ISSN 0001-4346, Mathematical Notes, 2026, Vol. 120, No. 5, pp. 877–886. © Pleiades Publishing, Ltd., 2026.
Russian Text © The Author(s), 2026, published in Matematicheskie Zametki, 2026, Vol. 120, No. 5, pp. 743–754.

On Petr Novikov's Problem about Ordered Systems of Uniform Sets

V. G. Kanovei[1]* and V. A. Lyubetsky[1]**

[1] *Institute for Information Transmission Problems of Russian Academy of Sciences (Kharkevich Institute), Moscow, 127051 Russia*

Received April 3, 2026; in final form, June 4, 2026; accepted June 10, 2026

Abstract—It is proved that every ordinal $\alpha < \omega_2$ is the order type of a system of uniform Borel sets in the plane well-ordered according to P. S. Novikov. This result gives a positive answer to a 1935 problem of N. N. Luzin.

DOI: 10.1134/S0001434626604375

Keywords: *uniform set, Borel set, representation of ordinals in the continuum.*

1. INTRODUCTION

In the first half of the 1930s, a number of deep and interesting results were obtained in descriptive set theory, which was then a comparatively new branch of mathematics. There results were presented and carefully analyzed is the book [1] by Luzin, who headed this research direction at the time. A significant part of the book was devoted to results included in the doctoral dissertation of Luzin's young student, the future Academician P. S. Novikov. In addition to those results of Novikov which, at the time of the publication of [1], had already been or were shortly published, e.g., in [2]–[5], Luzin paid attention to studies in Novikov's doctoral dissertation which have been published neither at that time nor afterwards and are known only from Luzin's presentation and analysis in [1].

In particular, Sec. 32 and Subsecs. I–IV of Sec. 33 in the book [1] contain an analysis of well-ordered families of uniform sets in the plane, which Novikov used to obtain a "geometric" representation of ordinals $\alpha < \omega_2$ (transfinite numbers of the third class and lower in the terminology of that time).

Considering order types of well-ordered collections of uniform sets, Luzin proved in [1, Sec. 33, Subsec. II] that these types are strictly less than ω_2 and asked whether it is true that, conversely, every ordinal $\mu < \omega_2$ is the length (order type) of some well-ordered sequence of uniform sets in the plane $\mathbb{R}^2$ [1, Sec. 33, Subsec. III].

Theorem 1 of the present paper (see Sec. 2) gives a positive answer to this question; this is our principal result. Note that the uniform sets constructed in the proof of the theorem are Borel, the ordinates of all points in these sets are rational, and the construction of each of them is completely effective; in particular, all of these uniform sets have Gödel-constructible Borel codes. The organization of the paper is described at the end of Sec. 2.

2. BASIC DEFINITIONS AND THE MAIN THEOREM

We consider sets in the real plane $\mathbb{R} \times \mathbb{R}$; we refer to them simply as *plane sets*. By the *projection* of a plane set $P \subseteq \mathbb{R} \times \mathbb{R}$ (on the first, horizontal axis) we mean the *linear* set

$$\mathbf{proj}\, P = \{x\colon \exists\, y\, (\langle x, y\rangle \in P)\} \subseteq \mathbb{R},$$

and given a point $x \in \mathbb{R}$, we define its (*vertical*) *section* by

$$P{\restriction}_x = \{y\colon \langle x, y\rangle \in P\} \subseteq \mathbb{R},$$

*E-mail: kanovei@iitp.ru
**E-mail: lyubetsk@iitp.ru

so that $\mathbf{proj}\, P = \{x\colon P{\restriction}_x \neq \varnothing\}$. A plane set P is said to be *uniform* if each of its sections $P{\restriction}_x$ contains at most one point. It is convenient to denote this unique point by $P(x)$. Thus, $P{\restriction}_x = \{P(x)\}$; we identify the uniform set P with the graph of the function $\mathbf{proj}\, P \to \mathbb{R}$.

Given uniform sets $P, Q \subseteq \mathbb{R} \times \mathbb{R}$, we write $P \lhd Q$ (informally, P is below Q) if the following two conditions are satisfied:

$$\mathbf{proj}\, P \subseteq \mathbf{proj}\, Q \quad \text{or} \quad \mathbf{proj}\, Q \subseteq \mathbf{proj}\, P; \tag{1}$$

$$P(x) < Q(x) \quad \text{for all} \quad x \in \mathbf{proj}\, P \cap \mathbf{proj}\, Q. \tag{2}$$

Note that, knowing only that $P \lhd Q$, we cannot determine which of the two mutually reverse inclusions in condition (1) holds for the sets P and Q.

Simple examples show that the relation $\lhd$ is not generally transitive, i.e., this is not even a partial order. Nevertheless, we consider families of nonempty uniform plane sets well-ordered by $\lhd$. We refer to such families as *chains* and consider ordinals which are *lengths*, i.e., order types, of such chains. For $\mu \in \mathrm{Ord}$, by a *μ-chain* we mean a chain of length μ. A notation of the form $\langle P_\alpha\rangle_{\alpha<\mu}$ is used for a well-ordered μ-chain of uniform sets P_α; this means that $P_\alpha \lhd P_\beta$ for all $\alpha < \beta < \mu$.

Note that the projections $\mathbf{proj}\, P_\alpha$ of elements of any chain are partially ordered by inclusion $\subset$, but this order is not necessarily compatible with the order $\lhd$ on the elements themselves.

The following proposition imposes a bound for the length of a chain well-ordered by $\lhd$.

Proposition 1 (Novikov, Luzin [1, Sec. 33-II]). *The length μ of any well-ordered chain $\langle P_\alpha\rangle_{\alpha<\mu}$ of uniform sets P_α is strictly less than ω_2.*

In light of this result, Luzin posed the inverse problem [1, Sec. 33-III].

