Models of set theory in which the separation theorem fails

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Abstract

We make use of a finite support product of the Jensen minimal forcing to define a model of set theory in which the separation theorem fails for projective classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, for a given $n \geq 3$.

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1 Introduction

The separation problem was introduced in descriptive set theory by Luzin [1]. In particular, Luzin asked whether (in modern notation for projective classes):

(I) any pair of disjoint $\mathbf{\Sigma}^1_n$ sets of reals can be separated by a $\mathbf{\Delta}^1_n$ set,

(II) the remainders of two $\mathbf{\Sigma}^1_n$ sets, obtained by the removal of their intersection, can be separated by disjoint $\mathbf{\Pi}^1_n$ sets, and

(III) there are two disjoint $\mathbf{\Pi}^1_n$ sets not separable by a $\mathbf{\Delta}^1_n$ set.

Luzin underlined the importance and difficulty of these problems. [1] Novikov characterized the separation problem as one of three main problems of descriptive set theory in [2], along with the measurability problem for $\mathbf{\Sigma}^1_2$ sets and the cardinality problem for $\mathbf{\Pi}^1_1$ sets. (See *e.g.* [3] on the two latter poblems.)

The problem is well known in descriptive set theory. In modern terms (see Moschovakis [4], Kechris [5]), the *(first) separation theorem* for a class Γ of pointsets (sets in Polish spaces) is the claim that any two disjoint sets in Γ (in the same space) can be separated by a set in $\Gamma \cap \Gamma^{\complement}$, where $\Gamma^{\complement}$ is the class of complements of Γ-sets. The *second separation theorem* for Γ claims that if X, Y are sets in Γ (in the same space) then the sets $X' = X \smallsetminus Y$ and $Y' = Y \smallsetminus X$ are separable by two disjoint sets in $\Gamma^{\complement}$. Thus the content of the problems (I), (II), (III) is as follows:

— does the (first) separation theorem hold for $\mathbf{\Sigma}^1_n$?

[1] L'un des problémes les plus importants de la théorie des ensembles projectifs et qui attend encore sa solution, est celui de leur *séparabilité*. On sait que deux ensembles analytiques quelconques sans point commun sont toujours séparables B. Il serait trés important de démontrer que deux ensembles (A_n) quelconques sans point commun sont séparables (B_n). De même, nous savons que si l'on supprime la partie commune à deux ensembles analytiques, les parties restantes sont séparables au moyen de deux complémentaires analytiques. La question se pose naturellement de savoir si ce principe subsiste quand on remplace les ensembles analytiques par (A_n) et les complémentaires analytiques par (CA_n). C'est un probléme qui mérite d'attirer l'attention des analystes malgré sa difficulté. D'ailleurs, il importe de savoir s'il existe deux ensembles (CA_n) qui ne soient pas séparables (B_n). (Lusin [1, p. 289].)

— does the second separation theorem hold for $\mathbf{\Sigma}^1_n$?

— does the (first) separation theorem fail for $\mathbf{\Pi}^1_n$?

Both separation theorems hold for $\mathbf{\Sigma}^1_1$ by Luzin [6, 1], but fail for $\mathbf{\Pi}^1_1$ by Novikov [7], and these results were known before the publication of the (French original) of [1] in 1930. Somewhat later, it was established by Novikov [8] that the picture changes at the second projective level: both separation theorems hold for $\mathbf{\Pi}^1_2$ but fail for $\mathbf{\Sigma}^1_2$.

In the same time Kuratowski [9] proved the *reduction theorem* for $\mathbf{\Sigma}^1_2$, that is, if X, Y are sets in $\mathbf{\Sigma}^1_2$ then there exist disjoint sets $X' \subseteq X$ and $Y' \subseteq Y$ in the same class $\mathbf{\Sigma}^1_2$, with the same union $X' \cup Y' = X \cup Y$. Kuratowski also observed that Luzin's arguments in the proof of the separation theorem for $\mathbf{\Sigma}^1_1$ yield the reduction theorem for $\mathbf{\Pi}^1_1$. Generally, if the reduction theorem holds for a projective class Γ then both separation theorems hold for the dual class $\Gamma^{\complement}$.

Generally, by classical studies, the reduction theorem holds for projective classes $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$ and fails for $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_2$, while the separation theorems hold for $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_2$ and fail for $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$. Note the inversion between the 1st and 2nd levels of the hierarchy.

As for the higher levels of projective hierarchy, all attempts made in classical descriptive set theory to solve the separation/reduction problems above the 2nd level did not work, until some additional set theoretic axioms were added. In particular, by Novikov [2] (see also Addison [10]), Gödel's *axiom of constructibility* $\mathbf{V} = \mathbf{L}$ implies that, for any $n \geq 3$, the reduction theorem holds for $\mathbf{\Sigma}^1_n$ and fails for $\mathbf{\Pi}^1_n$, while the separation theorems hold for $\mathbf{\Pi}^1_n$ and fail for $\mathbf{\Sigma}^1_n$ — pretty similar to 2nd level. On the contrary, by Addison and Moschovakis [11] and Martin [12], the *axiom of projective determinacy* **PD** implies that, for any $m \geq 1$, the reduction theorem holds for projective classes $\mathbf{\Pi}^1_{2m+1}$, $\mathbf{\Sigma}^1_{2m+2}$ and fails for $\mathbf{\Sigma}^1_{2m+1}$, $\mathbf{\Pi}^1_{2m+2}$, while the separation theorems hold for $\mathbf{\Sigma}^1_{2m+1}$, $\mathbf{\Pi}^1_{2m+2}$ and fail for $\mathbf{\Pi}^1_{2m+1}$, $\mathbf{\Sigma}^1_{2m+2}$ — pretty similar to what happens at the 1st and 2nd level corresponding to $n = 0$ in this scheme. Moreover, by Steel [13], it is true under the full *axiom of determinacy* **AD**, that if Γ is a class of pointsets closed under some simple operations and not self-dual (that is, $\Gamma \neq \Gamma^{\complement}$), then reduction holds for exactly one of the classes $\Gamma, \Gamma^{\complement}$, and the separation theorems hold for the other one. Conversely, Steel [14] proved that a more special form of $\mathbf{\Pi}^1_3$ separation implies otherwise impossible connections between some determinacy hypotheses. See also Hauser and Schindler [15] for other relevant results.

