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An Effectively Computable Projective Invariant

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Abstract. We consider a projective invariant of hypersurfaces over a field of characteristic zero. The invariant can be computed in polynomial time with generalized register machines. It has been computed for certain low-dimensional hypersurfaces. One can effectively recognize some plane cubic curves as well as some cubic surfaces. Our method allows to recognize some cubic hypersurfaces with reducible Hessian.

The aim of this work is to introduce an effectively computable projective invariant of hypersurfaces over the field of complex numbers. The simplest case of hypersurface is a plane curve. Every plane cubic curve is projectively equivalent to a curve whose affine part is given by a Weierstrass equation $y^2 = x^3 + px + q$. This curve is singular iff the discriminant of the right-hand univariate polynomial vanishes, that is, $-4p^3 - 27q^2 = 0$. Classification of cubic surfaces is more complicated. A cubic surface is cyclic when there exists a Galois cover of degree 3 over projective plane. A cyclic cubic surface is projectively equivalent to a surface defined by a form of the type $x_0^3 + x_1^3 + x_2^3 + x_3^3 + px_1x_2x_3$, where p is a parameter [1]. The general cubic surface depends on four parameters. It can be defined by the Emch normal form [2]. But this normal form has been found earlier [3, 4].

Let us consider generalized register machines over a field of characteristic zero $(\mathbb{K}, 0, 1, +, -, \times)$. Each register contains an element of $\mathbb{K}$. There exist index registers containing nonnegative integers. The running time is said polynomial, when the total number of operations performed before the machine halts is bounded by a polynomial in the number of registers occupied by the input. Initially, this number is placed in the zeroth index register [5]. If a polynomial serves as an input, then its coefficients are written into registers.

For $n \geq 2$, let us consider a square-free form $f(x_0, \ldots, x_n)$ of degree $d \geq 2$. Let us fix a point U with homogeneous coordinates $(u_0 : \cdots : u_n)$. Every straight line passing through the point U consists of points with homogeneous coordinates $((x_0 - u_0)t + u_0 s : \cdots : (x_n - u_n)t + u_n s)$, where $(s : t)$ are homogeneous coordinates inside the line. The restriction of the form f is a binary form denoted by $r(s, t)$. Let us denote by $D[f, U]$ the discriminant of the binary form $r(s, t)$. If $x_0 = 1$, then the discriminant is a inhomogeneous polynomial in affine coordinates x_k. In the general case, its degree is equal to $d^2 - d$. If a straight line either is tangent to the hypersurface or passes through a singular point, then the discriminant of the form $r(t, s)$ vanishes. So, if the point U is not any singular point of the hypersurface, then the polynomial $D[f, U](x_1, \dots, x_n)$ defines a cone with U as a vertex. If U is singular, then $D[f, U]$ vanishes identically.

The set of polynomials of the type $D[f, U]$ for all points U generates a linear subspace W_f of the ambient linear space of all inhomogeneous polynomials of degree $d^2 - d$ in n variables. The dimension of the ambient linear space is equal to

$$w(n, d) = \frac{(n + d^2 - d)!}{n!(d^2 - d)!}.$$

For every irreducible form f, the dimension $\dim W_f$ is projectively invariant. If $d \geq 3$ and n is sufficiently large, then $\dim W_f < w(n, d)$, that is, W_f is a proper subspace of the ambient linear space. If the rank of a quadratic form f is equal to n, then the equality $\dim W_f = w(n, 2)$ holds. For given n and d, the dimension $\dim W_f$ considered as a function of coefficients of f is lower semi-continuous [6]. Thus, if there exists a form $f(x_0, \dots, x_n)$ of degree d such that $\dim W_f = w(n, d)$, then for almost every form $f(x_0, \dots, x_n)$ of degree d, the equality $\dim W_f = w(n, d)$ holds too.

Let be given a square-free polynomial $f(x_1, \dots, x_n)$. In accordance with [6], in the expansion of the polynomial $D[f, U]$ in powers of coordinates of the point U, each coefficient belongs to the linear subspace W_f. These polynomials in variables $x_1, \dots, x_n$ span whole linear subspace W_f. Thus, there exists a polynomial time algorithm to compute the dimension of the linear subspace W_f.

It is sufficient to calculate the rank of a matrix whose order equals $w(n, d)$. It requires $O(w^\omega)$ multiplications, where ω denotes the matrix multiplication exponent [7, 8]. In small dimensions, the rank can be calculated with computer algebra system software.

