Gaps in partially ordered sets and related problems
Commentary to [H 1909a] and [H 1936b]

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According to its impact on the course of set theoretic investigations, HAUSDORFF's paper [H 1936b] is obviously one of the most valuable studies in the pre-forcing period of set theory. The results obtained, especially, the existence of (ω_1, ω_1^*)-gaps, concepts and methods introduced, and problems discussed in [H 1936b], have inspired numerous set theoretic studies, including those based on forcing and other technique fully unknown in HAUSDORFF's times. On the other hand, the content of [H 1936b], in its part related to the gaps, goes back to HAUSDORFF's early works in set theory in 1906 – 1909, especially to [H 1909a], while its motivation has roots in the mathematics of XIX Century. To adequately reflect these issues, we decided to write this Essay, which starts with a general review of the notions involved and structures and problems studied, and then expands into modern studies.

Introduction

Suppose that $\langle P; \leq \rangle$ is a partially quasi-ordered set, or PQOset [1], whose domain consists either of real functions defined on $[0, +\infty)$, or of infinite real sequences. Suppose also that the relation $\leq$ is compatible with the idea that $f \leq g$ means that a function (or sequence) g grows faster-or-equal than f does, and hence the degree of infinity represented by g is larger than, or equal to, the degree of infinity represented by f.

Following HAUSDORFF, we can call such a PQO structure $\langle P; \leq \rangle$ a *graduation method* on the domain P, meaning that it graduates objects in P (functions or sequences) in accordance to their rates of growth.

The history of studies related to graduation of real functions according to their rate of growth goes back to works of DU BOIS-REYMOND [DBR-1870, DBR-1873, DBR-1875, DBR-1882] on the degree of growth and divergence of real functions, later extended by HADAMARD, HARDY, and others (see comment [1] to [H 1909a]). In particular, DU BOIS-REYMOND was obviously inspired by the idea to define a *linear* graduation method, that would make

[1] By a (non-strict) *partial quasi-order*, or PQO for brevity, we mean any *transitive* ($x \leq y$ and $y \leq c$ implies $x \leq z$) and *reflexive* ($x \leq x$) binary relation $\leq$. If the relation $\leq$ is also *antisymmetric*, that is, $x \leq y \wedge y \leq x$ implies $x = y$, then it is called a partial *order*. If a PQO $\leq$ satisfies the condition that any two elements x, y in its domain are comparable, that is, at least one of the condition $x \leq y$ or $y \leq x$ holds, then $\leq$ is called a *linear* (or *total*) quasi-order, LQO for brevity.

Given a PQO $\leq$, we can define an associated *equivalence relation*

$$x \equiv y \quad \text{iff} \quad x \leq y \ \text{and} \ y \leq x\,,$$

and a *strict partial order* (transitive and *asymmetric* relation)

$$x < y \quad \text{iff} \quad x \leq y \ \text{but not} \ y \leq x\,,$$

on its domain. Conversely, given an equivalence relation $\equiv$ and an $\equiv$-invariant strict partial order $<$, we can define a non-strict PQO: $x \leq y$ iff $x < y$ or $x \equiv y$.

Given a PQOset $\langle P; \leq \rangle$, we define the equivalence relation $\equiv$ as above, and the *quotient set* $P/{\equiv} = \{[x]_\equiv : x \in P\}$, where $[x]_\equiv = \{y \in P : y \equiv x\}$, the *equivalence class* of x. The set $P/{\equiv}$ is ordered naturally so that $[x]_\equiv \leq [y]_\equiv$ iff $x \leq y$, and the induced relation $\leq$ on $P/{\equiv}$ is a partial order, not merely PQO.

comparable any two monotonic real functions with nonnegative values. Different families of functions, among them polynomials, exponents, logarithms, and their combinations, admitted such a graduation. But all attempts to extend those graduation methods to a truly representative family of real functions easily led to the existence of incomparable functions, hence, non-linearity of the graduation. This is why HAUSDORFF was quite skeptical regarding the existence of a reasonable "universal" graduation method with the property of linearity. HAUSDORFF approached the topic from another point of view, where the focal issue is: the existence and properties of certain *linearly* ordered substructures (gaps, limits, scales *etc.*) in different graduation PQOsets.

The main goal of this Essay will be to present main traits in the development of HAUSDORFF's ideas, concepts, results since the beginning of XX century, from the modern standpoint and as far as their set theoretic content is concerned. We begin with an outline of different graduation methods (dubbed as HAUSDORFF's ordered structures here), and their gap-like linearly ordered substructures (Sections 1 – 3), and discuss the HAUSDORFF gap theorem (Section 4). Then we discuss "the main problem" (Section 5), that is, the general problem of existence of different gap-like substructures in different HAUSDORFF's ordered structures. In spite of apparent multitude of concrete questions under this common title, it occurs that, at least in the case when the substructures considered are associated with the first uncountable cardinal ω_1, there exist only three really different existence problems, so that the rest of them are equivalent to one of these three. This is the content of our "main theorem" (Sections 6 – 8, with related independence results in Section 9). It occurs that problems of the existence of gap-like substructures in HAUSDORFF's ordered structures are intrincically connected with apparently different group of set theoretic problems, those related to partitions of the continuum; we discuss this in Sections 10, 11. Section 12 is devoted to five HAUSDORFF's problems of the existence of "pantachies" (maximal linearly ordered subsets in HAUSDORFF's ordered structures) with certain properties. One of them is still unsolved, and it seems to be the oldest yet unsolved set theoretic problem. Then we come back to some HAUSDORFF's ideas related to the graduation problem in its generality (Sections 13 and 14), to discuss them from the point of view of modern theory of Borel equivalence relations and Borel order relations. We finish (Section 15) with a brief review of HAUSDORFF's set theoretic problems.

1 HAUSDORFF's ordered structures

Now let us review PQOsets which HAUSDORFF considered in connection with the graduation problem. They are presented here in the *discrete* form, that is, being defined on the domain $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences $a = \{a(n)\}_{n\in\mathbb{N}}$ of reas $a(n)$. The relationship with the *continual* forms (for the domain of real functions instead of infinite sequences) will be discussed in Section 3.

Rate of growth order, RG:

$a \preccurlyeq b$ iff the limit $\mathtt{lim}_{n\to\infty}\big(b(n)-a(n)\big)$ exists and is >0,

with the associated equivalence relation and strict order:

$a \sim b$ iff $\mathtt{lim}_{n\to\infty}\big(a(n)-b(n)\big)$ exists and is finite,

$a \prec b$ iff $\mathtt{lim}_{n\to\infty}\big(b(n)-a(n)\big)=+\infty$.

This is clearly different from the Du Bois-Reymond original rate of growth ordering [2] $f \preccurlyeq g$ iff $\mathtt{lim}_{x\to+\infty}\frac{f(x)}{g(x)}<+\infty$, but the logarithm obviously induces an isomorphism between the latter (restricted to sequences with positive terms) and the former. On the other hand the "differential" definition is somewhat more convenient and more in custom in modern studies.

To get rid of problems related to the non-existence of the limit, we may follow Hausdorff [H 1909a, S. 299] in changing the limit to upper limit which always exists:

RG modified: $a \trianglelefteq b$ iff $\mathtt{lim\,sup}_{n\to\infty}\big(a(n)-b(n)\big)<+\infty$,

with the associated equivalence relation and strict order:

$a \bowtie b$ iff $\mathtt{lim\,sup}_{n\to\infty}\big|a(n)-b(n)\big|<+\infty$,

$a \triangleleft b$ iff $\mathtt{lim\,sup}_{n\to\infty}\big(b(n)-a(n)\big)=+\infty$, but

$\mathtt{lim\,sup}_{n\to\infty}\big(a(n)-b(n)\big)<+\infty$.

However non-comparable elements reappear in another form anyway.

Now we may note (and see [H 1909a, S. 308] for an equivalent argument) that for $f \preccurlyeq g$ to hold it is necessary and sufficient that $c+f \leqslant_{\mathtt{fro}} g$ for any real constant c ($c+f$ is the function $x \mapsto c+f(x)$), where $\leqslant_{\mathtt{fro}}$ is the following PQO:

Final Rangordnung: $a \leqslant_{\mathtt{fro}} b$ iff there is n_0 such that:

either $a(n)<b(n)$ for all $n\geq n_0$, or $a(n)=b(n)$ for all $n\geq n_0$,

with the associated equivalence relation and strict order:

$a \equiv_{\mathtt{fro}} b$ iff $\exists\, n_0\, \forall\, n\geq n_0\, \big(a(n)=b(n)\big)$,

$a <_{\mathtt{fro}} b$ iff $\exists\, n_0\, \forall\, n\geq n_0\, \big(a(n)<b(n)\big)$.

Finally, as none of the above orderings is suitable for the domain $2^{\mathbb{N}}$, Hausdorff studies one more ordering in [H 1936b]:

Eventual domination:

$a \leqslant^* b$ iff there is n_0 such that $a(n)\leq b(n)$ for all $n\geq n_0$,

with the associated equivalence relation and strict order:

$a \equiv^* b$ iff $\exists\, n_0\, \forall\, n\geq n_0\, \big(a(n)=b(n)\big)$,

[2] The latter formally consists of $\prec$ and $\sim$, of course, but this pair of a strict ordering and an equivalence is bi-reducible with the non-strict ordering, as explained in Footnote 1.

$$a <^* b \ \text{ iff } \ \exists\, n_0\, \forall\, n \ge n_0\, \big(a(n) \le b(n)\big), \ \text{ and}$$
$$\forall\, n_0\, \exists\, n \ge n_0\, \big(a(n) < b(n)\big).$$

The relations $\leqslant_{\mathtt{fro}}$ and $\leqslant^*$ are obviously different: in fact $\leqslant_{\mathtt{fro}} \subsetneqq \leqslant^*$. Yet they induce the same equivalence relation, $\equiv_{\mathtt{fro}}$ or $\equiv^*$, that is, $a \equiv_{\mathtt{fro}} b$ iff $a \equiv^* b$ iff $a(n) = b(n)$ for all but finite n. The corresponding strict relations are different: $<_{\mathtt{fro}} \subsetneqq <^*$.

Remark 1.1. The four quasi-order relations considered differ from the simple componentwise quasi-ordering, $a \leqslant_{\mathtt{cw}} b$ iff $a(n) \le b(n)$ for all n, in the following crucial detail: obviously there exists no strictly $<_{\mathtt{cw}}$-incteasing (or decreasing) sequences of length ω_1, while for any of the four PQOs considered such sequences do exist. For instance, to get a $<^*$-increasing ω_1-sequence in $\mathbb{N}^{\mathbb{N}}$, it suffices to show that for any countable collection $\{f_n : n \in \mathbb{N}\}$ of sequences $f_n \in \mathbb{N}^{\mathbb{N}}$ there exists a sequence $f \in \mathbb{N}^{\mathbb{N}}$ satisfying $f_n <^* f$ for all n. Put $f(k) = 1 + \mathtt{max}_{n \le k} f_n(k)$.

The relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathtt{fro}}$, $\leqslant^*$, together with special forms for the subdomains $2^{\mathbb{N}}$ (dyadic sequences) and $\mathbb{N}^{\mathbb{N}}$, lead us to the following:

Definition 1.2. *Hausdorff's ordered structure*, in brief HOS, is a PQOset of the form $\langle D; \le\rangle$, where the domain D is one of the sets $\mathbb{R}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$, and the relation $\le$ is one of $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathtt{fro}}$, $\leqslant^*$, except for the non-interesting trivial structures $\langle 2^{\mathbb{N}}; \preccurlyeq\rangle$, $\langle 2^{\mathbb{N}}; \trianglelefteq\rangle$, and $\langle 2^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$.

Thus we have the total number of 9 HOS, of them one *dyadic*, $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, four HOS are $\mathbb{N}$-type (*i. e.*, with $\mathbb{N}^{\mathbb{N}}$ as the ground set) and four HOS are $\mathbb{R}$-type (with $\mathbb{R}^{\mathbb{N}}$ as the ground set).

It remains to note that the basic definitions (of HOS, gaps, *etc.*) above are connected with the ideal Fin of all finite subsets of ω. Yet they remain equally meaningful for any other ideal $\mathscr{Z}$ on $\mathbb{N}$: one has only to replace the basic orderings $\preccurlyeq$, $\leqslant^*$, $\leqslant_{\mathtt{fro}}$ by their $\mathscr{Z}$-versions, for instance, $a \leqslant^*_{\mathscr{Z}} b$ iff $a(n) \le b(n)$ for $\mathscr{Z}^{\complement}$-many n, that is, the set $\{n : a(n) > b(n)\}$ belongs to $\mathscr{Z}$. It is demonstrated by Todorcevic [To-1998] that, under some assumptions, the "gap spectra" of $\mathscr{Z}$-HOS include those of the Fin-versions, but generally not much is clear. We refer to a substantial review in Farah [Far-2000] as a sourse of further information.

2 Gaps and related constructions

The following list presents some important types of linearly ordered subsets. Let $P = \langle P; \le\rangle$ be a PQOset, with $<$ and $\equiv$ being the associated strict partial order and equivalence relation, and κ, λ be any cardinals, usually infinite and regular, or otherwise finite and equal to 0 or 1.

pregaps: a (κ, λ^*)-*pregap* is a pair which consists of a $<$-increasing sequence $X = \{x_\alpha\}_{\alpha<\kappa}$ and a $<$-decreasing sequence $Y = \{y_\beta\}_{\beta<\lambda}$ of elements $x_\alpha, y_\beta \in P$ such that $X < Y$ (*i. e.*, $x_\alpha < y_\beta$ for all $\alpha < \kappa$, $\beta < \lambda$);

gaps: any z satisfying $X < z < Y$ is said to *fill in* a pregap $\langle X, Y \rangle$, and if such a z does **not** exist then a (κ, λ^*)-pregap is called a (κ, λ^*)-*gap*; [3]

limits: a κ-*limit* (or κ-*Element*, as in [H 1909a, p. 320]) is a $(\kappa, 1^*)$-gap, that is, a $<$-increasing sequence $\{x_\alpha\}_{\alpha<\kappa}$ together with an element $x \in P$ satisfying $x_\alpha < x$, $\forall\, \alpha$, and with no $y < x$ such that $x_\alpha < y$, $\forall\, \alpha$ — let us write $x = \mathtt{lim}_{\alpha\to\kappa}\, x_\alpha$ in this case; [4]

towers: a κ-*tower* is a $(\kappa, 0^*)$-gap, that is, a $<$-increasing κ-sequence unbounded from above; [5]

scales: a κ-*scale* is an increasing sequence $\{x_\alpha\}_{\alpha<\kappa}$ such that for any $x \in P$ we have $x < x_\alpha$ for some α;

pantachies: a *pantachy* is a maximal linearly ordered subset. [6] This concept has two distinct versions. By a *non-strict* pantachy we understand any maximal $\le$-ninearly ordered set $L \subseteq P$, and by a *strict* pantachy, correspondingly, any maximal $<$-ninearly ordered set $L \subseteq P$.

Any non-strict pantachy $L \subseteq P$ is necessarily $\equiv$-*saturated*, in the sense that if $x \in L$, $y \in P$, $x \equiv y$, then $y \in L$. On the contrary, any strict pantachy $L' \subseteq P$ is necessarily *pairwise* $\equiv$-*inequivalent*, in the sense that if $x \ne y$ belong to L then $x \not\equiv y$. But both types are in 1-1 connection. Indeed, given a non-strict pantachy $L \subseteq P$, pick an element in each $\equiv$-equivalence class C such that $C \cap L \ne \emptyset$ (and then $C \subseteq L$), and the set $L' \subseteq L$ of all chosen elements will be a strict pantachy. Conversely, if $L' \subseteq P$ is a strict pantachy then $L = \{x \in P : \exists\, y \in L'\, (x \equiv y)\}$ is a non-strict one.

Towers and scales are particular types of much wider categories:

unbounded sets: those $X \subseteq P$ for which there does not exist any $x \in P$ satisfying $X \le x$ (*i. e.*, $x' \le x$ for any $x' \in X$);

dominating sets: those $X \subseteq P$ satisfying $\exists\, x \in X\, (x' \le x)$ for every $x' \in P$.

Thus a *tower* in $P = \langle P; \le \rangle$ is a $<$-wellordered unbounded set while a *scale* is a $<$-wellordered dominating set. Each dominating set is unbounded (unless there exist largest elements).

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[3] Also called a (κ, λ)-gap, with the understanding that λ indicates the inverse order.

[4] In most cases considered below the partially ordered sets will be symmetric enough to prove that the existence of $(\kappa, 1^*)$-gaps is equivalent to the existence of $(1, \kappa^*)$-gaps, and the latter type will be called *decreasing* limits.

[5] The notion of tower is due to Rothberger, Hausdorff used “transzendente Reihe”. In most cases the existence of $(\kappa, 0^*)$-gaps is equivalent to the existence of $(0, \kappa^*)$-gaps, that is, decreasing κ-sequences unbounded from below, which will be called *decreasing* towers.