These achievements still leave open important questions about the status of the separation theorems for higher projective classes. For instance the following:

Problem 1.1 (Mathias [16] for $n = 3$)**.** Given a number $n \geq 3$, is it consistent with **ZFC** that the (first) separation theorem fails for both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$? □

Harrington solved the problem in the positive via a generic extension of $\mathbf{L}$ in which the (first) separation theorem fails for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$. The solution was obtained by the technique of almost-disjoint forcing of [17], and was sketched in unpublished handwritten notes [18, Part 2]. The result itself was mentioned, with a reference to Harrington, *e.g.* in Moschovakis [4, 5B.3]. Harrington also suggested in [18] some substantial changes in the construction of the generic extension indicated, supposedly leading to the failure of separation for both classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ for a given $n > 3$, or even for all n, but such a generalization has never been published in detail.

Our goal here is to prove the next theorem, which indeed solves Problem 1.1 in the positive for any given $n > 3$, albeit by a method different from the one involved in [18].

Theorem 1.2. *Let* $\mathtt{n} \geq 3$. *It is true, in a suitable generic extension of* $\mathbf{L}$, *that*

(i) *there is a pair of disjoint lightface* $\mathit{\Pi}^1_{\mathtt{n}}$ *sets* $X, Y \subseteq 2^\omega$, *not separable by disjoint* $\mathbf{\Sigma}^1_{\mathtt{n}}$ *sets, and hence separation fails for both* $\mathbf{\Pi}^1_{\mathtt{n}}$ *and* $\mathit{\Pi}^1_{\mathtt{n}}$;

(ii) *there is a pair of disjoint lightface* $\mathit{\Sigma}^1_{\mathtt{n}}$ *sets* $X, Y \subseteq 2^\omega$, *not separable by disjoint* $\mathbf{\Pi}^1_{\mathtt{n}}$ *sets, and hence separation fails for both* $\mathbf{\Sigma}^1_{\mathtt{n}}$ *and* $\mathit{\Sigma}^1_{\mathtt{n}}$.[2]

2 Outline of the proof

Given $\mathtt{n} \geq 3$, our plan is to define a sequence of forcing notions $\mathbb{P}_\xi$, $\xi < \omega_1$ in $\mathbf{L}$, whose finite-support product $\mathbb{P} = \prod_\xi \mathbb{P}_\xi$ satisfies the countable antichain condition CCC and adjoins a sequence of generic reals $x_\xi \in 2^\omega$, that are independent of each other in the sense that

(I) if $\eta < \omega_1$, then (a) the submodel $\mathbf{L}[\langle x_\xi \rangle_{\xi \neq \eta}]$ contains no reals $\mathbb{P}_\eta$-generic over $\mathbf{L}$, and moreover, (b) x_η is the only real in $\mathbf{L}[\langle x_\xi \rangle_{\xi < \omega_1}]$, $\mathbb{P}_\eta$-generic over $\mathbf{L}$,

and have the following definability property:

(II) the relation "$x \in 2^\omega$ is a real $\mathbb{P}_\xi$-generic over $\mathbf{L}$" (of arguments x, ξ) is $\mathit{\Pi}^1_{\mathtt{n}-1}$ in the whole extension and any its submodel.

Then, to define an example for Theorem 1.2(i), we can generically split ω_1 into three unbounded sets[3] $\omega_1 = \Omega_1 \cup \Omega_2 \cup \Omega_3$, define $\Delta = \{2\nu : \nu \in \Omega_1 \cup \Omega_3\} \cup \{2\nu + 1 : \nu \in \Omega_2 \cup \Omega_3\}$, and prove that Ω_1 and Ω_2 (more exactly, the sets of codes in 2^ω for ordinals in Ω_1 and Ω_2) are disjoint $\mathit{\Pi}^1_{\mathtt{n}}$ sets not separable by disjoint $\mathbf{\Sigma}^1_{\mathtt{n}}$ sets in the model $M = \mathbf{L}[\langle x_\xi \rangle_{\xi \in \Delta}]$. Indeed by (I) we have

$$\Omega_1 = \{\nu < \omega_1 : \neg \exists x \, (x \text{ is } \mathbb{P}_{2\nu+1}\text{-generic over } \mathbf{L})\}$$

in M, so Ω_1 is $\mathit{\Pi}^1_{\mathtt{n}}$ in M by (II), and accordingly so is Ω_2. The non-separability claim involves the following crucial property of $\mathbb{P}$-generic extensions:

(III) if a set $X \in \mathbf{L}$, $X \subseteq \omega_1$ is unbounded in ω_1, and a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic over $\mathbf{L}$ then $\mathbf{L}[\langle x_\xi \rangle_{\xi \in X}]$ is an elementary submodel of $\mathbf{L}[G]$ w.r.t. all $\mathit{\Sigma}^1_{\mathtt{n}-1}$ formulas.

Each factor forcing $\mathbb{P}_\xi$ in this scheme is a clone of Jensen's minimal forcing, defined in [19] (*Jensen's forcing* below, for the sake of brevity, see also [20, 28A] on this forcing). in particular, it consists of *perfect trees* in $2^{<\omega}$. The idea to use finite-support products of Jensen's forcing in order to obtain models with different definability effects belongs to

[2] Both (i) and (ii) in Theorem 1.2 are somewhat stronger than the nonseparability properties in [18], where only $\mathbf{\Delta}^1_{\mathtt{n}}$-nonseparability is asserted in both cases.

[3] In fact this partition is more complex because we have to simultaneously define and example for 1.2(ii) as well, see Section 19.

Enayat [21]. It was exploited to obtain generic models with: countable non-empty $\mathit{\Pi}^1_2$ sets (even E_0-classes) with no OD elements [22, 23], a countable $\mathit{\Pi}^1_2$ Groszek – Laver pair [24], planar $\mathit{\Pi}^1_2$ sets with countable cross-sections and OD-non-uniformizable [25, 26], counterexamples to the separation theorem for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$ [25], counterexamples to the axiom of choice [27], and an ordinal-definable partition of the real line onto two non-empty ordinal-undefinable sets [28].

