-1 correspondences between the three domains.

projective set. They also defined models [13] containing counterexamples related to the principles of separation and reduction at the third projective level. Golshani, Kanovei, and Lyubetsky [14] defined a model in which there is a Π^1_2 pair of countable disjoint sets $X, Y \subseteq 2^{\mathbb{N}}$, effectively indistinguishable in the sense that if $A \subseteq 2^{\mathbb{N}}$ is a projective set of any effective class Π^1_n, then $X \cap A \neq \varnothing$ is equivalent to $Y \cap A \neq \varnothing$.

2. PERFECT TREES

By $2^{<\omega}$ (the full dyadic tree) we denote the set of all *strings* (finite sequences) of numbers 0, 1, including the *empty sequence* Λ. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that t extends the string s (including the possibility of $t = s$), while $s \subsetneq t$ means proper extension ($s \subseteq t$ and $t \neq s$). If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t^\frown i$ denotes the extension of t by the rightmost term i. If $s \in 2^{<\omega}$, then $\mathrm{lh}(s)$ is the length of the string s, and we put

$$2^n = \{s \in 2^{<\omega} : \mathrm{lh}(s) = n\}$$

(all strings of length n).

A set $T \subseteq 2^{<\omega}$ is called a *tree* if, for any strings $s \subsetneq t$ in $2^{<\omega}$, $t \in T$, implies $s \in T$. In this case, if $s \in T$, then we let

$$T \restriction s = \{t \in T : s \subseteq t \vee t \subseteq s\}$$

(cutting at a string).

Any nonempty tree $T \subseteq 2^{<\omega}$ contains the empty string Λ.

By $\mathbf{PT}$ we denote the set of all *perfect trees* $\varnothing \neq T \subseteq 2^{<\omega}$. Thus, a nonempty tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ if it does not contain terminal nodes and isolated branches. In this case, there is a largest string $s = \operatorname{stem} T \in T$ satisfying $T = T \restriction s$, the *stem* of T, and then $s^\frown 1 \in T$ and $s^\frown 0 \in T$. If $T \in \mathbf{PT}$, then let

$$[T] = \{a \in 2^{\mathbb{N}} : \forall n\,(a \restriction n \in T)\} \subseteq 2^{\mathbb{N}},$$

the perfect set of all *branches of the tree* T.

Example 5. The full tree $2^{<\omega}$ belongs to $\mathbf{PT}$ and $[2^{<\omega}] = 2^{\mathbb{N}}$. If $u \in 2^{<\omega}$, then the tree

$$T[u] = \{s \in 2^{<\omega} : u \subseteq s \vee s \subseteq u\}$$

also belongs to $\mathbf{PT}$, and $[T[u]] = \{a \in 2^{\mathbb{N}} : u \subsetneq a\}$ is a *Baire interval* in $2^{\mathbb{N}}$.

Definition 6. *A perfect tree forcing* (briefly, PTF) is any nonempty set $\mathbb{P} \subseteq \mathbf{PT}$, satisfying the following condition: if $u \in T \in \mathbb{P}$, then $T \restriction u \in \mathbb{P}$.

Such a set $\mathbb{P}$ can be considered as a forcing (if $T \subseteq T'$, then T is a stronger condition). It adjoins a real in $2^{\mathbb{N}}$ to the ground model. Indeed, if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic, then the intersection $\bigcap_{T \in G}[T] = \{x\}$ contains a unique real $x = \mathbf{x}[G] \in 2^{\mathbb{N}}$, called the $\mathbb{P}$-*generic real*.

Definition 7. Let $\mathbb{P}$ be a PTF. By $\mathbb{P}^{<\omega}$ we denote the *product of ω copies of the forcing $\mathbb{P}$ with finite support*.

In other words, a typical element of the set $\mathbb{P}^{<\omega}$ — we call it a *multitree* (over $\mathbb{P}$) — is a sequence of the form $\boldsymbol{\tau} = \langle T_n \rangle_{n \in \omega}$, each term $T_n = \boldsymbol{\tau}(n)$ of which belongs to $\mathbb{P} \cup \{2^{<\omega}\}$ (usually $\mathbb{P}$ will contain $2^{<\omega}$), and the set $|\boldsymbol{\tau}| = \{n : T_n \neq 2^{<\omega}\}$ (the support of $\boldsymbol{\tau}$) is finite.

The set $\mathbb{P}^{<\omega}$ is ordered componentwise: $\boldsymbol{\sigma} \leq \boldsymbol{\tau}$ (i.e., $\boldsymbol{\sigma}$ is a stronger condition), if $\boldsymbol{\sigma}(n) \subseteq \boldsymbol{\tau}(n)$ in $\mathbb{P}$ for all n. As a forcing, $\mathbb{P}^{<\omega}$ adjoins a sequence $\langle x_n \rangle_{n<\omega}$ of $\mathbb{P}$-generic reals $x_n \in 2^{\mathbb{N}}$. Indeed, if a set $G \subseteq \mathbb{P}^{<\omega}$ is $\mathbb{P}^{<\omega}$-generic and $n < \omega$, then, by the product forcing theorem, the set $G_n = \{\boldsymbol{\tau}(n) : \boldsymbol{\tau} \in G\}$ is $\mathbb{P}$-generic, and hence the real $x_n = \mathbf{x}_n[G] = \mathbf{x}[G_n]$ is $\mathbb{P}$-generic by the above.

Remark 8. Strings $\langle T_0, \dots, T_n \rangle$ of trees $T_i \in \mathbb{P}$ will be used to denote the multitrees

$$\langle T_0, \dots, T_n, 2^{<\omega}, 2^{<\omega}, 2^{<\omega}, \dots \rangle \in \mathbb{P}^{<\omega}.$$

3. SPLITTING SYSTEMS OF TREES

Let us fix a PTF $\mathbb{P} \subseteq \mathbf{PT}$.

Definition 9. The set $\mathbf{SS}(\mathbb{P})$ (*splitting systems* over $\mathbb{P}$) consists of all systems $\varphi = \langle T_s \rangle_{s \in 2^{\leq n}}$, where $n = \operatorname{hgt}(\varphi) < \omega$ (the height of φ), and

(1) every $T_s = T_s^\varphi = \varphi(s)$ is a tree in $\mathbb{P}$,

(2) if $s^\frown i \in 2^{\leq n}$ $(i = 0, 1)$, then $T_{s^\frown i} \subseteq T_s$ and $\operatorname{stem} T_s{}^\frown i \subseteq \operatorname{stem} T_{s^\frown i}$, which easily implies that $[T_{s^\frown 0}] \cap [T_{s^\frown 1}] = \varnothing$.

We set that the *empty system* $\boldsymbol{\Lambda}$ also belongs to $\mathbf{SS}(\mathbb{P})$, and $\operatorname{hgt}(\boldsymbol{\Lambda}) = -1$. (There exist many systems $\varphi \in \mathbf{SS}(\mathbb{P})$ with $\operatorname{hgt}(\varphi) = 0$; each one of them contains a single tree $T = T_\Lambda^\varphi$, where Λ is the empty string.)