[6] The notion of pantachy, which Hausdorff owes to Du Bois-Reymond, was not accepted in set theory since descriptions like “maximal branch” are fully informative.

3 Discrete vs. continual structures

Note that each of the quasi-orderings, defined in Section 1 on $\mathbb{R}^{\mathbb{N}}$, has a meaningful *continual* [7] version defined on the collection $\mathbf{C}[0,+\infty)$ of all continuous functions $f:[0,+\infty)\to\mathbb{R}$, or even on the collection of all $f:[0,+\infty)\to\mathbb{R}$ not necessarily continuous but bounded on every bounded interval of $[0,+\infty)$, like say all increasing functions. Namely,

$$
\begin{array}{rcl}
f \preccurlyeq g & \text{iff} & \text{the limit } \mathtt{lim}_{x\to\infty}\big(g(x)-f(x)\big) \text{ exists and is } > 0;\\
f \trianglelefteq g & \text{iff} & \mathtt{lim\,sup}_{x\to\infty}\big(f(x)-g(x)\big) < +\infty;\\
f \leqslant_{\mathtt{fro}} g & \text{iff} & \text{there is } x_0 \text{ such that either } f(x)<g(x) \text{ for all } x\ge x_0,\\
 & & \text{or } f(x)=g(x) \text{ for all } x \ge x_0;\\
f \leqslant^* g & \text{iff} & \text{there is } x_0 \text{ such that } f(x)\le g(x) \text{ for all } x\ge x_0.
\end{array}
$$

The next theorem contains several rather elementary reductions between the discrete and continual cases of the relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathtt{fro}}$, $\leqslant^*$, w. r. t. the existence of various gaps and scales. Unfortunately the theorem does not cover the whole spectrum of anticipated reductions.

Theorem 3.1. *Suppose that $\le$ is any of the order relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathtt{fro}}$, $\leqslant^*$, and κ is an infinite regular cardinal. Then*

(i) *a κ-scale in $\langle\mathbb{R}^{\mathbb{N}};\le\rangle$ implies a κ-scale in $\langle\mathbf{C}[0,+\infty);\le\rangle$;*

(ii) *a κ-scale in $\langle\mathbf{C}[0,+\infty);\le\rangle$ implies a κ-scale in $\langle\mathbb{R}^{\mathbb{N}};\le\rangle$.*

If, in addition, either a) λ is an infinite regular cardinal, or b) $\lambda=1$ [8] and $\le$ is one of the relations $\preccurlyeq$, $\leqslant_{\mathtt{fro}}$, or finally c) $\lambda=0$, then

(iii) *a (κ,λ^*)-gap in $\langle\mathbb{R}^{\mathbb{N}};\le\rangle$ implies a (κ,λ^*)-gap in $\langle\mathbf{C}[0,+\infty);\le\rangle$.*

Proof. For any $f:[0,+\infty)\to\mathbb{R}$, let $f\restriction\mathbb{N}$ be the sequence $\{f(n)\}_{n\in\mathbb{N}}$. The following argument is valid for any choice of $\le$ in $\{\preccurlyeq, \trianglelefteq, \leqslant_{\mathtt{fro}}, \leqslant^*\}$.

(i) Suppose that $\{a_\xi\}_{\xi<\kappa}$ is a scale in $\langle\mathbb{R}^{\mathbb{N}};\le\rangle$. Let $f_\xi\in\mathbf{C}[0,+\infty)$ be defined so that $a_\xi=f_\xi\restriction\mathbb{N}$ and f_ξ is linear on every interval $[n,n+1]$. Easily $\{f_\xi\}_{\xi<\kappa}$ is $<$-increasing together with $\{a_\xi\}_{\xi<\kappa}$, where $<$ is the strict order associated with $\le$. To see that $\{f_\xi\}$ is a scale, consider any $f\in\mathbf{C}[0,+\infty)$. As f is continuous, $a(n)=n\,\mathtt{max}_{0\le x\le n+1}f(x)$ is finite for any n, and hence this defines a sequence $a\in\mathbb{R}^{\mathbb{N}}$. Then $a\le a_\xi$ for some ξ. Now we have $f\le f_\xi$.

(ii) Suppose that $\{f_\xi\}_{\xi<\kappa}$ is a scale in $\langle\mathbf{C}[0,+\infty);\le\rangle$. Put $a_\xi=f_\xi\restriction\mathbb{N}$. Take any $a\in\mathbb{R}^{\mathbb{N}}$. Let $f\in\mathbf{C}[0,+\infty)$ be any (continuous) function with $a=f\restriction\mathbb{N}$.

[7] The word "continual" reflects the character of the domain rather than of the functions considered. Continual versions of, say, $\preccurlyeq$ and $\trianglelefteq$ historically precede *discrete* versions (those for $\mathbb{R}^{\mathbb{N}}$). Hausdorff motivates his passage from continual to discrete versions, especially to $\langle\mathbb{R}^{\mathbb{N}};\leqslant_{\mathtt{fro}}\rangle$, in [H 1909a] by a specifically simple forms which the questions considered take in this case.

[8] Recall that $(\kappa,0^*)$-gaps = κ-towers (unbounded chains) while $(\kappa,1^*)$-gaps = κ-limits.

Then $f \preccurlyeq f_\xi$ for some $\xi < \kappa$, and hence $a \preccurlyeq a_\xi$, as required. It remains to note that $\{a_\xi\}_{\xi<\kappa}$ is still a $\le$-increasing sequence, but is not necessarily strictly $<$-increasing in the case when $\le$ is $\trianglelefteq$ or $\leqslant^*$ because in this case $f < g$ does not necessarily imply $f \restriction \mathbb{N} < g \restriction \mathbb{N}$. Thus it seems that reducing $\{a_\xi\}$ to a scale may result in a scale in $\langle \mathbb{R}^\mathbb{N}; \le\rangle$ shorter than κ. However, in this case we would have a shorter scale in $\langle \mathbf{C}[0,+\infty); \le\rangle$ by (i), which is impossible since two scales of different (transfinite regular) length cannot exist.

(iii) Let $\langle \{a_\xi\}_{\xi<\kappa}, \{b_\eta\}_{\eta<\lambda}\rangle$ be a (κ,λ^*)-gap in $\langle \mathbb{R}^\mathbb{N}; \le\rangle$. Let $f_\xi, g_\eta \in \mathbf{C}[0,+\infty)$ be functions linear on every interval $[n, n+1]$ and satisfying $a_\xi = f_\xi \restriction \mathbb{N}$, $b_\eta = g_\eta \restriction \mathbb{N}$. Then $f_\xi < f_{\xi'} < g_{\eta'} < g_\eta$ strictly whenever $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$. Suppose that $\langle \{f_\xi\}_{\xi<\kappa}, \{g_\eta\}_{\eta<\lambda}\rangle$ is not a gap. Let $h \in \mathbf{C}[0,+\infty)$ witness this, that is $f_\xi < h < g_\eta$ for all ξ, η. Then $c = h \restriction \mathbb{N}$ satisfies $a_\xi \le c \le b_\eta$ for all ξ, η.

If now κ and λ are limit ordinals then $a_\xi < a_{\xi+1} \le c \le b_{\eta+1} < b_\eta$, and hence $a_\xi < c < b_\eta$ strictly, contradiction.

If $\le$ is one of the relations $\preccurlyeq$, $\leqslant_{\mathtt{fro}}$ then it is clear that $f < g$ implies $f \restriction \mathbb{N} < g \restriction \mathbb{N}$, and hence we have $a_\xi < c < b_\eta$ strictly, contradiction.

Finally, assume that $\lambda = 0$, so that $\{a_\xi\}_{\xi<\kappa}$ is a tower (unbounded chain), and prove that so is $\{f_\xi\}$. Suppose towards the contrary that $h \in \mathbf{C}[0,+\infty)$ and $f_\xi \le h$ for all ξ. Then $c = h \restriction \mathbb{N}$ satisfies $a_\xi \le c$ for all ξ, which is a contradiction. $\square$

We end this section with two open questions.

Problem 3.2. (1) Is the inverse of (iii) of the theorem true?

(2) Does (iii) hold for $\lambda = 1$ and $\le$ being one of $\trianglelefteq$ or $\leqslant^*$?

To demonstrate the difficulty in (1), let $\langle \{f_\xi\}_{\xi<\kappa}, \{g_\eta\}_{\eta<\lambda}\rangle$ be a (κ,λ^*)-gap in $\langle \mathbf{C}[0,+\infty); \preccurlyeq\rangle$. Put $a_\xi = f_\xi \restriction \mathbb{N}$ and $b_\eta = g_\eta \restriction \mathbb{N}$. Then $a_\xi \prec a_{\xi'} \prec b_{\eta'} \prec b_\eta$ whenever $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$. Suppose towards the contrary that $c \in \mathbb{R}^\mathbb{N}$ satisfies $a_\xi \preccurlyeq c \preccurlyeq b_\eta$ for all ξ, η. And here we got stuck: it is not clear at all how to define a function $h \in \mathbf{C}[0,+\infty)$, with $c = h \restriction \mathbb{N}$, which fills in the gap $\langle \{f_\xi\}, \{g_\eta\}\rangle$. Regarding (2), suppose that a κ-limit $\langle \{a_\xi\}_{\xi<\kappa}, b_0\rangle$ in, say, $\langle \mathbb{R}^\mathbb{N}; \trianglelefteq\rangle$ has the following property: if ξ is even then $a_\xi(n) = b_0(n)$ for even n but $a_\xi(n) < b_0(n)$ and $b_0(n) - a_\xi(n) \to +\infty$ for odd n, while for ξ odd the other way around. Then $\langle \{f_\xi\}_{\xi<\omega_1}, g_0\rangle$ defined as in the proof of (iii) is not a κ-limit in the structure $\langle \mathbf{C}[0,+\infty); \trianglelefteq\rangle$.

Remark 3.3. The following alternative "discretization" may be of some use for these questions. Fix once and for all an enumeration $\{q_n\}_{n\in\mathbb{N}}$ of all rationals in $[0,+\infty)$. For any $f : [0,+\infty) \to \mathbb{R}$ let $\bar{a}_f$ be the sequence $\{f(q_n)\}_{n\in\mathbb{N}}$, where $\{q_n\}_{n\in\mathbb{N}}$ is any fixed enumeration of all rationals in $[0,+\infty)$. With this definition, it is quite clear that, for any $f, g \in \mathbf{C}[0,+\infty)$, $f \trianglelefteq g \Longleftrightarrow \bar{a}_f \trianglelefteq \bar{a}_g$. But this appears to be of little use since there does not seem to be any natural condition on $a \in \mathbb{R}^\mathbb{N}$ sufficient for there to exist $f \in \mathbf{C}[0,+\infty)$ with $a = \bar{a}_f$. In particular it is not clear how to convert a scale or gap in $\langle \mathbb{R}^\mathbb{N}; \trianglelefteq\rangle$ into that in $\langle \mathbf{C}[0,+\infty); \trianglelefteq\rangle$ using the map $f \mapsto \bar{a}_f$ in the opposite direction.

From now on, we consider only "discrete" structures.

4 The HAUSDORFF gap theorem

HAUSDORFF shows in [H 1909a] that (ω, ω^*)-gaps and ω-limits do not exist in structures of the type considered. The proof [9] utilizes the idea of a construction due to DU BOIS-REYMOND, see Comment [12] to [H 1909a]. The following theorem is much more difficult:

Theorem 4.1 (The HAUSDORFF gap theorem). *(ω_1, ω_1^*)-gaps do exist in all HAUSDORFF ordered structures (HOSs).*

The result for $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathtt{fro}} \rangle$ appeared in [H 1909a]. In the "*Nachlass*", the result first mentioned, without a proof, in Fasz. 116 under 21–23 January, 1909. The most known version, for dyadic sequences, was established in [H 1936b], which is a standard reference in modern set theoretic literature. [10] The proofs in [H 1909a] and [H 1936b] follow one and the same scheme, that also works for any of the nine HOSs with more or less obvious modifications, but such a generalization can also be established as a formal consequence of Theorem 4.1 by means of some rather transparent reductions, see Section 7.

Proof (for $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$: a sketch). If $a, b \in 2^{\mathbb{N}}$ and $a \leqslant^* b$ then let N_{ab} be the least number n_0 satisfying $n \geq n_0 \Longrightarrow a(n) \leq b(n)$. HAUSDORFF defines a $<^*$-increasing sequence $A = \{a_\xi\}_{\xi<\omega_1}$ and a $<^*$-decreasing sequence $B = \{b_\xi\}_{\xi<\omega_1}$ of $a_\xi, b_\xi \in 2^{\mathbb{N}}$, satisfying $a_\eta <^* b_\xi$ for all ξ, η (that is, $\langle A, B \rangle$ is a pre-gap), and the following key condition:

$$\text{for all } n \in \mathbb{N} \text{ and } \xi < \omega_1, \text{ the set } \{\eta < \xi : N_{a_\eta b_\xi} = n\} \text{ is finite.} \tag{4.2}$$

This means that b_ξ, although $<^*$-bigger than all a_η, is rather $<^*$-close to the set $\{a_\eta : \eta < \xi\}$. Suppose this has been done. To see that $\langle A, B \rangle$ is a (ω_1, ω_1^*)-gap, suppose towards the contrary that $a_\xi <^* c <^* b_\xi$ for all ξ. As ω_1 is uncountable, there are ξ and n such that $N_{a_\eta c} = n$ for infinitely many ordinals $\eta < \xi$. But this contradicts (4.2) as $c <^* b_\xi$.

The construction of a_ξ, b_ξ, goes on by transfinite induction on ξ.

The successor steps are rather trivial. Indeed if $a_\xi <^* b_\xi$ in $2^{\mathbb{N}}$ have been defined, then choose, as $a_{\xi+1}$ and $b_{\xi+1}$, any pair of $a, b \in 2^{\mathbb{N}}$ satisfying $a_\xi <^* a <^* b <^* b_\xi$.

[9] Following a similar construction in Remark 1.1, suppose that $a_0 <^* a_1 <^* \ldots <^* b_1 <^* b_0$. There are natural numbers $n_0 < n_1 < \ldots$ such that, for all k, we have $a_i(n) < b_j(n)$ whenever $i, j \leq k$ and $n_k \leq n < n_{k+1}$. Put $c(n) = \mathtt{max}_{i \leq k}\, a_i(n)$ for all n with $n_k \leq n < n_{k+1}$. Then $a_n <^* c <^* b_n$ for all n, as required.

[10] HAUSDORFF mentioned in a footnote in [H 1936b, S. 244] that his paper [H 1909a] remained "wenig bekannt" (little known). Clearly the gap construction was far ahead of the level of development and, perhaps, even the level of motivation of set theory in the early years of the century. In addition, the paper was published in a rather provincial journal. It is less clear why HAUSDORFF did not include the result in the monographs "*Grundzüge*", 1914 and "*Mengenlehre*", 1927.

The limit steps need more effort. Suppose that $\lambda < \omega_1$ is a limit ordinal, $a_\xi, b_\xi \in 2^{\mathbb{N}}$ have been defined for $\xi < \lambda$, and (4.2) holds for $\xi < \lambda$. A version of the argument that proves the absence of (ω, ω^*)-gaps (see Footnote 9) yields $c \in 2^{\mathbb{N}}$ satisfying $a_\xi <^* c <^* b_\xi$ for all ξ. Note that, for all n and $\xi < \lambda$, the set $\{\eta < \xi : N_{a_\eta\, c} = n\}$ is finite. In this case, a more tricky version of the same argument yields a dyadic sequence $c' <^* c$ such that still $a_\xi <^* c'$ for all $\xi < \lambda$, and in addition for any n the set $\{\eta < \lambda : N_{a_\eta\, c'} = n\}$ is finite. Put $b_\lambda = c'$ and take, as a_λ, any a satisfying $a_\xi <^* a <^* c'$ for all ξ. $\square$

5 The main problem and principal pre-forcing results

Our essay is largely devoted to different aspects of the following general problem, in connection with the PQOsets called HOS above.

Problem 5.1 (the main problem). Given a PQOset P, what is the spectrum of its gaps, limits, towers, scales, pantachies?

In particular (**the gap and scale problem**), if κ, λ be regular cardinals, does P have a (κ, λ^*)-gap? a κ-scale?

This problem includes a variety of more special questions, for instance, related to the second part of the main problem: the existence of gaps (including limits and towers) of certain cardinal characteristics in this or another HOS. In particular, Hausdorff, mainly interested in the case $\kappa = \omega_1$, asks in [H 1909a, S. 324] whether the structure $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ has ω_1-limits, ω_1-towers, and (ω_1, ω^*)-gaps. [11] Generally, Hausdorff considered all these problems as relevant to the continuum-hypothesis **CH** (usually expressed by the equality $\mathfrak{c} = \aleph_1$), or perhaps as questions which underline a certain dimension in connections between the cardinals $\aleph_1$ and $\mathfrak{c}$. He conjectured in [H 1936b, S. 320,324] that the problems may be as hard as **CH** itself.

