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# On Constructible Sets in Simpson's Set Theory

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**Abstract**—Simpson's set theory **SST** is obtained from **ZF** by removing the power set axiom and the replacement axiom schema (but keeping the separation schema) and adding the axioms of (1) transitive superset, (2) transitive collapse of well-founded relations, (3) countability, and (4) Cartesian product. Metamathematically, **SST** is the theory of the standard set-theoretic superstructure over the universe of second-order arithmetic $\mathbf{Z}_2$ (without the axiom of choice). Moreover, **SST** is equiconsistent with $\mathbf{Z}_2$ and is related to $\mathbf{Z}_2$ by mutually inverse interpretations. Simpson showed that the construction of the class $\mathbf{L}$ of all Gödel constructible sets is feasible in this theory in spite of the absence of the replacement axiom schema, which is usually employed to justify transfinite induction in **ZF**. Examining the class $\mathbf{L}$ in Simpson's theory, we prove that the class $\mathbf{L}$ satisfies **SST** without the axiom of countability; in particular, it satisfies the separation axiom schema.

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## 1. INTRODUCTION

In [1, Chap. VII], Simpson made a careful study of relationships between various subtheories of the second-order arithmetic $\mathbf{Z}_2$ and set theories without the power set axiom but with the axiom **Cntbl** of countability of all sets. As a kind of core of his set-theoretic schemata, Simpson used the theory $\mathbf{ATR}_0^{\rm set}$ obtained from Zermelo's theory **Z** by removing the power set axiom and the separation schema and adding (1) the axiom of the existence of transitive supersets, (2) the axiom of the existence of a transitive model for any well-founded binary relation (*Mostowski's collapse*), (3) the axiom **Cntbl** of countability of every set, and (4) the axiom of universal definability of a certain version of Gödel operations (which includes, e.g., the existence of $\{u,v\}$ and $u\times v$ for any sets $u$ and $v$). To the base theory Simpson added the separation schema for certain classes of $\in$-formulas; this version can be completed by adding the whole separation schema **Sep** (or $\Pi^{\rm set}_\infty$-comprehension, as in [1]). We refer to the theory $\mathbf{ATR}_0^{\rm set}+\mathbf{Sep}$ thus obtained as *Simpson's set theory* and denote it by **SST**. The axioms of this theory and some of their immediate consequences are discussed in detail in Secs. 2 and 3.

From the viewpoint of foundations of mathematics, the theory **SST** is important because, as shown in [1, Sec. VII.3], this is precisely the theory of the standard interpretation of set theory in the second-order arithmetic $\mathbf{Z}_2$ by means of well-founded trees; see also, e.g., [2]–[5] or Sec. 4 of the present paper.

In [1, Sec. VII.4], Simpson proved that **SST** (and even $\mathbf{ATR}_0^{\rm set}$ to some extent) is strong enough for implementing many typical set-theoretic constructions by transfinite induction, including an adequate construction of the class $\mathbf{L}=\bigcup_{\alpha\in\mathrm{Ord}}\mathbf{L}_\alpha$ of constructible sets and the derivation of a number of fundamental properties of the hierarchy of $\mathbf{L}_\alpha$ in spite of the absence of the replacement schema, which is usually employed for these purposes in **ZF**. In this direction, Simpson essentially obtained (in a different terminological system) the following result (see Theorem VII.5.4 of [1] and the statement about the axiom of choice in its proof).

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**Theorem (SST).** *The class or set* $\mathbf{HCL} \subseteq \mathbf{L}$ *of all sets* $x$ *belonging to a transitive set* $y \in \mathbf{L}$ *countable in* $\mathbf{L}$ *satisfies the axioms of* $\mathbf{ZFC}^-$.

Here $\mathbf{ZFC}^-$ is **ZFC** without the power set axiom (see Sec. 2).

A fundamental consequence of this theorem is that the theories $\mathbf{Z}^-$, $\mathbf{ZF}^-$, and $\mathbf{ZFC}^-$ (without the power set axiom) and the second-order arithmetic $\mathbf{Z}_2$ are *equiconsistent*. This consistency result has been known at least since the 1960s: it was mentioned in [6], and a complete proof was presented, e.g., in [7]. See also [8]–[12] for recent results concerning consistency in relation to $\mathbf{Z}_2$. These results were discussed in our recent paper [13].

In contrast to the subclass **HCL** of **L**, the class **L** itself in the **SST** theory was little studied in the monograph [1] in context of the axioms of set theory. It was only mentioned in [1] that **L** does not necessarily satisfy the axiom of countability (see Example 1 in the present paper). In the positive direction, only the special case of **Sep** for $\Sigma_1$ relations was considered [1, Exercise VII.5.5]. We fill this lacuna with the following theorem.

**Theorem 1 (SST).** *The class* **L** *satisfies the axioms of* **SST** (*except possibly the countability axiom* **Cntbl**) *and the following two axioms*:

(I) *Every set can be well ordered* (*this is a weakening of* **Cntbl**).

(II) *For every set* $X$, *there exists a superset* $Y \supseteq X$ *containing all finite subsets of itself.*

*In particular,* **L** *satisfies the separation schema* **Sep** *for formulas of any type. Therefore, the class* (*or set*) $\mathbf{L} \cap \mathscr{P}(\omega)$ *satisfies* $\mathbf{Z}_2$.

Theorem 1 is the main result of this paper. We prove it in Sec. 7 based on results of Secs. 5 and 6 concerning constructibility in the context of **SST**.

## 2. SIMPSON'S SET THEORY

The theory $\mathbf{ZFC}^-$ is obtained from the Zermelo–Fraenkel set theory **ZFC** as follows:

(I) The power set axiom **PS** is removed (this removal is symbolized by the minus sign in the notation).

(II) Zermelo's usual axiom of choice **AC** (as in **ZFC**) is removed, because it is ineffective without **PS**, and replaced by the *well-ordering axiom* **WOA** postulating the possibility of well ordering any set.

(III) The separation schema **Sep** is retained, but the replacement schema **Repl** (which is very weak in the absence of **PS**) is replaced by the *collection* schema

$$\mathbf{Coll}\colon \forall\, X\, \bigl(\forall\, x \in X\, \exists\, y\, \Phi(x,y) \Longrightarrow \exists\, Y\, \forall\, x \in X\, \exists\, y \in Y\, \Phi(x,y)\bigr).$$

It is known that $\mathbf{Coll} + \mathbf{Sep} \Longrightarrow \mathbf{Repl}$.

For details on $\mathbf{ZFC}^-$, see [14] and [15]. For various equivalent formulations of the collection schema, see, e.g., [16], [17], and [18, Sec. 2].

- $\mathbf{ZF}^-$ is $\mathbf{ZFC}^-$ without the well-ordering axiom **WOA**.
- $\mathbf{Z}^-$ is $\mathbf{ZF}^-$ without the collection schema **Coll**. In other words, $\mathbf{Z}^-$ contains the axioms of extensionality, pairing $\{x, y\}$, sum or union $\bigcup X$, infinity, and foundation and the **Sep** schema (see, e.g., [19] for these axioms).

Finally, we define *Simpson's set theory* **SST** as $\mathbf{Z}^-$ augmented by the following four axioms **Prod**, **TrCov**, **Beta**, and **Cntbl**:

*Cartesian product axiom* **Prod**: for any $X$ and $Y$, the set $X \times Y$ exists.

*Transitive covering axiom* **TrCov**: for any $X$, there exists a transitive set $Y \supseteq X$.

*Axiom* **Beta**, or *Mostowski's collapse*: for any well-founded relation $A$ on the set

$$D = \mathbf{fld}A := \operatorname{dom} A \cup \operatorname{ran} A,$$

there exists a transitive set $X$ and a function $\boldsymbol{\mu} : D$ onto $X$ satisfying the condition

$$\boldsymbol{\mu}(d) = \{\boldsymbol{\mu}(j) \colon j\ A\ d\} \qquad \text{for all} \quad d \in D. \tag{$*$}$$

The function $\boldsymbol{\mu}$ and the set $X$ are uniquely determined by this condition. The function $\boldsymbol{\mu} = \boldsymbol{\mu}_A$ is called the *collapsing function* for $A$.