The result of[25] corresponds to the case $\mathfrak{n} = 3$ of Theorem 1.2, in which case (III) is immediately true by Shoenfield. On the other hand, conditions similar to (I), (II) for $\mathfrak{n} = 3$, are involved in the forcing constructions in [24, 22, 25, 23, 26], and in [19] itself, where a CCC forcing $\mathbb{J} \in \mathbf{L}$ is defined to add a real $a \in 2^\omega$ so that a is the only $\mathbb{J}$-generic real in $\mathbf{L}[a]$, and "being a $\mathbb{J}$-generic real" is $\mathit{\Pi}^1_2$. These properties are implied by a special construction of $\mathbb{J} = \bigcup_{\alpha<\omega_1} \mathbb{J}_\alpha$ in $\mathbf{L}$ from countable sets $\mathbb{J}_\alpha$ of perfect trees. The construction can be viewed as a maximal branch in a certain mega-tree, say $\mathscr{P}$, whose nodes are countable sets of perfect trees, and each $\mathbb{J}_\alpha$ is chosen just as the $\leqslant_{\mathbf{L}}$-least appropriate extension. The complexity of this construction is $\mathit{\Delta}^1_2$ in the codes, that leads to the $\mathit{\Pi}^1_2$ definability of the property of being generic, while a suitable character of extension in the mega-tree allows to "kill" all possible competitors of a to be $\mathbb{J}$-generic.

Pretty similar ideas and constructions work in the mentioned papers, in particular in [25], where a model is defined in which $\mathbf{\Pi}^1_3$-separation fails.

A method of reproducing some generic counterexamples, originally defined on 2nd and 3rd projective level, at any given higher projective level $\mathfrak{n}$, was introduced by Harrington [18] on the base of the almost-disjoint forcing [17], and independently in [29] on the base of Jensen's forcing of [19]. In the terms above, the method requires to define a maximal branch in $\mathscr{P}$ that intersects all dense sets in $\mathscr{P}$ of descriptive complexity $\mathfrak{n}$ (or $\mathfrak{n} + c$, where c is a small entire constant depending on the nature of the problem). The method was recently applied to get models in which, for a given $\mathfrak{n} \geq 2$, there exists:

(a) a $\mathit{\Pi}^1_\mathfrak{n}$ E_0-equivalence class containing no OD elements, while every countable $\mathit{\Sigma}^1_\mathfrak{n}$-set of reals contains only OD reals [30],

(b) a $\mathit{\Pi}^1_\mathfrak{n}$ singleton $\{a\}$ such that a codes a cofinal map $f : \omega \to \omega_1^{\mathbf{L}}$ minimal over $\mathbf{L}$, while every $\mathit{\Sigma}^1_\mathfrak{n}$ set $X \subseteq \omega$ is constructible [31],

(c) a non-ROD-uniformizable $\mathit{\Pi}^1_\mathfrak{n}$ set with countable cross-sections, while all $\mathbf{\Sigma}^1_\mathfrak{n}$ sets with countable cross-sections are $\mathbf{\Delta}^1_{\mathfrak{n}+1}$-uniformizable [32],

as well as a model, in which the collection $\mathscr{P}(\omega) \cap \mathbf{L}$ of all constructible sets $x \subseteq \omega$ is equal to the collection of all $\mathit{\Delta}^1_\mathfrak{n}$-sets $x \subseteq \omega$, in a recent paper [33]. Here the method is used to prove Theorem 1.2.

Sections 3 to 7: perfect trees in $2^{<\omega}$, perfect tree forcing notions, multitrees (finite products of trees), multiforcings (countable products of forcings), splitting, the refinement relation, generic refinements by Jensen's splitting construction.

Sections 8 to 13: properties of generic refinements, sealing dense sets, sealing real names, and applications to generic extensions.

Sections 14 to 16: we define the set $\overrightarrow{\mathbf{MF}}$ of all countable sequences $\vec{\boldsymbol{\pi}}$ of small multiforcings, increasing in the sense of the refinement relation. Arguing in $\mathbf{L}$, we define a

$\varDelta^1_{\mathbb{n}-1}$ (in the codes) maximal branch $\vec{\Pi}$ in $\overrightarrow{\mathbf{MF}}$, which blocks all $\varSigma^1_{\mathbb{n}-2}$ sets in $\overrightarrow{\mathbf{MF}}$, where $\mathbb{n}$ is the number in Theorem 1.2, and $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ *blocks* a set $W \subseteq \overrightarrow{\mathbf{MF}}$ if either $\vec{\boldsymbol{\pi}} \in W$ or no extension of $\vec{\boldsymbol{\pi}}$ in $\overrightarrow{\mathbf{MF}}$ belongs to W. The forcing notion $\mathbb{P}$ for Theorem 1.2 is a derivate of $\vec{\Pi}$.

Sections 17 to 20: we show that $\mathbb{P}$ satisfies (I) and (II).

Sections 22 to 26: to achieve (III), we develop an auxiliary forcing relation $\texttt{forc}$, which approximates the truth in $\mathbb{P}$-generic extensions for $\varSigma^1_{\mathbb{n}-1}$-formulas and below, so that the relation $\texttt{forc}$ restricted to any class $\varSigma^1_m$ or $\varPi^1_m$, $m \geq 2$, is $\varSigma^1_m$, resp., $\varPi^1_m$. Using the invariance of the relation $\texttt{forc}$ under certain transformations (while the forcing notion $\mathbb{P}$ is not invariant!), we accomplish the proof of (III) and Theorem 1.2.