Beside Theorem 4.1, Hausdorff's main results in [H 1909a, § 4] regarding these particular problems, amount to the following:

Theorem 5.2. (i) *The problems of existence of ω_1-limits, ω_1-towers, and (ω_1, ω^*)-gaps in the structure $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$, are equivalent to each other: the existence of any such an object implies the existence of the two other objects.*

(ii) *The continuum-hypothesis* **CH** *implies the existence of ω_1-limits, ω_1-towers, and (ω_1, ω^*)-gaps, and also ω_1-scales.* $\square$

Rothberger later partially reproved this theorem in [Ro-1948].

Hausdorff's gap construction underwent further development in [Lu-1943] and [Lu-1946], where Luzin considered subsets of $\mathbb{N}$ ordered as follows:

[11] The (ω_1, ω^*)-gap existence problem was reformulated in [H 1936b] in terms of the structure $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$.

Almost–inclusion: $x \subseteq^* y$ iff the difference $x \setminus y$ is finite, and the corresponding strict relation: $x \subset^* y$ iff $x \subseteq^* y$ but not $y \subseteq^* x$. Thus, $\langle \mathscr{P}(\mathbb{N}); \subseteq^* \rangle$ is isomorphic to $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$.

Luzin proved the existence of a pair of $\subset^*$-increasing ω_1-sequences $\{x_\xi\}_{\xi<\omega_1}$ and $\{y_\xi\}_{\xi<\omega_1}$ of sets $x_\xi,\ y_\xi \subseteq \mathbb{N}$, which are *orthogonal* (*i. e.*, all intersections $x_\xi \cap y_\xi$ are finite) but *unseparable* (*i. e.*, there is no set z such that $x_\xi \subseteq^* z$ but $y_\xi \cap z$ is finite for all ξ) — which is equivalent to the existence of a (ω_1, ω_1^*)-gap in $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$. In addition Luzin obtained an uncountable set X of infinite subsets of $\mathbb{N}$ such that any two disjoint uncountable subsets X', X'' of X are orthogonal but unseparable. Luzin also posed problems of the existence of an ω_1-limit and an (ω_1, ω^*)-gap in $\langle \mathscr{P}(\mathbb{N}); \subseteq^* \rangle$ (in terms of the existence of a pair of orthogonal and unseparable sequences, one of length ω_1, one of length ω, called Luzin *pairs*), now associated with his name. [12]

Rothberger [Ro-1948] studied logical connections between the problems. In particular, he proved that, in the structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$, the existence of (ω_1, ω^*)-gaps is equivalent to the existence of ω_1-towers and implies the existence of ω_1-limits [Ro-1948, ch. II]. (Compare with Hausdorff's Theorem 5.2(i); limits have a somewhat different nature in the structures $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ and $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\texttt{fro}} \rangle$.) [13]

See Engelking [En-1972] for some other pre-forcing results related to gaps in Hausdorff's structures.

Todorcevic [To-1996] introduced a new dimension in the study of gaps. It occurs that not only cardinal characteristics but also the descriptive class of a gap may be an interesting issue. For instance, if $\mathscr{X}$ and $\mathscr{Y}$ are two orthogonal families of subsets of $\mathbb{N}$, and at least one of them is a Suslin set (as a subset of $\mathscr{P}(\mathbb{N})$) and both are σ-directed (*i. e.*, say, for any countable $\mathscr{X}' \subseteq \mathscr{X}$ there is $x \in \mathscr{X}$ such that $x' \subseteq^* x$ for all $x' \in \mathscr{X}'$) then they are separable.

6 Classification theorem

Quite surprisingly, for any regular cardinal $\kappa \geq \omega_1$ the whole variety of the existence problems of κ-scales, κ-towers, κ-limits, and (κ, ω^*)-gaps in different HOS of Definition 1.2 is reducible to a rather short list of really different problems — especialy if we consider only the following seven HOS of the original nine ones:

$$\left.\begin{array}{l} \langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle, \quad \langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\texttt{fro}} \rangle, \quad \langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle, \\ \langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle, \quad \langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\texttt{fro}} \rangle, \quad \langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle, \quad \text{and} \quad \langle 2^{\mathbb{N}}; \leqslant^* \rangle, \end{array}\right\} \tag{6.1}$$

[12] van Douwen [vD-1984, S. 127] wrote that "Luzin ... seems to have been unaware of Hausdorff" [H 1936b], which hardly can be the case as [H 1936b] is in the reference list of [Lu-1943] — actually the only item of the list. The reference was omitted in the Russian translation of [Lu-1943] in [Lu-1958], perhaps, accidentally. Anyway, the Editors' preface to [Lu-1958] mentions Hausdorff's influence on Luzin's papers [Lu-1943, Lu-1946].

[13] Readers of [Ro-1948] should be aware of Rothberger's non-traditional use of symbols $\prec$, $<$, $\subset$ to indicate non-strict orderings. For instance $\prec$ in [Ro-1948] is what we denote by $\leqslant^*$, $\subset$ is $\subseteq$, $<$ can be both $\subseteq^*$ (between sets) and $\leq$ (even between numbers).

that is, to the exclusion of the $\trianglelefteq$-structures

$$\langle \mathbb{N}^{\mathbb{N}} ; \trianglelefteq \rangle \quad \text{and} \quad \langle \mathbb{R}^{\mathbb{N}} ; \trianglelefteq \rangle . \tag{6.2}$$

Such a reduction is provided by the next theorem. It deals with the seven HOS of (6.1). As for the two remaining HOS of (6.2), see Section 8.

Theorem 6.3. *Let $\kappa, \lambda \geq \omega$ be infinite regular cardinals. Then*

(1) *All HOS in* (6.1)*, except for the dyadic structure $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$, are equivalent w. r. t. the existence of a κ-scale, and also equivalent w. r. t. the existence of a κ-tower.*[14]

(2) *The hypothesis of the existence of a (κ, λ^*)-gap is subject of the following diagram, where $A \Longrightarrow B$ means that* (in **ZFC**) *the existence of a (κ, λ^*)-gap in the structure A implies the existence of such a gap in the structure B, and $\Longleftrightarrow$ is understood accordingly*:

$$\begin{array}{ccccccc}
\langle 2^{\mathbb{N}} ; \leqslant^* \rangle & \Longleftrightarrow & \langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle & \Longleftrightarrow & \langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle & \Longleftrightarrow & \langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle \\
 & & \Downarrow & & \Downarrow & & \Updownarrow \\
 & & \langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle & \Longleftarrow & \langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle & \Longleftarrow & \langle \mathbb{R}^{\mathbb{N}} ; \preccurlyeq \rangle
\end{array}$$

If moreover $\lambda = \omega$ then all cases of $\Longrightarrow$ in the diagram can be changed to $\Longleftrightarrow$, and hence all seven HOS in (6.1) *are equivalent w. r. t. the existence of a (κ, ω^*)-gap.*

This allows us to introduce κ-**scales**, κ-**towers**, (κ, ω^*)-**gaps** as shorthands for the assertions that any/every non-dyadic HOS in (6.1) has a κ-scale, resp., κ-tower, resp., (κ, ω^*)-gap.

(3) *The assertion (κ, ω^*)-**gaps** is equivalent to κ-**towers**.*

(4) *The hypothesis of the existence of a κ-limit is subject of the following diagram, where $A \Longrightarrow B$ means that the existence of a κ-limit in the structure A implies the existence of such a limit in the structure B, $A \Longleftrightarrow B$ is understood accordingly, $A \longrightarrow B$ means that the existence of a κ-limit in A implies the existence of a κ'-limit in B for some regular cardinal $\kappa' \leq \kappa$, and the composite symbol $\Leftarrow\!\!\!\Updownarrow$ is understood in the sense of* (4)(vii).

---

[14] The scale and tower existence questions are vacuous for the dyadic structure $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$. Indeed let an *almost-1* sequence be any dyadic sequence in $2^{\mathbb{N}}$ having at most finitely many terms 0. Then $a <^* b$ whenever b is an almost-1 sequence and a is any dyadic sequence which is not almost-1. If one removes almost-1 sequences from $2^{\mathbb{N}}$ then there will be no scale at all, while any κ-tower will be just a κ-limit in the full structure $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$ (with the constant 1 as the limit element of $\mathbb{N}^{\mathbb{N}}$). It follows that, without any loss of generality, we can eliminate $2^{\mathbb{N}}$ and concentrate on $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}$ as the ground sets for the problems related to towers and scales.

$$
\begin{array}{cccc}
 & \langle \mathbb{N}^{\mathbb{N}} ; \leqslant^{*} \rangle & \Longleftrightarrow & \langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle \\
\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^{*} \rangle & \Leftarrow\!\updownarrow \quad \uparrow & & \\
 & \boxed{\kappa\text{-}\mathbf{towers}} & \Longleftrightarrow & \langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle \\
 & \downarrow \quad \Uparrow & & \\
 & \langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle & \Longleftrightarrow & \langle \mathbb{R}^{\mathbb{N}} ; \preccurlyeq \rangle
\end{array}
$$

More exactly:

(i) *There is no κ-limits in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$;*

(ii) *κ-limits exist in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^{*} \rangle$ iff they exist in $\langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle$;*

(iii) *κ-limits exist in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ iff κ-**towers**;*

(iv) *κ-limits exist in $\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$ iff they exist in $\langle \mathbb{R}^{\mathbb{N}} ; \preccurlyeq \rangle$, and each of these two existence claims implies κ-**towers**;*

(v) *if κ-**towers** then κ'-limits exist in $\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$ for some uncountable cardinal $\kappa' \le \kappa$;*

(vi) *if κ-**towers** then κ'-limits exist in $\langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle$ for some uncountable cardinal $\kappa' \le \kappa$;*

(vii) *κ-limits exist in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^{*} \rangle$ if and only if either κ-limits exist in $\langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle$ or κ-**towers**.*

(5) *every scale is a tower, therefore κ-**scales** implies κ-**towers**.*

(6) *If κ-**towers**, but there are no κ-limits in $\langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle$, then κ-**scales**.*

Remark 6.4. In the particular case $\lambda = \omega$, the theorem reduces the multitude of the existence problems related to κ-scales, κ-towers, κ-limits, and (κ, ω^{*})-gaps in different HOS in (6.1) to the following groups of hypotheses mutually equivalent within each group:

6.4a: the existence of a κ-limit in $\langle 2^{\mathbb{N}} ; \leqslant^{*} \rangle$ and/or in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^{*} \rangle$;

6.4b: κ-**towers**, (κ, ω^{*})-**gaps**, the existence of a κ-limit in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$;

6.4c: κ-**scales**;

6.4d: the existence of a κ-limit in $\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$ and/or in $\langle \mathbb{R}^{\mathbb{N}} ; \preccurlyeq \rangle$;

6.4e: the existence of a κ-limit in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^{*} \rangle$, equivalent to the disjunction 6.4a $\vee$ 6.4b by (4)(vii) of the theorem.

Recall that the existence of κ-limits in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ is impossible.

In the most important special case $\kappa = \omega_1$, the picture is even simpler because by necessity $\kappa' = \kappa$ in (4)(v) and (4)(vi) of the theorem (there are no

ω-limits), and hence 6.4d is equivalent to 6.4b, moreover 6.4b implies 6.4a, and 6.4e joins 6.4a. Thus we have

$$\left.\begin{array}{ccccc} \text{6.4a} & \Longleftarrow & \text{6.4b} & \Longleftarrow & \text{6.4c} \\ \Updownarrow & & \Updownarrow & & \\ \text{6.4e} & & \text{6.4d} & & \end{array}\right\} \quad \text{the case when } \kappa = \omega_1 \text{ and } \lambda = \omega .$$

The problems mentioned in (i) of Theorem 5.2 are of type 6.4b.

Another consequence is that, in any $\mathbb{N}$-type HOS, the existence of (κ, λ^*)-gaps implies the existence of (λ, κ^*)-gaps, since this clearly holds for $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$. (For $\mathbb{R}$-type HOS, *i. e.* those with $\mathbb{R}^{\mathbb{N}}$ as the underlying set, such a symmetry is rather clear.)

Problem 6.5. Which implications in the diagram in (2) of the theorem can be improved to equivalences in the general case of infinite (regular) cardinals κ, λ? For instance, it would be interesting to prove that a (κ, λ^*)-gap in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ implies such a gap in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$.

If $\kappa > \omega_1$ then what is the relationship between 6.4d and 6.4b ?

7 Proof of the classification theorem

Different parts of Theorem 6.3 are just a folklore (we take many of them from a substantial Scheepers' survey [Sch-1993]), but a few most interesting can be attributed to Hausdorff and Rothberger. In the course of the proof of the theorem below in this section, notation like (4)(ii) or (2) will identificate items of the theorem.

7a Towers and scales

Easily any $\preccurlyeq$-tower in $\mathbb{N}^{\mathbb{N}}$ of any length is a $\leqslant_{\texttt{fro}}$-tower and any $\leqslant_{\texttt{fro}}$-tower is a $\leqslant^*$-tower. Conversely, the map which sends any $a \in \mathbb{N}^{\mathbb{N}}$ to $a'(n) = \sum_{i=0}^{n} a(i)$ transforms any $\leqslant^*$-tower in $\mathbb{N}^{\mathbb{N}}$ into a $\preccurlyeq$-tower. The same for scales. In addition, changing negative real terms by 0 and positive real terms by nearest natural numbers, we convert any "real" tower or scale into a "natural" one. This proves (1) of Theorem 6.3 and justifies the definition of the abbreviations **κ-scales** and **κ-towers**.

Definition 7.1. From now on, κ is an uncountable regular cardinal while λ is either an infinite regular cardinal (including ω) or 1 (to include limits), unless explicitly specified otherwise.

7b Gaps

Here we prove the first part of claim (2) of Theorem 6.3: the diagram related to the hypotheses of the existence of (κ, λ^*)-gaps in different structures. The proof splits into a few simple lemmas.

Lemma 7.2. *The structures $\langle 2^{\mathbb{N}};\leqslant^*\rangle$ and $\langle \mathbb{N}^{\mathbb{N}};\leqslant^*\rangle$ are equivalent w. r. t. the existence of (κ,λ^*)-gaps.*

Proof. Any gap in $\langle 2^{\mathbb{N}};\leqslant^*\rangle$ obviously remains a gap in $\langle \mathbb{N}^{\mathbb{N}};\leqslant^*\rangle$. Conversely any gap in $\langle \mathbb{N}^{\mathbb{N}};\leqslant^*\rangle$ can be transformed into a gap of the same type in $\langle 2^{\mathbb{N}};\leqslant^*\rangle$. Indeed replace any sequence $a\in\mathbb{N}^{\mathbb{N}}$, which occurs in a given gap, first by the set $X_a=\{\langle i,n\rangle : i<a(n)\}\subseteq\mathbb{N}^2$, then by the image $Y_a=\{f(i,n):\langle i,n\rangle\in X_a\}$ of X_a via any fixed bijection $f:\mathbb{N}^2\xrightarrow{\text{onto}}\mathbb{N}$, and finally by the characteristic function of Y_a: this yields a gap in $2^{\mathbb{N}}$. □

Lemma 7.3. *If $\le$ is any of the orderings $\leqslant^*$, $\leqslant_{\mathtt{fro}}$, $\preccurlyeq$, then any (κ,λ^*)-gap in $\langle \mathbb{N}^{\mathbb{N}};\le\rangle$ remains a gap in $\langle \mathbb{R}^{\mathbb{N}};\le\rangle$.*

Proof. Consider, for example, a $(\kappa,1^*)$-gap $\langle\{a_\xi\}_{\xi<\kappa},a\rangle$, that is, a κ-limit, in, say, $\langle \mathbb{N}^{\mathbb{N}};\leqslant^*\rangle$. Suppose towards the contrary that $x\in\mathbb{R}^{\mathbb{N}}$ satisfies $a_\xi<^* x<^* a$ for all ξ. We can assume that $0\le x(n)<a(n)$ for all n. Let, for any n, $c(n)$ be the largest integer with $c(n)\le x(n)$. Then $c\in\mathbb{N}^{\mathbb{N}}$, $c<^* a$ (since $c(n)\le x(n)$, $\forall\, n$), and easily $a_\xi<^* c$ for all ξ because all terms of a_ξ belong to $\mathbb{N}$. □

Lemma 7.4. *The structures $\langle \mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$ and $\langle \mathbb{R}^{\mathbb{N}};\preccurlyeq\rangle$ are equivalent w. r. t. the existence of (κ,λ^*)-gaps.*

Proof. To pass from $\langle \mathbb{R}^{\mathbb{N}};\preccurlyeq\rangle$ to $\langle \mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$ cut terms to the nearest integers and note that $\preccurlyeq$ is stable under uniformly bounded changes. □

Lemma 7.5. *If $\lambda\ge\omega$ and D is $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$ then any (κ,λ^*)-gap in $\langle D;\preccurlyeq\rangle$ is a gap in $\langle D;\leqslant_{\mathtt{fro}}\rangle$, and any (κ,λ^*)-gap in $\langle D;\leqslant_{\mathtt{fro}}\rangle$ is a gap in $\langle D;\leqslant^*\rangle$.*

Proof. This claim is elementary: for instance, to prove the transition from $\preccurlyeq$ to $\leqslant_{\mathtt{fro}}$ note that $a\prec b<_{\mathtt{fro}} c$ implies $a\prec c$. □

The next lemma completes the short cirquit between the $\mathbb{N}$-type HOS w. r. t. the existence of arbitrary gaps (except for limits and towers).