If φ, ψ are systems in $\mathbf{SS}(\mathbb{P})$, then we say by definition that

- φ *extends* ψ (in symbol, $\psi \preccurlyeq \varphi$), if $n = \operatorname{hgt}(\psi) \leq \operatorname{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{<n}$ (and separately $\boldsymbol{\Lambda} \preccurlyeq \varphi$ for each φ);
- *strictly extends* ψ $(\psi \prec \varphi)$ if, in addition, $\operatorname{hgt}(\psi) < \operatorname{hgt}(\varphi)$;

and finally φ *refines* ψ, if

$$n = \operatorname{hgt}(\psi) = \operatorname{hgt}(\varphi), \qquad \varphi(s) \subseteq \psi(s), \quad s \in 2^{\operatorname{hgt}(\varphi)}, \qquad \varphi(s) = \psi(s), \quad s \in 2^{<\operatorname{hgt}(\varphi)}.$$

Thus, the refinement allows to shrink the trees at the highest level of the system, but does not change those trees which belong to lower levels.

Since $\mathbb{P}$ is a PTF, there exist strictly $\prec$-increasing sequences $\langle \varphi_n \rangle_{n<\omega}$ of systems φ_n in $\mathbf{SS}(\mathbb{P})$. The limit system

$$\varphi = \bigcup_n \varphi_n = \langle T_s \rangle_{s \in 2^{<\omega}}$$

of such a sequence obviously catisfies conditions (1) and (2) of Definition 9 on the whole domain $2^{<\omega}$.

Lemma 10. *In this case, both $T = \bigcap_n \bigcup_{s \in 2^n} T_s$ and all intersections of the form $T \cap T_s$ are trees in $\mathbf{PT}$ (not necessarily in $\mathbb{P}$), and we have*

$$[T] = \bigcap_n \bigcup_{s \in 2^n} [T_s] \qquad \textit{and} \qquad [T \cap T_s] = \bigcap_{n \geq \operatorname{lh}(s)} \bigcup_{v \in 2^n,\, s \subseteq v} [T_v].$$

If $u \in T$, then there is a string $s \in 2^{<\omega}$ such that $T \restriction u = T \cap T_s$.

Proof. To prove the last claim, it suffices to pick the least string $s \in 2^{<\omega}$ such that $s \subseteq \operatorname{stem} T_s$. □

We define $\mathbf{SS}^{<\omega}(\mathbb{P})$, *the product of* $\mathbf{SS}(\mathbb{P})$ *with finite support*, to be the set of all infinite sequences $\Phi = \langle \varphi_k \rangle_{k \in \omega}$, where each term $\varphi_k = \Phi(k)$ belongs to $\mathbf{SS}(\mathbb{P})$ and the set $|\Phi| = \{k : \varphi_k \neq \boldsymbol{\Lambda}\}$ (the support of Φ) is finite. Sequences in $\mathbf{SS}^{<\omega}(\mathbb{P})$ will be called *multisystems* (over $\mathbb{P}$).

We define $\Psi \preccurlyeq \Phi$, if $\Psi(k) \preccurlyeq \Phi(k)$ (in $\mathbf{SS}(\mathbb{P})$) for all k. Then $\Psi \prec \Phi$ means that

$$\Psi \preccurlyeq \Phi, \quad \Psi(k) \prec \Phi(k) \qquad \text{for at least one } k.$$

In addition, let $\Psi \prec\!\!\prec \Phi$, if $|\Psi| \subseteq |\Phi|$ and $\Psi(k) \prec \Phi(k)$ for all $k \in |\Phi|$.

A tree $T \in \mathbf{PT}$ *occurs in a system* $\varphi \in \mathbf{SS}(\mathbb{P})$, if $T = \varphi(s)$ for a string $s \in 2^{\leq \operatorname{hgt}(\varphi)}$, and *occurs in a multisystem* $\Phi = \langle \varphi_k \rangle$, if it occurs in one of the systems φ_k, $k \in |\Phi|$.

4. JENSEN'S EXTENSION OF A FORCING

The goal of the following key Definition 12 is to define a forcing $\mathbb{U}$, which *extends* a given PTF $\mathbb{P}$ in a certain way. This method of extension introduces important properties of genericity of the extended forcing. It was proposed by Jensen [11] in the context of perfect trees.

By $\mathbf{ZFC}'$ we denote the subtheory of the Zermelo–Fraenkel set theory $\mathbf{ZFC}$, including all axioms except for the power set axiom, which is replaced by the axiom of the existence of the power set $\mathscr{P}(X)$ for every at most countable set X. (This also implies the existence of the ordinal ω_1 and such continual sets as $\mathbf{PT}$.)

Definition 11. Let $\mathbb{P}$ be a PTF. A set $D \subseteq \mathbf{SS}(\mathbb{P})$ is *dense in* $\mathbf{SS}(\mathbb{P})$ if, for any system $\psi \in \mathbf{SS}(\mathbb{P})$, there is a system $\varphi \in D$ extending ψ, and *open dense in* $\mathbf{SS}(\mathbb{P})$ if, in addition, any system $\varphi' \in \mathbf{SS}(\mathbb{P})$, extending a system $\varphi \in D$ belongs to D itself.

Finally, D is *pre-dense in* $\mathbf{SS}^{<\omega}(\mathbb{P})$ if $\{\varphi' \in \mathbf{SS}(\mathbb{P}) : \exists\, \varphi \in D,\ \varphi \preccurlyeq \varphi'\}$ (the set of all extensions of systems in D) is dense in $\mathbf{SS}(\mathbb{P})$.

Definition 12. Let $\mathfrak{M}$ be a countable transitive model of the theory $\mathbf{ZFC}'$.

Assume that $\mathbb{P} \in \mathfrak{M}$ is a PTF containing the tree $2^{<\omega}$. Then the sets $\mathbb{P}^{<\omega}$, $\mathbf{SS}(\mathbb{P})$, $\mathbf{SS}^{<\omega}(\mathbb{P})$ belong to $\mathfrak{M}$ as well.