3 Trees and perfect-tree forcing notions

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\frown i$ is the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. $\texttt{lh}(t)$ is the length of t, and $2^n = \{t \in 2^{<\omega} : \texttt{lh}(t) = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* iff for any strings $s \subset t$ in $2^{<\omega}$, if $t \in T$ then $s \in T$. Thus every non-empty tree $T \subseteq 2^{<\omega}$ contains the *empty string* Λ.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T{\restriction}_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Definition 3.1. $\mathbf{PT}$ is the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a tree $\varnothing \neq T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ iff it has no endpoints and no isolated branches. If $T \in \mathbf{PT}$ then define a perfect set

$$[T] = \{a \in 2^\omega : \forall\, n\, (a{\restriction}n \in T)\} \subseteq 2^\omega.$$

Trees $T, S \in \mathbf{PT}$ are *almost disjoint*, AD for brevity, iff the intersection $S \cap T$ is finite; this is equivalent to just $[S] \cap [T] = \varnothing$. A set $\mathbb{A} \subseteq \mathbf{PT}$ is an *antichain* iff any two trees $T \neq T'$ in $\mathbb{A}$ are AD. □

We'll consider pairs of the form $\langle n, T\rangle$, where $n < \omega$ and $T \in \mathbf{PT}$. Following [34], the set $\omega \times \mathbf{PT}$ of such pairs is ordered by a special relation $\preccurlyeq$ so that $\langle n, T\rangle \preccurlyeq \langle m, S\rangle$ (reads: $\langle n, T\rangle$ *extends* $\langle m, S\rangle$) iff $m \leq n$, $T \subseteq S$, and $T \cap 2^m = S \cap 2^m$.[4] The role of the number m in a pair $\langle m, S\rangle$ is to preserve the value $S \cap 2^m$ under $\preccurlyeq$-extensions.

The implication $m > n \Longrightarrow \langle m, T\rangle \preccurlyeq \langle n, T\rangle$ (the same T!) always holds, but $S \subseteq T \Longrightarrow \langle n, S\rangle \preccurlyeq \langle n, T\rangle$ is not necessarily true: we also need $T \cap 2^n = S \cap 2^n$.

Lemma 3.2 (see [34]). *Let $\ldots \preccurlyeq \langle n_2, T_2\rangle \preccurlyeq \langle n_1, T_1\rangle \preccurlyeq \langle n_0, T_0\rangle$ be a decreasing sequence in $\omega \times \mathbf{PT}$, with $n_0 < n_1 < n_2 < \ldots$ strictly, minimally generic in the sense that it meets every set of the form*

$$D_t = \{\langle n, T\rangle \in \omega \times \mathbf{PT} : t \notin T \vee \exists\, s \in T\, (t \subseteq s \wedge s^\frown 0, s^\frown 1 \in T)\}\,,\ t \in 2^{<\omega}.$$

Then $T = \bigcap_n T_n \in \mathbf{PT}$, and if $i < \omega$ then $\langle n_i, T\rangle \preccurlyeq \langle n_i, T_i\rangle$. □

[4] This definition does not explicitly contain any splitting condition. This is why one needs the genericity condition in Lemma 3.2. An earlier definition in [35] stipulates that for any $s \in S \cap 2^m$ there exist two strings $s' \neq s''$ in $T \cap 2^n$ such that $s \subset s'$ and $s \subset s''$. With such an ordering, Lemma 3.2 holds without the genericity condition.

Definition 3.3. Let an *arboreal forcing* be any set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $u \in T \in \mathbb{P}$ then $T\restriction_u \in \mathbb{P}$. Let $\mathbf{AF}$ be the set of all such sets $\mathbb{P}$. A forcing $\mathbb{P} \in \mathbf{AF}$ is:

regular, if for any $S, T \in \mathbb{P}$, the intersection $[S] \cap [T]$ is clopen in $[S]$ or clopen in $[T]$ (or clopen in both $[S]$ and $[T]$);

special, if there is a finite or countable antichain $\mathbb{A} \subseteq \mathbb{P}$ such that $\mathbb{P} = \{T\restriction_s : s \in T \in \mathbb{A}\}$ — the antichain $\mathbb{A}$ is unique in this case, and the forcing $\mathbb{P}$ itself is obviously regular. □

Example 3.4. If $s \in 2^{<\omega}$ then the tree $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$ belongs to $\mathbf{PT}$ and $T[s] = (2^{<\omega})\restriction_s$, $\forall s$. The set $\mathbb{P}_{\text{coh}} = \{T[s] : s \in 2^{<\omega}\}$ (the Cohen forcing) is a regular and special arboreal forcing notion. □

Any set $\mathbb{P} \in \mathbf{AF}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition); such a forcing $\mathbb{P}$ obviously adds a real in 2^ω.

To carry out splitting constructions, as in Lemma 3.2, over a forcing $\mathbb{P} \in \mathbf{AF}$, we make use of a bigger forcing notion $\bigcup^{\texttt{fin}} \mathbb{P} \in \mathbf{AF}$, that consists of all finite unions of trees in $\mathbb{P}$. Then $\mathbb{P}$ is dense in $\bigcup^{\texttt{fin}} \mathbb{P}$, so the forcing properties of both sets coincide. Yet $\bigcup^{\texttt{fin}} \mathbb{P}$ is more flexible w.r.t. tree constructions.

The next lemma implies that the compatibility of conditions in a *regular* forcing notion is absolute.

Lemma 3.5. *Assume that $\mathbb{P} \in \mathbf{AF}$ is regular and $S, T \in \mathbb{P}$ are not* AD. *Then $S \cap T \in \bigcup^{\texttt{fin}} \mathbb{P}$, hence the trees S, T are compatible in $\mathbb{P}$.*

Proof. Because of the regularity, the intersection $[S] \cap [T]$ is clopen, say, in $[S]$. Then there exists a finite set $U \subseteq S$, such that $[S] \cap [T] = \bigcup_{u \in U} [S\restriction_u]$. Yet every tree $S\restriction_u$ belongs to $\mathbb{P}$ as the latter is an arboreal forcing. □

Lemma 3.6. *Let $\mathbb{P} \in \mathbf{AF}$ and $S, T \in \bigcup^{\texttt{fin}} \mathbb{P}$, $u \in S$, $n = \texttt{lh}(u)$, $T \subseteq S\restriction_u$. Then the tree $S' = T \cup \bigcup_{v \in S \cap 2^n, v \neq u} S\restriction_v$ belongs to $\bigcup^{\texttt{fin}} \mathbb{P}$, $\langle n, S' \rangle \preccurlyeq \langle n, S \rangle$, $S'\restriction_u = T$, and $S'\restriction_v = S\restriction_v$ whenever $v \in S$, $\texttt{lh}(v) = n$, $v \neq u$.* □