Lemma 7.6. *If $\lambda\ge\omega$ and a (κ,λ^*)-gap exists in $\langle 2^{\mathbb{N}};\leqslant^*\rangle$, then a (κ,λ^*)-gap exists in the structure $\langle \mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$, too.*

Proof. Indeed assume that $\{a_\xi\}_{\xi<\kappa}$, $\{b_\eta\}_{\eta<\lambda}$ is a gap in $\langle 2^{\mathbb{N}};\leqslant^*\rangle$. For any $a\in 2^{\mathbb{N}}$, we define $\widetilde{a}\in\mathbb{N}^{\mathbb{N}}$ by $\widetilde{a}(n)=\sum_{i=0}^{n}2^n a(n)$. Then the sequences $\{\widetilde{a}_\xi\}_{\xi<\kappa}$, $\{\widetilde{b}_\eta\}_{\eta<\lambda}$ are resp. $\prec$-increasing and $\prec$-decreasing, and $\widetilde{a}_\xi\prec\widetilde{b}_\eta$ for all ξ, η. To prove that this is a $\preccurlyeq$-gap, suppose that on the contrary $\widetilde{a}_\xi\preccurlyeq\widetilde{c}\preccurlyeq\widetilde{b}_\eta$ for some $\widetilde{c}\in\mathbb{N}^{\mathbb{N}}$. Define $c\in 2^{\mathbb{N}}$ as follows: $c(n)=1$ iff $\widetilde{c}(n)\ge 2^n$. Then easily $a_\xi\leqslant^* c\leqslant^* b_\eta$ for all ξ, η, which is a contradiction. □

7c Gaps and towers

Here we prove the remaining part of item (2) of Theorem 6.3: the "moreover" claim in the case $\lambda = \omega$, along with item (3) of the theorem.

In view of the results above, it suffices to prove that the structures $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ and $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$ are equivalent w. r. t. the existence of a (κ, ω^*)-gap. Our strategy will be to obtain a κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$ from the "weakest" gap and then obtain the "strongest" gap from such a tower. The plan is realized by the following two Hausdorff's reductions, demonstrated in [H 1909a, § 4] for the structure $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ in the case $\kappa = \omega_1$ (see (i) of Theorem 5.2) and reproduced by Rothberger [Ro-1941] for the structure $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$.

Lemma 7.7. *A (κ, ω^*)-gap in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ implies a κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$, and hence a κ-tower in any other non-dyadic HOS, e. g., $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$, see Subsection 7a.*

Proof. Let $\langle \{a_\xi\}_{\alpha<\kappa}, \{b_n\}_{n\in\mathbb{N}} \rangle$ be an (κ, ω^*)-gap in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$.

We can assume that $b_{n+1}(k) \le b_n(k)$ for all n, k. If $a \in \mathbb{R}^{\mathbb{N}}$ satisfies $a \leqslant^* b_n$ for every n, then let, for any n, $\widetilde{a}(n)$ denote the least natural number such that $a(k) \le b_n(k)$ for all $k \ge \widetilde{a}(n)$. The sequence $\{\widetilde{a}_\xi\}_{\xi<\kappa}$ is obviously $\leqslant^*$-increasing, thus it suffices to show that it is unbounded in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$. (Indeed, then it contains a strictly $<^*$-increasing cofinal subsequence.) Suppose towards the contrary that some $c \in \mathbb{N}^{\mathbb{N}}$ satisfies $\widetilde{a}_\xi \leqslant^* c$ for all $\xi < \kappa$.

Define $k_{-1} = 0$, and then, by induction, $k_n = \mathtt{max}\{c(n)+1, k_{n-1}\}$. Put $a(k) = b_n(k)$ whenever $k_n \le k < k_{n+1}$ (separately $a(k) = b_0(k)$ for $k < k_0$). It follows from our assumptions that then $a(k) \le b_n(k)$ for all $k \ge k_n$, and hence $a \leqslant^* b_n$, $\forall\, n$. Now it suffices to prove that $a_\xi \leqslant^* a$ for all ξ: indeed then a fills in the given gap, contradiction.

Recall that $\widetilde{a}_\xi \leqslant^* c$, therefore there is an index N such that $\widetilde{a}_\xi(n) \le c(n) \le k_n$ for all $n \ge N$. Take any semi-interval $I_n = (k_n, k_{n+1}]$, $n \ge N$. Then $a_\xi(k) \le b_n(k) = a(k)$ for all $k \in I_n$ because $\widetilde{a}_\xi(n) \le k_n$. Thus $a_\xi(k) \le a(k)$ for all $k > k_N$, and hence $a_\xi \leqslant^* a$, as required. □

Lemma 7.8. *A κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ implies a (κ, ω^*)-gap in the atructure $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$.*

Proof. Hausdorff's idea can be explained as follows. Let $\{c_\xi\}_{\xi<\kappa}$ be a $\leqslant_{\mathtt{fro}}$-tower in $\mathbb{N}^{\mathbb{N}}$. We can assume that each $c_\xi \in \mathbb{N}^{\mathbb{N}}$ is a strictly increasing sequence (otherwise put $c'_\xi(n) = n + \sum_{k\le n} c_\xi(k)$), so that $c_\xi(n) \ge n$. Define $a_\xi \in \mathbb{N}^{\mathbb{N}}$ for any ξ so that $a_\xi(k) = n$ whenever $c_\xi(n) \le k < c_\xi(n+1)$, so that a_ξ, as a map $\mathbb{N} \to \mathbb{N}$, is in a sense an inverse of c_ξ. Then $a_\eta \leqslant^* a_\xi$ for all $\xi < \eta < \kappa$. We claim that moreover $a_\eta <^* a_\xi$ strictly whenever $\xi < \eta$. Indeed if $c_\xi(n) < c_\eta(n)$ (this happens for infinitely many n since $c_\xi \leqslant^* c_\eta$) then by definition $n - 1 = a_\eta(c_\eta(n) - 1)) < a_\xi(c_\eta(n) - 1)) = n$.

Thus $\{a_\xi\}_{\xi<\kappa}$ is a strictly $<^*$-decreasing sequence in $\mathbb{N}^{\mathbb{N}}$. Note also that any a_ξ is increasing (as a map $\mathbb{N} \to \mathbb{N}$), perhaps non-strictly, and unbounded, that is $\mathbf{0} \prec a_\xi$, where $\mathbf{0} \in 2^{\mathbb{N}}$ is the constant 0, but $a_\xi(k) \le k$ for all k.

We assert that

7.9. *There is no* $a \in \mathbb{N}^{\mathbb{N}}$ *with* $\mathbf{0} \prec a$ *and* $a \leqslant^* a_\xi$ *for all* ξ.

Indeed, assuming otherwise, we can w. l. o. g. suppose that a is increasing (non-strictly) and $a(n+1) \le a(n)+1, \forall n$. Then there exists a unique strictly increasing $c \in \mathbb{N}^{\mathbb{N}}$ such that $a(k) = n$ whenever $c(n) \le k < c(n+1)$. Then $a \leqslant^* a_\xi$ implies $c_\xi \leqslant^* c$ for all ξ, a contradiction with the tower assumption. Thus 7.9 holds.

It follows that, with $b_n =$ the constant n, $\langle \{b_n\}_{n\in\mathbb{N}}, \{a_\xi\}_{\xi<\kappa}\rangle$ is a (ω,κ^*)-gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^*\rangle$. To obtain a (κ,ω^*)-gap, put $a'_\xi(k) = k - a_\xi(n)$ (recall that $a_\xi(k) \le k$) and $b'_n(k) = \mathtt{max}\,\{0, k-n\}$ for all ξ, k, n. □

This ends the proof of items (2) and (3) of Theorem 6.3 and justifies the definition of the abbreviation (κ,ω^*)-**gaps**.

7d Limits

Here we prove item (4) of Theorem 6.3.

To prove (4)(i) (there is no κ-limits in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$) note that any $a \in \mathbb{N}^{\mathbb{N}}$ has an exact $\leqslant_{\mathtt{fro}}$-predecessor $a_-(n) = \mathtt{max}\,\{a(n)-1, 0\}$. Further, (4)(ii) follows from Lemma 7.2. The remaining subitems of (4) need some work.

(4)(iii) Suppose that $\{a_\xi\}_{\xi<\kappa}$ is a tower in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$. We get a limit $\langle 0,0,0,...\rangle = \mathtt{lim}_{\xi\to\kappa}\, c_\xi$ in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$, where $c_\xi(n) = \frac{1}{a_\xi(n)}$. (For any ξ there may be finitely many cases of division by 0, the result of which can be set to be, say, 1.) The converse is similar.

(4)(iv) The equivalence follows from Lemma 7.4. The construction of a tower resembles the final part of the proof of Lemma 7.8. Consider a κ-limit $a = \mathtt{lim}_{\xi\to\kappa}\, a_\xi$ in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq\rangle$, where $a_\xi \prec a_\eta$ whenever $\xi < \eta < \kappa$. Put $b_n(k) = \mathtt{max}\,\{0, a(k)-n\}$, thus, $\{b_n\}_{n\in\mathbb{N}}$ is a $\leqslant_{\mathtt{fro}}$-descending sequence, and $\{a_\xi\}_{\xi<\kappa}$, $\{b_n\}_{n\in\mathbb{N}}$ form a (κ,ω^*)-gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$. To derive a κ-tower in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$ apply Lemma 7.7.

(4)(v) Say that a tower $\{c_\xi\}_{\xi<\kappa}$ is *regular*, if it satisfies

7.10. for any ξ there is $\eta > \xi$ and n_0 such that $c_\eta(n) \ge c_\xi(n+1)$ for all $n \ge n_0$ — in other words, it is required that $\forall\,\xi\ \exists\,\eta > \xi\ (c^+_\xi \leqslant^* c_\eta)$, where $c^+_\xi(n) = c_\xi(n+1), \forall\, n$. [15]

Now suppose κ-**towers**, in particular, there is a κ-tower $\{c_\xi\}_{\xi<\kappa}$ in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq\rangle$. Then there exist a regular cardinal $\kappa' < \kappa$ and a regular κ'-tower $\{c'_\xi\}_{\xi<\kappa'}$ in

[15] It does not seem that 7.10 follows from the fact that $\{c_\xi\}_{\xi<\kappa}$ is $\prec$-increasing. Accordingly, we are not able to follow the proof that κ-towers imply κ-limits in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq\rangle$ for the same κ in [Sch-1993] (Theorem 14, $2 \Longrightarrow 3$) in its key part, Claim 5 on S. 454. But we have no example of a tower not satisfying 7.10 either. Note that 7.10 holds provided $\{c_\xi\}$ is a scale, thus κ-scales imply κ-limits in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq\rangle$.

$\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$, satisfying $c_\xi \preccurlyeq c'_\xi$ for all $\xi < \kappa'$. (Indeed define c'_ξ by transfinite induction so that $c_{\xi+1} \preccurlyeq c'_{\xi+1}$ and $c'_{\xi+1}(n) \geq c'_\xi(n+1)$ for all ξ and n, and if λ is limit and $\{c'_\xi\}_{\xi<\lambda}$ is not yet a tower then $c_\lambda \preccurlyeq c'_\lambda$ and $c'_\xi \preccurlyeq c'_\lambda$ for all $\xi < \lambda$.)

Simulating the proof of Lemma 7.8, let us define $a_\xi \in \mathbb{R}^{\mathbb{N}}$ for all $\xi < \kappa'$ on the base of such a regular κ'-tower $\{c'_\xi\}_{\xi<\kappa'}$. Then, if $\xi < \eta < \kappa'$ and $c_\eta(n) \geq c_\xi(n+1)$ holds for all $n \geq n_0$, then we have $a_\eta <_{\mathtt{fro}} a_\xi$ in the proof of Lemma 7.8, not merely $a_\eta <^* a_\xi$. Therefore the sequence $\{a_\xi\}$ contains a cofinal strictly $<_{\mathtt{fro}}$-decreasing subsequence. Moreover, the limit members of such a subsequence form a cofinal strictly $\prec$-decreasing subsequence. Thus, by 7.9 in the proof of Lemma 7.8, we have a κ'-limit in $\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$, as required.

(4)(vi) Let, for any infinite set $x \subseteq \mathbb{N}$, φ_x be the unique increasing bijection $\mathbb{N} \xrightarrow{\text{onto}} x$. Suppose that $\{f_\alpha\}_{\alpha<\kappa}$ is a κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$. We can assume that every $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ is a strictly increasing function, for if not change it to $g_\alpha(k) = k + \sum_{n=0}^{k} f_\alpha(n)$. We are going to define a $\subset^*$-decreasing sequence $\{x_\alpha\}_{\alpha<\kappa'}$ of infinite sets $x_\alpha \subseteq \mathbb{N}$ such that $f_\alpha \leqslant^* \varphi_{x_\alpha}$ for all $\alpha < \kappa'$; the ordinal $\kappa' \leq \kappa$ is determined in the course of the construction.

Suppose that $\nu \leq \kappa$ and the sets x_α, $\alpha < \nu$, have been defined.

Case 1: there is an infinite set $x \subseteq \mathbb{N}$ such that $x \subseteq^* x_\alpha$ for all $\alpha < \nu$. Then $f_\alpha \leqslant^* \varphi_{x_\alpha} \leqslant^* \varphi_x$ for all α, therefore $\nu < \kappa$. Obviously there is an infinite set $y \subset^* x$ such that $f_\alpha \leqslant^* \varphi_y$. Put $x_\nu = y$.

Case 2: otherwise. Then the sequence $\{x_\alpha\}_{\alpha<\nu}$ can be easily converted to a κ'-limit in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$, where κ' is the cofinality of ν.

(4)(vii) Any κ-limit in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$ obviously remains such in the structure $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ by Lemma 7.3. A κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ can be transformed to a κ-limit in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ as follows. First of all we convert it to a κ-tower $\{a_\xi\}_{\xi<\kappa}$ in $\langle \mathbb{N}^{\mathbb{N}} ; \preccurlyeq \rangle$ such that any $a_\xi \in \mathbb{N}^{\mathbb{N}}$ is increasing. Then, following the proof of (4)(iii), put $c_\xi(n) = \frac{1}{a_\xi(n)}$. We claim that the sequence $\{c_\xi\}_{\xi<\kappa}$ is a κ-limit in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant^* \rangle$ (not only in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$, as in (4)(iii)), with $\mathtt{lim}_{\xi\to\kappa} c_\xi = \mathbf{0}$. (Where $\mathbf{0} \in \mathbb{R}^{\mathbb{N}}$ is the constant 0.)

Indeed suppose, towards the contrary, that $x \in \mathbb{R}^{\mathbb{N}}$ satisfies $\mathbf{0} <^* x \leqslant^* c_\xi$ for all ξ. The set $D = \{k : x(k) \neq 0\}$ is infinite as $\mathbf{0} <^* x$ strictly; let $D = j_0 < j_1 < j_2 < \dots$. Put $a(k) = \frac{1}{x(k)}$ for $k \in D$. Obviously $x \restriction D \leqslant^* c_\xi \restriction D$, and hence $a_\xi \restriction D \leqslant^* a$ for any ξ. Now take any strictly increasing $b \in \mathbb{N}^{\mathbb{N}}$ with $b(k) \geq a(j_{n+1})$ whenever $j_n \leq k < j_{n+1}$ — we have $a_\xi \leqslant^* b$ because a_ξ is also increasing. Therefore the sequence $\{a_\xi\}_{\xi<\kappa}$ is bounded, a contradiction to the tower assumption.