Let us fix a $\preccurlyeq$-increasing sequence $\Phi = \langle \Phi^j \rangle_{j<\omega}$ of multisystems $\Phi^j = \langle \varphi_k^j \rangle_{k\in\omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$, *generic over* $\mathfrak{M}$, i.e., a sequence that has a nonempty intersection with any set $D \in \mathfrak{M}$, $D \subseteq \mathbf{SS}^{<\omega}(\mathbb{P})$, dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$. Then, in particular, Φ intersects every set of the form

$$D_k = \{\Phi \in \mathbf{SS}^{<\omega}(\mathbb{P}) : \forall k' \le k,\ k \le \operatorname{hgt}(\Phi(k'))\}.$$

Thus, if $k < \omega$, then the sequence $\{\varphi_k^j\rangle_{j<\omega}\}$ of systems $\varphi_k^j \in \mathbf{SS}(\mathbb{P})$ *strictly increases with breaks*: $\varphi_k^j \prec \varphi_k^{j+1}$ for infinitely many indices j (and $\varphi_k^j = \varphi_k^{j+1}$ for all other j). Therefore, there exists a system of trees $\langle \mathbf{T}_k^{\Phi}(s) \rangle_{k<\omega \wedge s\in 2^{<\omega}}$ in $\mathbb{P}$ such that $\varphi_k^j = \langle \mathbf{T}_k^{\Phi}(s) \rangle_{s\in 2^{<h(j,k)}}$, where $h(j,k) = \operatorname{hgt}(\varphi_k^j)$. Then

$$\mathbf{U}_k^{\Phi} = \bigcap_n \bigcup_{s\in 2^n} \mathbf{T}_k^{\Phi}(s) \qquad \text{and} \qquad \mathbf{U}_k^{\Phi}(s) = \bigcap_{n\ge \operatorname{lh}(s)} \ \bigcup_{t\in 2^n,\, s\subseteq t} \mathbf{T}_k^{\Phi}(t)$$

are trees in $\mathbf{PT}$ (not necessarily in $\mathbb{P}$) by Lemma 10 for all k and $s \in 2^{<\omega}$; and we have

$$\mathbf{U}_k^{\Phi} = \mathbf{U}_k^{\Phi}(\Lambda), \qquad \mathbf{U}_k^{\Phi}(s) = \mathbf{U}_k^{\Phi} \cap \mathbf{T}_k^{\Phi}(s).$$

We let $\mathbb{U} = \{\mathbf{U}_k^{\Phi}(s) : k < \omega \wedge s \in 2^{<\omega}\}$.

We prove the basic properties of this set $\mathbb{U}$ in the following lemmas.

Lemma 13. *The sets* $\mathbb{U}$ *and* $\mathbb{P} \cup \mathbb{U}$ *are* PTFs. $\mathbb{P} \cap \mathbb{U} = \varnothing$.

Proof. To prove $\mathbb{P} \cap \mathbb{U} = \varnothing$, we let $T \in \mathbb{P}$ and $U = \mathbf{U}_K^{\Phi}(s) \in \mathbb{U}$, where $K \in \mathbb{N}$, $s \in 2^{<\omega}$. We must check that $T \ne U$.

The set $D(T,K)$ of all multisystems

$$\Phi = \langle \varphi_k \rangle_{k\in\omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$$

such that $K \in |\Phi|$, $\operatorname{lh}(s) \le h = \operatorname{hgt}(\varphi_K)$, and $T \setminus \bigcup_{t\in 2^h} \varphi_K(t) \ne \varnothing$, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$. Therefore there is an index j such that the multisystem Φ^j belongs to $D(T,K)$. Thus $K \in |\Phi^j|$, $\operatorname{lh}(s) \le h = \operatorname{hgt}(\varphi_K^j)$, and $T \setminus \bigcup_{t\in 2^h} \varphi_K(t) \ne \varnothing$. To prove $T \ne U$, it remains to check that $U \subseteq \bigcup_{t\in 2^h} \varphi_K^j(t)$.

Indeed, by definition,

$$U = \mathbf{U}_K^{\Phi}(s) \subseteq \bigcup_{t\in 2^h,\, s\subseteq t} \mathbf{T}_k^{\Phi}(t) \subseteq \bigcup_{t\in 2^h} \mathbf{T}_k^{\Phi}(t),$$

and hence we really have $U \subseteq \bigcup_{t\in 2^h} \varphi_K^j(t)$, because, by definition, $\mathbf{T}_k^{\Phi}(t) = \varphi_K^j(t)$. □

Lemma 14. *The set $\mathbb{U}$ is dense in $\mathbb{U} \cup \mathbb{P}$.*

Proof. Let $T \in \mathbb{P}$. The set $D(T)$ of all multisystems $\Phi = \langle \varphi_k \rangle_{k\in\omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that $\varphi_k(\Lambda) = T$ for some k, belongs to $\mathfrak{M}$, and is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$. It follows that $\Phi^j \in D(T)$ for some j, by the choice of Φ. Then $\mathbf{T}^{\Phi}_k(\Lambda) = T$ for some k. But $\mathbf{U}^{\Phi}_k(\Lambda) \subseteq \mathbf{T}^{\Phi}_k(\Lambda)$ by construction. □

Lemma 15. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$ and $U \in \mathbb{U}$, then there exists a finite set $D' \subseteq D$ satisfying $U \subseteq \bigcup D'$: formally, $U \subseteq^{\text{fin}} \bigcup D$.*

Proof. We suppose that D is open dense in $\mathbb{P}$. Indeed, otherwise, we replace D by the set $D_1 = \{T \in \mathbb{P} : \exists\, S \in D,\ T \subseteq S\}$. Assume that the lemma is established for the set D_1, which is obviously open dense. Thus, let $D_1' \subseteq D_1$ be a finite set satisfying $U \subseteq \cup D_1'$. However, by definition, there is a finite set $D' \subseteq D$ such that if $T \in D_1'$, then $\exists\, S \in D'$, $T \subseteq S$. Then $U \subseteq \cup D_1' \subseteq \cup D'$, as required.

Let $U = \mathbf{U}^{\Phi}_k(s) \in \mathbb{U}$, where $s \in 2^{<\omega}$ and $k \in \mathbb{N}$.

The set $\Delta \in \mathfrak{M}$ of all multisystems $\Phi = \langle \varphi_k \rangle_{k\in\omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that

$$K \in |\Phi|, \qquad \mathrm{lh}(s) \le h = \mathrm{hgt}(\varphi_K), \qquad \varphi_K(t) \in D \quad \text{for all } t \in 2^h,$$

is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$ by the choice of D. Therefore, there is an index j such that $\Phi^j \in \Delta$. Let $h(j) = \mathrm{hgt}(\varphi^j_K)$; $\mathrm{lh}(s) \le h(j)$. Then the tree $S_t = \varphi^j_K(t) = \mathbf{T}^{\Phi}_K(t)$ belongs to D for all $t \in 2^{h(j)}$. We conclude that

$$U = \mathbf{U}^{\Phi}_k(s) \subseteq \bigcup_{t \in 2^{h(j)},\, s \subsetneq t} \mathbf{T}^{\Phi}_K(t) \subseteq \bigcup_{t \in 2^{h(j)},\, s \subsetneq t} S_t = \cup D',$$

where $D' = \{S_t : t \in 2^{h(j)} \wedge s \subsetneq t\} \subseteq D$ is finite. □

Lemma 16. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}^{<\omega}$, is pre-dense in $\mathbb{P}^{<\omega}$, then it is pre-dense in $(\mathbb{P} \cup \mathbb{U})^{<\omega}$ as well.*

Proof. For a given multitree $\boldsymbol{\tau} \in (\mathbb{P} \cup \mathbb{U})^{<\omega}$, we must prove that $\boldsymbol{\tau}$ is compatible with a multitree $\boldsymbol{\sigma} \in D$ in $(\mathbb{P} \cup \mathbb{U})^{<\omega}$. Let $|\boldsymbol{\tau}| = \{0, 1\}$, for the sake of clarity, so that $\boldsymbol{\tau} = \langle U, V \rangle$ (see Remark 8), where the trees $U = \mathbf{U}^{\Phi}_k(s)$ and $V = \mathbf{U}^{\Phi}_\ell(t)$ belong to $\mathbb{U}$.