Problem. *Given an arbitrary ordinal $\mu < \omega_2$, determine whether there exists a well-ordered chain of uniform sets whose length is precisely μ.*

Assertion (I) of Theorem 1 below gives a positive solution of this problem. Moreover, our chains satisfy not only the condition in the problem as posed in [1] but also consist of Borel (uniform) sets, namely, of sets $P \subseteq \mathbb{R} \times \mathbb{Q}$ in which all points have rational ordinates.

Assertions (II) and (III) give additional information about the coding of these Borel sets and its effectiveness (in the form of Gödel constructibility). In particular, assertion (III),(a) is concerned with the case of $\mu < \boldsymbol{\Xi}$, where $\boldsymbol{\Xi}$ is the first $\mathbf{L}$-cardinal of cardinality strictly greater than the "true" ω_1; in other words, if $\omega_1 = \omega_\gamma^{\mathbf{L}}$, then $\boldsymbol{\Xi} = \omega_{\gamma+1}^{\mathbf{L}}$. We remind that $\mathbf{L}$ is the class of all constructible sets. Notions related to Gödel constructibility in set theory can be found, e.g., in [6], [7, Sec. 2], or [8, Chap. 8].

Theorem 1 (main theorem). *Let $\mu < \omega_2$. Then*

(I) *There exists a μ-chain of uniform Borel sets $P_\alpha \subseteq \mathbb{R} \times \mathbb{Q}$, $\alpha < \mu$, well-ordered by $\lhd$.*

(II) *All these sets P_α have constructible Borel codes.*[1]

(III) (a) *If $\mu < \boldsymbol{\Xi}$, then the Borel codes of the sets P_α can be chosen so that they form a constructible sequence.*

(b) *If $\mu \geq \boldsymbol{\Xi}$, then there exists no constructible μ-sequence of pairwise distinct Borel codes.*

[1] The coding of Borel sets used in this paper is specified in Sec. 6.

It follows from (III) that the ordinal $\mathbf{\Xi}$ separates the domain of effectiveness (understood in the sense of the existence of a constructible sequence of Borel codes for the sets in a given $\lhd$-chain) from the domain of noneffectiveness. Of course, its role depends on the true relation between $\mathbf{\Xi}$ and ω_2, and it is known from the practice of the forcing method in modern set theory that both relations $\mathbf{\Xi} < \omega_2$ and $\mathbf{\Xi} = \omega_2$ are consistent with the axioms of the Zermelo–Fraenkel theory **ZFC**.

The paper is organized as follows.

The proof of Proposition 1 is given in Sec. 3.

Assertion (I) of Theorem 1 is proved in Sec. 4 for the case $\mu = \omega_1$ and in Sec. 5 in the general case.

Borel coding is described in Sec. 6.

Assertions (II) and (III) of Theorem 1 are proved in Sec. 7. Consequences concerning a representation of ordinals $\mu < \omega_2$ in terms of Borel sets are discussed in Sec. 8.

The paper is concluded with a related problem posed in Sec. 9.

3. A BOUND FOR THE LENGTH OF AN INCREASING CHAIN

For the reader's convenience, we present a fairly short, though by no means obvious, proof of Proposition 1, which was given by Luzin in [1, Sec. 33–II].

Note that Proposition 1 is not used in the proof of Theorem 1.

Thus, consider a well-ordered chain $\langle P_\alpha\rangle_{\alpha<\mu}$ of uniform sets $P_\alpha \subseteq \mathbb{R}\times\mathbb{R}$, i.e., a family of sets satisfying the relation $P_\alpha \lhd P_\beta$ for $\alpha < \beta < \mu$. We must prove that $\mu < \omega_2$.

To this end, consider the set $P = \bigcup_{\alpha<\mu} P_\alpha$. Clearly, each section $P{\restriction}_x = \{y\colon \langle x, y\rangle \in P\} \subseteq \mathbb{R}$ is well-ordered and has order type $\mu_x < \omega_1$, because there are no strictly increasing ω_1-sequences in $\mathbb{R}$. In other words, $P{\restriction}_x = \{y_{x\xi}\colon \xi < \mu_x\}$, where the points $y \in P{\restriction}_x$ are numbered in ascending order according to the usual ordering of real numbers. We set $Q_\xi = \{\langle x, y_{x\xi}\rangle\colon \xi < \mu_x\}$ for all $\xi < \omega_1$; then

$$\begin{aligned}&\text{the sets } Q_\xi \text{ are uniform, } Q_\xi \lhd Q_\eta, \quad \mathbf{proj}\, Q_\eta \subseteq \mathbf{proj}\, Q_\xi \text{ for } \xi < \eta < \omega_1,\\ &\text{and } P = \bigcup_{\alpha<\mu} P_\alpha = \bigcup_{\xi<\omega_1} Q_\xi.\end{aligned} \tag{3}$$

We claim that

$$\text{if } \xi < \omega_1, \text{ then the set } A_\xi = \{\alpha < \mu\colon Q_\xi \cap P_\alpha \neq \varnothing\} \text{ is countable.} \tag{4}$$

Suppose that this is not so. Take the least ordinal $\xi_0 < \omega_1$ for which A_{ξ_0} is uncountable. Since each A_ξ with $\xi < \xi_0$ is countable, it follows that, removing countably many indices from A_{ξ_0}, we obtain an uncountable set $A \subseteq A_{\xi_0}$ such that

$$\text{if } \xi < \xi_0, \text{ then } Q_\xi \cap P_\alpha = \varnothing \text{ for all } \alpha \in A. \tag{5}$$