Let us prove the converse. We begin with a $\leqslant^*$-limit $\{c_\xi\}_{\xi<\kappa}$ in $\mathbb{R}^{\mathbb{N}}$. To simplify the argument, we assume w. l. o. g. that $\{c_\xi\}_{\xi<\kappa}$ is $\leqslant^*$-decreasing, the limit value $\mathtt{lim}_{\xi\to\kappa} c_\xi$ is $\mathbf{0}$ and that all terms $c_\xi(n)$ are non-negative. Let $D_\xi = \{n : c_\xi(n) = 0\}$ and let h_ξ be the characteristic function of D_ξ. The sequence of functions h_ξ is $\leqslant^*$-increasing, so that if $\mathtt{lim}_{\xi\to\omega_1} h_\xi = \mathbf{1}$ (the constant 1) in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$ then we are done. Thus suppose that the equality fails, so that there exists $h \in 2^{\mathbb{N}}$ with $h_\xi \leqslant^* h <^* \mathbf{1}$ for all ξ. Then the set $D = \{n : h(n) = 1\}$ is co-infinite in $\mathbb{N}$ and $D_\xi \subseteq^* D$ for all ξ since $h_\xi \leqslant^* h$.

It follows that the infinite set $C = \mathbb{N} \smallsetminus D$ has a finite intersection with each D_ξ. This allows us to define $a_\xi(k) = \frac{1}{c_\xi(a)}$ for all $k \in C$ and ξ. (For each ξ, a finite number of exceptions with division by 0 can be ignored.) The sequence of functions $a_\xi : C \to \mathbb{N}$ is $\leqslant^*$-increasing (at least non-strictly) because $\{c_\xi\}$ is $\leqslant^*$-decreasing. Moreover $\{a_\xi\}$ obviously is $\leqslant^*$-unbounded in the family $\mathbb{N}^C$ of all functions $a : C \to \mathbb{N}$ because $\{c_\xi\}$ is a limit (and remains such even after the restriction to C). It follows that $\{a_\xi\}$ has a strictly $<^*$-increasing subsequence. Thus we have a tower in $\langle C ; \leqslant^* \rangle$. To transform it into a tower in $\mathbb{N}^{\mathbb{N}}$ use any bijection of D onto $\mathbb{N}$.

It follows from (4)(vii) that $\leqslant^*$-limits in $\mathbb{R}^{\mathbb{N}}$ consist of at least two different “species”: those homological to towers in $\mathbb{N}^{\mathbb{N}}$ and those homological to $\leqslant^*$-limits in $2^{\mathbb{N}}$ (or $\mathbb{N}^{\mathbb{N}}$, that is equivalent).

7e Towers, limits, and scales

Here we prove the last items of Theorem 6.3, that is, (5) ans (6). Basically (5) is obvious, so it remains to establish (6). This is based on the the following claim (due to ROTHBERGER for $\kappa = \omega_1$).

Lemma 7.11. *If there is no κ-limits in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$ then any κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$ is a κ-scale.*

Proof. Suppose that $\{f_\alpha\}_{\alpha<\kappa}$ is a κ-tower in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$. Consider an arbitrary $f \in \mathbb{N}^{\mathbb{N}}$ and suppose towards the contrary that $f \not\leqslant^* f_\alpha$ for all $\alpha < \kappa$. We can assume that f is strictly increasing, together with all functions f_α. Then the sets $x_\alpha = \{n : f_\alpha(n) < f(n)\}$ are infinite, and we have $x_\beta \subseteq^* x_\alpha$ whenever $\alpha < \beta < \kappa$ since $f_\alpha \leqslant^* f_\beta$.

We claim that there is an infinite set $x \subseteq \mathbb{N}$ with $x \subseteq^* x_\alpha$ for all α. Indeed if the sequence $\{x_\alpha\}_{\alpha<\kappa}$ contains a cofinal strictly decreasing subsequence then such an x exists because otherwise the subsequence would produce a κ-limit in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$. If a cofinal strictly decreasing subsequence does not exist then for some $\gamma < \kappa$ we have $\forall\, \xi > \gamma \,(x_\xi \equiv^* x_\gamma)$, and hence $x = x_\gamma$ is as required.

Thus let x be as indicated. Then $f_\alpha \restriction x \leqslant^* f \restriction x$ (in the sense that the set $\{n \in x : f(n) < f_\alpha(n)\}$ is finite) for any α. Assuming that

$$x = \{0 = i_0 < i_1 < \cdots < i_n < \dots\},$$

we put $g(k) = f(i_{n+1})$ whenever $i_n \le k < i_{n+1}$. Then, as f and all f_α are increasing, we have $f_\alpha \leqslant^* g$ for all α, a contradiction with the assumption that the sequence of all f_α is a tower. □ (*Lemma*)

□ (*Theorem 6.3*)

8 The case of the modified rate of growth

The structures $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$, $\langle \mathbb{R}^{\mathbb{N}}\,;\trianglelefteq\rangle$ of (6.2), not covered by Theorem 6.3, require a separate consideration. Here the results remain less complete. The principal difficulty is connected with the fact that even the strict relation $f \vartriangleleft g$ is compatible with $g(n) < f(n)$ on the vast majority of numbers n, with $g(n) - f(n) \to +\infty$ only on a very scarse sequence of n's. This does not allow to convert $\trianglelefteq$-ordered constructions to, say, $\preccurlyeq$-ordered ones.

The transformation in the other direction is sometimes possible.

Proposition 8.1. *The structures $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$, $\langle \mathbb{R}^{\mathbb{N}}\,;\trianglelefteq\rangle$ are equivalent w. r. t. the existence of a scale, a tower, a gap, or a limit with any given cardinal characteristics. Moreover if κ is an uncountable regular cardinal then*:

(i) *the existence of κ-scales in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ is equivalent to κ-***scales***;*

(ii) *the existence of κ-towers in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ follows from κ-***towers*** and implies κ'-***towers*** for some regular uncountable cardinal $\kappa' \le \kappa$;*

(iii) (κ,ω^*)-**gaps** *implies the existence of (κ,ω^*)-gaps in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$.*

Proof. First of all, one easily proves that any scale, tower, or gap in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ is a scale, resp., tower, resp., gap in $\langle \mathbb{R}^{\mathbb{N}}\,;\trianglelefteq\rangle$, and conversely, any scale, tower, gap, limit in $\langle \mathbb{R}^{\mathbb{N}}\,;\trianglelefteq\rangle$ can be converted to resp. a scale, tower, gap, limit in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ of the same cardinal characteristics, by changing real terms of sequences to nearest bigger integers.

Note further that $\prec$ is a stronger order than $\vartriangleleft$ (in the sense that $f \prec g$ implies $f \vartriangleleft g$), and moreover $f \vartriangleleft g \prec h$ implies $f \prec h$. It easily follows that any scale, tower, gap (with both cardinals infinite) in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$ remains a scale, resp., tower, resp., gap in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$.

The converse holds for scales: if there is a κ-scale $\{f_\alpha\}_{\alpha<\kappa}$ in the structure $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ then there is a κ-scale in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$. Indeed by definition for any set $X \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality $\mathtt{card}\, X < \kappa$ there is $h \in \mathbb{N}^{\mathbb{N}}$ satisfying $f \preccurlyeq h$ for all $f \in X$. (Indeed as $\{f_\alpha\}$ is a scale there is $\alpha < \kappa$ such that $f \trianglelefteq f_\alpha$ for all $f \in X$. Put $h = f_\alpha$.) This enables us to define a $\prec$-increasing κ-sequence $\{h_\alpha\}_{\alpha<\kappa}$ of $h_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $f_\alpha \preccurlyeq h_\alpha$ for all α. It is clear that $\{h_\alpha\}$ is a κ-scale in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$.

As for towers, the inverse holds in a weaker form: if there is a κ-tower $\{f_\alpha\}_{\alpha<\kappa}$ in $\langle \mathbb{N}^{\mathbb{N}}\,;\trianglelefteq\rangle$ then there is a κ'-tower in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$ for some $\kappa' \le \kappa$. Indeed $\{f_\alpha\}$ remains an unbounded family in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$, of course, but it is not necessarily $\prec$-increasing. Now consider a maximal $\prec$-increasing sequence $\{h_\alpha\}_{\alpha<\kappa'}$ such that $x_\alpha \preccurlyeq y_\alpha$ for all $\alpha < \kappa'$. Clearly $\kappa' \le \kappa$ (otherwise $\{f_\alpha\}$ would not be a tower), while the maximality simply means that $\{h_\alpha\}$ is unbounded, *i. e.*, a tower in $\langle \mathbb{N}^{\mathbb{N}}\,;\preccurlyeq\rangle$. $\square$

9 Solutions of the problems

Hausdorff proved in [H 1909a] that Cantor's continuum-hypothesis **CH** (that is, $2^{\aleph_0} = \aleph_1$) implies hypotheses 6.4a, 6.4b, 6.4c of Remark 6.4 for $\kappa = \aleph_1$. On the contrary, the status of these hypotheses in the absense of **CH** was made fully clear only in the 1970s. It was established, by forcing technique, that there is no any connection between the hypotheses in the case $\kappa = \aleph_1$, provable in $\mathbf{ZFC} + \neg\,\mathbf{CH}$, except for the double implication 6.4c $\Longrightarrow$ 6.4b $\Longrightarrow$ 6.4a that follows from Theorem 6.3 in this case. This is summarized in the next theorem.

Theorem 9.1. *Each of the four statements is consistent with* $2^{\aleph_0} > \aleph_1$:

(1) 6.4c, 6.4b, 6.4a *hold for* $\kappa = \aleph_1$;

(2) 6.4c *fails but* 6.4b, 6.4a *hold for* $\kappa = \aleph_1$;

(3) 6.4c, 6.4b *fail but* 6.4a *holds for* $\kappa = \aleph_1$;

(4) 6.4c, 6.4b, 6.4a *fail for* $\kappa = \aleph_1$. □

In particular, it follows that hypotheses 6.4a, 6.4b, 6.4c (with $\kappa = \aleph_1$) are *undecidable* in $\mathbf{ZFC} + \neg\,\mathbf{CH}$. In other words, this theory is not strong enough to solve the problems. [16]

Proofs of different parts of the theorem appeared in the general frameworks of studies of "special subsets" of the reals and spaces like $2^{\mathbb{N}}$ (often in the form of $\mathscr{P}(\mathbb{N})$, the power set of $\mathbb{N}$) or $\mathbb{N}^{\mathbb{N}}$. Let us take some space to present an idea of the modern setup.

It has become plausible to associate, to each interesting type of "special subsets", a certain *cardinal invariant*, a cardinal (in the interval $[\aleph_1, 2^{\aleph_0}]$) equal to the least cardinality of a set of this type. Among the multitude of the cardinal invariants (see VAN DOUWEN [vD-1984] for a part of them) the following four are of interest because of their connection with the problems we discuss:

$\mathfrak{t} =$ the least cardinal κ such that κ-limits exist in $2^{\mathbb{N}}$;

$\mathfrak{b} =$ the least cardinality of an $\leqslant^*$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$ — easily equal to the least cardinal κ such that κ-towers exist in $\langle \leqslant^* ; \mathbb{N}^{\mathbb{N}} \rangle$;

$\mathfrak{b}_6 =$ the least cardinal κ such that (ω, κ)-LUZIN pairs exist in $2^{\mathbb{N}}$;

$\mathfrak{d} =$ the least cardinality of an $\leqslant^*$-dominating subset of $\mathbb{N}^{\mathbb{N}}$. [17]

(See Section 2 on dominating and unbounded sets.) Among them, we have $\mathfrak{t} \le \mathfrak{b} = \mathfrak{b}_6 \le \mathfrak{d}$ (VAN DOUWEN [vD-1984, 3.1, 3.3], but mainly due to ROTHBERGER [Ro-1948]). Thus hypotheses 6.4a, 6.4b, 6.4c of Remark 6.4 obtain

[16] It seems that long before any idea of the methods used in the proof of Theorem 9.1 was known, HAUSDORFF in certain way foresaw that the problems could not be solved in the usual sense of words "to solve the problem" — *i. e.*, to give a definite answer to the question by means of standard mathematical reasoning. See [H 1909a, p. 324].

[17] If $\mathfrak{d} = \omega_1$ then a $\mathfrak{d}$-scale exists. However note that for $\kappa > \aleph_1$ the equality $\mathfrak{d} = \kappa$ does not imply the existence of κ-scales, or to that extent of scales of any length. It is only in the assumption $\mathfrak{d} = \aleph_1$, or more generally $\mathfrak{b} = \mathfrak{d}$, that a dominating subset of cardinality $\mathfrak{d}$ can be converted to a scale.

compact formulations in the case $\kappa = \aleph_1$ as the equalities, resp., $\mathfrak{t} = \aleph_1$, $\mathfrak{b} = \aleph_1$, $\mathfrak{d} = \aleph_1$.

This theory includes a universal tool of getting non-existence results, *the Martin axiom*, or **MA**, which makes all reasonable "cardinal invariants" equal to $2^{\aleph_0}$. (See KUNEN [Kun-1980] as a general reference on **MA**.) It is known that **MA** is consistent with **ZFC** plus $2^{\aleph_0} > \aleph_1$, therefore every consequence of **MA** is consistent with $\neg$**CH**. In particular, as **MA** easily implies $\mathfrak{t} = 2^{\aleph_0}$ [18], hence, implies the absense of (ω, κ)- LUZIN pairs (= (ω, κ^*)-HAUSDORFF gaps) for $\kappa < 2^{\aleph_0}$, we have the consistency of item (4) of the theorem.

Consistency results of opposite character usually involve forcing construction of models of set theory. In particular HECHLER used in [He-1970, He-1974] some models to prove the consistency of $\mathfrak{t} = \mathfrak{b} = \aleph_1 < \mathfrak{d} = 2^{\aleph_0}$ and of $\mathfrak{t} = \mathfrak{b} = \mathfrak{d} = \aleph_1 < 2^{\aleph_0}$ (in fact in a rather generalized form), which proves the theorem in parts (1) and (2). See a more modern proof in [Bur-1997]. As for the consistency of statement (3) of the theorem, VAN DOUWEN credits the result (Theorem 5.3 in his survey, actually the consistency of the combination $\aleph_1 = \mathfrak{t} < \mathfrak{b} = 2^{\aleph_0}$) to SOLOMON [Slm-1977].

10 Uncountable sequences of Borel sets and partitions

It is known that in a separable space there is no wellordered uncountable increasing or decreasing sequences of open or closed sets. ZALCWASSER [Za-1922] proved that moreover, in any separable space any wellordered increasing or decreasing sequence of $\mathbf{\Delta}^0_2$ sets (*i. e.*, those simultaneously $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$) is at most countable. (HAUSDORFF gives a simplified proof that deals directly with reducible sets rather than $\mathbf{\Delta}^0_2$ sets in *"Mengenlehre"*, § 30.4. Another modification of ZALCWASSER's argument is presented in [Kur-1966, § 24.III].)

To define an increasing ω_1-sequence of $\mathbf{F}_\sigma$ (even countable) sets, take a set $X = \{x_\xi : \xi < \omega_1\} \subseteq \mathbb{R}$ of cardinality $\aleph_1$ and put $X_\xi = \{x_\eta : \eta < \xi\}$. It is a bit more difficult to obtain a strictly increasing sequence of $\mathbf{G}_\delta$ sets. A construction, again due to ZALCWASSER, (see KURATOWSKI [Kur-1966, § 40.III]), produces such a sequence of sets $X_\xi \subseteq \mathbb{R}$ by transfinite induction, so that X_ξ is a $\mathbf{G}_\delta$ set of Lebesgue measure 0 which covers $\bigcup_{\eta<\xi} X_\eta$ and contains at least one extra point. HAUSDORFF (Fasc. 281) suggested another construction: take a strictly $\leqslant_{\mathtt{fro}}$-increasing ω_1-sequence $\{a_\xi\}_{\xi<\omega_1}$ of elements $a_\xi \in \mathbb{N}^\mathbb{N}$ and put $X_\xi = \{x \in \mathbb{N}^\mathbb{N} : a_\xi \not\leqslant_{\mathtt{fro}} x\}$. Yet another construction was given by SIERPIŃSKI in [Si-1934], see also [Si-1956, § 68]. [19]

HAUSDORFF's Gap Theorem yields a much stronger result: there is a strictly increasing ω_1-sequence of $\mathbf{G}_\delta$ sets $X_\xi \subseteq 2^\mathbb{N}$ the union of which is the whole space $2^\mathbb{N}$. (Satz I in [H 1936b, S. 248]; this is true then for any uncountable Polish space.)

[18] See *e. g.* RUDIN [Ru-1977], Corollary 8, which is an elementary consequence of a more general fact, Theorem 7 there, first proved perhaps by MARTIN and SOLOVAY [MS-1970].

[19] By a theorem of LUZIN [Lu-1934] a strictly increasing ω_1-sequence of $\mathbf{G}_\delta$ sets X_α cannot be continuous in the sense that $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for any limit ordinal λ.