Consider the set $\Delta \in \mathfrak{M}$ of all multisystems $\Phi = \langle \varphi_k \rangle_{k\in\omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that there are strings $s', t' \in 2^{<\omega}$ with $s \subseteq s'$, $t \subseteq t'$, $\mathrm{lh}(s') \le \mathrm{hgt}(\varphi_k)$, $\mathrm{lh}(t') \le \mathrm{hgt}(\varphi_\ell)$, and trees $S, T \in \mathbb{P}$ such that

$$\langle S, T \rangle \in D, \qquad \varphi_k(s') \subseteq U \cap S, \quad \varphi_\ell(t') \subseteq V \cap T.$$

The set Δ is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$ by the pre-density of D. Therefore, there is an index j such that $\Phi^j = \langle \varphi^j_k \rangle_{k\in\omega} \in \Delta$.

Then there exist strings $s', t' \in 2^{<\omega}$ and a multitree $\langle S, T \rangle \in D$ satisfying

$$s \subseteq s', \quad t \subseteq t', \quad \varphi^j_k(s') \subseteq U \cap S, \quad \varphi^j_\ell(t') \subseteq V \cap T.$$

However, we have

$$U' = \mathbf{U}^{\Phi}_k(s') \subseteq \varphi^j_k(s'), \qquad V' = \mathbf{U}^{\Phi}_\ell(t') \subseteq \varphi^j_\ell(t')$$

by construction. It follows that the multitree $\langle U', V' \rangle \in \mathbb{U}^{<\omega}$ is "stronger" in $(\mathbb{P} \cup \mathbb{U})^{<\omega}$ than either of $\langle U, V \rangle$, $\langle S, T \rangle$, as required. □

5. NAMES AND DIRECT FORCING

It is assumed in this section that $\mathbb{P}$ is a PTF and $2^{<\omega} \in \mathbb{P}$. Recall that $\mathbb{P}$ adjoins a generic real $x = \mathbf{x}[G] \in 2^{\mathbb{N}}$, while $\mathbb{P}^{<\omega}$ adjoins an infinite sequence of $\mathbb{P}$-generic reals $x_k = \mathbf{x}_k[G]$; see Sec. 2.

Definition 17. By a $\mathbb{P}^{<\omega}$-*real name* we shall mean a system $\mathbf{c} = \langle C_n^i \rangle_{n<\omega,\, i<2}$ of sets $C_n^i \subseteq \mathbb{P}^{<\omega}$ such that every set $C_n = C_n^0 \cup C_n^1$ is pre-dense in $\mathbb{P}^{<\omega}$, and any pair of multitrees $S \in C_n^0$ and $T \in C_n^1$ is incompatible in $\mathbb{P}^{<\omega}$. In this case, if a set $G \subseteq \mathbb{P}^{<\omega}$ is $\mathbb{P}^{<\omega}$-generic, at least over the family of all sets C_n, then we define a real $\mathbf{c}[G] \in 2^{\mathbb{N}}$ such that $\mathbf{c}[G](n) = i$ iff $G \cap C_n^i \neq \varnothing$.

It is clear that $\mathbf{c} = \langle C_n^i \rangle$ is a $\mathbb{P}^{<\omega}$-name of a real $\mathbf{c}[G] \in 2^{\mathbb{N}}$.

Example 18. Let $k < \omega$. Define a $\mathbb{P}^{<\omega}$-real name $\dot{\mathbf{x}}_k = \langle C_{ni}^k \rangle$ such that every set C_{ni}^k contains all multitrees $\rho \in \mathbb{P}^{<\omega}$ satisfying

$$|\rho| = \{k\}, \qquad \rho(k) = T[s] = \{u \in 2^{<\omega} : s \subsetneq u \vee u \subseteq s\},$$

where $s \in 2^{n+1}$ and $s(n) = i$. Thus $\dot{\mathbf{x}}_k$ is a $\mathbb{P}^{<\omega}$-name of the real $x_k = \mathbf{x}_k[G]$, i.e., the kth term of the $\mathbb{P}^{<\omega}$-generic sequence $\langle x_k \rangle_{k<\omega}$.

Let $\mathbf{c} = \langle C_n^i \rangle$, $\mathbf{d} = \langle D_n^i \rangle$ are $\mathbb{P}^{<\omega}$-real names. The tree $T \in \mathbf{PT}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$, $i = 0, 1$, if $T \subseteq S$ for some tree $S \in C_n^i$;
- *directly forces* $s \subsetneq \mathbf{c}$, where $s \in 2^{<\omega}$ if, for each $n < \mathrm{lh}(s)$, the tree T directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \neq \mathbf{c}$ if there exist strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that T directly forces $s \subsetneq \mathbf{c}$ and $t \subsetneq \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [U]$, where $U \in \mathbf{PT}$, if there is a string $s \in 2^{<\omega} \setminus U$ such that the tree T directly forces $s \subsetneq \mathbf{c}$.

6. KEY THEOREM

Arguing in the conditions of Definition 12, the goal of the following Theorem 19 will be the proof that, whenever $\mathbf{c}$ is a $\mathbb{P}$-real name, the extended forcing $\mathbb{P} \cup \mathbb{U}$ forces that $\mathbf{c}$ does not belong to the sets $[U]$, where $U \in \mathbb{U}$, except for the case when $\mathbf{c}$ is the name $\dot{\mathbf{x}}_k$ for one of generic reals $x_k = \mathbf{x}_k[G]$.

Theorem 19. *Under the conditions of Definition* 12, *let* $\mathbf{c} = \langle C_m^i \rangle_{m<\omega,\, i<2} \in \mathfrak{M}$ *be a* $\mathbb{P}^{<\omega}$*-real name, and for each* k *the set*

$$D(k) = \{\boldsymbol{\tau} \in \mathbb{P}^{<\omega} : \boldsymbol{\tau} \text{ directly forces } \mathbf{c} \neq \dot{\mathbf{x}}_k\}$$

dense in $\mathbb{P}^{<\omega}$. *Let* $U \in \mathbb{U}$ *and* $\mathbf{u} \in (\mathbb{P} \cup \mathbb{U})^{<\omega}$. *Then there exists a multitree* $\mathbf{v} \in \mathbb{U}^{<\omega}$, $\mathbf{v} \leq \mathbf{u}$, *which directly forces* $\mathbf{c} \notin [U]$.