We claim that

$$\text{if } \alpha < \beta \text{ belong to } A, \text{ then } \mathbf{proj}\, P_\alpha \subseteq \mathbf{proj}\, P_\beta. \tag{6}$$

Indeed, since $\beta \in A$, it follows that there exists a point $\langle x, y\rangle \in Q_{\xi_0} \cap P_\beta$, whence $x \in \mathbf{proj}\, P_\beta$. Let us show that $x \notin \mathbf{proj}\, P_\alpha$. Suppose that, on the contrary, there exists a y' for which $\langle x, y'\rangle \in P_\alpha$. The inequality $y' < y$ cannot hold, because if it held, then we would have $\langle x, y'\rangle \in Q_\xi$ for some $\xi < \xi_0$ according to (3), which contradicts (5). The equality $y' = y$ cannot hold either, because $P_\beta \cap P_\alpha = \varnothing$ for $\alpha \neq \beta$. Finally, the inequality $y < y'$ is impossible, too, because it contradicts the condition $P_\alpha \lhd P_\beta$ for $\alpha < \beta$. Therefore, $x \notin \mathbf{proj}\, P_\alpha$. This implies $\mathbf{proj}\, P_\beta \not\subseteq \mathbf{proj}\, P_\alpha$ and hence $\mathbf{proj}\, P_\alpha \subseteq \mathbf{proj}\, P_\beta$ by virtue of (1). Thus, we have proved (6).

Continuing the proof of (4), we take the least ordinal $\alpha_0 \in A$ and any $x \in \mathbf{proj}\, A_{\alpha_0}$. According to (6), we have $x \in \mathbf{proj}\, A_\alpha$ for all $\alpha \in A$. This means that there exists a set of points y_α, $\alpha \in A$, such that $\langle x, y_\alpha\rangle \in P_\alpha$ and $y_\alpha < y_\beta$ for $\alpha < \beta$, because $\alpha < \beta$ implies $P_\alpha \lhd P_\beta$. Thus, we have obtained an uncountable strictly increasing sequence of points y_α, $\alpha \in A$, in $\mathbb{R}$, which cannot exist. This contradiction proves (4).

It follows from (4) that the given chain $\langle P_\alpha\rangle_{\alpha<\mu}$ contains at most $\aleph_1$ sets; therefore, $\mu < \omega_2$. This completes the proof of Proposition 1.

4. THE MAIN THEOREM FOR THE FIRST UNCOUNTABLE ORDINAL

In this section, we prove assertion (I) of Theorem 1 for $\mu = \omega_1$. This is an important step in the proof of the general case.

We begin with a standard lemma. As usual, $\mathbb{Q}$ denote the set of rational numbers.

Lemma 1. *There exists a Borel set $G \subseteq \mathbb{R} \times \mathbb{Q}$ such that for any $Z \subseteq \mathbb{Q}$ there exists a point $x \in \mathbb{R}$ for which $Z = G{\restriction}_x := \{q \in \mathbb{Q}\colon \langle x, q\rangle \in G\}$.*

Proof. Let us fix a recursive enumeration of rational numbers: $\mathbb{Q} = \{r_n\colon n < \omega\}$. Given $x \in \mathbb{R}$, let $A_x \subseteq \omega$ be the set of all n for which the binary fraction expansion of the fractional part of x with a deficiency has 0 at the $(2n+1)$st position. The set $G = \{\langle x, r_n\rangle\colon n \in A_x\}$ is as required. □

We fix such a set G for further considerations.

Note that G is one of the versions of the *Lebesgue binary sieve* [9], which is described, e.g., in Sec. 17 of Luzin's book [1]. For each ordinal $\xi < \omega_1$, we set

$$\left.\begin{aligned} D_\xi &= \{x \in \mathbb{R}\colon G{\restriction}_x \text{ has well-ordered} \\ &\qquad \text{initial interval of order type } > \xi\}; \\ U_\xi &= \{\langle x, r\rangle \in D_\xi \times \mathbb{Q}\colon r \text{ is the } \xi\text{th greatest point of } C{\restriction}_x\}; \end{aligned}\right\} \tag{7}$$

finally, we set $\mathscr{D} = \{D_\xi\colon \xi < \omega_1\}$.

The following lemma implies assertion (I) of Theorem 1 for the case $\mu = \omega_1$.

Lemma 2. *The sets $U_\xi \subseteq \mathbb{R} \times \mathbb{Q}$, $\xi < \omega_1$, are uniform and form a $\lhd$-increasing chain $\langle U_\xi\rangle_{\xi<\omega_1}$ of length ω_1; moreover, $D_\xi = \mathbf{proj}\, U_\xi$ and $D_\eta \subseteq D_\xi$ for $\xi < \eta$. All sets U_ξ and D_ξ are nonempty and Borel.*

Proof. Only the U_ξ and D_ξ being Borel needs to be proved; the other assertions are obvious. Note that, since we consider only plane Borel sets of the form $B \subseteq \mathbb{R} \times \mathbb{Q}$, we can partition every set under consideration into *horizontal sections*, that is, the Borel sets

$$[B]_r = \{x\colon \langle x, r\rangle \in B\} \subseteq \mathbb{R}, \text{ where } r \in \mathbb{Q} \text{ and } B = \bigcup_{r\in\mathbb{Q}} ([B]_r \times \{r\}); \tag{8}$$

to prove that B is Borel, it suffices to show that so are all $[B]_r$.

