On Some Submatrix of the Macaulay Matrix

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Abstract. We consider the generic-case complexity of the Gröbner basis of a zero-dimensional ideal in the ring of multivariate polynomials over a field. The ideal is generated by some linear functions as well as all univariate polynomials $x_k^2 - x_k$. The generic rank of an auxiliary matrix is also estimated.

Introduction

We assume three possible answers: the input may not only be accepted or rejected, but also an explicit notification of uncertainty of the choice is possible. In any case, the answer must be obtained in a finite time and without errors, and if an easily verifiable condition is met, then the notification of uncertainty can be issued only for a small fraction of inputs among all inputs of a given size. Such algorithms are called *generic* [1] or *errorless heuristics*.

Our algorithm can be considered as method to compute the Gröbner basis of a zero-dimensional ideal in the ring of multivariate polynomials over a field K. The ideal is generated by all univariate polynomials $x_k^2 - x_k$ and some linear functions.

On the Macaulay matrix definition as well as the Gröbner basis computation refer to [2, 3, 4, 5, 6]. In fact, we consider a submatrix of the Macaulay matrix.

Results

Let us consider a system of m linear equations in n variables:

$$\begin{cases} \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n + \alpha_{10} = 0 \\ \cdots \\ \alpha_{m1}x_1 + \cdots + \alpha_{mn}x_n + \alpha_{m0} = 0 \end{cases}.$$

Multiplying each linear equation by each of the variables and taking into account the equalities $x_k^2 = x_k$, which are satisfied with $\{0,1\}$-solutions, we obtain mn new equations of the second degree. In the general case, a new linearly independent linear equation can be derived from resulting quadratic equations.

Discarding the terms depending only on one variable, we obtain a set of mn bilinear forms, the coefficients of which form a matrix denoted by W. The rows correspond to the bilinear forms, and the columns correspond to monomials of the form $x_j x_k$ for $j < k$.

For $n = 3$ and $m = 1$, the 3×3 matrix

$$W = \begin{pmatrix} \alpha_{12} & \alpha_{13} & 0 \\ \alpha_{11} & 0 & \alpha_{13} \\ 0 & \alpha_{11} & \alpha_{12} \end{pmatrix}$$

is degenerate over a field of characteristic $\mathrm{char}(K) = 2$ because

$$\det(W) = -2\alpha_{11}\alpha_{12}\alpha_{13}.$$

Next, for $n = 5$ and $m = 2$, the 10×10 matrix W is degenerate over any field because $\mathrm{rank}(W) \leq 9$.

For $n = 7$ and $m = 3$, the 21×21 matrix W is also degenerate over any field because $\mathrm{rank}(W) \leq 18$. (The rank is computed with Maple.)

Theorem 1. *Let the matrix W be computed for m linear equations in n variables over a purely transcendental extension of the field K, where all coefficients α_{ij} are algebraically independent of each other. The rank of the matrix satisfies the inequality*

$$\mathrm{rank}(W) \geq mn - \frac{m(m+1)}{2}.$$

Hypothesis 1. *If the matrix W be computed for two linear equations in $n \geq 2$ variables, then the rank of the matrix satisfies the inequality* $\mathrm{rank}(W) \leq 2n - 1$.

For $2 \leq n \leq 9$, the hypothesis has been confirmed using Maple.

Theorem 2. *Let us assume Hypothesis 1 holds. If the matrix W be computed for m linear equations in $n \geq 2$ variables, then the rank of the matrix satisfies the inequality* $\mathrm{rank}(W) \leq mn - m + 1$.

Hypothesis 2. *If the matrix W be computed for m linear equations in n variables and the inequality $n \geq 2m + 1$ holds, then the rank of the matrix satisfies the inequality*

$$\mathrm{rank}(W) \leq mn - \frac{m(m-1)}{2}.$$

Theorem 3. *If $mn >$ rank(W) and the free terms α_{i0} are uniformly and independently distributed on the set $S \subset K$ of cardinality $\lceil 1/\varepsilon \rceil$, then there is no new linear equation with the probability not exceeding ε. Otherwise, the new linear equation can be found using $O(n^6)$ algebraic operations over the field K.*

Conclusion

So, for almost all systems of linear equations, if the number of equations is sufficiently large, then one can easily either find a $\{0,1\}$-solution, or prove that there is no such solution. The method is not applicable when the system has many $\{0,1\}$-solutions. Thus, we have a polynomial upper bound on the generic-case complexity, but not in the worst case.

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