Proof. By definition, we have $U = \mathbf{U}_K^{\Phi}(s_0)$, where $K \in \mathbb{N}$ and $s_0 \in 2^{<\omega}$, and, in the context of the theorem, it can be assumed that $s_0 = \Lambda$, thus $U = \mathbf{U}_K^{\Phi}$. Let, for the sake of clarity and simplicity, $K = 1$, i.e., $U = \mathbf{U}_1^{\Phi}$. By Lemma 14, it can also be assumed that $\mathbf{u} \in \mathbb{U}^{<\omega}$. And, for the sake of clarity, let

$$|\mathbf{u}| = \{0, 1, 2, 3\}, \qquad \mathbf{u} = \langle U_0, U_1, U_2, U_3 \rangle \in \mathbb{U}^{<\omega}$$

(see Remark 8), where

$$U_0 = \mathbf{U}_0^{\Phi}(t_0), \qquad U_1 = \mathbf{U}_0^{\Phi}(t_1), \qquad U_2 = \mathbf{U}_1^{\Phi}(t_2), \qquad U_3 = \mathbf{U}_1^{\Phi}(t_3),$$

and t_0, t_1, t_2, t_3 are strings in $2^{<\omega}$. Let $H = \max\{\mathrm{lh}(t_0), \mathrm{lh}(t_1), \mathrm{lh}(t_2), \mathrm{lh}(t_3)\}$.

Consider the set $\mathscr{D}$ of all multisystems $\Phi = \langle \varphi_k \rangle_{k\in\omega}$ in $\mathbf{SS}^{<\omega}(\mathbb{P})$, such that $0, 1 \in |\Phi|$,

$$H \leq h = \mathrm{hgt}(\varphi_0) = \mathrm{hgt}(\varphi_1),$$

and there is a multitree $\boldsymbol{\sigma} = \langle S_0, \ldots, S_N \rangle \in \mathbb{P}^{<\omega}$, $N \geq 3$, such that

(I) $\boldsymbol{\sigma}$ directly forces $\mathbf{c} \notin [T]$, where $T = \bigcup_{s \in 2^h} \varphi_1(s)$;

(II) every tree S_i occurs in Φ (see Sec. 3);

(III) in addition, $S_0 = \varphi_0(s_0)$, $S_1 = \varphi_0(s_1)$, $S_2 = \varphi_1(s_2)$, $S_3 = \varphi_1(s_3)$, where s_0, s_1, s_2, s_3 are strings in 2^h and $t_i \subseteq s_i$, $i = 0, 1, 2, 3$.

Lemma 20. *Under the conditions of the theorem, the set $\mathscr{D}$ is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$.*

Proof. Consider any multisystem $\Phi' \in \mathbf{SS}^{<\omega}(\mathbb{P})$. We must define a multisystem $\Psi \in \mathscr{D}$ such that $\Phi' \preccurlyeq \Psi$. Our plan is as follows. Take any multisystem $\Phi = \langle \varphi_k \rangle_{k \in \omega}$ in $\mathbf{SS}^{<\omega}(\mathbb{P})$ such that

$$\Phi' \prec\prec \Phi, \qquad H \le h = \operatorname{hgt}(\varphi_0) = \operatorname{hgt}(\varphi_1);$$

then every multisystem $\Psi \in \mathbf{SS}^{<\omega}(\mathbb{P})$ refining Φ satisfies $\Phi' \prec\prec \Psi$. Thus, the problem is reduced to the construction of a multisystem $\Psi \in \mathscr{D}$ which refines Φ.

Pick strings $s_0, s_1, s_2, s_3 \in 2^h$, satisfying $t_i \subseteq s_i$, $i = 0, 1, 2, 3$. Consider a multitree

$$\boldsymbol{\rho} = \langle R_0, R_1, R_2, R_3, R_4, \dots, R_N \rangle \in \mathbb{P}^{<\omega},$$

where $N = 1 + 2^h$ (here 2^h is the arithmetic exponent),

$$R_0 = \varphi_0(s_0), \qquad R_1 = \varphi_0(s_1), \qquad R_2 = \varphi_1(s_2), \qquad R_3 = \varphi_1(s_3),$$
$$R_j = \varphi_1(s_j), \quad j = 4, \dots, N,$$

where $\{s_4, \dots, s_N\}$ is an arbitrary enumeration of the set $\{s \in 2^h : s \neq s_2, s_3\}$.

By the density of the sets $D(k)$, there exists a multitree

$$\boldsymbol{\sigma} = \langle S_0, S_1, S_2, S_3, \dots, S_N, \dots, S_M \rangle \in \mathbb{P}^{<\omega}$$

such that $\boldsymbol{\sigma} \le \boldsymbol{\rho}$, thus $M \ge N$ and $S_i \subseteq R_i$ for all $i \le N$, which directly forces $\mathbf{c} \neq \dot{\mathbf{x}}_k$ for all $k = 2, \dots, N$. This means that there exist strings $u, v_2, \dots, v_N \in 2^{<\omega}$ such that $\boldsymbol{\sigma}$ directly forces every formula

$$u \subsetneq \mathbf{c}, \qquad \text{as well as} \quad v_2 \subsetneq \dot{\mathbf{x}}_2, \quad v_3 \subsetneq \dot{\mathbf{x}}_3, \quad \dots, \quad v_N \subsetneq \dot{\mathbf{x}}_N,$$

and u is incomparable in $2^{<\omega}$ with every v_k. This implies $v_k \subseteq \operatorname{stem} S_k$, $k = 2, \dots, N$. Therefore $\boldsymbol{\sigma}$ directly forces $\mathbf{c} \notin [S]$, where $S = \bigcup_{2 \le k \le N} S_k$.

We now define a required multisystem $\Psi = \langle \psi_k \rangle_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$.

Step 1. Recall that $R_0 = \varphi_0(s_0)$, $R_1 = \varphi_0(s_1)$, $R_2 = \varphi_1(s_2)$, $R_3 = \varphi_1(s_3)$ in Φ. Put

$$\psi_0(s_0) = S_0, \qquad \psi_0(s_1) = S_1, \qquad \psi_1(s_2) = S_2, \qquad \psi_1(s_3) = S_3.$$

We also let $\psi_0(s) = \varphi_0(s)$ for all $s \in 2^{\le h}$, $s \neq s_0, s_1$.

Step 2. Assume that $4 \le j \le N$. By construction the tree R_j is equal to $\varphi_1(s_j)$, where $s_j \in 2^h$, $s_j \neq s_2, s_3$; we let $\psi_1(s_j) = S_j$.