Taking into account this simplifying remark, we prove that the given sets $U_\xi \subseteq \mathbb{R} \times \mathbb{Q}$ are Borel by induction on ξ. For simplicity, we assume that p, q, and r denote only rational numbers. For $\xi = 0$,

$$x \in [U_0]_r \quad \Longleftrightarrow \quad x \in [G]_r \wedge \forall\, q < r\, (x \notin [G]_q), \tag{9}$$

which implies that all sections $[U_0]_r$ and the set U_0 itself are Borel, because the initial set G is Borel by Lemma 1, and the quantifier $\forall\, q$ on the right-hand side is restricted to the countable set $\mathbb{Q}$.

Now suppose that $\xi > 0$ and all sets U_η, $\eta < \xi$, are Borel. We have

$$\left.\begin{aligned} x \in [U_\xi]_r \quad \Longleftrightarrow \quad & x \in [G]_r \wedge \forall\, \eta < \xi\, \exists\, q < r\, (x \in [U_\eta]_q \\ & \wedge \forall\, p\, (q' < p < q \implies x \notin [G]_p)) \end{aligned}\right\} \tag{10}$$

for $\xi = \eta + 1$, and if ξ is a limit ordinal, then

$$\left.\begin{aligned} x \in [U_\xi]_r \quad \Longleftrightarrow \quad & x \in [G]_r \wedge \forall\, \eta < \xi\, \exists\, q < r\, (x \in [U_\eta]_r) \\ & \wedge \forall\, q < r\, \exists\, \eta < \xi\, (x \in [G]_q \implies x \in [U_\eta]_q). \end{aligned}\right\} \tag{11}$$

In both cases, since G and all sets U_η, $\eta < \xi$, are Borel, it follows that so is U_ξ from the same considerations as U_0.

Finally, every set $D_\xi = \mathbf{proj}\, U_\xi$ is Borel, because

$$x \in D_\xi \quad \Longleftrightarrow \quad \exists\, r \in \mathbb{Q}\, (x \in [U_\xi]_r). \tag{12}$$

This completes the proof of Lemma 2. □

5. PROOF OF THE MAIN THEOREM IN THE GENERAL CASE

We begin with the important special case of *transitive* relation $\lhd$.

Lemma 3. *For uniform sets $P \lhd U \lhd Q$, the conjunction of the following condition* (13) *and condition* (1) *in* Sec. 2 *is sufficient for the relation $P \lhd Q$ to hold*:

$$\mathbf{proj}\, P \subseteq \mathbf{proj}\, U \quad \textit{or} \quad \mathbf{proj}\, Q \subseteq \mathbf{proj}\, U. \tag{13}$$

Proof. To deduce (2), we take any point x in the intersection $\mathbf{proj}\, P \cap \mathbf{proj}\, Q$ and note that $x \in \mathbf{proj}\, U$ according to (13); the rest is obvious. □

We say that a given chain of uniform sets P_α has the *$\mathscr{D}$-projection property* if $\mathbf{proj}\, P_\alpha \in \mathscr{D}$ for all α. (Recall that the set $\mathscr{D} = \{D_\xi \colon \xi < \omega_1\}$ was defined before Lemma 2.) For example, the chain $\langle U_\xi \rangle_{\xi<\omega_1}$ has this property.

Lemma 4. *For any $\mu < \omega_2$, there exists a $\lhd$-increasing μ-chain $P \subseteq \mathbb{R} \times \mathbb{Q}$ of uniform Borel sets with the $\mathscr{D}$-projection property.*

Proof. According to Lemma 2, the already constructed chain $\langle U_\xi \rangle_{\xi<\omega_1}$ proves the lemma for $\mu \le \omega_1$. We proceed by induction.

Beginning of the induction step. Suppose given an ordinal μ, $\omega_1 < \mu < \omega_2$.

$$\begin{gathered}\text{We fix an enumeration } \mu = \{\nu_\xi \colon \xi < \omega_1\} \text{ of all smaller ordinals } \nu < \mu.\\ \text{(We will return to an analysis of this point later on.)}\end{gathered} \tag{14}$$

By the induction hypothesis, for each ordinal ν_ξ, there exists

$$\begin{gathered}a \lhd \text{-increasing } \nu_\xi\text{-chain } \langle F^\xi_\alpha \rangle_{\alpha<\nu_\xi} \text{ of uniform Borel}\\ \text{sets } F^\xi_\alpha \subseteq \mathbb{R} \times \mathbb{Q} \text{ with the } \mathscr{D}\text{-projection property.}\end{gathered} \tag{15}$$

We are going to insert each chain $\langle F^\xi_\alpha \rangle_{\alpha<\nu_\xi}$ between the uniform sets U_ξ and $U_{\xi+1}$ by a vertical shift. To this end, we set

$$\left.\begin{aligned}&Q^\xi_\alpha = F^\xi_\alpha \restriction D_{\xi+1} := P^\xi_\alpha \cap (D_{\xi+1} \times \mathbb{Q}) \subseteq \mathbb{R} \times \mathbb{Q},\\ &\text{or, equivalently,} \quad [Q^\xi_\alpha]_r = [F^\xi_\alpha]_r \cap D_{\xi+1}\ \forall\, r\end{aligned}\right\} \tag{16}$$

for all $\alpha < \nu_\xi$.[2] Since $D_{\xi+1} \in \mathscr{D}$ and the chain $\langle F^\xi_\alpha \rangle_{\alpha<\nu_\xi}$ has the $\mathscr{D}$-projection property, it follows that all reduced sets Q^ξ_α are nonempty and form a $\lhd$-increasing chain $\langle Q^\xi_\alpha \rangle_{\alpha<\nu_\xi}$ of the same length ν_ξ having the $\mathscr{D}$-projection property and $\mathbf{proj}\, Q^\xi_\alpha \subseteq D_{\xi+1} \subseteq D_\xi$ for all α.

Note also that all sets Q^ξ_α are Borel, because so are the F^ξ_α and D_ξ (by Lemma 2).

