P. S. Novikov's research on descriptive set theory

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Théorème (Lebesgue). — *Une fonction* uniforme *définie implicitement à l'aide d'expression analytique est exprimable analytiquement d'une manière explicite* [3]).

54. **Fonctions implicites multiformes. Les recherches de M. Novikoff.** — Les choses sont plus compliquées dans le cas des fonctions implicites *multiformes.*

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1. Investigation of multivalued Borel functions.

2. Separation and Inseparation for the first and second projective levels and **C**-sets, and multiple Separation.

3. The index comparizon and the minimal index technique.

4. Effective uniformization of **CA** sets.

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- Notation
- Planar Borel sets
- Borel $(\mathbf{A}, \mathbf{B})$-Separation and $(\mathbf{CA}, \mathbf{B})$-Inseparation
- Indices
- 2nd projective level Separation
- Minimal indices: the proof tool
- Separation for $\mathbf{C}$ sets
- Multiple Separation
- Earlier uniformization theorems
- Uniformization of $\mathbf{CA}$ sets
- Gödel: constructible sets
- Projective sets under constructibility
- Independent problems
- Novikov's most significant contributions, summary
- Novikov: classification of the effective
- Vertical ordering of Borel curves
- Representation of 3rd class ordinals, a problem
- End

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Examples: $\mathbb{R}$, the real line, $\mathcal{N} = \omega^\omega$ or $= \mathbb{R} \setminus \mathbb{Q}$, the Baire space, *etc*.

Borel sets = **B**-sets = $\mathbf{\Delta}^1_1$ sets (in Polish spaces).

A-sets = analytic sets = Suslin sets = $\mathbf{\Sigma}^1_1$ sets.

CA-sets = coanalytic sets = co-Suslin sets = $\mathbf{\Pi}^1_1$ sets.

Then by induction projective classes are defined:

$\mathbf{A}_2 = \mathbf{\Sigma}^1_2$ sets = continuous images of **CA** sets;

$\mathbf{CA}_2 = \mathbf{\Pi}^1_2$ sets = complements of $\mathbf{A}_2$ sets;

$\mathbf{B}_2 = \mathbf{\Delta}^1_2 = \mathbf{A}_2 \cap \mathbf{CA}_2$ = sets simultaneously $\mathbf{A}_2$ and $\mathbf{CA}_2$;

and so on: $\mathbf{A}_n = \mathbf{\Sigma}^1_n$, $\mathbf{CA}_n = \mathbf{\Pi}^1_n$, $\mathbf{B}_n = \mathbf{\Delta}^1_n = \mathbf{A}_n \cap \mathbf{CA}_n$, $n \geq 3$.

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$y = f(x)$ – explicit, or single-valued function
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Let's identify functions with their graphs, *i. e.*, planar sets in $\mathbb{R} \times \mathbb{R}$ or in $\mathbb{X} \times \mathbb{Y}$, the product of any two uncountable Polish spaces:

multi-valued Borel function $=$ *any* Borel set $P \subseteq \mathbb{R} \times \mathbb{R}$;
single-valued Borel function $=$ a *uniform* Borel set $P \subseteq \mathbb{R} \times \mathbb{R}$.

A set $P \subseteq \mathbb{R} \times \mathbb{R}$ is:

uniform iff $\forall\, x \,\exists$ at most one y $(\langle x, y\rangle \in P)$, *i. e.*, any vertical section $P_x = \{y : \langle x, y\rangle \in P\}$ of P contains ≤ 1 point.

ctble-form iff $\forall\, x \,\exists$ at most $\aleph_0$ y $(\langle x, y\rangle \in P)$, *i. e.*, any vertical section P_x of P contains $\leq \aleph_0$ points.

