Integer Matrices with Peculiar Leading Diagonal

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For simplicity, let integers be defined as rational integers.

Nevertheless, our results hold over any unique factorization domain, e. g.,

the ring of Gaussian integers $\{a + ib | a, b \in \mathbb{Z}\}$,

the ring of Eisenstein integers $\{a + \omega b | a, b \in \mathbb{Z}, \omega = (-1 + i\sqrt{3})/2\}$,

the ring of Kleinian integers $\{a + \lambda b | a, b \in \mathbb{Z}, \lambda = (-1 + i\sqrt{7})/2\}$,

the ring of polynomials over such a ring, e. g., $\mathbb{Z}[x, y]$, $\mathbb{Q}[x, y]$, $\mathbb{F}_p[x, y]$, and so on.

The rank of a matrix is defined over the field of fractions. It is sufficient to use symbolic computations over the ring **involving no fractions.**

The ring of multivariate polynomials over a field is not a Euclidean ring. There are hard methods involving either Gröbner bases or resultants. So, simple lower bounds on the rank are indispensable.

Let q denote an integer.

The first result uses auxiliary restriction on q, i. e., q is square-free.

Theorem 1. *Let prime numbers $p_1, \ldots, p_s$ be pairwise distinct, and $q = p_1 \cdots p_s$. Let $\boldsymbol{M}$ denote an $n \times n$ integer matrix in which no diagonal entry is divisible by q, and all entries above the leading diagonal are divisible by q. The rank of $\boldsymbol{M}$ is at least n/s.*

Proof. Any diagonal entry is not divisible by some number among $p_1, \ldots, p_s$. So, there is an index $1 \leq j \leq s$ such that at least n/s diagonal entries are not divisible by p_j. The corresponding principal submatrix is nondegenerate because its determinant does not vanish modulo p_j.

Thus, the rank of $\boldsymbol{M}$ is at least n/s.

Example. The rank of the matrix

$$\begin{pmatrix} p^{n-1} & p^n & \cdots & p^{2n-2} \\ p^{n-2} & p^{n-1} & \cdots & p^{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p & \cdots & p^{n-1} \end{pmatrix}$$

is equal to one.

Thus, for $q = p^n$, the naive generalization of Theorem 1 fails.

Next, let us set additional restrictions on the matrix $\boldsymbol{M}$.

Lemma 1. *Let p be prime, and $q = p^r$. If $\boldsymbol{M}$ is a square integer matrix in which no diagonal entry is divisible by q, and all entries outside the leading diagonal are divisible by q, then $\boldsymbol{M}$ is nondegenerate.*

Proof. Let us denote by m an integer such that the product of all diagonal entries is divisible by p^m, but not divisible by p^{m+1}.

$$0 \leq m \leq n(r-1).$$

The product of any n entries is divisible by p^m, and if at least one of these n entries does not belong to the leading diagonal, then the product is divisible by p^{m+1}. Thus, the determinant of the matrix $\boldsymbol{M}$ is equal to the sum of numbers of which exactly one is not divisible by p^{m+1}. Therefore, the determinant of the matrix $\boldsymbol{M}$ is not divisible by p^m, and it is nonzero.

Lemma 1 is an p-adic analogue of the Lévy–Desplanques theorem.

Any strictly diagonally dominant matrix is nondegenerate.

Theorem 2. *Let prime numbers $p_1, \ldots, p_s$ be pairwise distinct, and $q = p_1^{r_1} \cdots p_s^{r_s}$. Let $\boldsymbol{M}$ denote an $n \times n$ integer matrix in which no diagonal entry is divisible by q, and all entries outside the leading diagonal are divisible by q. The rank of M is at least n/s.*

Proof. Any diagonal entry is not divisible by some number among $p_1^{r_1}$, $\ldots$, $p_s^{r_s}$. So, there is an index $1 \leq j \leq s$ such that at least n/s diagonal entries are not divisible by $p_j^{r_j}$. The corresponding principal submatrix is nondegenerate according to Lemma 1.

Thus, the rank of $\boldsymbol{M}$ is at least n/s.

