[H 1936b]: Summen von $\aleph_1$ Mengen

„Summen von $\aleph_1$ Mengen" [H 1936b] is obviously HAUSDORFF's work of highest impact and probably one of the most cited and honored papers in the set theory of pre-forcing era.

Anmerkungen

[1] S. 241 k-konvergent … m-konvergent

These notions express the additivity of category and measure w. r. t. a given increasing $\aleph_1$-long sequence of Borel sets the union of which is equal to the whole space. In this paper, HAUSDORFF considers two general methods to define such sequences. The first method, the LUZIN – SIERPIŃSKI sums, goes back to indices and constituents of Suslin and co-Suslin sets, generally, to the early days of descriptive set theory, although the properties of k-konvergence and m-konvergence were finally established only in the early 1930s (see below). The other method employs HAUSDORFF's construction of (ω_1, ω_1^*)-gaps, originally invented in [H 1909a] and reintroduced in [H 1936b].

[2] S. 241 (3) and (4)

In the notation of our comment to S. 177 of *"Mengenlehre"* in Volume III, a LUZIN– SIERPIŃSKI sum is $\bigcup_{\xi<\omega_1} X^\xi$, where $X^\xi = A^\xi \cup B^\xi$ is the union of the approximation $A^\xi = \bigcup_{\eta<\xi} A_\eta$ of a Suslin subset A of a Polish space X and the approximation $B^\xi = \bigcup_{\eta<\xi} B_\eta$, with the same index ξ, of the complementary co-Suslin set $B = X \smallsetminus A$. The sets X^ξ increase when $\xi \to \omega_1$, and are Borel, in addition, $X = \bigcup_{\xi<\omega_1} X^\xi$. That LUZIN– SIERPIŃSKI sums are k- and m-konvergent, follows by argument of E. SELIVANOWSKI, *Sur les propriétés des constituantes des ensembles analytiques*, Fund. Math., **21** (1933), 20–28, from which HAUSDORFF extracts sufficient conditions (3) and (4). [1]

[1] SELIVANOWSKI proved that sequences of Borel approximations of Suslin sets, defined in terms of sieves (see comments to NL 426 in Volume III, pp. ???–???) rather than A-operation, are k- and m-convergent. In the subsequent note, W. SIERPIŃSKI, *Sur les constituantes des ensembles analytiques*, Fund. Math., **21** (1933), 29–34, adapted SELIVANOWSKI's method to Borel approximations of Suslin sets defined in terms of A-operation, *i. e.*, sets A^ξ. HAUSDORFF developed in NL 559 (see Volume III, pp. ???–???) a short argument which contains arguments of both SELIVANOWSKI and SIERPIŃSKI as particular cases. Both versions of the proof are valid for LUZIN - SIERPIŃSKI sums, where the reduction to HAUSDORFF's sufficient conditions (3), (4) can be carried out as follows. For brevity, we take the case of A-operation.

Arguing in the notation of *"Mengenlehre"*, § 34.2 (with E replaced by X), note that for any ξ the set T_ξ (*"Mengenlehre"*, S. 187) consists of all points $x \in X$ such that the set $R_\xi(x)$, obtained after ξ steps of cutting off all maximal elements, beginning with $R_0(x) = \{r \in \mathbb{N}^{<\omega} : x \in F(r)\}$, still contains maximal elements. Let T_n^ξ be the set of all $x \in X$ such that r_n is a maximal element in $R_\xi(x)$, where $\mathbb{N}^{<\omega} = \{r_n : n \in \mathbb{N}\}$ is a fixed enumeration of finite sequences. The sets T_n^ξ satisfy (3) and (4).

By the way, that sequences of Borel approximations of <u>co</u>-Suslin sets are k- and m-convergent, easily follows from the Index Restriction theorem (as in our comments to § 34 of *"Mengenlehre"* in Volume III).

[3] S. 242 Problems (A), (B)
are commented upon in our Essay *Gaps and partially ordered sets* in this Volume.

[4] S. 243 „finale“ Ordnung
The order relation $\leqq$ defined here is obviously different from anything considered in HAUSDORFF's early papers. For instance, HAUSDORFF defines in [H 1909a, S. 304] $a < b$ iff eventually $a_n < b_n$, and $a = b$ iff eventually $a_n = b_n$, so that $a \leq b$ would be that either eventually $a_n < b_n$ or eventually $a_n = b_n$ — which is not the same as $\leqq$ here. Accordingly, $<$ in the sense of [H 1909a, S. 304] is not the same as "$\leqq$ but not $\geqq$". Generally, the definitions in [H 1909a, S. 304] are not at all suitable for dyadic sequences.