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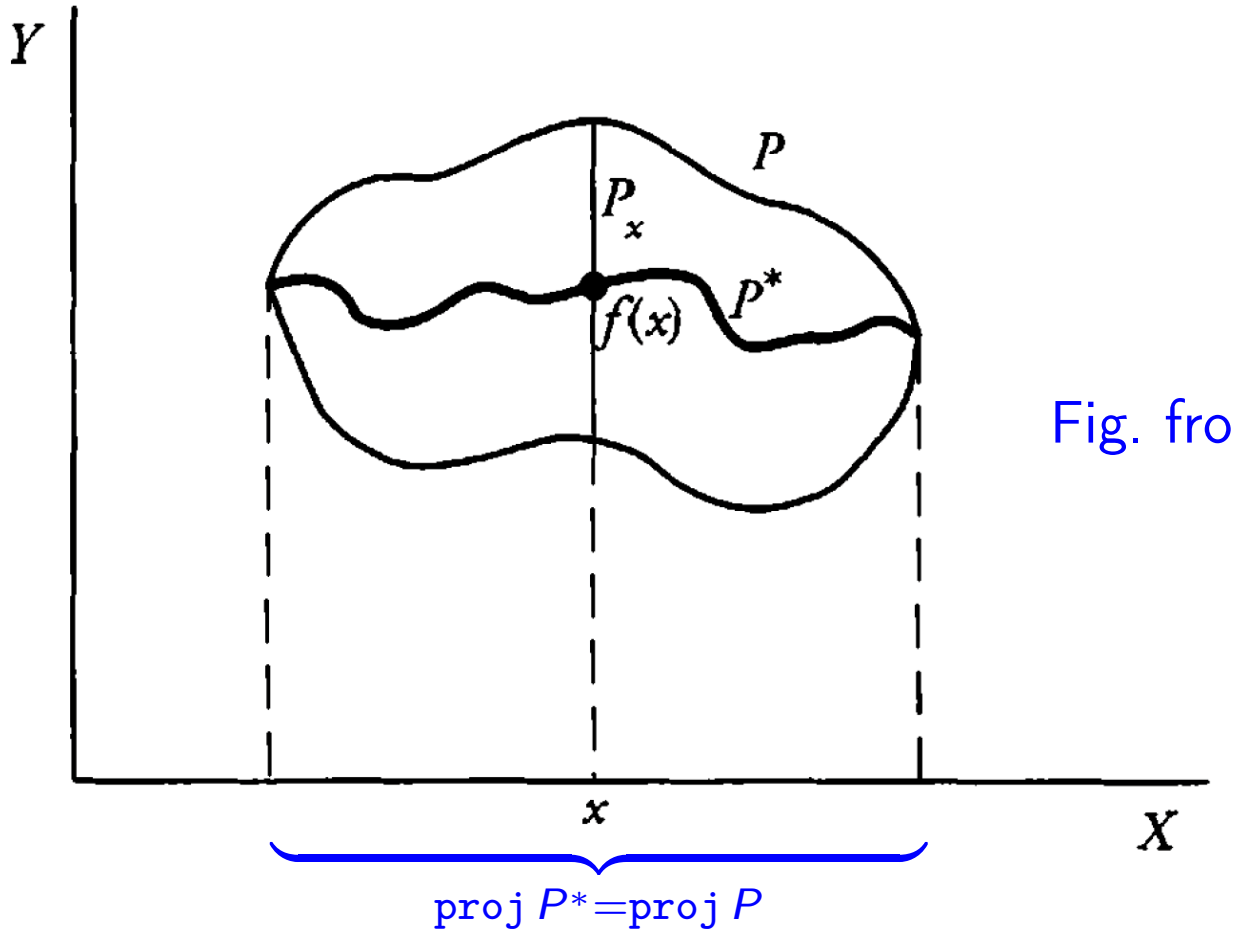


Fig. from Kechris *Class. DST*

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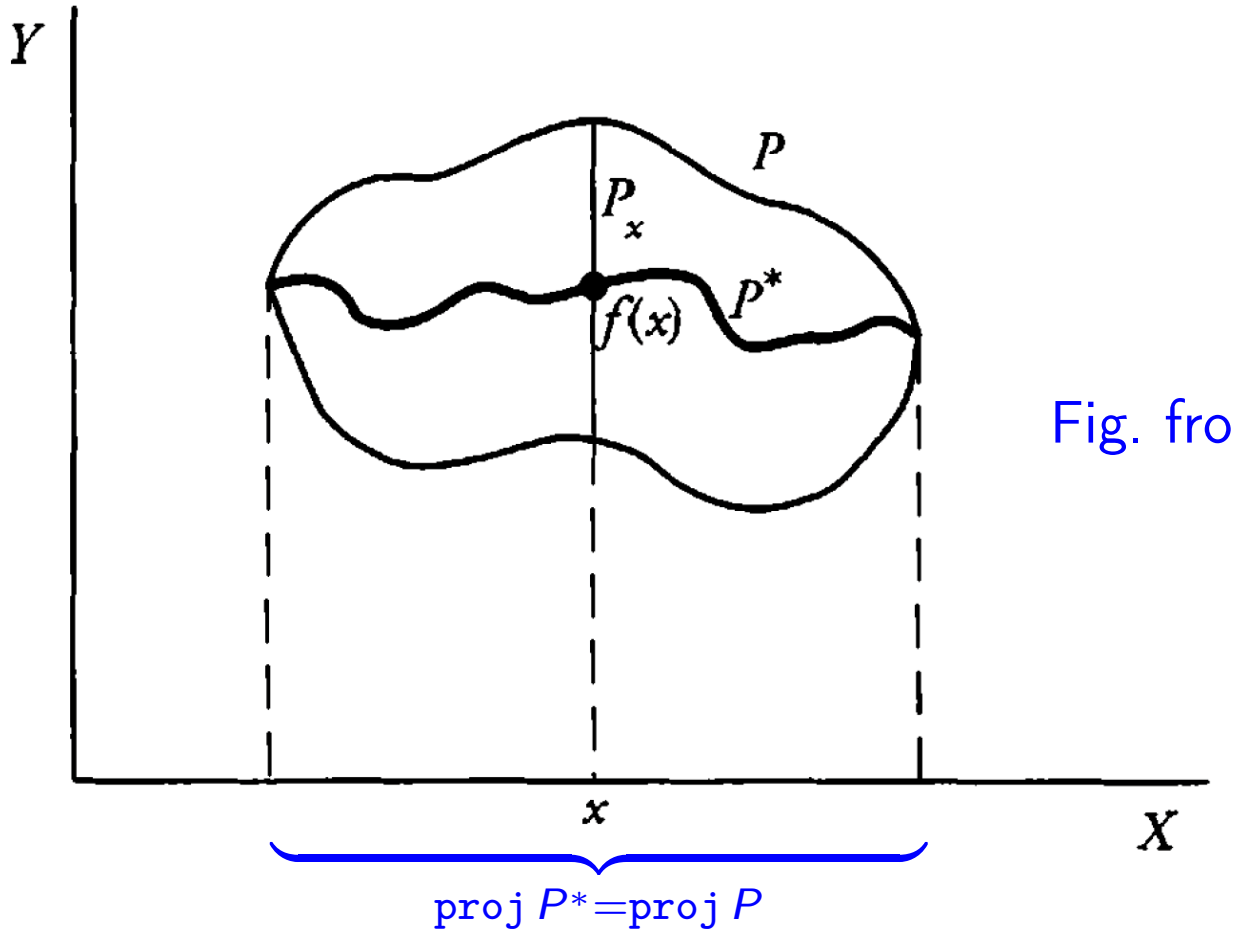