Take $\xi < \omega_1$ and $x \in D_{\xi+1}$. By Lemma 2, there exists a unique pair of points $\langle x, u^x_\xi \rangle \in U_\xi$ and $\langle x, u^x_{\xi+1} \rangle \in U_{\xi+1}$, because $U_\xi \lhd U_{\xi+1}$ and $u^x_\xi < u^x_{\xi+1}$ are rational.

For future use, for any pair $u < v$ of rational numbers, we define an *order-preserving* homeomorphism $H[u,v] \colon (u,v) \xrightarrow{\text{onto}} \mathbb{R}$ of the open interval (u,v) onto the real line $\mathbb{R}$ by setting

$$H[u,v](y) = \left\{\begin{aligned}&\frac{y-c}{v-y} && \text{for } c \le y < v,\\ &\frac{y-c}{u-y} && \text{for } u < y \le c,\end{aligned}\right|, \quad \text{where} \quad c = \frac{u+v}{2}. \tag{17}$$

Clearly, $H[u,v]$ preserves rationality, i.e., $r \in \mathbb{Q} \iff H[u,v](r) \in \mathbb{Q}$.

[2] Here and in what follows, we use the decomposition (8) of plane sets into horizontal sections.

Take $\xi<\omega_1$ again. It follows from the above considerations that if $\alpha<\nu_\xi$, then the uniform set $Q^\xi_\alpha\subseteq\mathbb{R}\times\mathbb{Q}$ satisfies the condition $\mathbf{proj}\,Q^\xi_\alpha\subseteq D_{\xi+1}$. We define the *vertical shift* of Q^ξ_α to between U_ξ and $U_{\xi+1}$ as

$$S^\xi_\alpha=\{\langle x,r\rangle\in D_{\xi+1}\times\mathbb{Q}\colon u^x_\xi<r<u^x_{\xi+1}\wedge\langle x,H[u^x_\xi,u^x_{\xi+1}](r)\rangle\in Q^\xi_\alpha\}. \tag{18}$$

Since the abscissas do not change and the ordinates remain rational and ordered in the same way as previously, it follows that the sets $S^\xi_\alpha\subseteq D_{\xi+1}\times\mathbb{Q}$, $\alpha<\nu_\xi$, are uniform and form a $\lhd$-increasing ν_ξ-chain $\langle S^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ with the $\mathscr{D}$-projections property; moreover,

$$\mathbf{proj}\,S^\xi_\alpha=\mathbf{proj}\,Q^\xi_\alpha\subseteq D_{\xi+1}\subseteq D_\xi\quad\text{and}\quad U_\xi\lhd S^\xi_\alpha\lhd U_{\xi+1} \tag{19}$$

for all $\alpha<\nu_\xi$ by construction.

Let us check that the new sets S^ξ_α are Borel. By construction, we have

$$[S^\xi_\alpha]_r=\bigcup_{u,v,w\in\mathbb{Q}\wedge u<v\wedge w=H[u,v](r)}[U_\xi]_u\cap[U_{\xi+1}]_v\cap[Q^\xi_\alpha]_w \tag{20}$$

for any $r\in\mathbb{Q}$. Since the sets $U_\xi,U_{\xi+1},Q^\xi_\alpha\subseteq\mathbb{R}\times\mathbb{Q}$ are Borel, it follows that so are their horizontal sections $[U_\xi]_u$, $[U_{\xi+1}]_v$, and $[Q^\xi_\alpha]_w$. Therefore, according to (20), the sections $[S^\xi_\alpha]_r$ are Borel. This implies that the sets S^ξ_α themselves are Borel (see (8)).

Now consider the entire family

$$\mathbf{S}_\mu=\{U_\xi\colon\xi<\omega_1\}\cup\{S^\xi_\alpha\colon\xi<\omega_1\wedge\alpha<\nu_\xi\} \tag{21}$$

of uniform subsets of $\mathbb{R}\times\mathbb{Q}$ involved in the induction step μ. Let us prove that

$$\mathbf{S}_\mu\text{ is a }\lhd\text{-increasing chain of length }\mu'=\sum_{\xi<\omega_1}(1+\nu_\xi)\geq\mu. \tag{22}$$

Fact 1. We already know that, for any $\xi<\omega_1$, $\langle U_\xi\rangle_{\xi<\omega_1}$ and $\langle S^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ are $\lhd$-increasing chains of uniform sets with the $\mathscr{D}$-projection property and (19) holds.

Fact 2. If $P,Q\in\mathbf{S}_\mu$, then the projections $\mathbf{proj}\,P$ and $\mathbf{proj}\,Q$ belong to $\mathscr{D}$ and hence one has either $\mathbf{proj}\,P\subseteq\mathbf{proj}\,Q$ or $\mathbf{proj}\,Q\subseteq\mathbf{proj}\,P$ (see condition (1) in Definition $\lhd$).

Fact 3. If $\xi<\eta<\omega_1$ and $\alpha<\nu_\xi$, then $S^\xi_\alpha\lhd U_{\xi+1}\lhd U_\eta$ by virtue of Fact 1 and $\mathbf{proj}\,U_\eta\subseteq\mathbf{proj}\,U_{\xi+1}$ by Lemma 2, so that $S^\xi_\alpha\lhd U_\eta$ in view of Lemma 3 and Fact 2.

Fact 4. If $\xi<\eta<\omega_1$ and $\beta<\nu_\eta$, then $U_\xi\lhd U_\eta\lhd S^\eta_\beta$ by virtue of Fact 1 and $\mathbf{proj}\,S^\eta_\beta\subseteq\mathbf{proj}\,U_\eta$ in view of (19), so that $S^\xi_\alpha\lhd U_\eta$ according to Lemma 3 and Fact 2.