Theorem 3. *Let q have at least s prime divisors. For any integer n divisible by s, there is a symmetric integer matrix $\boldsymbol{M}$ of rank n/s in which no diagonal entry is divisible by q, and all entries outside the leading diagonal are divisible by q.*

Proof. Let $q = q_1 \cdots q_s$ so that $\gcd(q_j, q_k) = 1$ for all $j > k$. Let us consider the $s \times s$ matrix $\boldsymbol{A}$ with integer entries $A_{jk} = q^2/(q_j q_k)$.
In particular, for $s = 2$, it is

$$\begin{pmatrix} q_1^2 & q_1 q_2 \\ q_1 q_2 & q_2^2 \end{pmatrix}$$

of rank one. In the general case, the rows are also proportional to each other because $A_{jk} = (q_j/q_\ell) A_{\ell k}$. Thus, $\boldsymbol{A}$ is a matrix of rank one.

Let us set a symmetric matrix $\boldsymbol{B}$ of order n/s such that each diagonal entry is mutually prime to q, and all entries outside the leading diagonal are divisible by q. $\boldsymbol{B}$ is nondegenerate. The Kronecker product $\boldsymbol{M} = \boldsymbol{B} \otimes \boldsymbol{A}$ satisfies the desired conditions. But its rank is n/s.

Example. Let $q = q_1 q_2 q_3$ so that $\gcd(q_1, q_2) = 1$, $\gcd(q_1, q_3) = 1$, and $\gcd(q_2, q_3) = 1$ hold. The rank-one 3×3 matrix

$$\boldsymbol{A} = \begin{pmatrix} q_2^2 q_3^2 & q_1 q_2 q_3^2 & q_1 q_2^2 q_3 \\ q_1 q_2 q_3^2 & q_1^2 q_3^2 & q_1^2 q_2 q_3 \\ q_1 q_2^2 q_3 & q_1^2 q_2 q_3 & q_1^2 q_2^2 \end{pmatrix}$$

contains large numbers.

For $q = 30 = 2 \cdot 3 \cdot 5$,

$$\boldsymbol{A} = \begin{pmatrix} 100 & 150 & 60 \\ 150 & 225 & 90 \\ 60 & 90 & 36 \end{pmatrix}$$

One can transfer our results to matrices over any unique factorization domain. But over a ring, one must consider the number of irreducible divisors instead of ordinary (i. e. rational) primes. Of course, irreducibility depends on the ring.

For example, **over the ring of Gaussian integers**

$$\{a + ib \mid a, b \in \mathbb{Z}\},$$

the matrix

$$\begin{pmatrix} 1+i & 2 \\ i & 1+i \end{pmatrix}$$

is degenerate because the number 2 can be decomposed as $2 = (1+i)(1-i)$. Nonetheless, there is no contradiction with Theorem 1.

If a positive rational prime is congruent to 3 modulo 4, then it is also a Gaussian prime.

In the same way, **over the ring of Eisenstein integers**

$$\left\{ a + \omega b \;\middle|\; a, b \in \mathbb{Z}, \omega = \frac{-1 + i\sqrt{3}}{2} \right\},$$

the matrix

$$\begin{pmatrix} 1 + i\sqrt{3} & 2 \\ 2 & 1 - i\sqrt{3} \end{pmatrix}$$

is degenerate. But there is no contradiction with Theorem 2.

Although the number 2 is an Eisenstein prime, the number $1 - i\sqrt{3}$ is divisible by 2.

If a positive rational prime is congruent to 2 modulo 3, then it is also an Eisenstein prime. Moreover, there are also non-real Eisenstein primes.

The rank of a matrix is important for solving a **system of ordinary differential equations** because the degeneracy of the matrix is fraught with difficulties.

Let $\boldsymbol{A}$ denote a nondegenerate $n \times n$ matrix.

The column $\boldsymbol{x}$ contains variables.