Corollary 3.7. *Assume that $\mathbb{P}, \mathbb{P}' \in \mathbf{AF}$. Then*

(i) *if $n < \omega$ and $T \in \bigcup^{\texttt{fin}} \mathbb{P}$, then there is a tree $S \in \bigcup^{\texttt{fin}} \mathbb{P}$ such that $\langle n, S \rangle \preccurlyeq \langle n, T \rangle$ and $S\restriction_t \in \mathbb{P}$* (not just $\in \bigcup^{\texttt{fin}} \mathbb{P}$!) *for all $t \in 2^n \cap S$,*

(ii) *if $T \in \mathbb{P}$ and $T' \in \mathbb{P}'$, then there are trees $S \in \mathbb{P}$, $S' \in \mathbb{P}'$ such that $S \subseteq T$, $S' \subseteq T'$, and $[S] \cap [S'] = \varnothing$;.*

(iii) *if $n < \omega$ and $T \in \bigcup^{\texttt{fin}} \mathbb{P}$, $T' \in \bigcup^{\texttt{fin}} \mathbb{P}'$, then there exist trees $S \in \bigcup^{\texttt{fin}} \mathbb{P}$, $S' \in \bigcup^{\texttt{fin}} \mathbb{P}'$ s.t. $\langle n, S \rangle \preccurlyeq \langle n, T \rangle$, $\langle n, S' \rangle \preccurlyeq \langle n, T' \rangle$, and $[S] \cap [S'] = \varnothing$.*

Proof. (ii) If $T = T'$ then pick a pair of strings $u \neq v$ in $T = T'$ with $\texttt{lh}(u) = \texttt{lh}(v)$, and let $S = T'\restriction_u$, $S' = T'\restriction_v$. If say $T \not\subseteq T'$ then let $u \in T \smallsetminus T'$, $S = T\restriction_u$, and simply $S' = T'$. To prove (iii) iterate (ii) and make use of Lemma 3.6. □

4 Multiforcings and multitrees

Call a **multiforcing** any map $\boldsymbol{\pi} : |\boldsymbol{\pi}| \to \mathbf{AF}$, where $|\boldsymbol{\pi}| = \mathtt{dom}\,\boldsymbol{\pi} \subseteq \omega_1$. Let $\mathbf{MF}$ be the collection of all multiforcings. Every $\boldsymbol{\pi} \in \mathbf{MF}$ will be typically presented as an indexed set $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, where $\mathbb{P}_\xi \in \mathbf{AF}$ for all $\xi \in |\boldsymbol{\pi}|$, so that each set $\mathbb{P}_\xi = \mathbb{P}_\xi^{\boldsymbol{\pi}} = \boldsymbol{\pi}(\xi)$, $\xi \in |\boldsymbol{\pi}|$, is an arboreal forcing notion. Such a $\boldsymbol{\pi}$ is:

- *small*, if both $|\boldsymbol{\pi}|$ and each forcing $\mathbb{P}_\xi^{\boldsymbol{\pi}}$, $\xi \in |\boldsymbol{\pi}|$, are countable;
- *special*, if each $\mathbb{P}_\xi^{\boldsymbol{\pi}}$ is special in the sense of Definition 3.3;
- *regular*, if each $\mathbb{P}_\xi^{\boldsymbol{\pi}}$ is regular, in the sense of Definition 3.3.

Let a *multitree* be any function $\boldsymbol{p} : |\boldsymbol{p}| \to \mathbf{PT}$, with a finite *support* $|\boldsymbol{p}| = \mathtt{dom}\,\boldsymbol{p}$; $\mathbf{MT}$ will be the collection of all multitrees. Every $\boldsymbol{p} \in \mathbf{MT}$ will be typically presented as an indexed set $\boldsymbol{p} = \langle T_\xi^{\boldsymbol{p}} \rangle_{\xi \in |\boldsymbol{p}|}$, where $T_\xi^{\boldsymbol{p}} = \boldsymbol{p}(\xi) \in \mathbf{PT}$ for all $\xi \in |\boldsymbol{p}|$.

Let $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ be a multiforcing. In this case, a $\boldsymbol{\pi}$-*multitree* is any multitree $\boldsymbol{p} \in \mathbf{MT}$ such that $|\boldsymbol{p}| \subseteq |\boldsymbol{\pi}|$, and if $\xi \in |\boldsymbol{p}|$ then the tree $\boldsymbol{p}(\xi) = T_\xi^{\boldsymbol{p}}$ belongs to $\mathbb{P}_\xi$. If $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ then the set

$$[\boldsymbol{p}] = \{x \in (2^\omega)^{|\boldsymbol{\pi}|} : \forall\, \xi \in |\boldsymbol{p}|\, (x(\xi) \in [T_\xi^{\boldsymbol{p}}])\}$$

is a cofinite-dimensional perfect cube in $(2^\omega)^{|\boldsymbol{\pi}|}$. We order $\mathbf{MT}$ and each $\mathbf{MT}(\boldsymbol{\pi})$ componentwise: $\boldsymbol{q} \leqslant \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger that $\boldsymbol{q}$) iff $|\boldsymbol{p}| \subseteq |\boldsymbol{q}|$ and $T_\xi^{\boldsymbol{q}} \subseteq T_\xi^{\boldsymbol{p}}$ for all $\xi \in |\boldsymbol{p}|$; this is equivalent to $[\boldsymbol{q}] \subseteq [\boldsymbol{p}]$. The empty multitree $\boldsymbol{\Lambda}$ defined by $|\boldsymbol{\Lambda}| = \varnothing$, belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and is the weakest condition.