Fact 5. If $\xi<\eta<\omega_1$, $\alpha<\nu_\xi$, and $\beta<\nu_\eta$, then $S^\xi_\alpha\lhd S^\eta_\beta$. By Fact 3, we have $S^\xi_\alpha\lhd U_\eta\lhd S^\eta_\beta$, and (19) implies $\mathbf{proj}\,S^\eta_\beta\subseteq\mathbf{proj}\,U_\eta$; again applying Lemma 3 and Fact 2, we obtain $S^\xi_\alpha\lhd S^\eta_\beta$.

It follows from Facts 1–5 that (22) holds for the family (21), i.e.,

$$\mathbf{S}_\mu=\{P_\alpha\colon\alpha<\mu'\} \tag{23}$$

is $\lhd$-increasing. Obviously, $\mu'\geq\mu$, so that a chain of length precisely μ is obtained by simply cutting off the "tail" of $\mathbf{S}_\mu$ in (23) from μ to μ'. This completes the induction step and proves Lemma 4. □

Clearly, Lemma 4 implies assertion (I) of Theorem 1.

6. CODING OF BOREL SETS

In this section, we recall basic notions related to Borel codes, which appear in the statements and are used in the proofs of assertions (II) and (III) of Theorem 1 in the next section.

Consider the set $\omega_1^{<\omega}$ of all tuples (finite sequences) of countable ordinals. In what follows,

- $s \subset t$ means that the tuple t is a proper extension of s.
- $\langle\,\rangle$ denotes the empty tuple and $\langle\alpha_1, \dots, \alpha_n\rangle$, the n-tuple with terms $\alpha_1, \dots, \alpha_n$.
- $s^\frown\alpha$ is the tuple obtained by attaching the term α to the tuple s on the right.
- A set $T \subseteq \omega_1^{<\omega}$ is called a *tree* if $s \subset t \in T \implies s \in T$.
- $\operatorname{Max}(T)$ denotes the set of all *terminal nodes* of T.
- A tree T is *well founded* if it has no *infinite branches*, i.e., there exists no function $b\colon \omega \to \omega_1$ such that $b \restriction n \in T$ for all n.

Following [8, Sec. 16.4], we define a *Borel code* (for the space $\mathbb{R}$) as a pair $\langle T, d\rangle$, where $\varnothing \neq T \subseteq \omega_1^{<\omega}$ is a finite or countable well-founded tree and $d \subseteq T \times \mathbb{Q} \times \mathbb{Q}$. Since T is well founded, for each $s \in T$, a Borel set $[T, d, s] \subseteq \mathbb{R}$ with the following properties is uniquely determined:

1*. $[T, d, s] = \mathbb{R} \setminus \bigcup_{\langle s,p,q\rangle \in d}(p, q)$ for $s \in \operatorname{Max}(T)$.

2*. $[T, d, s] = \mathbb{R} \setminus \bigcup_{s^\frown\alpha \in T}[T, d, s^\frown\alpha]$ for $s \in T \setminus \operatorname{Max}(T)$.

3*. $[T, d] = [T, d, \langle\,\rangle]$.

In 1*, as usual, $(p, q) = \{x \in \mathbb{R}\colon p < x < q\}$ denotes an open interval of the real line $\mathbb{R}$ with rational endpoints; it is empty for $p \geq q$.

Thus, the scheme 1*–3* constructs the set $[T, d] \subseteq \mathbb{R}$ from rational intervals by using the operation of taking the complement to a countable union, so that this set is Borel. Conversely, for any Borel set $X \subseteq \mathbb{R}$, there exists a Borel code $\langle T, d\rangle$ (with a countable tree T!) for which $X = [T, d]$.[3]

Fortunately, we do not have to consider Borel coding in full generality, because in the proofs of Lemma 4 and Theorem 1, we in fact deal with only *plane* sets $U \subseteq \mathbb{R} \times \mathbb{Q}$. As the *Borel multicode* of such a set we can take any indexed system of Borel codes $c = \langle T_r, d_r\rangle_{r\in\mathbb{Q}}$; setting

$$[c] = \bigcup_{r\in\mathbb{Q}}([T_r, d_r] \times \{r\}) \subseteq \mathbb{R} \times \mathbb{Q},$$

we obtain $[c] = U$ according to (8), provided that $[U]_r = [T_r, d_r]$ for all $r \in \mathbb{Q}$.

Note that the Borel codes and multicodes defined as above are *hereditarily countable*, which is essential for certain definability questions. If we used rational intervals instead of pairs of their endpoints in the sets d (which seems to be simpler and more natural), then the codes would not be hereditarily countable.

[3] This definition agrees with the structure of the space $\mathbb{R}$. To code Borel sets, e.g., in the *Baire space* $\mathbb{I} = \omega^\omega$, we should use sets $d \subseteq T \times \omega^{<\omega}$ and define the sets $[T, d, s]$ in Definition 1* as $[T, d, s] = \mathbb{I} \setminus \bigcup_{\langle s,\sigma\rangle\in d} I_\sigma$, where $I_\sigma = \{a \in \mathbb{I}\colon \sigma \subset a\}$.

7. CONSTRUCTING CODES FOR THE PROOF OF THE MAIN THEOREM

Having given the standard definitions related to Borel codes, we return to Theorem 1. Our goal in this section is to prove assertions (II) and (III) of Theorem 1 for the Borel sets (21)/(23) constructed in the proof of assertion (I) in Sec. 5. The following intermediate result gives Borel codes for the sets U_ξ and D_ξ in (7), which belong to the class $\mathbf{L}$ of all *Gödel constructible* sets.

