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SHORT COMMUNICATIONS

On the Equality Relation Modulo a Countable Set

V. G. Kanovei[1*] **and V. A. Lyubetsky**[1**]

[1] *Institute for Information Transmission Problems of Russian Academy of Sciences (Kharkevich Institute), Moscow, 127051 Russia*

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The equivalence relation $\mathsf{E}_{\aleph_0}$ (the equality relation modulo a countable set) is defined as follows:

$$X \mathrel{\mathsf{E}_{\aleph_0}} Y, \qquad \text{if the symmetric difference } X \Delta Y \text{ is at most countable.}$$

Here X and Y are sets in the Baire space ω^ω, although all that is stated below remains true for the real line and, in general, for any perfect Polish space. The axiom of choice AC allows us to choose a particular element in each $\mathsf{E}_{\aleph_0}$-equivalence class; that is, there exists a function $s\colon \mathscr{P}(\omega^\omega) \to \mathscr{P}(\omega^\omega)$ satisfying $s(X) \mathrel{\mathsf{E}_{\aleph_0}} X$ for all sets $X \subseteq \omega^\omega$ and $s(Y) = s(X)$ for all sets $X, Y \subseteq \omega^\omega$ satisfying $X \mathrel{\mathsf{E}_{\aleph_0}} Y$. Such a function S is called a *selector* for the relation $\mathsf{E}_{\aleph_0}$; see [1, Section 12.D].

However, the application of the axiom of choice does not resolve the question of the existence of an *effectively defined* selector s, that is, the choice of a concrete well-defined set in every $\mathsf{E}_{\aleph_0}$-class of point sets. The answer to this question depends on which point sets we consider. For instance, every class of $\mathsf{E}_{\aleph_0}$-equivalence of *closed* sets $X \subseteq \omega^\omega$ contains a unique perfect set, which we can take as $s(X)$, obtaining an effectively defined selector. The following theorem of ours extends this result to the much broader class $\mathbf{\Delta}^0_2$ of those sets that are simultaneously $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$.

Theorem. *There is an effectively defined selector for the relation $\mathsf{E}_{\aleph_0}$ on the $\mathbf{\Delta}^0_2$ sets in the Baire space.*

The theorem gives the best possible result, since already for the next (according to the volume of sets) Borel class $\mathbf{F}_\sigma$, there are generally no effectively definable selectors. This is a consequence of the result, recently obtained in [2, 5.5], that **ZFC** is not strong enough to define an effectively definable selector for the relation $\mathsf{E}_{\aleph_0}$ on the class of all $\mathbf{F}_\sigma$ sets, which is wider than $\mathbf{\Delta}^0_2$.[1]

The proof of the theorem uses the following Lemma.

As usual, $\overline{X}$ denotes the topological closure of a set X.

Lemma. *If X is a countable $\mathbf{G}_\delta$ set in a Polish space, then the closure $\overline{X}$ of X is countable. Therefore, if the $\mathbf{\Delta}^0_2$ sets X and Y satisfy $X \mathrel{\mathsf{E}_{\aleph_0}} Y$, then $\overline{X} \mathrel{\mathsf{E}_{\aleph_0}} \overline{Y}$.*

Proof. Otherwise, X would be a countable dense $\mathbf{G}_\delta$ set in the uncountable Polish space $\overline{X}$, which is a contradiction. □

[*] E-mail: kanovei@iitp.ru
[**] E-mail: lyubetsk@iitp.ru

[1] To be more exact, the result in [2, 5.5] claims that the relation $\mathsf{E}_{\aleph_0}$ on the $\mathbf{F}_\sigma$ sets does not admit a Baire measurable selector. However, it is true in the well-known Solovay model [3] of **ZFC** that all **ROD** maps are Baire measurable. The class **ROD** (real-ordinal definable sets) consists of all sets definable by set-theoretic formulas with ordinals and points of ω^ω (that is, reals, in the terminology of modern descriptive set theory) as parameters. The class **ROD** contains, with a margin, all the sets that can be considered effectively definable in the broadest sense; see [4, Section 7]. Therefore, the relation $\mathsf{E}_{\aleph_0}$ on the $\mathbf{F}_\sigma$ sets has no **ROD** selectors in the Solovay model, and hence has no selectors effectively definable in any reasonable sense.