Recall that a binary relation $A$ on a set $D = \mathbf{fld}A$ is *well-founded* if any nonempty set $Y \subseteq D$ contains an element $y \in Y$ for which $\forall\, x \in Y\ \neg\,(x\ A\ y)$.

*Countability axiom* **Cntbl**: $\forall\, x\ \exists\, f$ ($f \colon x \to \omega$ 1–1 function), i.e., all sets are countable.

Clearly, $\mathbf{SST} \subseteq \mathbf{ZF}^- + \mathbf{Cntbl}$.

The definition of the axiom **Beta** can be found in [1, Definition VII.3.8] and historical remarks on it, at the end of Sec. VII.3 of [1]. The derivation of **Beta** from **Repl** is given, e.g., in [16, Theorem 6.15].

The theory **SST** in the setting presented above was considered in [7] under a different name and with some minor changes in its axioms.

Simpson himself approached an axiomatics equivalent to our **SST** in [1, VII.3.3 and VII.3.8] and earlier in [20] from positions related to formal systems in the language of second-order arithmetic and their reflection in set theory. Namely, he introduced the theory $\mathbf{ATR}_0^{\mathrm{set}}$ in the $\in$-language, which contained the following axioms:

(a) *Equality axioms*: $=$ is an equivalence relation and $\in$ is $=$-invariant.

(b) *Extensionality, infinity, and foundation* in their usual **ZF**-forms.

(c) *Rudimentary closedness axiom*, which states that, for any sets $u$ and $v$, the following sets exist:

$$\mathcal{F}_1(u,v) = \{u,v\}, \qquad \mathcal{F}_2(u,v) = u \setminus v, \qquad \mathcal{F}_3(u,v) = u \times v, \qquad \mathcal{F}_4(u,v) = \bigcup u,$$

$$\begin{aligned}
&\mathcal{F}_5(u,v) = {\in} \restriction u = \{\langle x,y\rangle \colon x,y \in u \wedge x \in y\},\\
&\mathcal{F}_6(u,v) = u^{-1} = \{\langle x,y\rangle \colon \langle y,x\rangle \in u\},\\
&\mathcal{F}_7(u,v) = \{\langle y,\langle x,z\rangle\rangle \colon \langle y,x\rangle \in v \wedge z \in u\},\\
&\mathcal{F}_8(u,v) = \{\langle y,\langle z,x\rangle\rangle \colon \langle y,x\rangle \in v \wedge z \in u\},\\
&\mathcal{F}_9(u,v) = \{z \colon \exists\, x\,(x \in u \wedge z = v''\{x\}\},
\end{aligned}$$

where $v''\{x\} = \operatorname{ran}(v \restriction \{x\})$; this list of Gödel operations somewhat differs from that given, e.g., in the book [16] (see Sec. 6 of the present paper).

(d) Finally, the axioms **TrCov**, **Beta**, and **Cntbl** defined as above; the axiom **Prod** is not needed, because it is replaced by the operation $\mathcal{F}_3$.

**Lemma 1.** *The inclusion* $\mathbf{ATR}_0^{\mathrm{set}} \subseteq \mathbf{SST}$ *holds, and hence the theory* **SST** *is identical to* $\mathbf{ATR}_0^{\mathrm{set}} + \mathbf{Sep}$.

**Proof.** Let $u$ and $v$ be arbitrary sets. According to Lemma 5 in Sec. 3, there exists a set $Y$ containing the sets $u$ and $v$ and all of their elements and closed with respect to the formation of finite subsets and hence of ordered pairs, triples, etc. Therefore, the application of each of the operations $\mathcal{F}_1$–$\mathcal{F}_8$ in (c) to $u$ and $v$ yields a part of $Y$ and hence is a set by virtue of **Sep**. As to $\mathcal{F}_9$, we have $\mathcal{F}_9(u,v) \subseteq \mathscr{P}(Y)$ under our assumptions, so that Lemma 4, (ii) of Sec. 3 gives the desired result for $D = u$ and $F(x) = v''\{x\}$. □

## 3. DEVELOPMENT OF SIMPSON'S THEORY

This section collects a few fairly simple results in **SST**. In particular, the axiom **TrCov** and the schema **Sep** imply the following result.

**Lemma 2 (SST).** *For any set $X$, its transitive closure* $\operatorname{TC} X = \bigcap\{Y \supseteq X\colon Y \text{ is transitive}\}$ *exists.*

**Lemma 3 (SST**, transitive collapse**).** *For any set $D$, there exists a unique transitive set $X$ and a unique collapsing function $\tau : D$ onto $X$ such that $\tau(x) = \{\tau(y)\colon y \in x \cap D\}$ for all $x \in D$.*

**Proof.** It suffices to apply the **Beta** axiom to the relation $A = \in \restriction D$, which is well-founded by the axiom of foundation. □

For brevity, we define a *class-function* as an arbitrary (definable) class satisfying the usual conditions in the definition of a function (consisting of ordered pairs and so on). In this terminology, the replacement schema **Repl** postulates in **ZF** that any class-function whose domain is a set is itself a set (of pairs). This schema is absent from **SST**, but its important special case, namely, replacement with a transitive domain, is valid.

**Lemma 4 (SST).** *Let $F$ be a class-function on a set $D$. Then $F$ and its range*

$$R = F''D = \{F(x)\colon x \in D\}$$

*are sets in the following two cases*:

(i) *$R$ is a transitive class.*

(ii) *There exists a set $Y$ for which $R \subseteq \mathscr{P}(Y)$.*

**Proof.** (i) The relation $A = \{\langle j, d\rangle\colon j, d \in D \wedge F(j) \in F(d)\}$ is a legitimate set in view of the axioms **Prod** and **Sep**. Next, $A$ is well-founded, being the preimage of the class-relation $\in \restriction R$, whose well-foundedness easily follows from the axiom of foundation. Hence there exists a transitive set $X$ and a function $\boldsymbol{\mu} : D$ onto $X$ satisfying condition $(*)$ in the definition of the axiom **Beta**. However, $F$ satisfies the same condition by the definition of $A$. It follows that $F = \boldsymbol{\mu}$ and $R = X$ is a set.

(ii) The axiom **TrCov** suggests that the set $Y$ is transitive. Moreover, we may assume that $D \cap Y = \varnothing$; otherwise, we set $D' = D \times \{Y\}$ and change $F$ accordingly. Under these assumptions, we extend $F$ to a function $F_1$ on $D_1 = D \cup Y$ by the identity function on $Y$. According to (i), the image $F_1''D_1 = R \cup Y$ is a transitive set. Finally, $R \subseteq F_1''D_1$ is a set by virtue of **Sep**. □

We say that a set $Y$ is *closed with respect to the formation of finite subsets* if $\forall\, z \subseteq Y$ ($z$ is finite $\Longrightarrow z \in Y$) or, for short, $\mathscr{P}_{\text{fin}}(Y) \subseteq Y$. Given any $X$, we introduce its *closure* $\operatorname{FSC}(X)$ as the least superset $Y \supseteq X$ closed with respect to the formation of finite subsets (if it exists).

**Lemma 5 (SST).** *The set* $\operatorname{FSC}(X)$ *exists for any set $X$.*

**Proof.** Suppose that $X = \omega$. Let $p_k$ denote the $k$th prime: $p_1 = 2$, $p_2 = 3$, and so on. According to **Prod** and **Sep**, $A = \{\langle k, n\rangle\colon k \geq 1 \wedge p_k \text{ divides } n\}$ is a set. Moreover, $\mathbf{fld}A = \omega \setminus \{0\}$, $A$ is well-founded (because $k\,A\,n \Longrightarrow k < n$), and

(†) If $u \subseteq \mathbf{fld}A$ is finite, then there is an $n \in \mathbf{fld}A$ for which $u = \{k\colon k\,A\,n\}$.

By axiom **Beta**, there exists a function $\boldsymbol{\mu} : \mathbf{fld}A$ onto a transitive set $R$ such that $\boldsymbol{\mu}(n) = \{\boldsymbol{\mu}(k)\colon k\,A\,n)\}$ for all $n \in \mathbf{fld}A$. Clearly, (†) implies $R = \operatorname{FSC}(\omega)$.