Lemma 5. *For the sets U_ξ and D_ξ, $\xi<\omega_1$, defined by* (7)*, the following assertions hold*:

(i) *There exist Borel multicodes $c_\xi=\langle T^\xi_r,d^\xi_r\rangle_{r\in\mathbb{Q}}\in\mathbf{L}$ for U_ξ and Borel codes $\langle T'_\xi,d'_\xi\rangle\in\mathbf{L}$ for D_ξ.*

(ii) *The ω_1-sequences of these codes are constructible.*

Proof. The proof (sketch). Using relations (9)–(11), we can effectively define, by transfinite induction, Borel multicodes $c_\xi=\langle T^\xi_r,d^\xi_r\rangle_{r\in\mathbb{Q}}$ for the sets U_ξ, i.e., multicodes such that $U_\xi=[c_\xi]$, after which, using (12), we can also define Borel codes $\langle T'_\xi,d'_\xi\rangle$ for the sets $D_\xi=\mathbf{proj}\,U_\xi$. However, first, we should construct a constructible multicode for the initial set G constructed in the proof of Lemma 1; we leave this as a simple exercise. The inductive construction of all these codes is absolute for the class $\mathbf{L}$ of all constructible sets, i.e., gives the same result in $\mathbf{L}$ as in the universe of all sets. □

The application of Lemma 5, together with Lemma 2, completes the proof of all assertions of Theorem 1 for $\mu=\omega_1$. The proof of the general case $\omega_1\le\mu<\omega_2$ is based on the following result, which is similar to Lemma 5, (i), but refers to sequences of Borel sets in the proof of Lemma 4.

Lemma 6. *Let $\omega_1<\mu<\omega_2$. Suppose that the assumptions of the induction step in the proof of Lemma* 4 *hold and, in addition,*

(∗) *There exist Borel multicodes $\varphi^\xi_\alpha\in\mathbf{L}$ for the sets F^ξ_α in* (15).

Then there exist Borel multicodes $\pi_\alpha\in\mathbf{L}$ for the resulting sets P_α in (23) *in the proof of Lemma* 4.

Sketch of proof. First, we define intermediate multicodes $\wp^\xi_\alpha\in\mathbf{L}$ for the sets Q^ξ_α defined by (16) in Sec. 5 by using codes in (∗), and then we define multicodes $\sigma^\xi_\alpha\in\mathbf{L}$ for the sets S^ξ_α by using relations (20); these multicodes and those for U_ξ given by Lemma 5, being renumbered according to the passage from (21) to (23), are the final multicodes $\pi_\alpha\in\mathbf{L}$ for the sets P_α. The constructibility of these multicodes follows from obvious absoluteness. □

This completes the proof of assertion (II) of Theorem 1.

Proceeding to assertion (III) of the theorem, note that the constructions of Secs. 4 and 5 involve only one "noneffective" action employing an arbitrary choice, namely, the choice of a particular enumeration $\mu=\{\nu_\xi\colon\xi<\omega_1\}$ of all ordinals $\nu<\mu$ (see (14) in the proof of Lemma 4). Thus, we cannot strengthen Lemma 6 by requiring the μ-sequence of codes for the sets (21)/(23) obtained in the proof of Lemma 4 to be constructible.

Fortunately, this does not affect the constructibility of these codes themselves, because, for a fixed $\xi<\omega_1$, relations (18) and (20) depend only on ν_ξ for this ξ rather than on the entire sequence $\langle\nu_\xi\rangle_{\xi<\omega_1}$.

As regards assertion (III), (a) of Theorem 1, which deals with the case $\mu<\mathbf{\Xi}$, the inequality $\mu<\mathbf{\Xi}$ allows us to choose a *particular* enumeration $\mu=\{\nu_\xi\colon\xi<\omega_1\}$, namely, the enumeration *least* in the canonical well-ordering $<_{\mathbf{L}}$ of the constructible universe $\mathbf{L}$. Then the construction of the sequence of sets (21)/(23) and, accordingly, the codes π_α in the proof of Lemma 6 become completely effective and absolute for $\mathbf{L}$.

This completes the proof of assertion (III), (a) of Theorem 1.

We conclude this section with the proof of assertion (III), (b) of Theorem 1.

Consider the set $\mathbf{BMC}$ of all Borel multicodes defined as in Sec. 6. If $\omega_1=\omega^{\mathbf{L}}_\gamma$, then $\mathbf{\Xi}=\omega^{\mathbf{L}}_{\gamma+1}$. On the other hand, the set $\mathbf{BMC}\cap\mathbf{L}$ of all *constructible* multicodes (this is not the same as $(\mathbf{BMC})^{\mathbf{L}}$!) has cardinality $\aleph^{\mathbf{L}}_\gamma$ in $\mathbf{L}$, because GCH holds in $\mathbf{L}$. Hence there exist no *constructible* $\mathbf{\Xi}$-sequences of pairwise different Borel multicodes in $\mathbf{BMC}\cap\mathbf{L}$.

This completes the proof of assertion (III) of Theorem 1.

8. A CONSEQUENCE CONCERNING AN EFFECTIVE REPRESENTATION OF ORDINALS

In this short section, we discuss a fundamental, although somewhat uncertain, consequence of Theorem 1 or, rather, of the effectiveness of the construction of an $\lhd$-increasing μ-sequence of uniform Borel sets for $\mu < \omega_2$ used to prove Theorem 1.

It is easy to realize that the von Neumann ordinals correspond to, so to speak, vertical transfinite increase in the hierarchy of set-theoretic universe, while the real numbers, their subsets (say Borel), and, e.g., transfinite sequences of these sets correspond to the expansion of the universe at several initial steps of the "vertical" hierarchy. This suggest the problem of representing, modeling, or, if desired, naming ordinals of various cardinalities by means of objects related to the continuum of the real numbers, which is quite natural for the foundations of mathematics.