We have computed $\dim W_f$ for certain plane curves ($n = 2$). In this case, the linear subspace W_f can be improper. But it is small for the Fermat type curves, where $F_2 = x_0^d + x_1^d + x_2^d$.

d	2	3	4	5	6	7	8	9	10
$w(2, d)$	6	28	91	231	496	946	1653	2701	4186
$\dim W_{F_2}$	6	26	82	207	446	856	1506	2477	3862

If $f(x_0, x_1, x_2)$ defines a singular curve, then the strict inequality $\dim W_f < w(2, d)$ holds. For almost every f of degree $d \leq 7$, the equality $\dim W_f = w(2, d)$ holds. For all $d \leq 7$, the equality $\max\limits_{f(x_0, x_1, x_2)} \dim W_f = w(2, d)$ holds. In particular, the equality holds at forms of the type $f = x_0^d + x_1^d + x_2^d + (x_0 + x_1 + x_2)^d$. We guess that it holds for every larger degree too.

Let us consider cubic forms of the Fermat type $F_n = x_0^3 + \cdots + x_n^3$. The polynomial $D[F_n, U](x_1, \dots, x_n)$ is equal to the discriminant of a binary form of the type $at^3 + bt^2 s + pts^2 + qs^3$ whose coefficients are sums of univariate polynomials, that is, $a = a_1(x_1)+\cdots+a_n(x_n)$, $b = b_1(x_1)+\cdots+b_n(x_n)$, $p = p_0+p_1x_1+\cdots+p_nx_n$, and q is a constant. So, every monomial of $D[F_n, U]$ depends on at most four variables. Thus, $\dim W_{F_n} = O(n^4)$.

For $n \le 9$, the equation $\dim W_{F_n} = \frac{1}{4}n^4 + \frac{5}{6}n^3 + \frac{9}{4}n^2 + \frac{8}{3}n + 1$ holds.

We have also computed $\dim W_f$ for certain cubic hypersurfaces. For $n \le 3$, this result found by symbolic computations with parameters, where every parameter can be considered as a transcendental number.
For $n \ge 4$, $\dim W_f$ was only computed for certain cubic forms. They provide the lower bound on the maximum value of $\dim W_f$ for given n. For cubic forms $f(x_0, \dots, x_n)$, we guess that the maximum dimension is

$$\max_f \dim W_f = n + \dim W_{F_n} = \frac{1}{12}(n+1)(3n^3 + 7n^2 + 20n + 12)$$

n	2	3	4	5	6	7	8	9
$w(n,3)$	28	84	210	462	924	1716	3003	5005
$\max_f \dim W_f$	28	75	≥ 169	≥ 336	≥ 608	≥ 1023	≥ 1625	
$\dim W_{F_n}$	26	72	165	331	602	1016	1617	2455

Let us consider cubic curves. In the general case, $\dim W_f = 28$ except the Fermat type curves and all singular curves. We computed the determinant of a matrix composed by coefficients of polynomials generating the linear space W_f. For the Weierstrass normal form $f = x_2^2x_0 + x_1^3 + px_1x_0^2 + qx_0^3$, the determinant is proportionate to the expression $p^4(4p^3+27q^2)^8$. If $p = 0$ and $q \ne 0$, then the curve is projectively equivalent to a curve of the Fermat type. If $4p^3+27q^2 = 0$, then the curve is singular, else it is smooth. For the Fermat cubic curve, $\dim W_{F_2} = 26$. In this case, the Hessian curve is the union of three straight lines. For an irreducible cubic curve with a node, $\dim W_f = 25$. For a cubic curve with a cusp, $\dim W_f = 21$. Therefore, one can distinguish between nodal and cuspidal curves.

For the general cubic surface, $\dim W_f = 75$. For the general cyclic cubic surface, $\dim W_f = 74$. For the Fermat cubic surface, $\dim W_{F_3} = 72$. So, if the Hessian surface contains a plane, then $\dim W_f$ is small. These results found by symbolic computations with parameters, where every parameter can be considered as a transcendental number. For some singular surfaces, the equality $\dim W_f = 75$ holds too. For example, it holds for $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + (x_0^2+x_1^2+x_2^2)x_3$, where p is transcendental; the point $(0:0:0:1)$ is singular. Therefore, the approach does not allow one to decide whether a given cubic surface is smooth.

If $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + x_1x_2x_3$, where p is transcendental, then $\dim W_f = 73$; the point $(0:0:0:1)$ is singular too. If $f = x_0^3 + px_0^2x_1 + x_1^3 + x_0x_2^2 + x_0^2x_3$, where p is transcendental, then $\dim W_f = 48$; the point $(0:0:0:1)$ is singular too.

Conjecture. *For every cubic form g with reducible Hessian, the inequality holds* $\dim W_g < \max_f \dim W_f$. *Moreover, the more factors exists in Hessian, the more gap is between two values* $\dim W_g$ *and* $\max_f \dim W_f$.

The computational results show that one can easily verify smoothness of almost every plane quartic curve as well as almost every quartic surface in $\mathbb{P}^3$ by means of computing $\dim W_f$. The method is also applicable to other plane curves. On the other hand, the same problem for cubic surfaces is hard enough because the proposed projective invariant is useless in this case. Nevertheless, one can recognize singularities of some types. We also assume that our method allows to recognize cubic hypersurfaces with reducible Hessian in deterministic polynomial time.

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