Now consider the general case. In view of **TrCov**, we assume $X$ to be transitive. According to **Cntbl**, there exists a function $h : \omega$ onto $X$. It extends to the class-function $H$ defined on the larger set $R = \operatorname{FSC}(\omega)$ by $H \restriction \omega = h$ and $H(u) = \{H(v)\colon v \in u\}$ for $u \in R \setminus \omega$. We have $\operatorname{ran} H = \operatorname{FSC}(X)$ (this is a class), and hence $\operatorname{ran} H$ and $X$ are transitive. However, both $H$ and $\operatorname{ran} H = \operatorname{FSC}(X)$ are sets by Lemma 4, (i). □

**Lemma 6 (SST).** *For any sets $U$ and $V$, $\mathscr{P}_{\mathrm{fin}}(U)$ and $U^{<\omega}$ exist (as sets), and so does $U\times V$ by the axiom* **Prod**.

**Proof.** Note that $X=U\cup V=\bigcup\{U,V\}$ is a set even in $\mathbf{Z}^-$, so that $\mathrm{FSC}(X)$ is a set by Lemma 5. Clearly, $\mathscr{P}_{\mathrm{fin}}(U),U^{<\omega}\subseteq\mathrm{FSC}(U)$. Now the desired result follows from **Sep**. □

Thus, **SST** proves the existence of the sets listed above. Note that $\mathbf{Z}^-$ is not strong enough even to derive the existence of $\omega\times\omega$!

**Lemma 7 (SST).** *Let $E$ be a strict well-ordering of a set $U$. Then there exists an ordinal $\lambda$ and an order isomorphism of $\langle U;E\rangle$ onto $\langle\lambda;\in\rangle$.*

**Proof.** The set $E$ is, in particular, a well-founded relation on $U$. The axiom **Beta** gives a transitive set $\lambda=X$ well-ordered by $\in$ (i.e., an ordinal) order isomorphic to $\langle U;E\rangle$. □

**Corollary 1 (SST).** *For any ordinals $\alpha$ and $\beta$, the ordinals $\alpha+\beta$, $\alpha\cdot\beta$, and $\alpha^\beta$ exist (as sets); here the operations are in the standard arithmetic of ordinals.*

**Proof.** It suffices to define well-ordered sets of the required order types. The Cartesian product $\alpha\times\beta$ (which is a set by Lemma 6) ordered lexicographically is of type $\alpha\cdot\beta$. The exponential $\alpha^\beta$ is represented by the set

$$W=\{f\colon D\to\alpha\smallsetminus\{0\}\colon D\subseteq\beta\text{ is finite}\}$$

with the lexicographic order understood in the sense that each $f\in D$ is extended by setting $f(\xi)=0$ for all $\xi\in\beta\smallsetminus D$. Note that $W\subseteq\mathrm{FSC}(\beta\times\alpha)$ is a set according to Lemma 5. □

## 4. METAMATHEMATICS OF SIMPSON'S SET THEORY

The choice of the **SST** axioms seems to be accidental and poorly justified beyond the purely technical needs of certain proofs. Our goal in this section is to show that this is not quite true and that, in fact, **SST** can be regarded as a kind of a set-theoretic "cognate" of the second-order arithmetic $\mathbf{Z}_2$ (we remind the reader that it does not include the axiom of choice). The main results are as follows.

1. The theory **SST** is a conservative extension of $\mathbf{Z}_2$ in the sense that a proposition $P$ in the language of $\mathbf{Z}_2$ is a theorem of $\mathbf{Z}_2$ if and only if the relativization $(P)^{\mathscr{P}(\omega)}$ is a theorem of **SST**.

2. There exists a canonical interpretation **wft** of **SST** in $\mathbf{Z}_2$ (see Definition 3) such that the structure $(\langle\omega;\mathscr{P}(\omega)\rangle)^{\mathbf{wft}}$ of this interpretation is isomorphic to a given universe of $\mathbf{Z}_2$.

3. Conversely, in a universe of **SST**, the interpretation $(\mathbf{wft})^{\langle\omega;\mathscr{P}(\omega)\rangle}$ defined in the structure $\langle\omega;\mathscr{P}(\omega)\rangle$ is isomorphic to this universe of **SST**.

4. Statements 2 and 3 can be summarized as follows: the interpretation **wft** of the theory **SST** in $\mathbf{Z}_2$ and the trivial interpretation $\langle\omega;\mathscr{P}(\omega)\rangle$ of the theory $\mathbf{Z}_2$ in **SST** are mutually inverse.

These results can be extracted from calculations in [1, VII.3], but in fact, they have been known since the 1960s. To deduce them, consider the collection WFT of all nonempty well-founded trees $T\subseteq\mathrm{Seq}=\omega^{<\omega}$. Recall that

- $\mathrm{Seq}=\omega^{<\omega}=$ all tuples of positive integers.
- $\langle\rangle\in\mathrm{Seq}$ is the empty tuple and $\langle k\rangle$ is the tuple whose only member is $k$.
- $s^\frown j$ is obtained by adding the term $j\in\omega$ to $s\in\mathrm{Seq}$ on the right, and if $s,t\in\mathrm{Seq}$, then $s^\frown t\in\mathrm{Seq}$ denotes the *concatenation* of $s$ and $t$.
- $T\subseteq\mathrm{Seq}$ is a *tree* if $s^\frown j\in T\Longrightarrow s\in T$.

- $\operatorname{Max}(T)=\{s\in T\colon \neg\,\exists\, j<\omega\,(s^\frown j\in T)\}$ (this is the set of *terminal vertices* of the tree $T$).
- A tree $T\subseteq \operatorname{Seq}$ is *well-founded*, or $T\in \mathrm{WFT}$, if $\neg\,\exists\, g\colon \omega\to\omega\,\forall\, m\,(g\restriction m\in T)$.
- If $T$ is a tree and $s\in T$, then $T^s=\{t\in \operatorname{Seq}\colon s^\frown t\in T\}$ is a tree as well, and if $T$ is well-founded, then so is $T^s$.

**Definition 1 (SST).** Let $T\in\mathrm{WFT}$. Then there exists a unique function $\mathbf{set}_T$ defined on $T$ and satisfying the condition $\mathbf{set}_T(u)=\{\mathbf{set}_T(u^\frown j)\colon u^\frown j\in T\}$ for all $u\in T$; in particular, $\mathbf{set}_T(u)=\varnothing$ for $u\in\operatorname{Max}(T)$. We set $\mathbf{Set}_T=\mathbf{set}_T(\langle\rangle)$.

A drawback of the second-order arithmetic $\mathbf{Z}_2$ and it versions is that tuples of positive integers, trees, functions $\operatorname{Seq}\to\operatorname{Seq}$, and similar objects are not formally included in the scope of study. This can be remedied by a suitable *encoding*; see Chap. I of the book [1] and especially Sec. II.2 of that book, in which a series of examples are given.

To encode tuples, we set $s_0=\langle\rangle$ (the empty tuple) and $s_n=s_m^\frown j$ for $n=2^m(2j+1)-1\geq 1$. Then $\operatorname{Seq}=\{s_n\colon n<\omega\}$ and $n\mapsto s_n$ is a bijection. Accordingly, $n=n(s)$ serves as the *code* of the tuple $s=s_n\in\operatorname{Seq}$ and every set $x\subseteq\omega$ serves as the *code* of the set of tuples $\{s_n\colon n\in x\}\subseteq\operatorname{Seq}$. Following Simpson [1, Sec. II.2], we can freely consider tuples and sets of tuples, remaining in the framework of the theory $\mathbf{Z}_2$. This also refers to $X\subseteq\omega\times\omega$, $H\subseteq\operatorname{Seq}\times\operatorname{Seq}$, and similar sets.

With this caveat, remaining in the framework of $\mathbf{Z}_2$, we can consider tuples and trees, give an adequate $\mathbf{Z}_2$-definition of well-foundedness, and even determine whether the relations $\mathbf{Set}_S=\mathbf{Set}_T$ and $\mathbf{Set}_S\in\mathbf{Set}_T$ hold for given $S,T\in\mathrm{WFT}$ without defining the sets $\mathbf{Set}_S$ and $\mathbf{Set}_T$ themselves. We give the following definition in $\mathbf{Z}_2$.