For example, the set $G \subseteq \mathbb{R} \times \mathbb{Q}$ constructed in the proof of Lemma 1 (it is called the *binary Lebesgue sieve*) determines the representation of countable ordinals $\xi < \omega_1$ by Borel sets $U_\xi \subseteq \mathbb{R} \times \mathbb{Q}$ defined by formula (7); by the way, these sets are nonempty and pairwise disjoint. Countable ordinals can also be represented by the *linear* Borel sets $D_\xi = \mathbf{proj}\, U_\xi \subseteq \mathbb{R}$. However, these sets, although nonempty, are not disjoint; on the contrary, they are embedded in one another. We can render them disjoint by taking the differences $E_\xi = D_{\xi+1} \setminus D_\xi$, which are Borel and nonempty as well.

Thus, there exists an effective representation of countable ordinals by Borel sets; a different but related version of such a representation goes back to Lebesgue's old paper [9].

Assertion (III) of our Theorem 1 makes it possible to obtain an effective representation of ordinals in the substantially larger domain $\mu < \mathbf{\Xi}$. (Recall that the ordinal $\mathbf{\Xi}$ was defined in Sec. 2 and belongs to the interval $\omega_1 < \mathbf{\Xi} \leq \omega_2$). Namely, an effective representation of an ordinal $\mu < \mathbf{\Xi}$ is given by a $\lhd$-increasing μ-sequence of uniform Borel sets and the constructive sequence of their codes, whose existence for this μ follows from assertion (III) of Theorem 1.

As to ordinals μ in the interval $\mathbf{\Xi} \leq \mu < \omega_2$ (provided that $\mathbf{\Xi} < \omega_2$), it appears that the most reasonable effective code for such a μ in our context is the set of all μ-sequences of constructible multicodes for $\lhd$-increasing μ-chains provided by assertion (I) of Theorem 1.

9. CONCLUDING REMARKS

Theorem 1 solves the long-known classical Novikov–Luzin problem about the lengths of transfinite sequences of uniform plane sets; moreover, it gives a solution in the strongest form of uniform *Borel* sets of the space $\mathbb{R} \times \mathbb{Q}$ (consisting of points with rational ordinates) and constructible Borel codes. We expect that the obtained results will find application in modern research in descriptive set theory.

We also expect that the methods of effective transfinite constructions used in this paper will make some contribution to the modern theory of generalized computability on uncountable structures and to the theory of information transmission between structures on ordinals and Borel structures related to the real line, as in the recent papers [10] and [11].

In conclusion, we pose a problem. Let is return to assertion (III) of Theorem 1. The ordinal $\mathbf{\Xi}$ considered there can be regarded as a separator between the lengths of effectively existing $\lhd$-increasing chains of uniform Borel sets and those of effectively nonexisting ones. Here effectiveness is understood in terms of the existence of constructible sequences of Borel codes for these chains. However, there also exist other versions of effectiveness, such as, e.g., ordinal definability. The following problem arises.

Problem. *Construct a model of set theory in which $\mathbf{\Xi} < \omega_2$ and there are no ordinal definable (or even real-ordinal definable) chains of uniform Borel sets of length $\geq \mathbf{\Xi}$.*

ACKNOWLEDGMENTS

The authors are grateful to Mirna Džamonja for interesting discussions and valuable comments. The authors thank the anonymous referee for valuable comments and suggestions, which helped to substantially improve the exposition.

FUNDING

This work was performed in the framework of state assignment of the Ministry of Science and Higher Education of the Russian Federation to the Institute for Information Transmission Problems of Russian Academy of Sciences (Kharkevich Institute).

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. N. N. Luzin, *On Some New Results of Descriptive Function Theory* (Izd. Akad. Nauk SSSR, Moscow–Leningrad, 1935) [in Russian].
2. N. Lusin and P. Novikoff, "Choix effectif d'un point dans un complémentaire analytique arbitraire, donné par un crible," Fundam. Math. **25**, 559–560 (1935).
3. P. Novikoff, "Sur la séparabilité des ensembles projectifs de seconde classe," Fundam. Math. **25**, 459–466 (1935).
4. P. Novikoff, "Les projections des complémentaires analytiques uniformes," Rec. Math. [Mat. Sbornik] N. S. **2(44)** (1), 3–16 (1937).
5. P. Novikoff, "Sur quelques relations entre les families des ensembles projectifs de classe 2 et des projections des complémentaires analytiques uniformes," Izv. Akad. Nauk SSSR Ser. Mat. **1** (2), 231–252 (1937).
6. T. Jech, *Set Theory*, in *Springer Monogr. Math.* (Springer-Verlag, Berlin–Heidelberg–New York, 2003).
7. V. G. Kanovei and V. A. Lyubetskii, "On some classical problems of descriptive set theory," Russian Math. Surveys **58** (5), 839–927 (2003).
8. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory: Absolutely Unsolvable Classical Problems* (MTsNMO, Moscow, 2013) [in Russian].
9. H. Lebesgue, "Sur les fonctions représentables analytiquement," J. de Math. **1**, 139–216 (1905).
10. J. D. Hamkins and A. Lewis, "Infinite time Turing machines," J. Symbolic Logic **65** (2), 567–604 (2000).
11. *Effective Mathematics of the Uncountable*, in *Lect. Notes Log.*, Ed. by N. Greenberg, J. D. Hamkins, D. Hirschfeldt, and R. Miller (Cambridge Univ. Press, Cambridge, 2013), Vol. 41.

Publisher's Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations. AI tools may have been used in the translation or editing of this article.