Step 3. If $\ell \in |\Phi|$, $\ell \ge 2$, then let $\psi_\ell = \varphi_\ell$.

Step 4. We put $\mu = \max |\Phi|$. Assume that $N + 1 \le \ell < M$. Then $S_\ell \in \mathbb{P}$. Define a system $\psi_{\mu+\ell} \in \mathbf{SS}(\mathbb{P})$ such that $\operatorname{hgt}(\psi_{\mu+\ell}) = 0$ and $\psi_{\mu+\ell}(\Lambda) = S_\ell$.

One easily verifies that $\Psi \in \mathscr{D}$ is a multisystem required. $\square$

We return to Theorem 19. By Lemma 20, there is an index J such that the multisystem $\Phi^J = \langle \varphi_k^J \rangle_{k\in\omega}$ belongs to $\mathscr{D}$, thus $0, 1 \in |\Phi|$, $H \le h = \operatorname{hgt}(\varphi_0^J) = \operatorname{hgt}(\varphi_1^J)$, and there exists a multitree $\boldsymbol{\sigma} = \langle S_0, \dots, S_N \rangle \in \mathbb{P}^{<\omega}$ satisfying (I), (II), (III) for systems $\varphi_0 = \varphi_0^J$, $\varphi_1 = \varphi_1^J$.

Consider a multitree $\mathbf{v} = \langle V_0, V_1, V_2, V_3, \dots, V_n \rangle \in \mathbb{U}^{<\omega}$ defined so that

$$V_0 = \mathbf{U}_0^{\Phi}(s_0), \qquad V_1 = \mathbf{U}_0^{\Phi}(s_1), \qquad V_2 = \mathbf{U}_1^{\Phi}(s_2), \qquad V_3 = \mathbf{U}_1^{\Phi}(s_3),$$

and if $4 \le j \le n$, then V_j is any tree in $\mathbb{U}$ satisfying $V_k \subseteq S_k$ (see Lemma 14). Since $t_i \subseteq s_i$ holds for $i = 0, 1, 2, 3$, we have $\mathbf{v} \le \mathbf{u}$. And on the other hand, $\mathbf{v} \le \boldsymbol{\sigma}$, and hence $\mathbf{v}$ directly forces $\mathbf{c} \notin [T]$ by (I), where

$$T = \bigcup_{s\in 2^h} \varphi_1^J(s) = \bigcup_{s\in 2^h} \mathbf{T}_1^{\Phi}(s).$$

We finally have $\mathbf{U}_1^{\Phi} \subseteq \bigcup_{s\in 2^h} \varphi_1^J(s)$ by construction; therefore, $\mathbf{v}$ directly forces $\mathbf{c} \notin [\mathbf{U}_1^{\Phi}]$, as required. □

7. THE JENSEN TRANSFINITE FORCING CONSTRUCTION

We argue in the constructible universe $\mathbf{L}$ in this section. We let $\le_{\mathbf{L}}$ denote the canonical wellordering of the class $\mathbf{L}$; see [4, 8.1.6].

Definition 21 (in $\mathbf{L}$). Following a construction in [11, Sec. 3] (with appropriate modifications), we define a countable PTF $\mathbb{U}_\xi \subseteq \mathbf{PT}$ (see Sec. 2) by induction on $\xi < \omega_1$, as follows.

Let $\mathbb{U}_0$ denote the set of all trees of the form $T[s]$, see Example 5, including the full tree $2^{<\omega} = T[\Lambda]$ itself.

Now suppose that $0 < \lambda < \omega_1$, and countable PTFs $\mathbb{U}_\xi \subseteq \mathbf{PT}$ are already defined for $\xi < \lambda$. Let $\mathfrak{M}_\xi$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing the sequence $\langle \mathbb{U}_\xi \rangle_{\xi<\lambda}$ and such that $\lambda < \omega_1^{\mathfrak{M}}$ and all sets $\mathbb{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$. Then the set $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{U}_\xi$ is a PTF as well and is countable in $\mathfrak{M}$. We define $\langle \Phi^j \rangle_{j<\omega}$ to be the $\le_{\mathbf{L}}$-least sequence of systems $\Phi^j \in \mathbf{SS}^{<\omega}(\mathbb{P}_\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$. To accomplish the induction step, define $\mathbb{U}_\lambda = \mathbb{U}$ as in Definition 12.

We finally put $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$.

The set $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$ is equal to Jensen's forcing in [11] modulo technical details. We make use of the finite support product $\mathbb{P}^{<\omega}$ in the proof of Theorem 4. The following results are typical of this type of inductive forcing constructions.

Proposition 22 (in $\mathbf{L}$). *The sequence $\langle \mathbb{U}_\xi \rangle_{\xi<\omega_1}$ belongs to the definability class Δ_1^{HC}*[3].

Lemma 23 (in $\mathbf{L}$). *If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi^{<\omega}$ is pre-dense in $\mathbb{P}_\xi^{<\omega}$, then it remains pre-dense in $\mathbb{P}^{<\omega}$. Therefore, if $\xi < \omega_1$, then $\mathbb{U}_\xi^{<\omega}$ itself is pre-dense in $\mathbb{P}^{<\omega}$.*

Proof. Induction on $\lambda \ge \xi$ proves that if D is pre-dense in

$$\mathbb{P}_\lambda^{<\omega},$$

then it remains pre-dense in $\mathbb{P}_{\lambda+1}^{<\omega} = (\mathbb{P}_\lambda \cup \mathbb{U}_\lambda)^{<\omega}$ by Lemma 16. The limit step is obvious. To prove the second claim, note that $\mathbb{U}_\xi^{<\omega}$ is dense in $\mathbb{P}_{\xi+1}^{<\omega}$ by Lemma 14, and $\mathbb{U}_\xi^{<\omega} \in \mathfrak{M}_{\xi+1}$. □

[3] See [8, 3.4] or [15, Secs. 8 and 9] on the set HC of all hereditarily countable sets; note that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in $\mathbf{L}$. See [15, Chap. 5, Sec. 4] on the definability classes Σ_n^X, Π_n^X, Δ_n^X.

Lemma 24 (in $\mathbf{L}$). *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$, then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of the structure $\langle \mathbf{L}_{\omega_1}; X \rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1.*

Generally, if $X_n \subseteq \mathrm{HC}$ holds for all n, then the set W of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi; \langle X_n \cap \mathbf{L}_\xi \rangle_{n<\omega} \rangle$ is an elementary submodel of the structure

$$\langle \mathbf{L}_{\omega_1}; \langle X_n \rangle_{n<\omega} \rangle \qquad \textit{and} \qquad \langle X_n \cap \mathbf{L}_\xi \rangle_{n<\omega} \in \mathfrak{M}_\xi$$

is unbounded in ω_1.