These constructions make use of the axiom of choice, of course. On the other hand, LUZIN – SIERPIŃSKI decompositions and sequences of constituents of Suslin and co-Suslin sets (see our comment to § 34 of *"Mengenlehre"* in Volume III) do not use choice, but fail to produce ω_1-sequences of Borel sets of bounded rank (say, of $\mathbf{G}_\delta$ sets or $\mathbf{F}_\sigma$ sets). Yet all of them appear to be both k- and m-convergent in the sense of [H 1936b], *i. e.*, additive w. r. t. both measure and category. This observation allowed HAUSDORFF to recall the following problem of SIERPIŃSKI [Si-1920p] (problems (A) and (B) in [H 1936b]). It belongs to a type of questions known as *partition problems*:

Question 10.1. In a Polish space, is there a strictly increasing sequence $\{B_\xi\}_{\xi<\omega_1}$ of Borel sets of measure 0 (resp. meager sets) such that the union $\bigcup_{\xi<\omega_1} B_\xi$ is <u>not</u> a measure 0 set (resp. <u>not</u> a meager set) ?

In either case, one can also require that the union $\bigcup_{\xi<\omega_1} B_\xi$ is equal to the whole space. This will be referred to as *the strong version.*

The problem, in its "category" part, has one and the same solution in each perfect Polish space because every such a space contains a dense $\mathbf{G}_\delta$ subset homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$. The "measure" part is also independent of the choice of both the (Borel non-atomic σ-additive) measure and the uncountable Polish space.

Question 10.2 ([H 1936b, (P) on p. 250])**.** Is there a Polish space equal to the union of a strictly increasing ω_1-sequence of $\mathbf{F}_\sigma$ sets ?

Such a Polish space has to be uncountable, of course. Somewhat surprisingly, the solution also does not depend on the choice of the Polish space (in the sense that if there exists a Polish space equal to the union of a strictly increasing ω_1-sequence of $\mathbf{F}_\sigma$ sets, then any uncountable Polish space hac this property), see below.

Question 10.3 ([Si-1945], [Kur-1966, § 39.II])**.** Is there a Polish space equal to the union of $\aleph_1$ pairwise disjoint non-empty $\mathbf{G}_\delta$ (or closed) sets ?

These (and some related) problems attracted a lot of interest among set theorists, both in the "classical" period of set theory (for instance, ROTHBERGER [Ro-1948], SIERPIŃSKI [Si-1945]), and especially, in the modern set theory — as a typical field of application of advanced methods of forcing.

11 Solutions of the partition problems

Questions 10.1, 10.2, 10.3 (in any of the versions indicated) clearly are solved affirmatively assuming **CH**. But, similarly to the gap existence problems, they are undecidable in **ZFC**+¬**CH**. In this case there are interesting interrelations between the problems which certainly deserve a discussion in our comments.

We start with Question 10.1. The following associated "cardinal invariants" are considered (see VAUGHAN [Va-1990]):

$\mathtt{add}(\mathrm{K})$**:** the least cardinality of a family of meager subsets of $\mathbb{R}$, the real line, the union of which is not meager;

$\mathtt{add}(\mathrm{L})$**:** the same for sets of Lebesgue measure 0;

$\mathtt{cov}(\mathrm{K})$**:** the least cardinality of a family of meager subsets of $\mathbb{R}$ the union of which equals $\mathbb{R}$;

$\mathtt{cov}(\mathrm{L})$**:** the same for Lebesgue measure 0.

All of them belong to $[\aleph_1, 2^{\aleph_0}]$, of course, and obey certain rules, for instance obviously $\mathtt{add}(\mathrm{K}) \le \mathtt{cov}(\mathrm{K})$ and $\mathtt{add}(\mathrm{L}) \le \mathtt{cov}(\mathrm{L})$ [20]. Question 10.1 consists, in this notation, in the questions whether $\mathtt{add}(\mathrm{L}) = \aleph_1$ and $\mathtt{add}(\mathrm{K}) = \aleph_1$ (or resp. $\mathtt{cov}(\mathrm{L}) = \aleph_1$ and $\mathtt{cov}(\mathrm{K}) = \aleph_1$ in the strong version). The following theorem demonstrates that the questions cannot be answered in **ZFC** $+ \neg$ **CH**.

Theorem 11.1. *The next statements are consistent with* $2^{\aleph_0} > \aleph_1$:

(i) $\mathtt{add}(\mathrm{L}) = \mathtt{add}(\mathrm{K}) = 2^{\aleph_0}$;

(ii) $\mathtt{cov}(\mathrm{L}) = \aleph_1$;

(iii) $\mathtt{cov}(\mathrm{K}) = \aleph_1$. □

The consistency of (i) was established by Martin and Solovay [MS-1970] with the help of **MA**. To be more exact, they proved that **MA** implies (i). Parts (ii) and (iii) were granted in [MS-1970] to unpublished works of Solovay in the early period of forcing [21].

Questions 10.2 and 10.3 turn out to be closely related to Question 10.1. This is based on the following key result of Fremlin and Shelah [FS-1979].

Theorem 11.2. *If* $\mathbb{R}$ *is the union of* $\aleph_1$*-many pairwise disjoint non-empty* $\mathbf{G}_\delta$ *sets then* $\mathbb{R}$ *is the union of* $\aleph_1$*-many meager sets.* □

(The converse is trivial.)

Corollary 11.3. *Questions 10.1 (strong version, category), 10.2, and 10.3 (*$\mathbf{G}_\delta$*) are equivalent, hence undecidable in* **ZFC** $+ \neg$ **CH**.

Proof. Suppose that $\mathbb{R}$ (then any perfect Polish space as well, see above) is a union of $\aleph_1$-many meager sets. This can easily be converted to a union of $\aleph_1$-many closed nowhere dense sets and then both to a strictly increasing union of $\mathbf{F}_\sigma$ sets and to a pairwise disjoint union of non-empty $\mathbf{G}_\delta$'s.

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[20] Some other rules are rather nontrivial, for instance it is known that $\mathtt{add}(\mathrm{K}) \le \mathtt{add}(\mathrm{L})$ but the inequality $\mathtt{add}(\mathrm{K}) < \mathtt{add}(\mathrm{L})$ is consistent, so that there is no full symmetry, see Vaughan [Va-1990] or Bartoszyński *e. a.* [BJS-1993].

[21] The most elementary models that can be used to prove the consistency of (ii) and (iii) with $\neg$ **CH** are those obtained by adding resp. $\aleph_2$ Cohen-generic reals and $\aleph_2$ Solovay-random reals to the constructible model. See Hechler [He-1973] on details and related questions and Bartoszyński *e. a.* [BJS-1993] on the modern state in this area, including some interesting interrelations between the partition and scale/gap/tower/limit existence problems and, correspondingly, between the associated cardinal invariants.

Suppose that a Polish space X is a strictly increasing union of $\aleph_1$-many $\mathbf{F}_\sigma$ sets. Then X is a nontrivial union of $\aleph_1$-many closed sets, hence a disjoint union of $\aleph_1$-many non-empty $\mathbf{G}_\delta$ sets. Now, as X is a continuous image of the Baire space $\mathbb{N}^\mathbb{N}$, homeomorphic to the irrationals, $\mathbb{R}$ turns out to be a disjoint union of $\aleph_1$-many non-empty $\mathbf{G}_\delta$'s. It remains to apply Theorem 11.2. □

Finally few words on partitions onto closed sets. Question 10.3 for closed sets also does not depend on the choice of the (uncountable) Polish space (Miller [Mi-1980]). Stern [St-1978] proved that the problem is undecidable in $\mathbf{ZFC} + \neg\,\mathbf{CH}$. As the models used by Stern were pretty the same as those proving the undecidability of Question 10.1 (strong version, category) [22], there was a question whether the problems are in fact equivalent. That this is not the case was demonstrated by Miller [Mi-1980]: it is consistent with $2^{\aleph_0} > \aleph_1$ that Question 10.1 (strong version, category) – and then Question 10.2 and Question 10.3 ($\mathbf{G}_\delta$) by the above – are solved affirmatively but Question 10.3 (closed) – negatively.

12 On five Hausdorff's problems on pantachies

The following is a list of problems on pantachies in [H 1907a, S. 151–152], actually for the structure $\langle \mathbb{R}^\mathbb{N}; \leqslant_{\mathtt{fro}} \rangle$. Recall that a pantachy is a maximal set strictly linearly ordered by a given PQO. Hausdorff's formulations are slightly modernized here.

Problem 12.1. Does there exist a pantachy not containing a (ω_1, ω_1^*)-gap? [23]

Problem 12.2. Does there exist a non-homogeneous pantachy? [24] Or, saying it differently, do there exist different order types of pantachies?

Problem 12.3. What is the least possible cofinality of a pantachy?

Problem 12.4. Do all pantachies have the same cofinality?

Problem 12.5. For a given uncountable cardinal κ, does there exist a κ-scale?

Problem 12.1 remains open, in fact it seems to be the oldest open problem in set theory explicitly stated in a suitable mathematical publication! Let us call *gapless* any pantachy that does not have a (ω_1, ω_1^*)-gap. If a gapless pantachy exists then $2^{\aleph_0} = 2^{\aleph_1}$ (see Comment [15] to [H 1909a]), therefore, the continuum-hypothesis fails! Gödel used this fact in his attempt to prove

[22] Stern proved that a partition of the reals onto $\aleph_1$ non-empty closed sets is possible in an $\aleph_2$-Solovay extensions and impossible in $\aleph_2$-Cohen extensions, and also impossible under the Martin axiom $\mathbf{MA}$.

[23] The existence of pantachies which do contain (ω_1, ω_1^*)-gaps immediately follows from the Hausdorff gap theorem — explicitly observed in [H 1909a, S. 323].

[24] By Hausdorff, a linear order type is *homogeneous* iff it is similar (order isomorphic) to any its nonempty open interval, including initial and final segments. Otherwise it is *non-homogeneous*. The existence of a homogeneous pantachy is established in [H 1907a, § 4] (see S. 146).

$2^{\aleph_0} = \aleph_2$ in [Go-1970] from a plausible list of axioms. See some remarks on SOLOVAY's analysis of GÖDEL's argument in [Slv-1995] in Comment [15] to [H 1909a]. KANAMORI [Kana-2007] gives additional information.

It can hardly be expected that the theory **ZFC** plus $2^{\aleph_0} = 2^{\aleph_1}$ outright proves or refutes 12.1. In such a case, the practice of the forcing era in set theory leads to consistency questions. Thus one can ask:

(A) is the existence of gapless pantachies consistent with **ZFC** + $2^{\aleph_0} = 2^{\aleph_1}$?

(B) is the absence of gapless pantachies consistent with **ZFC** + $2^{\aleph_0} = 2^{\aleph_1}$?

Both questions remain unanswered. A somewhat stronger (?) form of (A) (with the additional requirement that the pantachy does not contain strictly increasing or decreasing sequences of cardinality $\le \aleph_2$) is in the list of (two) open questions in [Slv-1995, § 7]. SOLOVAY notes that the major problem in the construction of a model satisfying (A) is to avoid HAUSDORFF's (ω_1, ω_1^*)-gaps.

Now on Problem 12.2. Its two forms are equivalent to each other. Indeed, given two pantachies $\mathfrak{A}_1, \mathfrak{A}_2$, there is a pantachy $\mathfrak{B}$ having open intervals similar to $\mathfrak{A}_1, \mathfrak{A}_2$, thus if $\mathfrak{A}_1, \mathfrak{A}_2$ are not similar then $\mathfrak{B}$ is not homogeneous. Conversely, if a pantachy $\mathfrak{B}$ is not homogeneous, then it contains an open interval $\mathfrak{B}'$ non-similar to $\mathfrak{B}$, thus there is a pantachy $\mathfrak{A}$ similar to $\mathfrak{B}'$ and hence not similar to $\mathfrak{B}$.

The negative answer to 12.2 (that is, all pantachies are similar) easily follows from **CH**, and hence is consistent with **ZFC**. It is perhaps an open problem whether the negative answer is consistent with the negation of **CH**, for instance, with $2^{\aleph_0} = 2^{\aleph_1}$. Nevertheless, **MA** plus not-**CH** implies that at least all pantachies have the same cofinality. Does **MA** imply that moreover all pantachies are order isomorphic? As for the positive answer (there exist non-similar pantachies), its consistency with **ZFC** (by necessity with the negation of **CH**) follows from some results related to 12.4, see below.

Problem 12.3: again **ZFC** does not have much to say. **CH** answers the question, the answer is $\aleph_1$, simply because any pantachy has both the cardinality and the cofinality equal to $\omega_1 = \aleph_1 = 2^{\aleph_0}$. On the other hand, there is a model of **ZFC** where $2^{\aleph_0} = 2^{\aleph_1} \ge \aleph_2$ but no pantachy has increasing ω_2-chains. Such a model can be obtained by a construction known as *adding many random reals*, see SOLOVAY [Slv-1995, 6].

Many consistency results related to 12.4 are known. In particular, it is consistent with **ZFC** that there exist towers of different cofinalities (even of *many* different cofinalities), see DORDAL [Do-1989]. Therefore, as any tower can be extended to a pantachy of the same cofinality, the existence of pantachies of different cofinalities is also consistent. "Gluing" together a pair of pantachies of different cofinalities, we can obtain a pantachy of some cofinality $\kappa \ge \omega_1$ containing an increasing (but non-cofinal) λ-sequence for some cardinal $\lambda > \kappa$.

Finally, on 12.5, the scale existence problem. Unlike towers, scales can exist in only one regular cardinality. An obvious transfinite construction yields an ω_1-scale in the assumption of **CH**. If **CH** fails then all three logically possible possibilities are consistent with **ZFC**. In particular, each of the following three hypotheses is consistent:

1) $\aleph_1 < 2^{\aleph_0}$, *and there is an* ω_1*-scale.*

In fact this hypothesis is a consequence of the sentence $\mathfrak{d} = \aleph_1 < 2^{\aleph_0}$, proved to be consistent by HECHLER, see Section 9.

2) $\aleph_1 < \mathfrak{c}$, *and there is a* $\mathfrak{c}$*-scale.*

This is a consequence of the Martin axiom **MA** plus $\aleph_1 < 2^{\aleph_0}$, because such a hypothesis implies $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} > \aleph_1$.

3) $\aleph_1 < \mathfrak{c}$, *and there is no scale of any length.*

This holds for instance in any model of **ZFC** obtained by adding $\aleph_2$ Cohen generic reals to a model of **ZFC + CH**.

13 HAUSDORFF's equivalence relations in the structure of Borel reducibility

The study of Borel reducibility is one of the most exiting topics in modern descriptive set theory. (See *e.g.* [Kec-1999] for motivation.) Given equivalence relations E and F on Borel sets resp. X and Y (sets in Polish spaces [25]), E is said to be *Borel reducible to* F, symbolically $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$, if there is a Borel map $\vartheta : X \to Y$ (called a reduction) such that

$$x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y) \qquad \text{for all } x, y \in X.$$

In this case ϑ can be lifted to an embedding $\Theta : X/\mathsf{E} \to Y/\mathsf{F}$ between the quotients, defined so that $\Theta([x]_{\mathsf{E}}) = [\vartheta(x)]_{\mathsf{F}}$, and the existence of such an embedding is interpreted as the fact that the *Borel cardinality* of the quotient X/E is $\le$ than that of the quotient Y/F. They define the associated relations of *Borel equivalence*, or *Borel bi-reducibility*:

$$\mathsf{E} \approx_{\mathrm{B}} \mathsf{F} \quad \text{iff} \quad \text{both} \quad \mathsf{E} \le_{\mathrm{B}} \mathsf{F} \quad \text{and} \quad \mathsf{F} \le_{\mathrm{B}} \mathsf{E},$$

and *Borel strict reducibility*: $\mathsf{E} <_{\mathrm{B}} \mathsf{F}$ iff $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ but $\neg\, \mathsf{F} \le_{\mathrm{B}} \mathsf{E}$.

The $\le_{\mathrm{B}}$-structure of the family of all Borel equivalence relations has been subject of intense study in descriptive set theory since the late 1980s. The key role of several mathematically meaningful Borel equivalence relations in the $\le_{\mathrm{B}}$-structure has been established, including:

E_0, defined on the set $\mathbb{N}^{\mathbb{N}}$ of all infinite sequences of natural numbers so that, for $a, b \in \mathbb{N}^{\mathbb{N}}$, $a \mathrel{\mathsf{E}_0} b$ iff $a(n) = b(n)$ for all but finite n;

E_1, defined on the set $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences of reals similarly;

[25] That is, separable topological spaces metrizable by a complete metric, like $\mathbb{R}$.

$\boldsymbol{\ell}^\infty$, defined on $\mathbb{R}^{\mathbb{N}}$ so that, for $a, b \in \mathbb{R}^{\mathbb{N}}$, $a \,\boldsymbol{\ell}^\infty\, b$ iff there exists a real x such that $|a(n) - b(n)| < x$ for all n;

$\mathbf{c}_0$, defined on $\mathbb{R}^{\mathbb{N}}$ so that $a \,\mathbf{c}_0\, b$ iff $\mathtt{lim}_{n\to\infty}(a(n) - b(n)) = 0$;

and many more.[26] In particular, it is known that

$$\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1 <_{\mathrm{B}} \boldsymbol{\ell}^\infty \quad \text{and} \quad \mathsf{E}_0 <_{\mathrm{B}} \mathbf{c}_0\,,$$

but $\mathbf{c}_0$ is $\le_{\mathrm{B}}$-incomparable with either of E_1 and $\boldsymbol{\ell}^\infty$, see *e.g.* [Kano-2008] for a survey of basic results in this area.