**Definition 2** (see [21, 7.8] or [1, VII.3.13]). Let $S,T\in\mathrm{WFT}$. A set $H\subseteq S\times T$ is called an *$S,T$-bisimulation* if, for all $s\in S$ and $t\in T$,

$$s\,H\,t\quad\Longleftrightarrow\quad\begin{cases}\quad\forall\, s'=s^\frown j\in S\;\exists\, t'=t^\frown k\in T\;(s'\,H\,t')\;\wedge\\ \wedge\;\forall\, t'=t^\frown k\in T\;\exists\, s'=s^\frown j\in S\;(s'\,H\,t').\end{cases}$$

We write $S=^*T$ if there exists an $S,T$-bisimulation $H$ for which $\langle\rangle\,H\,\langle\rangle$.

We write $S\in^*T$ if $S=^*T^{\langle k\rangle}$ for some $k\in\omega$.

**Definition 3.** The *structure* $\mathbf{wft}=\langle\mathrm{WFT};=^*,\in^*\rangle$ is considered in $\mathbf{Z}_2$.

The **wft**-*interpretation* $\lceil\Phi\rceil^{\mathbf{wft}}$ of any $\in$-formula $\Phi$ (with parameters in WFT) is naturally defined by interpreting $=$ and $\in$ as $=^*$ and $\in^*$, respectively, and relativizing all quantifiers to WFT. For example, $\lceil x=y\rceil^{\mathbf{wft}}$ is $x=^*y$.

**Exercise 1 (SST).** Let $S,T\in\mathrm{WFT}$. Prove that $\mathbf{Set}_S=\mathbf{Set}_T$ if and only if $S=^*T$ in the $\mathbf{Z}_2$-structure $\langle\omega;\mathscr{P}(\omega)\rangle$. The same is true for $\in$ and $\in^*$.

The bisimulation relation $=^*$ between trees in WFT and, accordingly, the derived relation $\in^*$ is naturally formalized in $\mathbf{Z}_2$ by using the encoding described above. It follows that the **wft**-interpretation $\lceil\Phi\rceil^{\mathbf{wft}}$ of any $\in$-formula $\Phi$ with parameters in WFT is a formula in the language $\mathbf{Z}_2$.

**Theorem 2 ($\mathbf{Z}_2$).** (i) *The structure* **wft** *is well defined*; *i.e.*, $=^*$ *is an equivalence relation on* WFT *and the binary relation* $\in^*$ *on* WFT *is* $=^*$*-invariant.*

(ii) *The structure* **wft** *satisfies the* **SST** *axioms*: *if* $\Phi$ *is an axiom in* **SST**, *then* $\lceil\Phi\rceil^{\mathbf{wft}}$ *is a theorem of* $\mathbf{Z}_2$.

(iii) *There is a canonical isomorphism between the substructure* $\langle\omega;\mathscr{P}(\omega)\rangle^{\mathbf{wft}}$ *of the interpretation and the proper structure* $\langle\omega;\mathscr{P}(\omega)\rangle$ *of the universe of* $\mathbf{Z}_2$.

Results close to certain parts of this theorem were already known to Kreisel [6]; such results can also be found in [2], [4], [3, Theorem 1.1 and Corollary 1.1], and other papers.

**Proof** (sketch). The theorem was essentially proved in [1, Sec. VII.3] (and, as mentioned, parts of it were presented in the papers cited above). Namely, if the role of the base theory of second-order arithmetic is played by the subtheory $\mathbf{ATR}_0 \subseteq \mathbf{Z}_2$ rather than by the whole theory $\mathbf{Z}_2$, then it follows from Lemma VII.3.20 of [1] that if $\Phi$ is an axiom of $\mathbf{ATR}_0^{\mathrm{set}}$, then $\ulcorner\Phi\urcorner^{\mathbf{wft}}$ is a theorem of $\mathbf{ATR}_0$ (in which case it is also a theorem of $\mathbf{Z}_2$). Thus, to prove (ii), it suffices to check that **Sep** holds in **wft**.

Arguing in $\mathbf{Z}_2$, we suppose that $S \in \mathrm{WFT}$, $X = \{k\colon \langle k\rangle \in S\}$, and $\Phi(x)$ is a $\in$-formula with parameters in WFT and one free variable $x$. By definition, trees of the form $S^k = \{t \in \mathrm{Seq}\colon k^\frown t \in S\}$, $k \in X$, belong to **wft** and exhaust (up to $=^*$) all $\in^*$-elements $S$ in **wft**. Using the collapse schema in $\mathbf{Z}_2$, we put $Y = \{k \in X\colon \ulcorner\Phi(S^{\langle k\rangle})\urcorner^{\mathbf{wft}}\}$. The set $T = \{\langle\rangle\} \cup \{t \in S\colon t(0) \in X\}$ is a tree in WFT. It remains to check that $\ulcorner T = \{x \in S\colon \Phi(x)\}\urcorner^{\mathbf{wft}}$.

Suppose that $C \in \mathrm{WFT}$, $C \in^* S$, and $\ulcorner\Phi(C)\urcorner^{\mathbf{wft}}$ holds. Then $C =^* S^{\langle k\rangle}$ for some $k \in X$, so that $\ulcorner\Phi(S^{\langle k\rangle})\urcorner^{\mathbf{wft}}$ holds and hence $k \in Y$. This implies $C =^* T^{\langle k\rangle} = S^{\langle k\rangle} \in^* T$. The proof of the reverse implication is similar. □

**Corollary 2** (of Theorem 2). *Statements* 1–3 *at the beginning of this section are valid. Therefore, the theories* $\mathbf{Z}_2$, $\mathbf{Z}^-$, *and* **SST** *are mutually interpretable and hence equiconsistent.*

**Exercise 2.** We omitted some details of the proofs of Theorem 2 and Corollary 2, because these results are not related to the proof of the main Theorem 1 of this paper. However, it would be a good *exercise* for the reader to reconstruct the complete proof by following arguments in the papers [1]–[4], [6] cited above and to deduce statements 1–3 from the theorem.

**Remark 1.** Let $\mathbf{AC}_\omega$ be the countable axiom of choice in $\mathbf{Z}_2$, and let $\mathbf{Z}_2^+ := \mathbf{Z}_2 + \mathbf{AC}_\omega$. Under the replacement of $\mathbf{Z}_2$ by $\mathbf{Z}_2^+$ in the theorem, assertion (ii) takes the form

$$\text{if } \Phi \text{ is an axiom of } \mathbf{ZFC}^-, \text{ then } \ulcorner\Phi\urcorner^{\mathbf{wft}} \text{ is a theorem of } \mathbf{Z}_2^+.$$

This version is even better known and better reflected in the literature; see [3].

## 5. CONSTRUCTIBLE SETS IN SIMPSON'S THEORY

In this section, we give the key definitions and some important results of constructibility theory as presented in [1, Sec. VII.4]. Simpson proved these results in the theory $\mathbf{ATR}_0^{\mathrm{set}}$ and some of its extensions inside **SST** obtained by adding certain subshemata of **Sep**. Hence these results are true in our base set theory **SST**.

Note that, in this paper, we consider only the special case $\mathbf{L} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha$ of the more general *relative* constructible hierarchy $\mathbf{L}[u] = \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha[u]$ used in [1] ($\mathbf{L}$ corresponds to $u = \varnothing$). Therefore, in what follows, we reduce all definitions and results of [1, Chap. VII] to this case.

**Lemma 8** (**SST**; see [1, Lemma VII.4.1]). *For any transitive set* $X$, *there exists a* (*unique*) *set* $\mathbf{Def}\,X$ *consisting of all sets* $Y \subseteq X$ *definable over* $X$ *by* $\in$*-formulas with parameters in* $X$. *This set* $\mathbf{Def}\,X$ *is transitive and satisfies the condition* $X \cup \{X\} \subseteq \mathbf{Def}\,X$.

The proof of Lemma 8 given in [1] reduces to the deduction of the existence of an *evaluation function* for all $\in$-formulas with parameters in the given transitive set $X$. Another approach to define $\mathbf{Def}\,X$ (which is due to Gödel, as well as the first one) is eliminating $\in$-definability by means of Gödel operations; see Lemma 11 of the present paper.