[5] S. 244 Erster Einschaltungssatz
In spite of the differences just mentioned, HAUSDORFF easily accomodates the (ω_1, ω_1^*)-gap existence proof to obtain such a gap in $\langle 2^{\mathbb{N}}; \leqq\rangle$. In particular, Erster Einschaltungssatz is analogous to Satz 1 in [H 1909a, S. 304].

[6] S. 244 Footnote
Another difference with [H 1909a], and also of rather technical character, is that the partial order considered here is different.

[7] S. 245 Zweiter Einschaltungssatz
Analogous to II in [H 1909a, S. 321].

[8] S. 247 Es ist nicht bekannt...
In the modern notation (see our essay in this Band), HAUSDORFF' question is the question of existence of a (ω_1, ω^*)-limit in the structure $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, where $\leqslant^*$ coincides with the order $\leqq$ on S. 243 while the corresponding strict order $<^*$ coincides with $<$ on S. 243. This problem is not solvable in **ZFC**, see our Essay *Gaps and partially ordered sets* in this Volume. The question "in dem schärferen Sinne" is, perhaps, an oversight: ROTHBERGER easily answered it in the negative in *On some problems of Hausdorff and Sierpiński*, Fund. Math., **35** (1948), 29–46.

[9] S. 247 $T^\xi = \underset{x}{\mathrm{E}}\,[a^\xi \leqq x \leqq b^\xi]$
Thus $T^\xi = \{x : a^\xi \leqq x \leqq b^\xi\}$ in modern notation. The idea to use transfinite sequences of poins of spaces like $2^{\mathbb{N}}$ to define transfinite sequences of Borel sets goes back to the 1920s. For instance, HAUSDORFF employed an ω_1-sequence $\{a^\xi\}_{\xi<\omega_1}$, increasing in the sense of the final Rangordnung $a < b$ iff eventually $a_n < b_n$, to define a decreasing ω_1-sequence of $\mathbf{F}_\sigma$ sets, in NL 281 (see Volume III, pp. ???–???). An (ω_1, ω_1^*)-gap yields a much stronger result!

[10] S. 250 Problem (P): will be commented upon in our Essay *Gaps and partially ordered sets* in this Volume.

[11] S. 251 Raum der abgeschlossenen Mengen
Given a compact metric space X, the "hyperspace" $\mathfrak{X} = 2^X = \mathbf{F}(X)$ of all

closed sets $\emptyset \neq A \subseteq X$, with the HAUSDORFF metric, is also compact (*"Mengenlehre"*, § 28, in Volume III).

[12] S. 251 Ableitung
Recall that the derived set $A' = A_\beta$ is the set of all limit points of a set A situated in a topological space, § 23 in *"Mengenlehre"*. Then A' is closed, and $A' \subseteq A$ provided A itself is closed, moreover, $A' = A$ iff A is perfect. The sequence of iterated derivatives A^ξ is defined by induction, namely, $A^0 = A$, $A^{\xi+1} = (A^\xi)'$, and $A^\lambda = \bigcap_{\xi<\lambda} A^\xi$ for all limit ordinals λ. The sets A^ξ are closed and decrease, hence, by LINDELÖF, if the space has countable base (for instance, is Polish) then there is a least countable ordinal $\xi = |A|_{\mathrm{CB}}$, the *Cantor – Bendixson rank* of A, such that $A^\xi = A^{\xi+1} = A^\eta$ for all $\eta > \xi$.

HAUSDORFF defines $\mathfrak{X}^\xi = \{A \in \mathfrak{X} : |A|_{\mathrm{CB}} \leq \xi\}$ — then $\mathfrak{X} = \bigcup_{\xi<\omega_1} \mathfrak{X}^\xi$, in addition, $\mathfrak{X}^\xi \subseteq \mathfrak{X}^{\xi+1}$ for every ξ, and simple examples show that $\mathfrak{X}^\xi \subsetneqq \mathfrak{X}^{\xi+1}$ provided X itself (a given compact metric space) is uncountable. Finally, by Satz II (S. 252), the sets $\mathfrak{X}^\xi$ are Borel and form a k- and m-convergent sequence.

[13] S. 253 Herr HUREWICZ … hat …
HAUSDORFF notes, following W. HUREWICZ, *Zur Theorie der analytischen Mengen*, Fund. Math., **15** (1930), 4–17, that the sequence of sets $\mathfrak{X}^\xi$ is mutually cofinal with a LUZIN – SIERPIŃSKI sum for the space $\mathfrak{X}$, so that the k- and m-convergence part (but not the Borelness part) of Satz II follows from the general fact that LUZIN – SIERPIŃSKI sums are convergent (see the introductory part of the paper). This argument appears in detail in Part 1 of Fasz. 558 in this Volume.

[14] S. 254 Satz III: thus in this case the k-convergence is settled at the very first term of the sequence of sets $\mathfrak{X}^\xi$.