The system of differential equations

$$\frac{d^2\boldsymbol{x}}{dt^2} + \boldsymbol{A}\boldsymbol{x} = 0$$

with initial conditions $\boldsymbol{x}|_{t=0} = \boldsymbol{a}$ and $\frac{d\boldsymbol{x}}{dt}\big|_{t=0} = \boldsymbol{b}$ has the solution

$$\boldsymbol{x} = \cos\left(\sqrt{\boldsymbol{A}}t\right)\boldsymbol{a} + \left(\sqrt{\boldsymbol{A}}\right)^{-1}\sin\left(\sqrt{\boldsymbol{A}}t\right)\boldsymbol{b}.$$

If the matrix $\boldsymbol{A}$ is degenerate, then additional transformations are required. Finite terms are useful for symbolic computations using computer algebra systems.

Refer to F. R. Gantmacher, *Theory of Matrices*, 1959.

Let us consider a ring of multivariate polynomials.

Let $\boldsymbol{M}$ denote an $n \times n$ matrix in which no diagonal entry is divisible by a monomial f in s variables, but all entries outside the leading diagonal are divisible by f. In accordance with the straightforward generalization of Theorem 2, the rank of M is at least n/s. In particular, over the ring of univariate polynomials, such a matrix $\boldsymbol{M}$ is nondegenerate.

It is similar to the **strong nonsingularity result** by S. A. Abramov and A. A. Ryabenko, *Program. Comput. Soft.* **51** (2025), 117–122. `https://doi.org/10.1134/S0361768824700920`

Irreducible polynomials play the role of prime numbers in the theorems.

If a polynomial f is decomposed into a linear combination of monomials, then all terms of $\sqrt{f}$ can be easily found using dynamic programming, starting with the free term and moving step by step to terms of higher degree. Such calculations were performed by al-Karajī in the 11th century as well as by al-Samaw'al in the 12th century.

Conversely, the low rank of a specially chosen matrix can confirm the existence of many irreducible factors of entries on the leading diagonal. However, choosing such a matrix seems to be as difficult as the factorization problem.

Example. Let us consider polynomials in two variables x and y:

$$f = 6x^2 + 13xy + 22x + 6y^2 + 23y + 20,$$
$$g = 4x^2 + 12xy + 16x + 9y^2 + 24y + 16,$$
$$h = 9x^2 + 12xy + 30x + 4y^2 + 20y + 25.$$

It is easy to check that the matrix

$$\begin{pmatrix} g & f \\ f & h \end{pmatrix}$$

is degenerate. On the other hand, neither g nor h are divisible by f.

Thus, the polynomial f is reducible. In fact,

$$f = (2x + 3y + 4)(3x + 2y + 5).$$

Discussion

We have an p-adic analogue of the Lévy–Desplanques theorem as well as similar results over other unique factorization domains. Some applications of **p-adic analysis to physics** is discussed elsewhere, refer to
N. Makhaldiani, New Physics, p-adic nature of the vacuum energy, and cosmological constant, *Phys. Part. Nuclei*, 54 (2023), 1053–1055.
`https://doi.org/10.1134/S1063779623060199`

Many **similar results** have been published elsewhere. So, there are known lower bounds on the rank of complex matrices in which all but the diagonal entries lie in a multiplicative group of small rank, refer to
N. Alon, J. Solymosi, Rank of matrices with entries from a multiplicative group, *International Mathematics Research Notices*, 14 (2023)
`https://doi.org/10.1093/imrn/rnac183`

In contrast, we consider a similar problem with an additive group.

Conclusions

- We propose a simple method for proving nondegeneracy or obtaining a lower bound on the rank of a matrix over any unique factorization domain. It is useful for solving systems of differential equations.
- The results can be useful for solving problems related to integer linear programming. Furthermore, for planning purposes, it is convenient to have an easily verifiable criterion.
- Unfortunately, the application of the criterion under consideration presupposes knowledge of the prime decomposition. It seems difficult over the ring of rational integers as well as over the ring of multivariate polynomials.
- On the other hand, for a given either number or polynomial q, any low-rank matrix of the type under consideration confirms the existence of sufficiently many distinct prime divisors of q.

Thank you