Lemma 8 gives the following result.

**Lemma 9** (**SST**; see [1, VII.4.2]). *For any* $\beta \in \mathrm{Ord}$, *there exists a unique function* $f = \mathbf{f}_\beta$ *such that* $\operatorname{dom} f = \beta$, $f(0) = \varnothing$, $f(\alpha+1) = \mathbf{Def}\,f(\alpha)$ *for* $\alpha + 1 < \beta$, *and* $f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha)$ *for limit ordinals* $\lambda < \beta$.

The function $\mathbf{f}_\beta$ is called the *constructing function of height* $\beta$.

**Definition 4** (**SST**). We set $\mathbf{L}_\alpha = \mathbf{f}_{\alpha+1}(\alpha)$ for all $\alpha \in \mathrm{Ord}$.

Thus, the theory **SST** (in fact, even $\mathbf{ATR}_0^{\mathrm{set}}$) is strong enough to construct the standard constructible hierarchy:

$$\begin{gathered}\mathbf{L}_0=\varnothing,\\ \mathbf{L}_{\alpha+1}=\mathbf{Def}\,\mathbf{L}_\alpha\quad\text{for all }\ \alpha,\\ \mathbf{L}_\lambda=\bigcup_{\alpha<\lambda}\mathbf{L}_\alpha\quad\text{for all limit }\ \lambda,\\ \mathbf{L}=\bigcup_{\alpha\in\mathrm{Ord}}\mathbf{L}_\alpha=\text{all constructible sets.}\end{gathered}$$

**Theorem 3 (SST).** *The following assertions hold*:

(i) *Each* $\mathbf{L}_\alpha$ *is a transitive set and* $\alpha\subseteq\mathbf{L}_\alpha$.

(ii) *If* $\alpha<\beta$, *then* $\mathbf{L}_\alpha\in\mathbf{L}_\beta$ *and* $\mathbf{L}_\alpha\subseteq\mathbf{L}_\beta$.

(iii) *If* $\lambda$ *is limit, then* $\mathbf{L}_\lambda$ *is closed with respect to the operations listed in* (c) *in Sec.* 2.

(iv) *If* $\beta\in\mathrm{Ord}$, *then* $\mathbf{f}_\beta\in\mathbf{L}_{\alpha+3}$.

(v) *The class-function* $\alpha\mapsto\mathbf{L}_\alpha$ $(\alpha\in\mathrm{Ord})$ *is definable over* $\mathbf{L}$ *without parameters*.

(vi) *There exists a well-ordering* $<_{\mathbf{L}}$ *of* $\mathbf{L}$ *definable over* $\mathbf{L}$ *without parameters*.

(vii) *The class* $\mathbf{L}$ *satisfies all* **SST** *axioms except possibly* **Sep** *and* **Cntbl**.

**Proof.** Proofs of assertions (i)–(iii) can be found in [1, Theorem VII.4.3]. For proofs of (iv) and (v), see Theorem VII.4.8 and its proof given in [1], or, e.g., Lemma 4.1 in [22, Sec. B.5], or the proof of Lemma 2.6(ii) in [18]. Concerning (vi), see [1, Lemma VII.4.21]. For (vii), see [1, Corollary VII.4.15]. The most difficult part is the verification that the axiom **Beta** holds in $\mathbf{L}$ (see Theorem VII.4.12 of [1]); it is based on Lemma VII.4.10, according to which the collapse function $\boldsymbol{\mu}_A$ (see the definition of the axiom **Beta** in Sec. 2 above) of any well-founded relation $A\in\mathbf{L}$ belongs to $\mathbf{L}$. □

The truth of **Sep** in $\mathbf{L}$ is proved in Sec. 7. On the other hand, for the **Repl**/**Coll** schemata and the axiom **Cntbl**, there is the following counterexample.

**Example 1** (see [1, VII.4.17]). Arguing in **ZF**, we set $\mathfrak{M}=\mathbf{L}_\vartheta$, where $\vartheta=(\aleph_\omega)^{\mathbf{L}}$. Consider the generic extension $\mathfrak{N}$ of the model $\mathfrak{M}$ by means of a generic sequence of functions $f_n:\omega$ onto $(\aleph_n)^{\mathbf{L}}$. The extension $\mathfrak{N}$ is a model of **SST**; on the other hand, $(\mathbf{L})^{\mathfrak{N}}=\mathfrak{M}$, and the schemata **Repl**/**Coll** and the axiom **Cntbl** are certainly violated in $\mathfrak{M}=\mathbf{L}_\vartheta$.

## 6. CONSTRUCTIBILITY AND THE GÖDEL OPERATIONS

In this section, we prove the following theorem, which plays the key role in the proof of Theorem 1 in Sec. 7.

**Theorem 4.** *There exists a closed parameter-free* $\in$*-formula* $\boldsymbol{\sigma}$ *such that* **SST** *proves the following statements*:

(i) $\boldsymbol{\sigma}$ *is true in* $\mathbf{L}$.

(ii) *If a set* $\boldsymbol{K}$ *is transitive and* $\boldsymbol{K}\models\boldsymbol{\sigma}$, *then* $\boldsymbol{K}=\mathbf{L}_\lambda$, *where* $\lambda=\boldsymbol{K}\cap\mathrm{Ord}$.

Apparently, the first similar result was obtained in [23]. Another similar result was mentioned in the book [16] (see (13.13) before Lemma 13.17 in Chap. 13).

We begin the proof of the theorem with the list of *Gödel operations* as given in [16, Chap. 13]:

$$\begin{aligned}
\mathcal{G}_1(u,v) &= \{u,v\},\\
\mathcal{G}_2(u,v) &= u\times v,\\
\mathcal{G}_3(u,v) &= {\in}\restriction u = \{\langle x,y\rangle\colon x,y\in u\wedge x\in y\},\\
\mathcal{G}_4(u,v) &= u\smallsetminus v,\\
\mathcal{G}_5(u,v) &= u\cap v \quad \text{reduces to } \mathcal{G}_4, \text{ because } u\cap v = u\smallsetminus(u\smallsetminus v),\\
\mathcal{G}_6(u,v) &= \bigcup u,\\
\mathcal{G}_7(u,v) &= \operatorname{dom} u = \{x\colon \exists\, y\,(\langle x,y\rangle\in u\},\\
\mathcal{G}_8(u,v) &= u^{-1} = \{\langle x,y\rangle\colon \langle y,x\rangle\in u\},\\
\mathcal{G}_9(u,v) &= \{\langle x,y,z\rangle\colon \langle x,z,y\rangle\in u\},\\
\mathcal{G}_{10}(u,v) &= \{\langle x,y,z\rangle\colon \langle y,z,x\rangle\in u\}.
\end{aligned}$$

These operations can be compared with Simpson's *rudimentary operations* $\mathcal{F}_1$–$\mathcal{F}_9$ listed in Sec. 2. In particular, the operations $\mathcal{F}_1$–$\mathcal{F}_6$ are identical to $\mathcal{G}_1$, $\mathcal{G}_4$, $\mathcal{G}_2$, $\mathcal{G}_6$, $\mathcal{G}_3$, and $\mathcal{G}_8$, respectively. There are certain relationships between other operations as well.

**Lemma 10 (SST).** *For any sets $u$ and $v$ and any $i=1,2,\dots,10$, $\mathcal{G}_i(u,v)$ is a set.*

**Proof.** The required assertion for $\mathcal{G}_1$ follows from the pairing axiom, for $\mathcal{G}_3$, from Lemma 6, for $\mathcal{G}_3$, from the same lemma (for $u\times u$) and the separation schema, for $\mathcal{G}_{4,5}$, from the separation schema, and for $\mathcal{G}_6$, from Lemma 2 and the separation schema (because $\bigcup u\subseteq \operatorname{TC} u$). For the operations $\mathcal{F}_7$–$\mathcal{F}_{10}$, we have $\mathcal{G}_i(u,v)\subseteq \operatorname{FSC}(\operatorname{TC} u)$, so that the required result follows from Lemmas 2 and 5 and the schema **Sep**. □

An important application of the Gödel operations to constructibility theory is the possibility of eliminating the definition of **Def** by means of the following lemma, whose original proof (in **ZF**) is easy to formalize in **SST**.