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Theorem (Novikov – [Luzin 1927])

Let $P \subseteq \mathbb{R} \times \mathbb{R}$ be a planar ctble-form Borel set. Then: **I**, **II**, **III**:

I *the horizontal projection $\mathtt{proj}\, P$ is Borel — not true for arbitrary Borel planar sets P as then $\mathtt{proj}\, P$ can be any* **A***-set*;

II *P is a ctble union $P = \bigcup_n P^n$ of uniform Borel sets P^n*;

III Borel "ctble-form" uniformization *P can be uniformized by a uniform Borel subset $P^* \subseteq P$, i.e., $\mathtt{proj}\, P^* = \mathtt{proj}\, P$.*

In addition,

IV *there exists a Borel set $P \subseteq \mathbb{R} \times \mathbb{R}$ satisfying $\mathtt{proj}\, P = \mathbb{R}$ and* **non***-Borel-uniformizable — then P is* **not** *ctble-form by* **III**.

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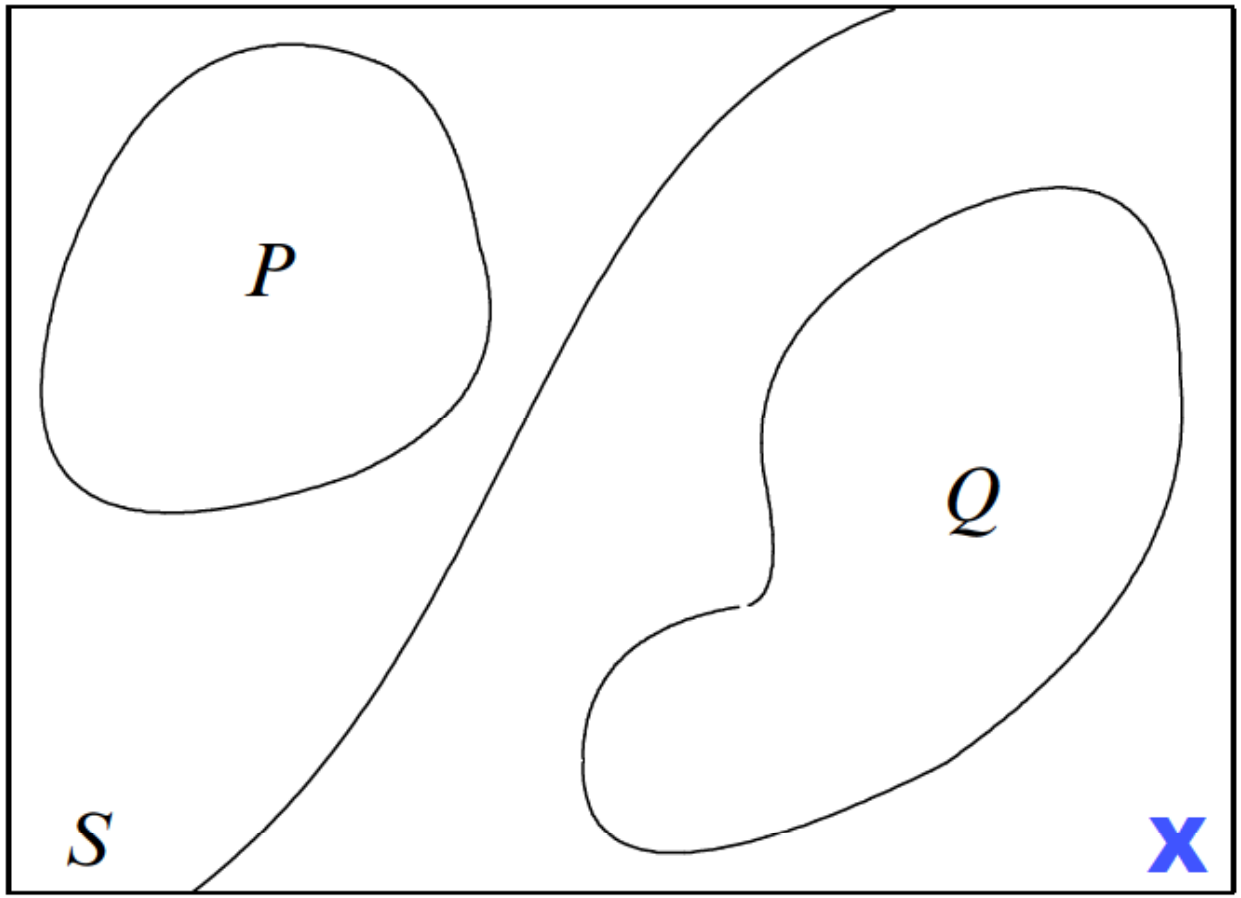
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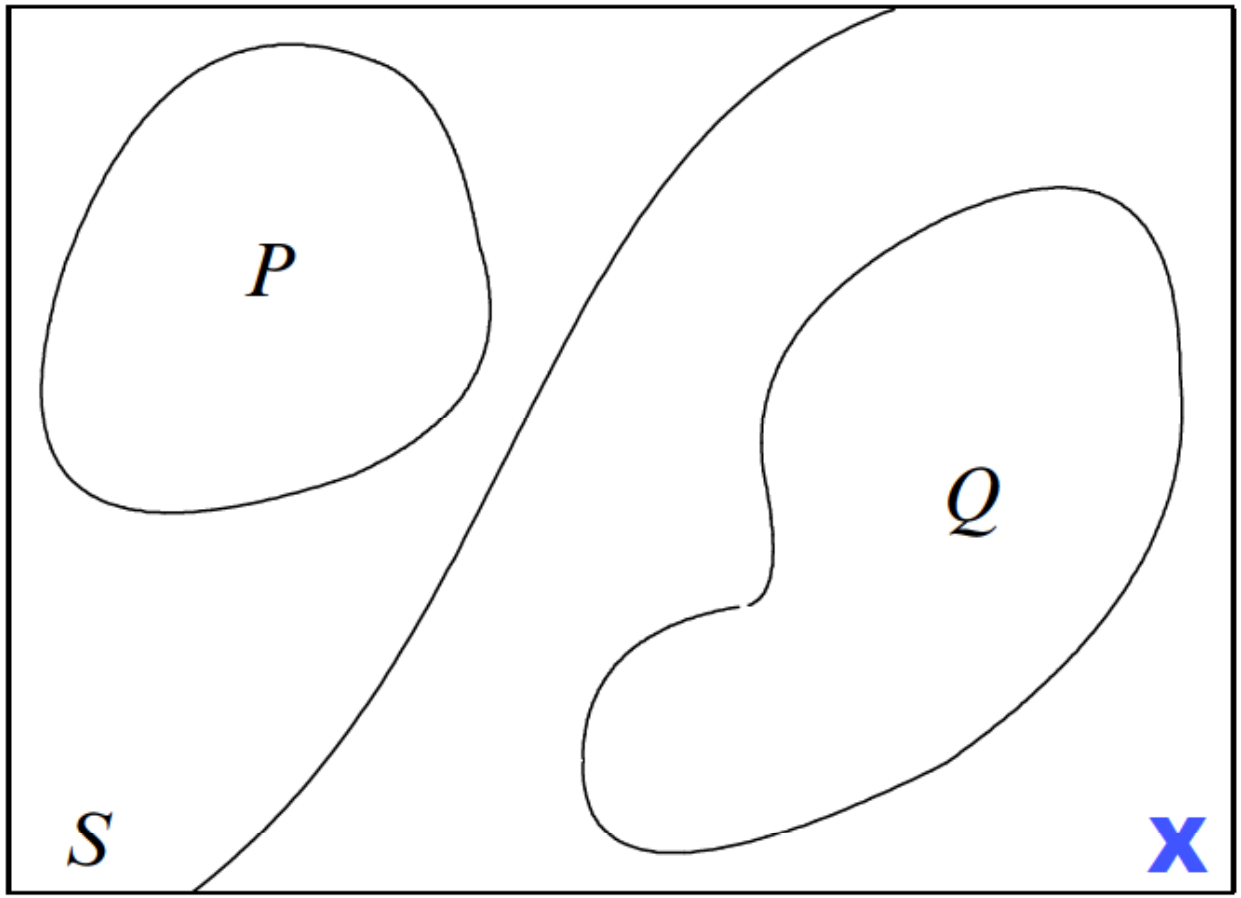
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Sketch of (A, B)-Sep. Consider the complementary **CA**-set $C = Q^{\complement} = \mathbb{R} \setminus Q$. There is an index function $\mathbf{ind}_C : C \to \omega_1$ s. t.:

1. the preimage $C_\alpha = \mathbf{ind}_C^{-1}(\alpha) = \{x \in C : \mathbf{ind}_C(x) = \alpha\}$ (called the αth constituent of C) is a Borel set for each $\alpha < \omega_1$;
2. index boundedness: any **A**-set $P \subseteq C$ is covered by ctbly many constituents, *i. e.*, $P \subseteq C_{<\lambda} = \bigcup_{\alpha<\lambda} C_\alpha$ for some $\lambda < \omega_1$.