Remark 4.1. If $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ be a multiforcing, then the set $\mathbf{MT}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-multitrees can be identified with the *finite support* product $\prod_{\xi \in |\boldsymbol{\pi}|} \mathbb{P}_\xi$ of the arboreal forcings $\mathbb{P}_\xi$ involved. □

Definition 4.2. Multitrees $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are *somewhere almost disjoint* (SAD) if there is $\xi \in |\boldsymbol{p}| \cap |\boldsymbol{q}|$ such that the trees $T_\xi^{\boldsymbol{p}}$ and $T_\xi^{\boldsymbol{q}}$ are AD. Being SAD is equivalent to $[\boldsymbol{p}] \cap [\boldsymbol{q}] = \varnothing$, and, in the case of regular multiforcings $\boldsymbol{\pi}$, equivalent to the *incompatibility* in $\mathbf{MT}(\boldsymbol{\pi})$ by the following result. □

Corollary 4.3 (of Lemma 3.5)**.** *Assume that $\boldsymbol{\pi}$ is a regular multiforcing and $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are not* SAD*. Then there is a finite set $R \subseteq \mathbf{MT}(\boldsymbol{\pi})$ such that $[\boldsymbol{p}] \cap [\boldsymbol{q}] = \bigcup_{\boldsymbol{r} \in R} [\boldsymbol{r}]$. Therefore $\boldsymbol{p}, \boldsymbol{q}$ are compatible in $\mathbf{MT}(\boldsymbol{\pi})$, that is, there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r} \leqslant \boldsymbol{q}$.* □

Definition 4.4. The *componentwise union* of multiforcings $\boldsymbol{\pi}, \boldsymbol{\varrho}$ is a multiforcing $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho}$ satisfying $|(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})| = |\boldsymbol{\pi}| \cup |\boldsymbol{\varrho}|$ and

$$(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})(\xi) \;=\; \boldsymbol{\pi}(\xi) \;\text{ or }\; \boldsymbol{\varrho}(\xi) \;\text{ or }\; \boldsymbol{\pi}(\xi) \cup \boldsymbol{\varrho}(\xi)$$

in cases resp. $\xi \in |\boldsymbol{\pi}| \smallsetminus |\boldsymbol{\varrho}|$, $\xi \in |\boldsymbol{\varrho}| \smallsetminus |\boldsymbol{\pi}|$, $\xi \in |\boldsymbol{\varrho}| \cap |\boldsymbol{\pi}|$.

If $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \lambda}$ is a sequence in $\mathbf{MF}$ then define $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathtt{cw}}_{\alpha < \lambda} \boldsymbol{\pi}_\alpha \in \mathbf{MF}$ so that $|\boldsymbol{\pi}| = \bigcup_{\alpha < \lambda} |\boldsymbol{\pi}_\alpha|$ and $\boldsymbol{\pi}(\xi) = \bigcup_{\alpha < \lambda,\, \xi \in |\boldsymbol{\pi}_\alpha|} \boldsymbol{\pi}_\alpha(\xi)$ for $\xi \in |\boldsymbol{\pi}|$. □

Remark 4.5. Any forcing of the form $\mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathbf{MF}$, adds a generic sequence $\langle x_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, where each $x_\xi = x_\xi[G] \in 2^\omega$ is a $\mathbb{P}_\xi$-generic real. Reals of the form $x_\xi[G]$ will be called *principal generic reals* in $\mathbf{V}[G]$. □

5 Refining arboreal forcings

If $T \in \mathbf{PT}$ (a tree) and $D \subseteq \mathbf{PT}$ then $X \subseteq^{\mathtt{fin}} \bigcup D$ will mean that there is a finite set $D' \subseteq D$ such that $T \subseteq \bigcup D'$, or equivalently $[T] \subseteq \bigcup_{S \in D'} [S]$.

Definition 5.1. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{AF}$ be arboreal forcing notions. Say that $\mathbb{Q}$ is a *refinement* of $\mathbb{P}$ (symbolically $\mathbb{P} \sqsubset \mathbb{Q}$) if

(1) the set $\mathbb{Q}$ is dense[5] in $\mathbb{P} \cup \mathbb{Q}$: if $T \in \mathbb{P}$ then $\exists\, Q \in \mathbb{Q}\,(Q \subseteq T)$;

(2) if $Q \in \mathbb{Q}$ then $Q \subseteq^{\mathtt{fin}} \bigcup \mathbb{P}$;

(3) if $Q \in \mathbb{Q}$ and $T \in \mathbb{P}$ then $[Q] \cap [T]$ is clopen in $[Q]$ and $T \not\subseteq Q$. □

Lemma 5.2. (i) *If* $\mathbb{P} \sqsubset \mathbb{Q}$ *and* $S \in \mathbb{P}$*,* $T \in \mathbb{Q}$*, then* $[S] \cap [T]$ *is meager in* $[S]$*, therefore* $\mathbb{P} \cap \mathbb{Q} = \varnothing$ *and* $\mathbb{Q}$ *is open dense in* $\mathbb{P} \cup \mathbb{Q}$ *;*

(ii) *if* $\mathbb{P} \sqsubset \mathbb{Q} \sqsubset \mathbb{R}$ *then* $\mathbb{P} \sqsubset \mathbb{R}$*, thus* $\sqsubset$ *is a strict partial order;*

(iii) *if* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{AF}$ *and* $0 < \mu < \lambda$ *then* $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset \mathbb{Q} = \bigcup_{\mu \le \alpha < \lambda} \mathbb{P}_\alpha$ *;*

(iv) *if* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{AF}$ *and each* $\mathbb{P}_\alpha$ *is special then* $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha \in \mathbf{AF}$*,* $\mathbb{P}$ *is regular, and each* $\mathbb{P}_\gamma$ *is pre-dense in* $\mathbb{P}$*.*

Proof. (i) Otherwise there is a string $u \in S$ such that $S{\restriction}_u \subseteq [T] \cap [S]$. But $S{\restriction}_u \in \mathbb{P}$, which contradicts to 5.1(3).

(ii), (iii) Make use of claim (i) just proved, to establish 5.1(3).

(iv) To check the regularity, let $S \in \mathbb{P}_\alpha$, $T \in \mathbb{P}_\beta$, $\alpha \le \beta$. If $\alpha = \beta$ then, as $\mathbb{P}_\alpha$ is special, the trees S, T are either AD or $\subseteq$-comparable by Lemma 3.5. If $\alpha < \beta$ then $[S] \cap [T]$ is clopen in $[T]$ by 5.1(3).