**Lemma 11** (see [16, Corollary 13.8]). *If $X$ is a transitive set, then $\mathbf{Def}X=\mathbf{Def}^*X$, where $\mathbf{Def}^*X=\mathbf{cl}(X\cup\{X\})\cap\mathscr{P}(X)$ and $\mathbf{cl}(Y)$ denotes the closure of $Y$ with respect to the operations $\mathcal{G}_1$–$\mathcal{G}_{10}$.*

An important role is also played by *absoluteness* issues related to the Gödel operations; in discussing them, we will follow the book [19, Sec. 11]. Let $\boldsymbol{K}$ be a transitive set or class. Consider the following requirements (A)–(G) on $\boldsymbol{K}$, which imply useful absoluteness properties (see Lemma 12).

(A) The class $\boldsymbol{K}$ is closed with respect to the operations $\mathcal{G}_i$, $i=1,\dots,10$.

(B) The class $\boldsymbol{K}$ is closed with respect to the operations

$$\mathcal{G}_i''(X) := \{\mathcal{G}_i(u,v)\colon u,v\in X\},\qquad i=1,\dots,10,$$

and the operation

$$\mathcal{G}^*(X) = \bigcup_{1\le i\le 10}\mathcal{G}_i''X.$$

(C) The condition $\omega\in\boldsymbol{K}$ holds.

Recall that the *Gödel closure function* $\mathbf{gcl}_X$ for a set $X$ is defined by $\operatorname{dom}(\mathbf{gcl}_X)=\omega$, $\mathbf{gcl}_X(0)=X$, and

$$\mathbf{gcl}_X(n+1)=\mathcal{G}^*(\mathbf{gcl}_X(n))=\bigcup_{1\le i\le 10}\{\mathcal{G}_i(u,v)\colon u,v\in\mathbf{gcl}_X(n)\}.$$

Clearly, $\mathbf{Def}^*X=\bigcup_n\mathbf{gcl}_X(n)$.

(D) If $X\in\boldsymbol{K}$, then the closure function $\mathbf{gcl}_X$ belongs to $\boldsymbol{K}$.

(E) If $X\in\boldsymbol{K}$, then $\mathbf{Def}^*X\in\boldsymbol{K}$.

In addition to the constructing sequence $\mathbf{f}_\beta$ of length $\beta$ (see Lemma 9), we define the *modified constructing sequence* (MCS) of length $\beta\in\operatorname{Ord}$ as the function $\mathfrak{U}^*_\beta$ defined by $\operatorname{dom}(\mathfrak{U}^*_\beta)=\beta$, $\mathfrak{U}^*_\beta(0)=\varnothing$, $\mathfrak{U}^*_\beta(\lambda)=\bigcup_{\alpha<\lambda}\mathfrak{U}^*_\beta(\alpha)$ if $\lambda<\beta$ is limit, and

$$\mathfrak{U}^*_\beta(\alpha+1)=\mathbf{Def}^*(\mathfrak{U}^*_\beta(\alpha))$$

if $\alpha+1<\beta$. Accordingly, we set $\mathbf{L}^*_\alpha=\mathfrak{U}^*_{\alpha+1}(\alpha)$. Then $\mathfrak{U}^*_\beta=\mathbf{f}_\beta$ and $\mathbf{L}^*_\alpha=\mathbf{L}_\alpha$ in **SST** by Lemma 11; however, we do not assert that these relations are preserved under relativization to any transitive set. The last two properties are as follows.

(F) If $\alpha\in\operatorname{Ord}\cap\boldsymbol{K}$, then $\mathfrak{U}^*_{\alpha+1}\in\boldsymbol{K}$.

(G) If $x\in\boldsymbol{K}$, then $\exists\,\alpha\in\operatorname{Ord}\cap\boldsymbol{K}$ $(x\in\mathbf{L}^*_\alpha)$.

**Lemma 12** (see [19, Sec. 11]). *Let $\boldsymbol{K}$ be a transitive set or class; then*

(1) *The formulas $x=\mathcal{G}_i(u,v)$, $1\le i\le 10$, are absolute for $\boldsymbol{K}$.*

(2) *If* (A) *holds, then the formulas $Y=\mathcal{G}_i''(X)$, $1\le i\le 10$, and $Y=\mathcal{G}^*(X)$ are absolute for $\boldsymbol{K}$.*

(3) *If* (A)–(C) *hold, then the formula $C=\mathbf{gcl}_X$ is absolute for $\boldsymbol{K}$.*

(4) *If* (A)–(D) *hold, then the formula $Y=\mathbf{Def}^*X$ is absolute for $\boldsymbol{K}$.*

(5) *If* (A)–(E) *hold, then the formula $f=\mathfrak{U}^*_{\alpha+1}$ is absolute for $\boldsymbol{K}$.*

(6) *If* (A)–(F) *hold, then the formulas $X=\mathbf{L}^*_\alpha$ and $x\in\mathbf{L}^*_\alpha$ are absolute for $\boldsymbol{K}$.*

**Proof.** See elementary calculations in [19, Sec. 11]. In particular, (1) follows from the fact that the formulas in question can be written as bounded $\in$-formulas and are therefore absolute for any transitive domains (sets or classes). To prove (2), we note that the formula $Y=\mathcal{G}_i''(X)$ can be written as

$$\forall\,y\in Y\,\exists\,u,v\in X\,\bigl(y=\mathcal{G}_i(u,v)\bigr)\ \wedge\ \forall\,u,v\in X\,\exists\,y\in Y\,\bigl(y=\mathcal{G}_i(u,v)\bigr),$$

so that its absoluteness follows from assumption (A) and assertion (1) proved above.

And so on. □

**Lemma 13 (SST).** *The class $\mathbf{L}$ satisfies conditions* (A)–(G). *In other words, $\omega\in\mathbf{L}$ and the corresponding functions, namely,*

$$\mathcal{G}_i,\quad \mathcal{G}_i'',\quad \mathcal{G}^*,\quad X\mapsto\mathbf{gcl}_X,\quad X\mapsto\mathbf{Def}^*X,\quad \beta\mapsto\mathfrak{U}^*_\beta,\quad \alpha\mapsto\mathbf{L}^*_\alpha,$$

*and*

$$x\mapsto\alpha_x:=\text{ the least ordinal }\alpha\text{ for which }x\in\mathbf{L}^*_\alpha,$$

*take values in $\mathbf{L}$ at arguments in $\mathbf{L}$.*

**Proof.** This is again a well-known fundamental result of constructibility theory. Therefore, we only outline its proof.

Condition (A) is proved by a direct verification. Consider, for example, $\mathcal{G}_6$. Let $u, v \in \mathbf{L}_\alpha$. Then $\mathcal{G}_6(u,v) = \bigcup u \subseteq \mathbf{L}_\alpha$ by the transitivity of $\mathbf{L}_\alpha$, and hence

$$\bigcup u = \{z \in \mathbf{L}_\alpha \colon \mathbf{L}_\alpha \models \exists\, y \in u\, (z \in y)\} \in \mathbf{Def}\, \mathbf{L}_\alpha = \mathbf{L}_{\alpha+1} \subseteq \mathbf{L}.$$

In the case of $\mathcal{G}_8(u,v) = u^{-1}$, note that if $\langle x, y\rangle \in u \in \mathbf{L}_\alpha$, then $x, y \in \mathbf{L}_\alpha$ by transitivity and hence the sets $\{y\}$ and $\{y, x\}$ belong to $\mathbf{L}_{\alpha+1}$, $\langle y, x\rangle = \{\{y\}, \{y, x\}\} \in \mathbf{L}_{\alpha+2}$, and, finally, $u^{-1} \in \mathbf{L}_{\alpha+3}$. The remaining operations $\mathcal{G}_i$ are considered in a similar way.

In essence, if $u, v \in X \in \mathbf{L}_\alpha$ and $i \leq 10$, then $\mathcal{G}_i(u,v) \in \mathbf{L}_{\alpha+5}$ and hence $\mathcal{G}_i''(X)$ and $\mathcal{G}^*(X)$ belong to $\mathbf{L}_{\alpha+6}$, which implies (B).