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Let $\mathbb{Q}$ = rationals (discrete), and $\mathcal{P}(\mathbb{Q}) = \{u : u \subseteq \mathbb{Q}\} = 2^{\mathbb{Q}}$, a Polish (compact) space homeomorphic to the Cantor discontinuum 2^ω.

Let $C \subseteq \mathbb{X}$ be a **CA** set in a Polish $\mathbb{X}$. There is a Borel map $f_C : \mathbb{X} \to \mathcal{P}(\mathbb{Q})$ such that $x \in C$ $\iff$ $f_C(x)$ is wellordered in $\mathbb{Q}$. (Luzin, Suslin 1917.)

If $x \in C$ then put $\mathbf{ind}_C(x) =$ the order type of the set $f_C(x) \subseteq \mathbb{Q}$.

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Theorem (Novikov 1935)

($\mathbf{CA}_2, \mathbf{B}_2$)-Separation: *If $P, Q \subseteq \mathbb{R}$ are disjoint* $\mathbf{CA}_2$*-sets, then there is a* $\mathbf{B}_2$ *set S which separates P, Q.*

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The result was highly surprising: the classes $\mathbf{A}_2$ and $\mathbf{CA}_2$ switch places, relative to the separation properties, with respect to the classes **A** and **CA**.

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To establish the $(\mathbf{CA}_2, \mathbf{B}_2)$-Separation theorem, Novikov developed the minimal index technique. It works as follows.

Consider disjoint $\mathbf{CA}_2$ sets $P, Q \subseteq \mathbb{R}$ and the complementary $\mathbf{A}_2$ sets $P^{\complement} = \mathtt{proj}\, U$, $Q^{\complement} = \mathtt{proj}\, V$, projections of $\mathbf{CA}$-sets $U, V \subseteq \mathbb{R} \times \mathbb{R}$.

The index functions $\mathbf{ind}_U : U \to \omega_1$, $\mathbf{ind}_V : V \to \omega_1$ satisfy 1, 2 above.

If $x \in P^{\complement}$ then the vertical section $U_x = \{y : \langle x, y\rangle \in U\} \neq \varnothing$.
Put $\mathbf{Mind}_U(x) = \min_{y \in U_x} \mathbf{ind}_U(x, y)$: the minimal index.

If $x \in P$ then put $\mathbf{Mind}_U(x) = \omega_1$, so overall $\mathbf{Mind}_U : \mathbb{R} \to \omega_1 \cup \{\omega_1\}$.

Define $\mathbf{Mind}_V(x) : \mathbb{R} \to \omega_1 \cup \{\omega_1\}$ the same way. Then

I the set $S = \{x \in \mathbb{R} : \mathbf{Mind}_V(x) < \mathbf{Mind}_U(x)\}$ is $\mathbf{B}_2$ — difficult, **minimal index comparizon – Novikov**!

II the set S separates P from Q — easy by construction.

Therefore the $\mathbf{B}_2$ set S separates P from Q, as required. □

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The hierarchy $\mathbf{C} = \bigcup_{\alpha<\omega_1} \mathbf{C}_\alpha$:

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Theorem (Novikov, generalization of Luzin's (A, B)-Separation)

Assume that $\alpha < \omega_1$. Let P, Q be $\mathbf{C}_\alpha$-sets in a fixed Polish space, and $P \cap Q = \varnothing$. Then there is a set S in $\mathbf{C}_\alpha \cap \overline{\mathbf{C}}_\alpha$ which separates P, Q.

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Several multiple versions of Luzin's $(\mathbf{A}, \mathbf{B})$-Separation theorem were established in the 1930s. Some of them were rather immediate corollaries, as *e. g.* the following theorem.

Theorem

Let $P_0, P_1, P_2, \dots$ be pairwise-disjoint $\mathbf{A}$-sets in any given Polish space. Then there exist pairwise-disjoint Borel supersets $S_n \supseteq P_n$.

Indeed let a Borel S_{nk} separate P_n from P_k, that is, $P_n \subseteq S_{nk}$ and $P_k \cap S_{nk} = \varnothing$. Put $S_n = \bigcap_k S_{nk}$.

The next result was much more tricky.

Theorem (Novikov, 1934)

Assume that $P_0, P_1, P_2, \dots$ are $\mathbf{A}$-sets in any given Polish space, and $\bigcap_n P_n = \varnothing$. Then there exist Borel supersets $X_n \supseteq P_n$ still satisfying $\bigcap_n X_n = \varnothing$.