To check the pre-density, let $S \in \mathbb{P}_\alpha$, $\alpha \ne \gamma$. If $\alpha < \gamma$ then by 5.1(1) there is a tree $T \in \mathbb{P}_\gamma$, $T \subseteq S$. Now let $\gamma < \alpha$. Then $S \subseteq^{\mathtt{fin}} \bigcup \mathbb{P}_\gamma$ by 5.1(2), hence there is a tree $T \in \mathbb{P}_\gamma$ such that $[S] \cap [T] \ne \varnothing$. However $[S] \cap [T]$ is clopen in $[S]$ by 5.1(3). Therefore $S{\restriction}_u \subseteq T$ for a string $u \in S$. Finally $S{\restriction}_u \in \mathbb{P}_\alpha$ since $\mathbb{P}_\alpha \in \mathbf{AF}$. □

Note that if $\mathbb{P}, \mathbb{Q} \in \mathbf{AF}$ and $\mathbb{P} \sqsubset \mathbb{Q}$ then a dense set $D \subseteq \mathbb{P}$ is not necessarily dense or even pre-dense in $\mathbb{P} \cup \mathbb{Q}$. Yet there is a special type of refinement which preserves at least pre-density.

Definition 5.3. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{AF}$ and $D \subseteq \mathbb{P}$. Say that $\mathbb{Q}$ *seals* D *over* $\mathbb{P}$, symbolically $\mathbb{P} \sqsubset_D \mathbb{Q}$, if $\mathbb{P} \sqsubset \mathbb{Q}$ holds and every tree $S \in \mathbb{Q}$ satisfies $S \subseteq^{\mathtt{fin}} \bigcup D$. Then simply $\mathbb{P} \sqsubset \mathbb{Q}$ is equivalent to $\mathbb{P} \sqsubset_{\mathbb{P}} \mathbb{Q}$. □

[5] If $\mathbb{P} \subseteq \mathbb{R} \subseteq \mathbf{PT}$ then, as usual, the set $\mathbb{P}$ is 1) *dense* in $\mathbb{R}$ iff $\forall T \in \mathbb{R}\, \exists S \in \mathbb{P}\, (S \subseteq T)$, 2) *open dense* in $\mathbb{R}$ iff in addition $\forall T \in \mathbb{R}\, \forall S \in \mathbb{P}\, (T \subseteq S \implies T \in \mathbb{P})$, and 3) *pre-dense* in $\mathbb{R}$ iff the derived set $\mathbb{P}' = \{T \in \mathbb{R} : \exists S \in \mathbb{P} (T \subseteq S)\}$ is dense in $\mathbb{R}$.

As we'll see now, a sealed set has to be pre-dense both before and after the refinement. The additional importance of sealing refinements lies in fact that, once established, it preserves under further simple refinements, that is, $\sqsubset_D$ is transitive in a combination with $\sqsubset$ in the sense of (ii) of the following lemma:

Lemma 5.4. (i) *If* $\mathbb{P} \sqsubset_D \mathbb{Q}$ *then* D *is pre-dense in* $\mathbb{P} \cup \mathbb{Q}$*, and if in addition* $\mathbb{P}$ *is regular then* D *is pre-dense in* $\mathbb{P}$ *as well;*

(ii) *if* $\mathbb{P} \sqsubset_D \mathbb{Q} \sqsubset \mathbb{R}$ (note: the second $\sqsubset$ is not $\sqsubset_D$!) *then* $\mathbb{P} \sqsubset_D \mathbb{R}$;

(iii) *if* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* **AF**, $0 < \mu < \lambda$*, and* $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset_D \mathbb{P}_\mu$*, then* $\mathbb{P} \sqsubset_D \mathbb{Q} = \bigcup_{\mu \le \alpha < \lambda} \mathbb{P}_\alpha$.

Proof. (i) To see that D is pre-dense in $\mathbb{P} \cup \mathbb{Q}$, let $T_0 \in \mathbb{P} \cup \mathbb{Q}$. By 5.1(1), there is a tree $T \in \mathbb{Q}$, $T \subseteq T_0$. Then $T \subseteq^{\mathtt{fin}} \bigcup D$, in particular, there is a tree $S \in D$ with $X = [S] \cap [T] \neq \varnothing$. However X is clopen in $[T]$ by 5.1(3). Therefore there is a tree $T' \in \mathbb{Q}$ with $[T'] \subseteq X$, thus $T' \subseteq S \in D$ and $T' \subseteq T \subseteq T_0$. We conclude that T_0 is compatible with $S \in D$ in $\mathbb{P} \cup \mathbb{Q}$.

To see that D is pre-dense in $\mathbb{P}$ (assuming $\mathbb{P}$ is regular), let $S_0 \in \mathbb{P}$. It follows from the above that S_0 is compatible with some $S \in D$, hence, S and S_0 are not AD. It remains to use Lemma 3.5.

To prove (ii) on the top of Lemma 5.2(ii), let $R \in \mathbb{R}$. Then $R \subseteq^{\mathtt{fin}} \bigcup \mathbb{Q}$, but each $T \in \mathbb{Q}$ satisfies $T \subseteq^{\mathtt{fin}} \bigcup D$. The same for (iii). □

6 Refining multiforcings

Let $\boldsymbol{\pi}, \boldsymbol{\varphi}$ be multiforcings. Say that $\boldsymbol{\varphi}$ is an *refinement* of $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, if $|\boldsymbol{\pi}| \subseteq |\boldsymbol{\varphi}|$ and $\boldsymbol{\pi}(\xi) \sqsubset \boldsymbol{\varphi}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}|$.

Corollary 6.1 (of Lemma 5.2)**.** *If* $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi} \sqsubset \boldsymbol{\rho}$ *then* $\boldsymbol{\pi} \sqsubset \boldsymbol{\rho}$*.*

If $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$ *then the set* $\mathbf{MT}(\boldsymbol{\varphi})$ *is open dense*[6] *in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varphi})$*.* □

Our next goal is to introduce a version of Definition 5.3 suitable for multiforcings; we expect an appropriate version of Lemma 5.4 to hold.