Let us review the place of HAUSDORFF's equivalence relations $\sim$, $\bowtie$, $\equiv_{\mathtt{fro}}$, $\equiv^*$ (Section 1) in the $\le_{\mathrm{B}}$-structure of Borel equivalence relations. Recall that, for $a, b \in \mathbb{R}^{\mathbb{N}}$,

$$\begin{array}{rll} a \sim b & \text{iff} & \text{the limit } \mathtt{lim}_{n\to\infty}(a(n) - b(n)) \text{ exists and is finite;} \\ a \bowtie b & \text{iff} & \text{the limit superior } \mathtt{lim}_{n\to\infty}(a(n) - b(n)) \text{ is finite;} \\ a \equiv^* b & \text{iff} & \exists\, n_0\, \forall\, n \ge n_0\; (a(n) = b(n)); \end{array}$$

that is, resp., the rate of growth, rate of growth modified, and the eventual domination equivalence relations on $\mathbb{R}^{\mathbb{N}}$, and the the final rangordnung equivalence relation $\equiv_{\mathtt{fro}}$ coincides with $\equiv^*$.

It does not seem that $\le_{\mathrm{B}}$-properties of $\sim$ have ever been studied with any success. For instance it is not known if either $\boldsymbol{\ell}^\infty$ or $\mathbf{c}_0$ is Borel reducible to $\sim$. On the other hand, the restricted version $\sim\restriction\mathbb{N}^{\mathbb{N}}$ (the restriction of $\sim$ to sequences of natural numbers) is a rather easy case. Let $\mathbb{Z}$ denote the entire numbers; $\mathbb{Z} \subseteq \mathbb{R}$. For $x, y \in \mathbb{Z}^{\mathbb{N}}$, we have $x \sim y$ iff there exist $n_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$ such that $x(n) = y(n) + j$ for all $n \ge n_0$. It easily follows that the equivalence relation $\sim\restriction\mathbb{Z}^{\mathbb{N}}$ is induced by a Borel action of a countable abelian group $\mathbb{Z} \times (\mathbb{Z})^{<\omega}$. Therefore, by a recent result of SU GAO and JACKSON [GJ-2007], that the restricted equivalence relation $\sim\restriction\mathbb{Z}^{\mathbb{N}}$ is Borel reducible to E_0. In fact we have

$$\mathsf{E}_0 \approx_{\mathrm{B}} (\sim\restriction\mathbb{Z}^{\mathbb{N}}) \approx_{\mathrm{B}} (\sim\restriction\mathbb{N}^{\mathbb{N}})\,.$$

Equivalence relations $\approx_{\mathrm{B}}$-equivalent to E_0 are called *essentially hyperfinite* in modern descriptive set theory.

The relation $\bowtie$ is obviously the same as $\boldsymbol{\ell}^\infty$ on $\mathbb{R}^{\mathbb{N}}$. The relation $\equiv^*$ is E_1 in the domain $\mathbb{R}^{\mathbb{N}}$ and E_0 in the domain $\mathbb{N}^{\mathbb{N}}$ by obvious reasons.

Let us now discuss *continual* versions of the same relations, defined on the

[26] Note that $\mathbb{N}^{\mathbb{N}}$ is a Polish space as a countable product of $\mathbb{N}$ with discrete topology, and $\mathbb{R}^{\mathbb{N}}$ is a Polish space as a countable product of $\mathbb{R}$, the real line. That E_0 is a Borel equivalence relation on $\mathbb{N}^{\mathbb{N}}$, while E_1, $\boldsymbol{\ell}^\infty$, $\mathbf{c}_0$, as well as the relations $\sim$, $\bowtie$, $\equiv^*$ discussed below, are Borel equivalence relations on $\mathbb{R}^{\mathbb{N}}$ is an easy exercise.

domain $\mathbf{C}[0,+\infty)$ of all continuous $f:[0,\infty)\to\mathbb{R}$ as follows: [27]

$$\begin{array}{lll} f\sim g & \text{iff} & \text{the limit } \mathtt{lim}_{x\to\infty}(f(x)-g(x)) \text{ exists and is finite;}\\ f\bowtie g & \text{iff} & \text{the limit superior } \mathtt{lim}_{x\to\infty}(f(x)-g(x)) \text{ is finite;}\\ f\equiv^* g & \text{iff} & \exists\, x_0\,\forall\, x\ge x_0\,(f(x)=g(x)). \end{array}$$

The relation $\bowtie\restriction\mathbf{C}[0,+\infty)$ is Borel-equivalent to $\bowtie\restriction\mathbb{R}^{\mathbb{N}}$. Indeed, given $f\in\mathbf{C}[0,+\infty)$, define $x_f\in\mathbb{R}^{\mathbb{N}}$ so that $x_f(n)=f(q_n)$, where $\{q_n\}_{n\in\mathbb{N}}$ is a fixed enumeration of $\mathbb{Q}^+$. Then clearly $f\bowtie g$ iff $x_f\bowtie x_g$ for all $f,g\in\mathbf{C}[0,+\infty)$, so that $f\mapsto x_f$ is a reduction of $\bowtie\restriction\mathbf{C}[0,+\infty)$ to $\bowtie\restriction\mathbb{R}^{\mathbb{N}}$. And the smaller domain $\mathbb{N}^{\mathbb{N}}$ does not change the picture: cut all values $x(n)$ of $x\in\mathbb{R}^{\mathbb{N}}$ to the nearest entire numbers and then work a bit more to convert from entire values to natural values. Formally,

$$(\bowtie\restriction\mathbf{C}[0,+\infty))\approx_{\mathrm{B}}(\bowtie\restriction\mathbb{R}^{\mathbb{N}})\approx_{\mathrm{B}}(\bowtie\restriction\mathbb{N}^{\mathbb{N}})\approx_{\mathrm{B}}\boldsymbol{\ell}^{\boldsymbol{\infty}}\,.$$

The relation $\equiv^*\restriction\mathbf{C}[0,+\infty)$ is Borel-equivalent to E_1, similarly to $\equiv^*\restriction\mathbb{R}^{\mathbb{N}}$. Indeed given $f\in\mathbf{C}[0,+\infty)$ define $h^n_f=f\restriction[n,n+1]$, so that h^n_f belongs to the Polish space $\mathbf{C}[n,n+1]$ of all continuous functions $h:[n,n+1]\to\mathbb{R}$. And obviously $f\equiv^* g$ iff $h^n_f=h^n_g$ for all but finite n. With the help of suitable Borel bijections $b_n:\mathbb{R}\xrightarrow{\text{onto}}\mathbf{C}[n,n+1]$, the map $f\mapsto\{h^n_f\}_{n\in\mathbb{N}}$ reduces the $\mathbf{C}[0,+\infty)$-version of $\equiv^*$ to E_1.

Problem 13.1. Study the place of the rate of growth equivalence relations $\sim\restriction\mathbb{R}^{\mathbb{N}}$ and $\sim\restriction\mathbf{C}[0,+\infty)$ in the $\le_{\mathrm{B}}$-structure of Borel equivalence relations. (See [Gao-2006] on some partial results.)

14 On the graduation problem

The goal of this Section is to demonstrate that the problem of "infinitary pantachy", or the problem of universal graduation as presented in Comment [6] to [H 1909a] in this Volume, can be solved, in the positive or in the negative, under different suitable clarifications of the setup. In brief, we'll show that a reasonable positive solution (and quite an elementary one) is possible within general set theoretic frameworks including the axiom of choice, but if a concrete definition or an "effective" construction of a graduation method is required then the problem is solved rather in the negative.

In parallel to this problem, we consider the question of existence of a usual pantachy as defined by Hausdorff, that is, a maximal branch in, say, the structure $\langle\mathbb{R}^{\mathbb{N}};\leqslant_{\mathtt{fro}}\rangle$. In this case, the positive solution on the base of the axiom of choice is clear, but the "effective" existence of a pantachy has not

[27] One can view $\mathbf{C}[0,+\infty)$ as a Borel set in the Polish space $X=\prod_n\mathbf{C}[n,n+1]$, where each $\mathbf{C}[n,n+1]$ is a Polish space of all continuous $f:[n,n+1]\to\mathbb{R}$ with the maximal distance metric $\rho(f,g)=\mathtt{max}_{n\le x\le n+1}|f(x)-g(x)|$. The equivalence relations $\sim$, $\bowtie$, $\equiv^*$ on $\mathbf{C}[0,+\infty)$ are Borel relations on X.

been studied in detail. The difference between the abstract existence and the "effective" existence of a pantachy was underlined by HAUSDORFF. [28]

14a What is an "infinitary pantachy"

First of all, let us find a reasonable exact formulation of the problem of universal graduation. Focusing on HAUSDORFF's favorite domain $\mathbb{R}^{\mathbb{N}}$ of countable real sequences, we shall be looking for

(1) a partial quasi-order (PQO) $\le$ on as set $D \subseteq \mathbb{R}^{\mathbb{N}}$, which is *linear*, that is, any two sequences $a, b \in D$ are $\le$-comparable, and

(2) $\le$ is based on the comparison of the behaviour of sequences at $+\infty$.

As the graduation is supposed to be universal, we assume for now that

(3) the domain D of $\le$ is equal to $\mathbb{R}^{\mathbb{N}}$,

but reserving the right to consider the case when $D \subsetneqq \mathbb{R}^{\mathbb{N}}$ in a suitable moment.

Condition (2) needs further clarification. As any finite number of values $a(n), b(n)$ must have no influence on the definition of $a \le b$, we stipulate that

(2a) the relation $\le$ is $\equiv_{\mathtt{fro}}$-invariant, that is, if $a, b, a', b' \in D$, $a \le b$, $a \equiv_{\mathtt{fro}} a'$, and $b \equiv_{\mathtt{fro}} b'$, then $a' \le b'$ as well.

Basically this means that $a \equiv_{\mathtt{fro}} b \Longrightarrow a \equiv b$, where $\equiv$ is the associated equivalence relation, that is, $a \equiv b$ iff $a \le b$ and $b \le a$.

Still the combination (1)+(3)+(2a) is compatible with $a \equiv b$ for all a, b. This triviality can be avoided by different extra conditions. One of them simply requires that the equivalence relation $\equiv$ has no "large" equivalence classes.

(2b) if $a \in D$ then the equivalence class $[a]_{\equiv}$ is a meager set.

And there is another reasonable condition, essentially saying that $\le$ respects a part of HAUSDORFF's *Fundamentalsatz*:

(2c) if a subset X of the domain D of $\le$ is at most countable then there is $b \in D$ such that $a < b$ (that is, $a \le b$ but $b \not\le a$) for all $a \in X$.

Let us call **a universal graduation** any relation $\le$ satisfying every condition in the list (1), (3), (2a), (2b), (2c). Note that the relation $\leqslant_{\mathtt{fro}}$ itself satisfies all these conditions, except, of course, for the linearity of $\le$ in (1).

All results below in this Section are explicitly related to the structure $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$, but they remain equally true for other HAUSDORFF's ordered structures defined in Section 1.

[28] "Since the attempt to actually legitimately construct a pantachie seems completely hopeless, it would now be a matter of gathering information *without further assumptions* about the order type of any pantachie" ([H 1907a], p. 110, HAUSDORFF's italics). Thus HAUSDORFF makes a clear distinction between the "effective" construction of a pantachy and the investigation of pantachies produced merely by the axiom of choice.

14b Linear extensions

We continue to consider HAUSDORFF's favorite graduation order $\leqslant_{\mathtt{fro}}$ on the set $\mathbb{R}^{\mathbb{N}}$ of all infinite real sequences. By a *linear extension* of $\leqslant_{\mathtt{fro}}$ we understand any quasi-linear ordering $\leq$ on $\mathbb{R}^{\mathbb{N}}$ such that

(A) $a \leqslant_{\mathtt{fro}} b \Longrightarrow a \leq b$, and

(B) the associated equivalence relation $a \equiv b$ iff $a \leq b$ and $b \leq a$ satisfies $a \equiv b \Longleftrightarrow a \equiv_{\mathtt{fro}} b$ for all $a, b \in \mathbb{R}^{\mathbb{N}}$.

Theorem 14.1. *There is a linear extension* $\leq$ *of* $\leqslant_{\mathtt{fro}}$. *Any such extension* $\leq$ *is a universal graduation, i. e., it satisfies* (1), (3), (2a), (2b), (2c).

Proof. The existence of a linear extension is one of basic facts in order theory; in this case it requires the axiom of choice, of course.

Any linear extension $\leq$ of $\leqslant_{\mathtt{fro}}$ is compatible with $\leqslant_{\mathtt{fro}}$, even preserves the $\equiv_{\mathtt{fro}}$-equivalence classes (that is, does not break or glue them), preserves the $<_{\mathtt{fro}}$-order between any two $<_{\mathtt{fro}}$-comparable classes, and in addition makes all elements in the domain $\mathbb{R}^{\mathbb{N}}$ comparable. After this remark, the proof of all five conditions becomes a triviality. □

Thus the problem of universal graduation in such a liberal form admits a rather simple positive solution on the base of the axiom of choice.

Yet a closer inspection shows that quasi-linear extensions of $\leqslant_{\mathtt{fro}}$ given by a formal application of the axiom of choice lack any concrete mathematical meaning, since the decision, which one of relations $a < b$, $a > b$, $a \equiv b$ is assigned for any given pair of $\leqslant_{\mathtt{fro}}$-incomparable sequences $a, b \in \mathbb{R}^{\mathbb{N}}$, is taken not really on the base of comparison of their behaviour at the infinity, but rather on an arbitrary choice in a transfinite sequence of arbitrary choices behind the very construction of a linear extension of $\leqslant_{\mathtt{fro}}$. Therefore we may ask whether there is any concrete, well-defined, or, as it is customary to say, "effective" universal graduation method.

14c The Borel domain

It turns out that the graduation problem solves flatly in the negative in the Borel domain, that is, we have the following non-existence result.

Theorem 14.2. *There is no Borel* PQO $\leq$ *on* $\mathbb{R}^{\mathbb{N}}$ *satisfying conditions* (1), (2a), (2b), (3).

Proof. We apply an old argument which SIERPIŃSKI [Si-1918] designed to prove that there is no Borel linear ordering of Vitali classes. Suppose towards the contrary that a Borel PQO $\leq^\circ$ on $\mathbb{R}^{\mathbb{N}}$ satisfies (1), (2a), (2b). (We add $^\circ$ in order not to mix this relation with the usual ordering of the real line $\mathbb{R}$ that will also participate in the argument.)

Consider the following Borel sets in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$:

$$P = \{\langle a, b\rangle : a <^\circ b\}, \quad Q = \{\langle a, b\rangle : b <^\circ a\}, \quad E = \{\langle a, b\rangle : a \equiv^\circ b\},$$

where $<^\circ$ and $\equiv^\circ$ are the associated strict order and equivalence relation. The cross-sections $E_a = \{b : \langle a, b\rangle \in E\}$ are equal to the equivalence classes $[a]_{\equiv^\circ}$. Therefore it follows from (2b) by the Ulam – Kuratowski theorem that E is meager in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Thus at least one of the sets P, Q is not meager. Let say P be a comeager set on a non-empty open set $U \times V \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, where

$$\begin{aligned} U &= \{a \in \mathbb{R}^{\mathbb{N}} : \forall\, i \le m\, (p_i < a(i) < q_i)\}, \\ V &= \{b \in \mathbb{R}^{\mathbb{N}} : \forall\, i \le m\, (r_i < b(i) < s_i)\}, \end{aligned}$$

m is a natural number, and $p_i < q_i$, $r_i < s_i$ are rationals for all $i \le m$. Then $\langle a, b\rangle \in U \times V$ iff $\langle b, a\rangle \in V \times U$, and hence Q is comeager on $V \times U$.

For every i, let f_i be the increasing linear map from the real interval (r_i, s_i) onto (p_i, q_i). Define a homeomorphism $f : U \xrightarrow{\text{onto}} V$ so that $f(a) = a'$ iff $a'(i) = f_i(a(i))$ for all $i \le m$, and $a'(i) = a(i)$ for all $i > m$. Then $a \equiv_{\texttt{fro}} f(a)$, and hence $a \equiv^\circ f(a)$ by (2a), for all $a \in U$, and accordingly $b \equiv^\circ f^{-1}(b)$, for all $b \in V$. Therefore if $\langle a, b\rangle \in U \times V$ then $\langle f(a), f^{-1}(b)\rangle \in V \times U$, and $a <^\circ b$ iff $f(a) <^\circ f^{-1}(b)$. We conclude that P is comeager on $V \times U$, but this contradicts to the fact that Q is comeager on $V \times U$. □

The proof of Theorem 14.2 works also on the base of measure instead of the category, provided a σ-additive measure on $\mathbb{R}^{\mathbb{N}}$ is given and it is invariant under the transformations $\langle a, b\rangle \mapsto \langle b, a\rangle$ and $\langle a, b\rangle \mapsto \langle f(a), f^{-1}(b)\rangle$.