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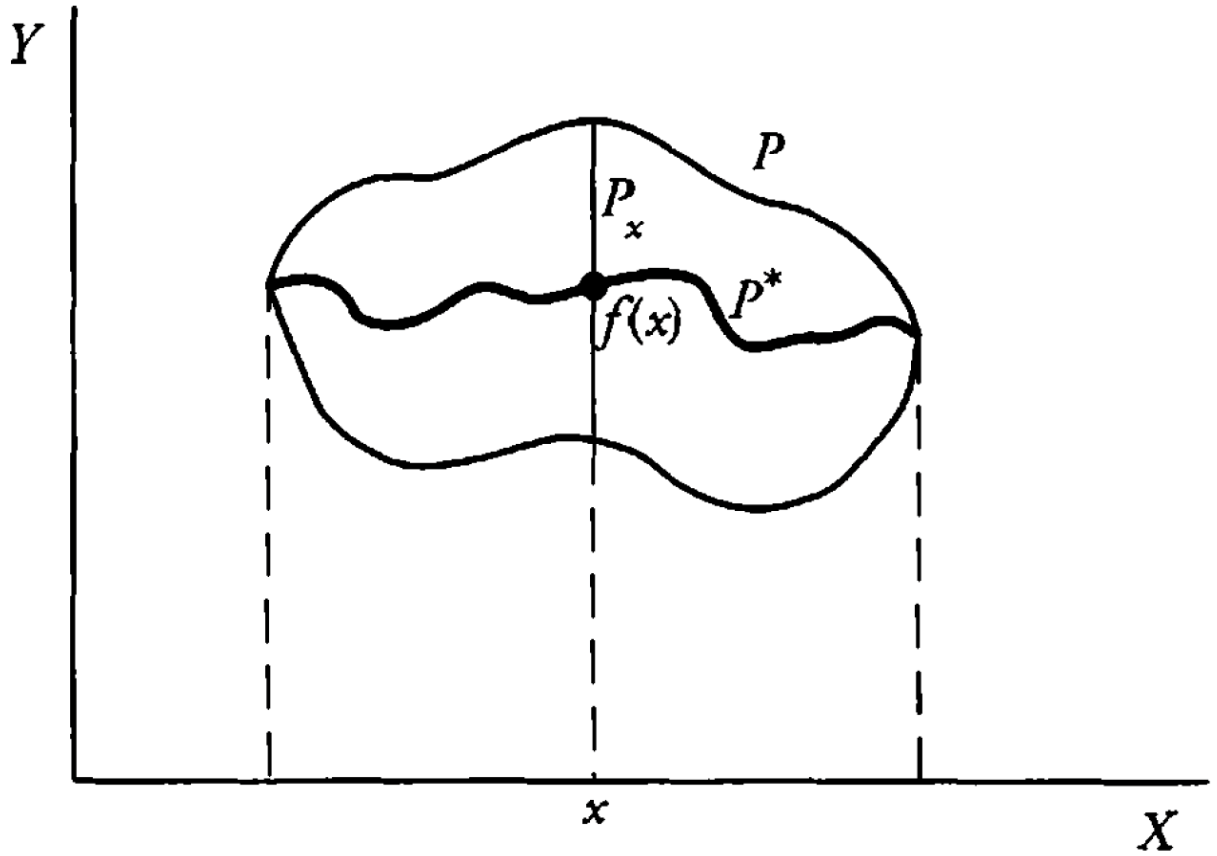
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A set $P^* \subseteq P$ uniformizes P iff

1) P^* is uniform (the graph of $y = f(x)$), and

2) $\mathtt{proj}\, P^* = \mathtt{proj}\, P$.

1 Closed sets P in the Baire plane $\mathcal{N} \times \mathcal{N}$ can be uniformized by taking the lowest branch in each vertical section P_x, and the uniformizing set is **CA** in this case. Recall that $\mathcal{N} = \omega^\omega$ is the Baire space.

2 Any Borel set can be uniformized by a **CA** set by 1 and since Borel sets are continuous 1–1 images of closed subsets of the Baire space.

3 (Luzin–Yankov–von Neumann) Any **A** set can be uniformized by a set of the form $\bigcap_k \bigcup_n (A_{kn} \smallsetminus A'_{kn})$, where all A_{kn}, A'_{kn} are **A** sets – by 1.

Open problem: *is* 3 *still true with* $\bigcap_k \bigcup_n$ *changed to* $\bigcup_k \bigcap_n$?
See 4G.19 and 4F.22 in Moschovakis *DST* for a partial negative answer.

4 (Novikov – Luzin) Any planar Borel set with countable sections can be uniformized by a Borel set.

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1 Closed sets P in the Baire plane $\mathcal{N} \times \mathcal{N}$ can be uniformized by taking the lowest branch in each vertical section P_x, and the uniformizing set is **CA** in this case. Recall that $\mathcal{N} = \omega^\omega$ is the Baire space.

2 Any Borel set can be uniformized by a **CA** set by 1 and since Borel sets are continuous 1–1 images of closed subsets of the Baire space.