Next, $\omega = \{k \in \mathbf{L}_\omega \colon k \text{ is an ordinal}\} \in \mathbf{L}_{\omega+1}$, which implies (C).

To prove (D), which says that $\mathbf{gcl}_X \in \mathbf{L}$ for $X \in \mathbf{L}$, take $X \in \mathbf{L}_\alpha$; then $X \subseteq \mathbf{L}_\alpha$, because $\mathbf{L}_\alpha$ is transitive. Therefore, $\mathbf{gcl}_X(1) = \mathcal{G}^*(X) \in \mathbf{L}_{\alpha+6}$ (see above). Arguing by induction, we see that $\mathbf{gcl}_X(n) \in \mathbf{L}_{\alpha+6n}$ for all $n$; in particular, $\mathbf{gcl}_X(n) \in \mathbf{L}_{\alpha+\omega}$ for all $n$. The formula $Y = \mathcal{G}^*(X)$ is absolute for $\mathbf{L}_{\alpha+\omega}$ by Lemma 12. It follows that the function $\mathbf{gcl}_X$ treated as a set of pairs belongs to $\mathbf{L}_{\alpha+\omega}$ and is $\in$-definable in $\mathbf{L}_{\alpha+\omega}$. Thus, $\mathbf{gcl}_X \in \mathbf{L}_{\alpha+\omega+1}$, as required.

The rest of the proof is similar and follows the practice of dealing with the constructible hierarchy in **ZF**. □

**Corollary 3 (SST).** *The $\forall\exists$-closures of all formulas in Lemma 12, i.e.,*

$$\forall\, u, v\, \exists\, x\, (x = \mathcal{G}_i(u,v)), \qquad \forall\, X\, \exists\, Y\, (Y = \mathcal{G}_i''(X)), \qquad \forall\, X\, \exists\, Y\, (Y = \mathcal{G}^*(X)),$$
$$\forall\, X\, \exists\, C\, (C = \mathbf{gcl}_X), \qquad \forall\, X\, \exists\, Y\, (Y = \mathbf{Def}^* X), \qquad \forall\, \alpha \in \mathrm{Ord}\, \exists\, f\, (f = \mathcal{U}^*_{\alpha+1}),$$
$$\forall\, \alpha \in \mathrm{Ord}\, \exists\, X\, (X = \mathbf{L}^*_\alpha), \qquad \forall\, x\, \exists\, \alpha \in \mathrm{Ord}\, (x \in \mathbf{L}^*_\alpha),$$

*are true in* **L**.

**Proof.** Consider, for example, the first formula. Let $u, v \in \mathbf{L}$. Then $x = \mathcal{G}_i(u,v) \in \mathbf{L}$ by Lemma 13. But the formula $x = \mathcal{G}_i(u,v)$ is absolute for **L** by Lemma 12. Therefore, the proposition $x = \mathcal{G}_i(u,v)$ is true in **L**.

Consider the second formula. Let $X \in \mathbf{L}$. Then $Y = \mathcal{G}_i''(X) \in \mathbf{L}$ by Lemma 13. But the formula $Y = \mathcal{G}_i''(X)$ is absolute for **L** by Lemma 12. Therefore, the proposition $Y = \mathcal{G}_i''(X)$ is true in **L**.

And so on for all formulas in the list. □

**Lemma 14 (SST).** *Let $\boldsymbol{K}$ be a transitive set in which the eight formulas in Corollary 3 are true, and let $\omega \in \boldsymbol{K}$. Then $\boldsymbol{K} = \mathbf{L}_\lambda$, where $\lambda = \mathrm{Ord} \cap \boldsymbol{K}$.*

**Proof.** Since $\boldsymbol{K} \models \forall\, u, v\, \exists\, x\, (x = \mathcal{G}_i(u,v))$, and, by Lemma 12, (1), the formula $x = \mathcal{G}_i(u,v)$ is absolute for $\boldsymbol{K}$, it follows at once that $\boldsymbol{K}$ has property (A).

Similarly, since $\forall\, X\, \exists\, Y\, (Y = \mathcal{G}_i''(X))$ and $\forall\, X\, \exists\, Y\, (Y = \mathcal{G}^*(X))$ in $\boldsymbol{K}$ and the formulas $Y = \mathcal{G}_i''(X)$ and $Y = \mathcal{G}^*(X)$ are absolute for $\boldsymbol{K}$ by Lemma 12, (2) (note that (A) is already proved), it follows that $\boldsymbol{K}$ has property (B).

The class $\boldsymbol{K}$ has property (C), i.e., $\omega \in \boldsymbol{K}$, by assumption.

Let us check (D), which says that $\mathbf{gcl}_X \in \boldsymbol{K}$ whenever $X \in \boldsymbol{K}$. Since the formula $\forall\, X\, \exists\, C\, (C = \mathbf{gcl}_X)$ of Corollary 3 is true in $\boldsymbol{K}$, we see that there exists a set $C \in \boldsymbol{K}$ such that $\boldsymbol{K} \models C = \mathbf{gcl}_X$. However, the formula $C = \mathbf{gcl}_X$ is absolute for $\boldsymbol{K}$ by Lemma 12 (note that (A)–(C) are already proved). Thus, we indeed have $C = \mathbf{gcl}_X$, and hence $C = \mathbf{gcl}_X \in \boldsymbol{K}$, as required.

Continuing in this manner step by step, we check that the given set $\boldsymbol{K}$ has properties (A)–(G), using Lemma 12 at each step. The last property (G) implies $\boldsymbol{K} = \bigcup_{\alpha \in \mathrm{Ord} \cap \boldsymbol{K}} \mathbf{L}^*_\alpha = \bigcup_{\alpha \in \mathrm{Ord} \cap \boldsymbol{K}} \mathbf{L}_\alpha = \mathbf{L}_\lambda$, where $\lambda = \mathrm{Ord} \cap \boldsymbol{K}$ is the least ordinal not belonging to the (transitive) set $\boldsymbol{K}$. □

**Proof of Theorem 4.** The sought formula is the conjunction of the eight formulas of Corollary 3 and the formula $\exists\, x\, (x = \omega)$. Corollary 3 and Lemma 14 prove the theorem. □

## 7. THE SEPARATION SCHEMA IN THE CONSTRUCTIBLE UNIVERSE OF SIMPSON'S THEORY

**Proof of Theorem 1.** We argue in **SST**. Thanks to assertion (vii) of Theorem 3, it suffices to check that, in $\mathbf{L}$, the separation schema **Sep** and statements (I) and (II) of Theorem 1 are true.

Suppose that, on the contrary, **Sep** is *violated* in $\mathbf{L}$, i.e., there exists a transitive set $B \in \mathbf{L}$, say, $B = \mathbf{L}_\alpha$ for some $\alpha$, and a formula $\varphi(p, x)$ with a parameter $p \in \mathbf{L}$ for which the set

$$Y = \{b \in B \colon \varphi^{\mathbf{L}}(p, b)\}$$

does not belong to $\mathbf{L}$. (Note that $Y$ is a set in the base universe of the theory **SST**, because this theory includes **Sep**.) Using the well-ordering $<_{\mathbf{L}}$ provided by Theorem 3, (vi), we take the $<_{\mathbf{L}}$-least pair of sets $B, p \in \mathbf{L}$ with the same properties as above. This reduces the general case to the case where

(I) $\boldsymbol{\varphi}(\cdot)$ is a parameter-free formula not containing $\forall$ (which is replaced by $\neg\,\exists\,\neg$) and

$$Y = \{b \in B \colon \boldsymbol{\varphi}^{\mathbf{L}}(b)\} \notin \mathbf{L}. \tag{‡}$$

Together with $\boldsymbol{\varphi}$, we consider

(II) The formula $\boldsymbol{\sigma}$ in Theorem 4 written without $\forall$.