Proof. Let $\xi_0 < \omega_1$. There exists a countable elementary submodel M of the structure $\mathbf{L}_{\omega_2}$, containing ξ_0, ω_1, X and such that $M \cap \mathrm{HC}$ is transitive. Let $\phi\colon M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\xi = \phi(\omega_1)$. Then $\xi_0 < \xi < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\xi$ by the choice of M. This implies that $\langle \mathbf{L}_\xi; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, therefore $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. We conclude that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$, because $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction. The proof of the general claim is similar. □

Corollary 25 (compare to [11, Lemma 6]). *The forcing $\mathbb{P}^{<\omega}$ satisfies the condition of countability of antichains in $\mathbf{L}$.*

Proof. Consider a maximal antichain $A \subseteq \mathbb{P}^{<\omega}$. By Lemma 24 there is an ordinal ξ such that $A' = A \cap \mathbb{P}_\xi$ is a maximal antichain in $\mathbb{P}_\xi^{<\omega}$ and $A' \in \mathfrak{M}_\xi$. But then A' remains a pre-dense set, hence still a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 23. It follows that the antichain $A = A'$ itself is countable. □

8. GENERIC EXTENSION

We consider the sets $\mathbb{P}$, $\mathbb{P}^{<\omega}$ in $\mathbf{L}$ (see Definition 21) as forcings for generic extensions of the universe $\mathbf{L}$. Recall that $\mathbb{P}$ (as any PTF in general) adjoins a generic real in $2^{\mathbb{N}}$, while $\mathbb{P}^{<\omega}$ adjoins an infinite sequence of such reals.

Lemma 26 (Lemma 7 in [11]). *A real $x \in 2^{\mathbb{N}}$ is $\mathbb{P}$-generic over $\mathbf{L}$ if and only if*

$$x \in Z = \bigcap_{\xi<\omega_1^{\mathbf{L}}} \bigcup_{U \in \mathbb{U}_\xi} [U].$$

Proof. If $\xi < \omega_1^{\mathbf{L}}$, then the set $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$ by Lemma 23. Therefore, each real $x \in 2^{\mathbb{N}}$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $\bigcup_{U \in \mathbb{U}_\xi}[U]$. Conversely, suppose that $x \in Z$, and prove that the real x is $\mathbb{P}$-generic over $\mathbf{L}$. Consider a maximal antichain $A \subseteq \mathbb{P}$ in $\mathbf{L}$; we need to check that $x \in \bigcup_{T \in A}[T]$. Indeed, $A \subseteq \mathbb{P}_\xi$ holds for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 25. But then every tree $U \in \mathbb{U}_\xi$ satisfies $U \subseteq^{\mathrm{fin}} \bigcup A$ by Lemma 15, so that

$$\bigcup_{U \in \mathbb{U}_\xi} [U] \subseteq \bigcup_{T \in A} [T],$$

and hence $x \in \bigcup_{T \in A}[T]$, as required. □

Corollary 27 (compare to Corollary 9 in [11]). *In any generic extension of the class $\mathbf{L}$, the set of all reals in $2^{\mathbb{N}}$, $\mathbb{P}$-generic over $\mathbf{L}$, is a set of classes Π_1^{HC} and Π_2^1.*

Proof. Apply Lemma 26 and Proposition 22. □

Definition 28. We fix a set $G \subseteq \mathbb{P}^{<\omega}$, $\mathbb{P}^{<\omega}$-generic over $\mathbf{L}$. If $k < \omega$, then the set $G_k = \{\boldsymbol{\tau}(k) : \boldsymbol{\tau} \in G\}$ is accordingly $\mathbb{P}$-generic over $\mathbf{L}$, and the intersection $X_k = \bigcap_{T \in G_k}[T]$ contains a single element $x_k \in 2^{\mathbb{N}}$ and the real $x_k = \mathbf{x}[G_k]$ is $\mathbb{P}$-generic over $\mathbf{L}$.

The entire extension $\mathbf{L}[G]$ is then equal to $\mathbf{L}[\langle x_k \rangle_{k<\omega}]$, and our goal will be to prove that it does not contain any other $\mathbb{P}$-generic reals.

Lemma 29 (compare to Lemma 10 in [11]). *If $y \in \mathbf{L}[G] \cap 2^{\mathbb{N}}$ and $y \notin \{x_k : k < \omega\}$, then y is not a $\mathbb{P}$-generic real over $\mathbf{L}$.*

Proof. Assume the converse: there exists a $\mathbb{P}^{<\omega}$-real name $\mathbf{c} = \langle C_n^i \rangle_{n<\omega,\, i=0,1} \in \mathbf{L}$ and a condition $\boldsymbol{\tau} \in \mathbb{P}^{<\omega}$ which $\mathbb{P}^{<\omega}$-forces that the real $\mathbf{c}$ is $\mathbb{P}$-generic, while the whole forcing $\mathbb{P}^{<\omega}$ forces that $\mathbf{c} \neq \dot{\mathbf{x}}_k$ for each k. (Recall that $\dot{\mathbf{x}}_k$ is a $\mathbb{P}^{<\omega}$-name of the real x_k.)

All sets $C_n = C_n^0 \cup C_n^1$ are pre-dense in $\mathbb{P}^{<\omega}$. Therefore, by Lemma 24 there is an ordinal $\lambda < \omega_1$ such that the sets $C'_n = C_n \cap \mathbb{P}_\lambda$ are pre-dense in $\mathbb{P}_\lambda^{<\omega}$, and the sequence $\langle C'_{ni} \rangle_{n<\omega,\, i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni} = C'_n \cap C_n^i$. Then the sets C'_n are pre-dense in $\mathbb{P}^{<\omega}$ by Lemma 23. Thus, we can assume that $C_n = C'_n$, i.e., $\mathbf{c} \in \mathfrak{M}_\lambda$ and $\mathbf{c}$ is a $\mathbb{P}_\lambda^{<\omega}$-real name.

Further, since $\mathbb{P}^{<\omega}$ forces $\mathbf{c} \neq \dot{\mathbf{x}}_k$, the set D_k of all multitrees $\boldsymbol{\sigma} \in \mathbb{P}^{<\omega}$ which directly force $\mathbf{c} \neq \dot{\mathbf{x}}_k$, is dense in $\mathbb{P}^{<\omega}$ for all k. Therefore, once again by Lemma 24, we can assume that, for the same ordinal λ, each set $D'_k = D_k \cap \mathbb{P}_\lambda^{<\omega}$ is dense in $\mathbb{P}_\lambda^{<\omega}$.