3 (Luzin–Yankov–von Neumann) Any **A** set can be uniformized by a set of the form $\bigcap_k \bigcup_n (A_{kn} \setminus A'_{kn})$, where all A_{kn}, A'_{kn} are **A** sets – by 1.

Open problem: *is* 3 *still true with* $\bigcap_k \bigcup_n$ *changed to* $\bigcup_k \bigcap_n$?
See 4G.19 and 4F.22 in Moschovakis *DST* for a partial negative answer.

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Every planar **CA** *set can be effectively uniformized — in fact can be uniformized by an* $\mathbf{A}_2$ *set.*

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$$\left.\begin{array}{lcl} \mathbf{L}_0 & = & \varnothing \\ \mathbf{L}_{\alpha+1} & = & \mathbf{Def}(\mathbf{L}_\alpha) \qquad (^*) \\ \mathbf{L}_\lambda & = & \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha, \text{ for every limit } \lambda \\ \mathbf{L} & = & \bigcup_{\alpha\in \mathtt{Ord}} \mathbf{L}_\alpha \ - \ \text{all constructible sets} \end{array}\right\}$$

Gödel 1939–40: the constructible hierarchy

$(^*)$ $\mathbf{Def}(\mathbf{L}_\alpha) =$ all sets $X \subseteq \mathbf{L}_\alpha$ definable over $\mathbf{L}_\alpha$ by $\in$-formulas with parameters from $\mathbf{L}_\alpha$.

Theorem (Gödel 1939-40, translation УМН 1948)

In **ZFC**, *the class* **L** *of all constructible sets still satisfies* **ZFC**.

"$\mathbf{V} = \mathbf{L}$" is the extra axiom saying that every set belongs to **L**.

Corollary

If Φ *is a consequence of "*$\mathbf{V} = \mathbf{L}$*" in* **ZFC** *then* Φ *does not contradict* **ZFC**, *or in other words, the negation* $\neg\Phi$ *is not provable.*

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Theorem (Novikov 1951; ZFC + "V = L")

1. *There exists a* $\mathbf{B}_2$*-set* $X \subseteq \mathbb{R}$*, not Lebesgue measurable.*
2. *There exists an uncountable* $\mathbf{CA}$*-set* $X \subseteq \mathbb{R}$*, without perfect subsets.*
3. *Every projective level* n *from some* $n_0 \geq 3$ *on, satisfies* $(\mathbf{CA}_n, \mathbf{B}_n)$*-Separation and* $(\mathbf{A}_n, \mathbf{B}_n)$*-Inseparation – similar to level* 2*.*

Corollary

1 + 2 + 3 *is consistent with* **ZFC***.*

1 + 2 Gödel 1940 sans proofs. In 3 $n_0 = 3$ by Addison 1959.

Theorem (Solovay 1970)

$\exists$ *a model of* **ZFC** *in which:* 1, 2 *fail, moreover, all projective sets are Lebesgue measurable, all uncountable projective sets have perfect subsets.*

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Corollary (from Gödel–Novikov and Solovay)

Each of the following two hypotheses is **independent** *of* **ZFC** :

1. *there exists a* $\mathbf{B}_2$*-set* $X \subseteq \mathbb{R}$*, not Lebesgue measurable*;
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In other words, we can neither prove nor disprove each of the two hypotheses by means of the ordinary set theoretic arguments. (This independence was predicted by N. N. Luzin in 1925.)

However rather unexpectedly the hypotheses 1 and 2 are **not** independent from each other!

Theorem (Lyubetsky 1970)

$1 \Longrightarrow 2$ *in* **ZFC**,
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1. Investigation of many-valued Borel functions.

2. Principles of separability and inseparability for the first and second projective levels.

3. The index comparizon and the minimal index technique.

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32. *Попытка классификации эффективного.* Изложенные результаты П. С. Новикова составляют содержание трех первых глав его диссертации, представленной в рукописи Академии Наук СССР на соискание звания доктора чистой математики.

последняя четвертая глава работы

Основная идея этой главы состоит в розыскании такой классификации множеств, которая охватила бы не только проективные, но и вообще все множества, которые можно индивидуально определить [1].

3rd class ordinals $= \{\alpha : \alpha < \omega_2\}$, or sometimes more formally $\{\alpha : \omega_1 \leq \alpha < \omega_2\}$.

Novikov proposed a geometric representation of such ordinals, as the basis of his "classification of the effective".

Novikov apparently planned to use this representation of ordinals as a tool for a classification of pointsets that can be individually defined (classification of the effective).

A summary by Luzin (1935) demonstrates some semblance of those Novikov's ideas and the theory of constructible sets by Gödel, subsequently appeared in 1939–1940.

Borel *curves* (*i. e.*, **uniform** sets) $P \subseteq \mathbb{R} \times \mathbb{R}$ are considered.