If $\exists\, y\, \psi\,(y, x_1, \dots, x_k)$ is an existential subformula of one of the formulas $\boldsymbol{\varphi}$ and $\boldsymbol{\sigma}$ and $x_1, \dots, x_k \in \mathbf{L}$, then we set $S_\psi(x_1, \dots, x_k)$ equal to the $<_{\mathbf{L}}$-least element $y \in \mathbf{L}$ for which $\psi^{\mathbf{L}}(y, x_1, \dots, x_k)$ if such a $y$ exists; otherwise, we set $S_\psi(x_1, \dots, x_k) = \varnothing$. Thus, each $S_\psi : \mathbf{L}^k \to \mathbf{L}$ is a *Skolem class-function* of arity $k = k(j) < \omega$ definable over $\mathbf{L}$ without parameters. We augment the list of functions $S_\psi$ by the function $S(x) = x$ and let $\boldsymbol{h}_0, \dots, \boldsymbol{h}_n$ be the complete list of class-functions thus obtained. Each $\boldsymbol{h}_j$ is a class-function defined over $\mathbf{L}$ without parameters.

The next step is performing the *Gödel collapse* procedure. In the framework of **ZFC**, it consists in choosing an elementary submodel $\mathfrak{M} \subseteq M$ of given cardinality $\kappa$ in a given transitive set, e.g., of the form $M = \mathbf{L}_\Omega$, where $\Omega > \kappa$ is a certain cardinal, and taking the transitive collapse of $\mathfrak{M}$ according to Lemma 3. However, in **SST**, the existence of cardinals above $\omega$ cannot be proved, and we must adapt the Gödel procedure. Namely, first, we take the entire class $\mathbf{L}$ as the initial structure $M$; second, we construct $\mathfrak{M} \subseteq \mathbf{L}$ as an elementary submodel of $\mathbf{L}$ with respect to a *finite* number of formulas ($\mathfrak{M}$ may be a proper class); and finally, we consider the transitive collapse $\mathfrak{N}$ of $\mathfrak{M}$. By virtue of Lemma 4, $\mathfrak{N}$ is a set even if $\mathfrak{M}$ is a proper class.

Following this plan, we consider the closure $\mathfrak{M} \subseteq \mathbf{L}$ of the set $B \cup \omega \cup \{\omega, B\} \in \mathbf{L}$ in $\mathbf{L}$ with respect to multiple applications of the class-functions $\boldsymbol{h}_0, \dots, \boldsymbol{h}_n$. Thus, $\mathfrak{M}$ is a class (possibly proper) definable without parameters.

**Lemma 15.** *The following assertions hold*:

(i) $\omega, B \in \mathfrak{M}$ *and* $\omega \cup B \subseteq \mathfrak{M}$.

(ii) $\mathfrak{M}$ *is an elementary substructure in* $\mathbf{L}$ *for* $\boldsymbol{\varphi}(b)$ *in* (I) *for any* $b \in B$, *and hence*

$$Y = \{b \in B \colon \boldsymbol{\varphi}^{\mathfrak{M}}(b)\}$$

*by virtue of* (‡) *in* (I).

(iii) *The formula* $\boldsymbol{\sigma}$ *is true in* $\mathfrak{M}$.

**Proof.** The assertions of the lemma are standard model-theoretic consequences of the construction of $\mathfrak{M}$ by means of Skolem functions. □

A simple combinatorial argument gives a class-function $F$ defined on an appropriate set $P$ for which

$$\mathfrak{M} = F''P = \{F(p)\colon p \in P\}.$$

Namely, we define $P$ as the set of all pairs $p = \langle T, f\rangle$, where $T = T_p \subseteq \omega^{<\omega}$ is a finite tree and $f = f_p$ is a function such that $\operatorname{dom} f = T$, $f(t) \in B \cup \{B\}$ if $t \in \operatorname{Max}(T)$ (the set of all terminal vertices), and $f(t) \in \{0, 1, \dots, n\}$ if $t \in T \setminus \operatorname{Max}(T)$; moreover, if $f(t) = j$ and $k = k(j)$ (the arity of the function $\boldsymbol{h}_j$), then

$$\{1, \dots, k\} = \{\ell\colon t^\frown\ell \in T\}.$$

We set

$$F_p(t) = \begin{cases} f(t) & \text{if } t \in \operatorname{Max}(T), \\ \boldsymbol{h}_j(F_p(t^\frown 1), \dots, F_p(t^\frown k)) & \text{if } t \notin \operatorname{Max}(T), \quad j = f(t), \quad k = k(j) \end{cases}$$

for all $t \in T$ and $F(p) = F_p(\langle\rangle)$ for the empty tuple $\langle\rangle \in T$. It is obvious from the construction that $\mathfrak{M} = F''P = \{F(p)\colon p \in P\}$.

Note that $P$ is indeed a set (in **SST**), because it is defined by means of the schema **Sep** and the operations of Lemma 6.

Now suppose that a class-function $\tau : \mathfrak{M}$ onto a transitive set or class $\mathfrak{N}$ is a transitive collapse, i.e., $\tau(x) = \{\tau(y)\colon y \in x \cap \mathfrak{M}\}$ for all $x \in \mathfrak{M}$. To construct $\mathfrak{N}$ and $\tau$, it suffices to apply Lemma 3 to sets of the form $M_\alpha = \mathfrak{M} \cap \mathbf{L}_\alpha$, $\alpha \in \operatorname{Ord}$, and define $\tau$ as the union of all transitive collapses $\tau_\alpha : M_\alpha$ onto transitive sets $N_\alpha$ thus obtained.

Applying Lemma 4, (i) to the superposition of $F$ and $\tau$, we see that $\mathfrak{N}$ is a set.

**Corollary 4.** *The following assertions hold*:

(i) $\omega, B \in \mathfrak{N}$ *and* $x = \tau(x) \in \mathfrak{N}$ *for all* $x \in B \cup \omega$.

(ii) $Y = \{b \in B\colon \boldsymbol{\varphi}^{\mathfrak{N}}(b)\}$, *and hence* $Y \in \mathbf{Def}\mathfrak{N}$.

(iii) *The formula* $\boldsymbol{\sigma}$ *is true in* $\mathfrak{N}$.

**Proof.** Assertion (i) holds because the sets $B$ and $\omega$ are transitive and $B \cup \omega \cup \{\omega, B\} \subseteq \mathfrak{M}$ by construction. To prove (ii), we note that $\tau : \mathfrak{M}$ onto $\mathfrak{N}$ is a $\in$-isomorphism and apply Lemma 15, (ii). Finally, (iii) follows from Lemma 15, (iii), because $\tau : \mathfrak{M}$ onto $\mathfrak{N}$ is a $\in$-isomorphism. □

However, by the choice of the formula $\boldsymbol{\sigma}$, the set $\mathfrak{N}$ has the form $\mathbf{L}_\alpha$ for $\alpha \in \operatorname{Ord}$. Therefore, $Y \in \mathbf{Def}\,\mathbf{L}_\alpha = \mathbf{L}_{\alpha+1}$, whence $Y \in \mathbf{L}$. This contradicts the choice of the set $Y$ at the beginning of Sec. 7, which proves that $\mathbf{L}$ satisfies **Sep**.

To complete the proof of the theorem, it remains to note that axiom (I) in Theorem 1 is true in $\mathbf{L}$, because each $\mathbf{L}_\lambda$ with limit $\lambda$ is obviously closed with respect to the formation of finite subsets. In view of Lemma 6, this entails the existence of Cartesian products in $\mathbf{L}$, which, together with Theorem 3, (vi), implies the well-orderability of all sets in $\mathbf{L}$. □

## 8. CONCLUSIONS

The content of Sec. 3 suggests the fairly interesting question of whether Lemmas 5 and 6 and Corollary 1 can be proved in **SST** without the axiom **Cntbl** or with **Cntbl** replaced by the weaker well-orderability axiom **WOA**. This question is important because, according to Theorem 1, the constructible part $\mathbf{L}$ of the **SST** universe satisfies **SST** in which the axiom **Cntbl** is replaced by **WOA**; moreover, **Cntbl** is not necessarily true in $\mathbf{L}$ (see Example 1 at the end of Sec. 5).

In general, it is interesting to consider the mutual independence of the special axioms **Prod**, **TrCov**, **Beta**, and **Cntbl** of the theory **SST** over the base formed by the remaining axioms of this theory.

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## CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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