Applying Theorem 19 for $\mathbb{P} = \mathbb{P}_\lambda$, $\mathbb{U} = \mathbb{U}_\lambda$, and $\mathbb{P} \cup \mathbb{U} = \mathbb{P}_{\lambda+1}$, we infer that if $U \in \mathbb{U}_\lambda$, then the set Q_U of all multitrees $\mathbf{v} \in \mathbb{P}_{\lambda+1}^{<\omega}$ directly forcing $\mathbf{c} \notin [U]$ is dense in $\mathbb{P}_{\lambda+1}^{<\omega}$. And since obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we conclude that Q_U is pre-dense in the entire forcing $\mathbb{P}^{<\omega}$ by Lemma 23. It follows that the forcing $\mathbb{P}^{<\omega}$ forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}_\lambda} [U]$, and hence forces that $\mathbf{c}$ is not a $\mathbb{P}$-generic real, by Lemma 26. But this contradicts the choice of T. □

The next lemma expresses a usual property of product forcings.

Lemma 30 (in the assumptions of Definition 28). *If $k < \omega$, then the real x_k is not* OD *in* $\mathbf{L}[G]$.

Proof. Assume the converse, and let $\varphi(\alpha, x)$ be a formula containing a parameter $\alpha \in \mathrm{Ord}$, and such that it is true in $\mathbf{L}[G]$ that: $\exists!\, x\varphi(\alpha, x) \wedge \varphi(\alpha, x_k)$. This sentence is $\mathbb{P}^{<\omega}$-forced by a certain condition, a multitree $\boldsymbol{\sigma} \in \mathbb{P}^{<\omega}$, i.e., $\boldsymbol{\sigma}$ forces $\exists!\, x\varphi(\alpha, x) \wedge \varphi(\alpha, \dot{\mathbf{x}}_k)$. By definition, the set $u = |\sigma| \subseteq \mathbb{N}$ is finite; let $m = \max u$ be its largest element.

Consider a bijection $b \colon \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ mapping the segment $[0, m]$ onto $[m+1, 2m+1]$ in order-preserving way, and equal to the identity on $[m+2, +\infty)$. It obviously induces an order automorphism of the whole set $\mathbb{P}^{<\omega}$. However, the multitree $\boldsymbol{\sigma}' = b(\boldsymbol{\sigma})$ satisfies $|\boldsymbol{\sigma}'| \subseteq [m+1, 2m+1]$, so that $|\boldsymbol{\sigma}| \cap |\boldsymbol{\sigma}'| = \varnothing$. Therefore $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are compatible, in fact $\boldsymbol{\tau} = \boldsymbol{\sigma} \cup \boldsymbol{\sigma}'$ is a multitree and a condition stronger than either of $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$.

Moreover, the bijection b also induces a transformation of names such that $b(\dot{\mathbf{x}}_k) = \dot{\mathbf{x}}_{k'}$, where $k' \neq k$. By the invariance of the forcing relation (see, for instance, [5, I.3.5]), it follows that $\boldsymbol{\sigma}'$ forces $\exists!\, x\varphi(\alpha, x) \wedge \varphi(\alpha, \dot{\mathbf{x}}_{k'})$, so that $\boldsymbol{\tau}$ forces

$$\exists!\, x\varphi(\alpha, x) \wedge \varphi(\alpha, \dot{\mathbf{x}}_k) \wedge \varphi(\alpha, \dot{\mathbf{x}}_{k'}),$$

which contradicts the inequality $k \neq k'$. □

Now it remains to finalize the proof of our main theorem.

Proof of Theorem 4. Arguing in the $\mathbb{P}^{<\omega}$-generic model $\mathbf{L}[G] = \mathbf{L}[\langle x_k \rangle_{k<\omega}]$, it is clear that the countable set $X = \{x_k : k < \omega\}$ is equal to the set of all $\mathbb{P}$-generic reals by Lemma 29, therefore, it belongs to Π_2^1 by Corollary 27, and finally it does not contain OD reals by Lemma 30. □

Remark 31. It is a special feature of Jensen's forcing $\mathbb{P}$ that its construction assumes that the ground universe (where the forcing is defined) is the Goedel constructible universe $\mathbf{L}$. It would be interesting to generalize this method, for instance, to the ground universes in which the coding methods introduced in [4] and [17] are applicable.

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REFERENCES

1. J. Hadamard, R. Baire, H. Lebesgue, and E. Borel, "Cinq lettres sur la théorie des ensembles," Bull. Soc. Math. France **33**, 261–273 (1905).
2. A. Tarski, "Der Wahrheitsbegriff in den formalisierten Sprachen," Stud. Philos. **1**, 261–405 (1936).
3. J. Myhill and D. Scott, "Ordinal definability," in 1971 *Axiomatic Set Theory* (Amer. Math. Soc., Providence, RI, 1971), pp. 271–278.
4. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory: Absolutely Unsolvable Classical Problems* (MTsNMO, Moscow, 2013) [in Russian].
5. R. M. Solovay, "A model of set theory in which every set of reals is Lebesgue measurable," Ann. of Math. (2) **92**, 1–56 (1970).
6. *A Question About Ordinal Definable Real Numbers*, in *Mathoverflow* (2010), `http://mathover-flow.net/questions/17608`.
7. A. Enayat, *Ordinal Definable Numbers*, in *Foundations of Mathematics* (2010), `http://cs.nyu.edu/pipermail/fom/2010-July/014944.html`.
8. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory: Borel and Projective Sets* (MTsNMO, Moscow, 2010) [in Russian].
9. V. Kanovei, *Borel Equivalence Relations. Structure and Classification*, in *Univ. Lecture Ser.* (Amer. Math. Soc., Providence, RI, 2008), Vol. 44.
10. A. Enayat, "On the Leibniz–Mycielski axiom in set theory," Fund. Math. **181** (3), 215–231 (2004).
11. R. Jensen, "Definable sets of minimal degree," in *Mathematical Logic and Foundations of Set Theory* (North-Holland, Amsterdam, 1970), pp. 122–128.
12. V. G. Kanovei and V. A. Lyubetsky, "Non-uniformizable sets of second projective level with countable cross-sections in the form of Vitali classes," Izv. Ross. Akad. Nauk, Ser. Mat. **82** (1) (2018) (in press) [Izv. Math. **82** (1) (2018), in press].
13. V. Kanovei and V. Lyubetsky, "Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation," Ann. Pure Appl. Logic **167** (3), 262–283 (2016).
14. M. Golshani, V. Kanovei, and V. Lyubetsky, "A Groszek–Laver pair of undistinguishable E_0 classes," MLQ Math. Log. Q. **63** (1-2), 19–31 (2017).
15. V. G. Kanovei, "The projective hierarchy of N. N. Luzin: The current state-of-the-art in the theory," in *Handbook of Mathematical Logic*, Part II: *Set Theory*, Supplement to the Russian translation (Nauka, Moscow, 1982), pp. 273–364 [in Russian].
16. V. G. Kanovei and V. A. Lyubetsky, "An effective minimal encoding of uncountable sets," Sib. Math. Zh. **52** (5), 1074–1086 (2011) [Sib. Math. J. **52** (5), 854–863 (2011)].
17. J. Bagaria and V. Kanovei, "On coding uncountable sets by reals," MLQ Math. Log. Q. **56** (4), 409–424 (2010).