If P, Q are such curves then define $P \lhd Q$ (P is below Q) iff

1. if $\langle x, y\rangle \in P$ and $\langle x, y'\rangle \in Q$ then $y < y'$, and
2. $\mathtt{proj}\, P \subseteq \mathtt{proj}\, Q$ **or** $\mathtt{proj}\, Q \subseteq \mathtt{proj}\, P$.

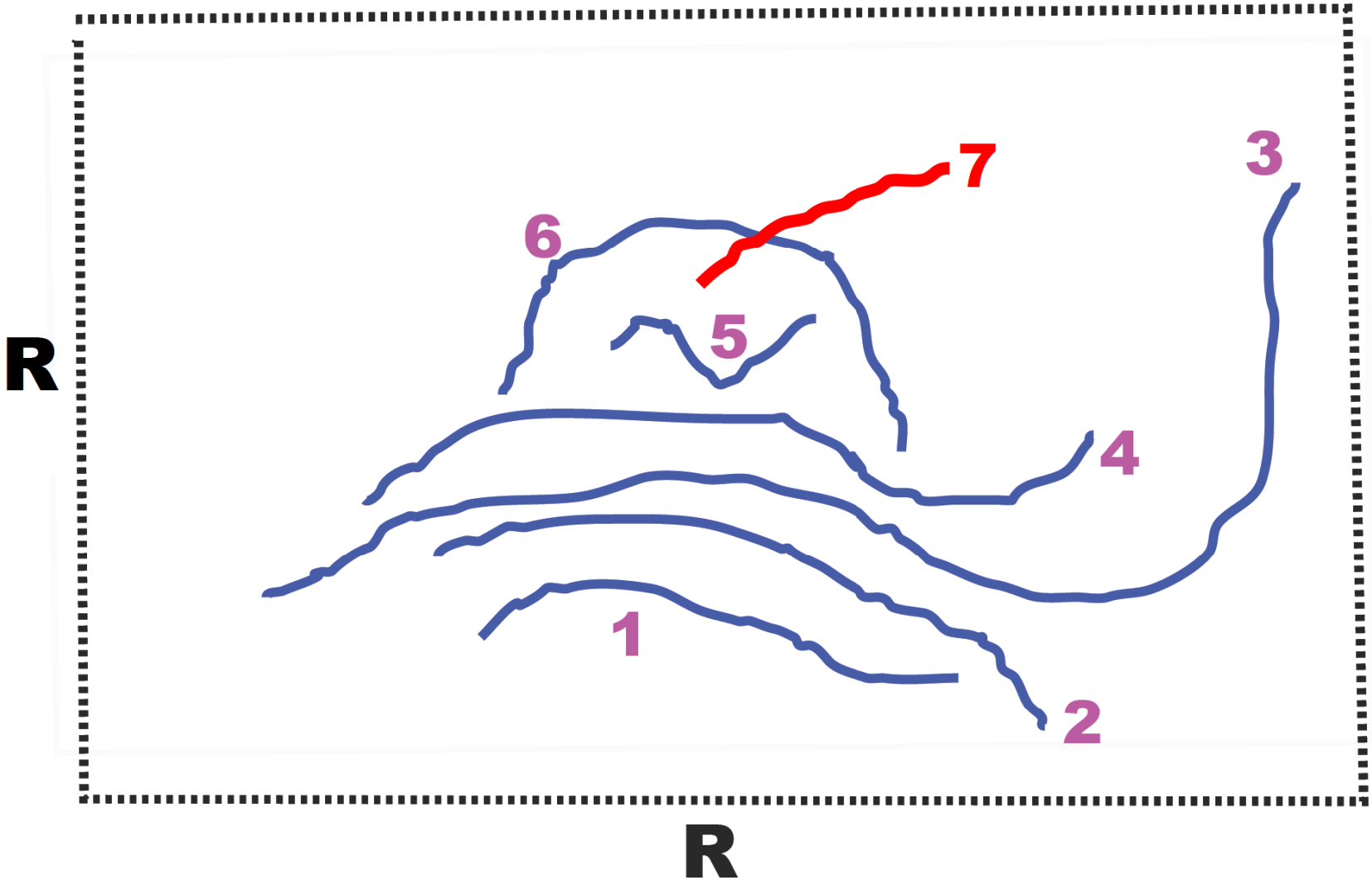
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$1 \lhd 2 \lhd 3 \lhd 4 \lhd 5 \lhd 6$

7 is $\lhd$-incomparable with 5 and 6.

Problem (Novikov–Luzin 1935)

Given $\alpha < \omega_2$, is there a collection $\mathcal{B}$ of Borel curves in $\mathbb{R} \times \mathbb{R}$, well-ordered by $\lhd$ similar to α ?

Theorem (K–Lyubetsky 2026 to appear)

If $\alpha < \omega_2$ then there exists a collection $\mathcal{B}$ of Borel curves in $\mathbb{R} \times \mathbb{R}$, even in $\mathbb{R} \times \mathbb{Q}$ with only rational ordinates, well-ordered by $\lhd$ similar to α.

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The speaker thanks
the Novikov-125 Organizers
for the opportunity to give this talk

The speaker thanks
everybody
for their interest and patience

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