We also make use of the *difference hierarchy of* $\mathbf{\Delta}^0_2$ *sets*. It is well known (see [1, 22.E] or [5, Sec. 34.VI]) that every $\mathbf{\Delta}^0_2$ set A in a Polish space $\mathbb{X}$ can be represented in the following form:

$$A = \bigcup_{\eta<\vartheta} (F_\eta \setminus H_\eta),$$

where $\vartheta < \omega_1$ and $F_0 \supseteq H_0 \supseteq F_1 \supseteq H_1 \supseteq \cdots \supseteq F_\eta \supseteq H_\eta \supseteq \cdots$ is a decreasing sequence of closed sets in $\mathbb{X}$ defined by transfinite induction so that

$$F_0 = \mathbb{X}, \qquad H_\eta = \overline{F_\eta \setminus A}, \qquad F_{\eta+1} = H_\eta \cap \overline{F_\eta \cap A},$$

and at the limit steps, the intersection is taken. It follows from separability that $F_\vartheta = \varnothing$ for some ordinal $\vartheta < \omega_1$, at which the construction is completed.

Proof of the theorem. It suffices to check that if two $\mathbf{\Delta}^0_2$ sets $A, B \subseteq \omega^\omega$ satisfy $A \mathsf{E}_{\aleph_0} B$, then the corresponding decreasing sequences of closed sets

$$\left.\begin{array}{l} F^A_0 \supseteq H^A_0 \supseteq F^A_1 \supseteq H^A_1 \supseteq \cdots \supseteq F^A_\eta \supseteq H^A_\eta \supseteq \cdots \\ F^B_0 \supseteq H^B_0 \supseteq F^B_1 \supseteq H^B_1 \supseteq \cdots \supseteq F^B_\eta \supseteq H^B_\eta \supseteq \cdots \end{array}\right\} \qquad (\eta < \vartheta = \vartheta^A = \vartheta^B),$$ [2]

which satisfy the equalities $A = \bigcup_{\eta<\vartheta}(F^A_\eta \setminus H^A_\eta)$ and $B = \bigcup_{\eta<\vartheta}(F^B_\eta \setminus H^B_\eta)$, satisfy also the relation

$$F^A_\eta \;\mathsf{E}_{\aleph_0}\; F^B_\eta \quad \text{and} \quad H^A_\eta \;\mathsf{E}_{\aleph_0}\; H^B_\eta \qquad \text{for all} \quad \eta < \vartheta. \tag{1}$$

If this is established, then the perfect kernels[3] $\mathbf{PK}(F^A_\eta)$ and $\mathbf{PK}(F^B_\eta)$ of the sets F^A_η and F^B_η are equal to each other: $\mathbf{PK}(F^A_\eta) = \mathbf{PK}(F^B_\eta)$, and similarly $\mathbf{PK}(H^A_\eta) = \mathbf{PK}(H^B_\eta)$. We conclude that the sets

$$s(A) = \bigcup_{\eta<\vartheta} (\mathbf{PK}(F^A_\eta) \setminus \mathbf{PK}(H^A_\eta)) \qquad \text{and} \qquad s(B) = \bigcup_{\eta<\vartheta} (\mathbf{PK}(F^B_\eta) \setminus \mathbf{PK}(H^B_\eta))$$

are equal to each other (under the assumption that A and B are $\mathbf{\Delta}^0_2$ sets satisfying $A \;\mathsf{E}_{\aleph_0}\; B$). In addition, we have $A \;\mathsf{E}_{\aleph_0}\; s(A)$ for each $\mathbf{\Delta}^0_2$ set A. Therefore, s is a required selector, which completes the proof of the theorem.

The proof of relation (1) itself is performed by induction.

We have $F^A_0 = F^B_0 = \omega^\omega$; this is the base case of induction.

Suppose that $F^A_\eta \;\mathsf{E}_{\aleph_0}\; F^B_\eta$. Then

$$(F^A_\eta \setminus A) \;\mathsf{E}_{\aleph_0}\; (F^B_\eta \setminus B)$$

holds as well. (Because it is assumed that $A \;\mathsf{E}_{\aleph_0}\; B$.) Therefore, $H^A_\eta \;\mathsf{E}_{\aleph_0}\; H^B_\eta$ by the lemma. And then we similarly obtain $F^A_{\eta+1} \;\mathsf{E}_{\aleph_0}\; F^B_{\eta+1}$, which completes the induction step $\eta \to \eta + 1$.

The limit step in the proof of (1) does not raise any questions. □

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[2] If necessary, the shorter of these two decreasing sequences is extended by empty sets to the length of the longer sequence.
[3] The perfect kernel $\mathbf{PK}(X)$ is the largest perfect subset of a given closed set X.