However both the category and the measure versions of the proof are limited by the assumption that the whole domain of ordering admits a suitable notion of category or measure. For instance the argument does not seem to work if one wants to show that there is no Borel pantachy in $\mathbb{R}^{\mathbb{N}}$ (in Hausdorff's sense, that is, a maximal $<_{\texttt{fro}}$-branch in $\mathbb{R}^{\mathbb{N}}$). Fortunately there is an entirely different argument based on the following advanced theorem of Harrington, Marker, Shelah [HMS-1989]:

Theorem 14.3. *If $\le$ is a Borel **thin** quasi-order on a Borel set X then there is an ordinal $\alpha < \omega_1$ and a Borel map $h : X \to 2^\alpha$ such that*

$$x \le y \Longrightarrow h(x) \le_{\texttt{lex}} h(y) \quad \textit{and} \quad x \equiv y \Longleftrightarrow h(x) = h(y)$$

for all $x, y \in X$, where $\le_{\texttt{lex}}$ is the lexicographic order on 2^α and $x \equiv y$ iff both $x \le y$ and $y \le x$. □

A *thin* quasi-order is any PQO $\le$ such that there is no perfect set of pairwise $\le$-incomparable elements. In particular, any linear quasi-order (LQO) is such, and if $\le$ is a LQO then the properties of h as in the cited theorem are equivalent to just the equivalence $x \le y \Longleftrightarrow h(x) \le_{\texttt{lex}} h(y)$ for all $x, y \in X$.

Corollary 14.4. *If $\le$ is a Borel LQO on a Borel set X then there is no strictly $<$-monotone ω_1-sequence, where $<$ is the associated strict order.*

Proof. There is no strictly $<_{\texttt{lex}}$-monotone ω_1-sequence in any 2^α, $\alpha < \omega_1$. □

This allows us to replace conditions (2a) and (2b) in Theorem 14.2 by (2c).

Theorem 14.5. *There is no Borel* PQO $\le$ *on a Borel set* $D \subseteq \mathbb{R}^{\mathbb{N}}$ *satisfying conditions* (1) *and* (2c). *In particular, there is no Borel pantachy in the structure* $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$.

Proof. Suppose towards the contrary that a Borel PQO $\le$ on D satisfies (1) and (2c). Then $\le$ is a Borel LQO, and hence by Corollary 14.4 there is no $<$-monotone ω_1-sequence in D. On the other hand, the existence of such a sequence easily follows from (2c). □

It follows from Theorem 14.3 that such Borel PQOs as $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathtt{fro}}$, $\leqslant^*$ do not admit a *Borel linearization*, that is, *e.g.*, for $\leqslant_{\mathtt{fro}}$, there does not exist a Borel LQO $\le$ on $\mathbb{R}^{\mathbb{N}}$ such that $a \leqslant_{\mathtt{fro}} b \Longrightarrow a \le b$ and $a \equiv_{\mathtt{fro}} b \Longleftrightarrow a \equiv b$ for all $a, b \in \mathbb{R}^{\mathbb{N}}$, where $\equiv$ is the associated equivalence relation.

14d Larger "effective" domains

The concept of "effectivity" in set theory includes, of course, much wider domains than the Borel domain, for instance Suslin sets and their complements, projective sets, Gödel-constructible sets, and so on. And as long as only sets related to Polish spaces are studied, the widest class of "effective" sets is considered to be the class ROD of *real-ordinal definable* sets [29], see Solovay [Slv-1970]. What about the state of the universal graduation problem in these wider non-Borel domains?

As it is typical in modern set theory, the answer depends on the background set theoretic environment, that is, the type of the set universe we consider, or the axioms we add to **ZFC** in order to solve the problem.

For instance, adding Gödel's *axiom of constructibility* [30] $\mathbf{V} = \mathbf{L}$, we obtain an "effective" wellordering of the reals, of class Δ^1_2, and this easily leads to the following rather routine result.

Theorem 14.6. *Assuming the axiom of constructibility*:

1) *there is a linear extension* $\le$ *of* $\leqslant_{\mathtt{fro}}$ *in the class* Δ^1_2, *and*

2) *there is a pantachy in* $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$ *in the class* Δ^1_2. □

As any such extension $\le$ satisfies (1), (3), (2a), (2b), (2c) by Theorem 14.1, we obtain a universal graduation in the class Δ^1_2. The proof of Theorem 14.6 consists in a direct and absolutely "effective" construction of $\le$ (under the assumption $\mathbf{V} = \mathbf{L}$), so we conclude that this axiom implies the "effective" existence of a universal graduation (in the sense of Subsection 14b) and of a

[29] The class ROD contains those sets which admit a definition by an arbitrary set theoretic formula which contains only ordinals and reals as parameters.

[30] It postulates that every set belongs to the *constructible universe* $\mathbf{L}$. The latter contains all sets obtained in the course of a certain transfinite inductive construction. $\mathbf{L}$ is a small part of the class ROD mentioned in the previous footnote. See [Kana-2007, Kun-1980] for more on this topic.

pantachy in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\texttt{fro}} \rangle$. In particular the "effective" existence of a universal graduation and such a pantachy is consistent with the axioms of **ZFC** because so is the axiom of constructibility.

Now consider the opposite side of the problem. SOLOVAY [Slv-1970] defined a model of set theory in which all real-ordinal definable (class ROD, see above) sets of reals are Lebesgue measurable, have the Baire property, and if uncountable then contain perfect subsets. This model is known as the SOLOVAY model in modern set theory, and they often prove that this model has no typical counterexamples related to pointsets in Polish spaces among projective sets, and generally in the class ROD — which is interpreted as the non-existence of "effective" counterexamples. The case of universal graduation and pantachies is no difference.

Theorem 14.7. *The following is true in the* SOLOVAY *model*:

1) *there is no* ROD PQO $\le$ *on* $\mathbb{R}^{\mathbb{N}}$ *satisfying* (1), (2a), (2b), (3), *and*
2) *there is no* ROD *pantachy in* $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\texttt{fro}} \rangle$. □

The proof of the first claim is quite similar to the proof of Theorem 14.2, with the only difference that, instead of the Baire property for Borel sets, SOLOVAY's theorem, that in this model all ROD sets have the Baire property, is applied. As for the second claim, the proof (yet unpublished) is much more complicated, and based on the analysis of ROD linear orders in the SOLOVAY model in KANOVEI [Kano-2000] and the analysis of Borel sets of bounded rank by STERN [St-1984].

In particular, it follows from Theorem 14.7 that there is no any "effective" construction in **ZFC** of either a universal graduation or a HAUSDORFF's pantachy in $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\texttt{fro}} \rangle$.

14e On graduation scales vs direct comparison

Theorem 14.3 also contributes to the analysis of a different aspect of the universal graduation problem, namely the dilemma **graduation scale vs direct comparison**, see Comment [2] to [H 1909a].

Suppose that $P = \langle P ; \le \rangle$ is an arbitrary partially ordered set considered as a *graduation scale* (and it is assumed that $x \le y \wedge y \le x$ implies $x = y$ for $x, y \in P$). Let a *graduation by means of* P be any map $\pi : \mathbb{R}^{\mathbb{N}} \to P$. It induces a PQO $\le_\pi^P$ on $\mathbb{R}^{\mathbb{N}}$ such that $a \le_\pi^P b$ iff $\pi(a) \le \pi(b)$. A question that could be asked here is to what extent PQOs of the form $\le_\pi^P$ exhaust the domain of all PQOs $\le'$ on $\mathbb{R}^{\mathbb{N}}$. (Here we discard any further requirements to $\le'$ connected *e.g.* with comparison by final behaviour; hence, $\mathbb{R}^{\mathbb{N}}$ below can be replaced by any Borel set in a Polish space.)

In the frameworks of pure set theory this question is rather vacuous. Indeed, given a PQO $\le'$ of any kind on $\mathbb{R}^{\mathbb{N}}$, we let $P = \mathbb{R}^{\mathbb{N}}/{\equiv'}$ (where $\equiv'$ is the equivalence relation associated with $\le'$) with the induced quotient order, as in Footnote 1 on p. 2, which we denote by $\le$, and let $\pi(a) = [a]_{\equiv'}$ for all $a \in \mathbb{R}^{\mathbb{N}}$ — then $\le'$ coincides with $\le_\pi^P$, of course, where $P = \langle P ; \le \rangle$.

However the picture changes in the domain of Borel sets and relations. Indeed, suppose that $\le'$ is a Borel PQO on $\mathbb{R}^{\mathbb{N}}$. Is there any Borel partially ordered set $P = \langle P; \le \rangle$ (P being a Borel set in a Polish space, and $\le$ being a Borel order on P), and a Borel map $\pi : \mathbb{R}^{\mathbb{N}} \to P$, such that $\le'$ is equal to $\le_{\pi}^{P}$? And the answer is that this may be not the case, and this is definitely not the case for $\le'$ being typical graduation orders, *e.g.*, those defined in Section 1. Indeed, if π *etc.* are as indicated then we have $a \equiv' b$ iff $\pi(a) = \pi(b)$, that is, the equivalence relation $\equiv'$ is Borel reducible to the equality on a Borel set. Equivalence relations satisfying this property are called *smooth*. On the other hand, it is known that all equivalence relations associated with HAUSDORFF's graduation orderings (see Section 13) are non-smooth. (See, *e.g.*, KANOVEI [Kano-2008].) Thus, in the Borel domain, graduation methods (not necessarily linear) form a wider category then those based on Borel scales.

But the picture again turns upside down if we consider, still in the Borel domain, only *linear* graduation methods, that is, basically, Borel LQOs $\le'$, and accordingly linear Borel orders $P = \langle P; \le \rangle$ as scales. But if $\le'$ is a Borel LQO on $\mathbb{R}^{\mathbb{N}}$ then by Theorem 14.3 there exist an ordinal $\alpha < \omega_1$ and a Borel map $h : \mathbb{R}^{\mathbb{N}} \to 2^{\alpha}$ such that we have for all $a, b \in \mathbb{R}^{\mathbb{N}}$: $a \le' b$ iff $h(a) \le_{\texttt{lex}} h(b)$. Therefore it suffices now to take 2^{α} (a Borel set and a Polish space itself) as P and the lexicographical order $\le_{\texttt{lex}}$ as $\le$. Thus, in the Borel domain, *linear* graduation methods are exactly those based on Borel linear scales.

15 Review of HAUSDORFF's problems in set theory

This Section is written to bring together different problems on descriptive set theory which we found in HAUSDORFF's books, printed papers, and the *"Nachlass"*. Some of them have been discussed above, but we consider it to the convenience of the reader to collect some other problems, not necessarily related to gaps and similar topics, in a common list, with brief remarks.

Problem 15.1 ([H 1933b]). Does there exist a countable system $\{A_n\}_{n\in\mathbb{N}}$ of sets $A_n \subseteq \omega_1$ such that every set $X \subseteq \omega_1$ can be presented in the form $X = \overline{\texttt{lim}}_{k\to\infty} A_{n_k}$ for some increasing sequence $\{n_k\}$?

This problem makes sense only if $2^{\aleph_1} = 2^{\aleph_0}$, of course. Under this assumption, Martin's axiom **MA** implies the positive answer, and hence the positive answer is consistent. The negative answer is consistent either, and this can be demonstrated by a model obtained by adding $\aleph_2$ Cohen generic reals (with finite support). See more on this in Commentary on [H 1933b] in Volume III.

Problem 15.2 ([H 1935c]). A function $f : \mathbb{R} \to \mathbb{R}$ is *symmetrically continuous* if for all x we have $\texttt{lim}_{h\to\infty}[f(x+h) - f(x-h)] = 0$. Consider the set D_f of all points of discontinuity of such a function f. Can D_f be uncountable? Can D_f be any given $\mathbf{F}_\sigma$ set?

Problem 15.2 may still be open. It is known that D_f is meager and null, and can be uncountable assuming **CH**. PONOMAREV [Pon-1973] proved that D_f

cannot be both closed and uncountable. See further comments in Kommentary on NL 601 and NL 602 in Volume III (pp. 733–735).

Problem 15.3. Let κ, λ be infinite cardinals. Are all 7 HOS of list 6.1 equivalent to each other with respect to the existence of (κ, λ^*)-gaps?

Theorem 6.3 above resolves the question in the affirmative provided one of the cardinals is countable. The answer is also positive when $\kappa = \lambda = \aleph_1$, just because each HOS contains a (ω_1, ω_1^*)-gap by the HAUSDORFF gap theorem. As for the general case, Problem 15.3 can be re-formulated as follows: which implications in the diagram related to (2) of Theorem 6.3 can be changed to equivalences in the general case of infinite (regular) κ, λ? The solution may be not entirely elementary. FARAH observed that a (κ, λ^*)-gap in $\langle \mathbb{N}^{\mathbb{N}} ; \leqslant^* \rangle$ implies a (κ, λ^*)-gap in $2^{\mathbb{N}\times\mathbb{N}}/\mathtt{Fin} \times 0$ while the latter implies a (κ, λ^*)-gap in $\langle 2^{\mathbb{N}} ; \leqslant^* \rangle$ *after adding a single Cohen real.*

Problem 15.4. For a function f to be of the 1st Baire class it is necessary and sufficient that any non-empty closed set contains a point of continuity of f. Is there any generalization for higher Baire classes?

Problem 15.5. In a complete spaces, meager dense sets are not $\mathbf{G}_\delta$. Are there similar characterizations for higher classes ?

These two rather vague problems were formulated in a short note NL 1002 published in Volume III, pp. 590–591. HAUSDORFF conjectures that any dense set $A = \bigcap_n A_n$, where $A_n \subseteq A_{n-1}$ for all n and each A_n is everywhere comeager in itself but meager in A_{n-1}, is not $\mathbf{\Sigma}^0_3$. See further remarks in our Kommentary in Volume III, pp. 618–621.

Problem 15.6 ([NL 629]). Let $\mathfrak{t}$ be a countable order type, not necessarily an ordinal. Is the set of all sets $X \subseteq \mathbb{Q}$ of order type $\mathfrak{t}$ Borel?

Problem 15.6 was solved in the affirmative by a rather nontrivial argument of SCOTT [Sco-1964]. Is there a really elementary proof? See further remarks in our Kommentary in Volume III, pp. 735–736.

Problem 15.7. By a HUREWICZ's result the union U of any increasing ω_1-sequence of $\mathbf{G}_\delta$ sets is $\mathbf{F}_{\mathrm{II}}$ (*i. e.*, any relatively closed $X \subseteq U$ is not meager in itself).

Conversely, is any $\mathbf{F}_{\mathrm{II}}$ set equal to such a union U?

Problem 15.8. Say that a decreasing sequence of Borel sets $\{X_\xi\}_{\xi<\omega_1}$ is *canonical* if, for any Borel set B, if $\bigcap_\xi X_\xi \subseteq B$ then there is $\xi < \omega_1$ such that $X_\xi \subseteq B$. Any Suslin set is known to be equal to the intersection of a canonical sequence. Conversely, is any intersection of a canonical sequence of Borel sets a Suslin set ?

These two problems were formulated in NL 1002, a note published in Volume III, pp. 701–703. And both remain open. ZAPLETAL noted that, under

CH, Problem 15.8 solves in the negative: any Bernstein set is the intersection of a canonical sequence of co-countable sets. See more substantial remarks in our Kommentary in Volume III, pp. 710–712.

Problem 15.9. When a sum of $\aleph_1$ Borel sets is a Suslin set ?

This question appeared in [NL 380], a half-page list of assorted mathematical problems dated between 1928 and 1931. The original text is as follows:

Wenn ist die Summe von $\aleph_1$ Borelschen Mengen eine Suslinsche?

This is a question perhaps rather vague to expect any direct answer. Moreover it is known that questions of this kind may lead to very strong hypotheses of axiomatic set theory. For instance, hypothetical Proposition II of Luzin [Lu-1935, Section 9], that any union of (perhaps, uncountably many) Borel constituents of a given $\mathbf{\Pi}^1_1$ set is $\mathbf{\Pi}^1_1$ itself, implies that $\aleph_1$ is a measurable cardinal (this is mentioned in [MS-1970]), which is incompatible with the axiom of choice. The only known way to prove Proposition II is from the axiom of determinacy AD [Mos-1980, Chapter 7]. Coming back to Problem 15.9, we may ask whether the hypothesis, that any union of Borel constituents of a given $\mathbf{\Sigma}^1_1$ set is $\mathbf{\Sigma}^1_1$ itself, is consistent with **ZF**.

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