First of all, we accomodate the definition of the relation $\subseteq^{\mathtt{fin}}$ in Section 5 for multitrees. Namely if $\boldsymbol{u}$ is a multitree and $\boldsymbol{D}$ a collection of multitrees, then $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}$ will mean that there is a finite set $\boldsymbol{D}' \subseteq \boldsymbol{D}$ satisfying 1) $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in \boldsymbol{D}'$, and 2) $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v} \in \boldsymbol{D}'} [\boldsymbol{v}]$.

Definition 6.2. Let $\boldsymbol{\pi}, \boldsymbol{\varphi}$ be multiforcings, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$. Say that $\boldsymbol{\varphi}$ *seals a set* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ *over* $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$, if the following condition holds:

($*$) if $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, then there is $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$, where

$$\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}\} . \quad \square$$

[6] If $\boldsymbol{P} \subseteq \boldsymbol{R} \subseteq \mathbf{MT}$ then, similarly to Footnote 5, the set $\boldsymbol{P}$ is 1) *dense* in $\boldsymbol{R}$ iff $\forall\, \boldsymbol{r} \in \boldsymbol{R}\, \exists\, \boldsymbol{p} \in \boldsymbol{P}\, (\boldsymbol{p} \leqslant \boldsymbol{r})$, 2) *open dense* in $\boldsymbol{R}$ iff in addition $\forall\, \boldsymbol{r} \in \boldsymbol{R} \forall\, \boldsymbol{p} \in \boldsymbol{P}\, (\boldsymbol{p} \leqslant \boldsymbol{r} \implies \boldsymbol{p} \in \boldsymbol{R})$, and 3) *pre-dense* in $\boldsymbol{R}$ iff the set $\boldsymbol{P}' = \{\boldsymbol{r} \in \boldsymbol{R} : \exists\, \boldsymbol{p} \in \boldsymbol{P} (\boldsymbol{r} \leqslant \boldsymbol{p})\}$ is dense in $\boldsymbol{R}$.

Note that if $\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{q}$ are as indicated then still $\boldsymbol{u} \cup \boldsymbol{q} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}'$, where $\boldsymbol{D}' = \{\boldsymbol{u}' \cup \boldsymbol{q} : \boldsymbol{u}' \in \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}\} \subseteq \boldsymbol{D}$. Anyway the definition of $\sqsubset_{\boldsymbol{D}}$ in 6.2 looks somewhat different and more complex than the definition of $\sqsubset_D$ in 5.3, which reflects the fact that finite-support products of forcing notions in **AF** behave differently (and in more complex way) than single arboreal forcings. Accordingly, the next lemma, similar to Lemma 5.4, is somewhat less obvious.

Lemma 6.3. *Let $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ be multiforcings and $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$. Then:*

(i) *if $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$ then $\boldsymbol{D}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$ and pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$;*

(ii) *if $\boldsymbol{\pi}$ is regular, $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}_i} \boldsymbol{\varrho}$ for $i = 1, \dots, n$, all sets $\boldsymbol{D}_i \subseteq \mathbf{MT}(\boldsymbol{\pi})$ are open dense in $\mathbf{MT}(\boldsymbol{\pi})$, and $\boldsymbol{D} = \bigcap_i \boldsymbol{D}_i$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$;*

(iii) *if $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$ then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\sigma}$;*

(iv) *if $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \lambda}$ is a $\sqsubset$-increasing sequence in **MF**, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha < \mu} \boldsymbol{\pi}_\alpha$, $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, and $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_\mu$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho} = \bigcup^{\mathtt{cw}}_{\mu \le \alpha < \lambda} \boldsymbol{\pi}_\alpha$.*

Proof. (i) To check that $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$, let $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$. Due to the product character of $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$, we can assume that $|\boldsymbol{r}| \subseteq |\boldsymbol{\pi}|$. Let

$$X = \{\xi \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_\xi \in \mathbf{MT}(\boldsymbol{\varrho})\}, \quad Y = \{\xi \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_\xi \in \mathbf{MT}(\boldsymbol{\pi})\}.$$

Then $\boldsymbol{r} = \boldsymbol{u} \cup \boldsymbol{p}$, where $\boldsymbol{u} = \boldsymbol{r} \restriction X \in \mathbf{MT}(\boldsymbol{\varrho})$, $\boldsymbol{p} = \boldsymbol{r} \restriction Y \in \mathbf{MT}(\boldsymbol{\pi})$. As $\boldsymbol{\varrho}$ seals $\boldsymbol{D}$, there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$. By an easy argument, there is a multitree $\boldsymbol{u}' \in \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$ compatible with $\boldsymbol{u}$ in $\mathbf{MT}(\boldsymbol{\varrho})$; let $\boldsymbol{w} \in \mathbf{MT}(\boldsymbol{\varrho})$, $\boldsymbol{w} \leqslant \boldsymbol{u}$, $\boldsymbol{w} \leqslant \boldsymbol{u}'$, $|\boldsymbol{w}| = |\boldsymbol{u}'| = |\boldsymbol{u}|$. Then the multitree $\boldsymbol{r}' = \boldsymbol{w} \cup \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi} \vee \boldsymbol{\varrho})$ satisfies $\boldsymbol{r}' \leqslant \boldsymbol{r}$ and $\boldsymbol{r}' \leqslant \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}$.

To check that $\boldsymbol{D}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$, suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$. Let $\boldsymbol{u} = \boldsymbol{\Lambda}$ (the empty multitree) in $(*)$ of Definition 6.2, so that $|\boldsymbol{u}| = \varnothing$ and $\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|} = \boldsymbol{D}$.

(ii) Let $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. Iterating $(*)$ for $\boldsymbol{D}_i$, $i = 1, \dots, n$, we find a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee (\boldsymbol{D}_i)_{\boldsymbol{q}}^{|\boldsymbol{u}|}$ for all i, where

$$(\boldsymbol{D}_i)_{\boldsymbol{q}}^{|\boldsymbol{u}|} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}_i\}.$$

Thus there are finite sets $U_i \subseteq (\boldsymbol{D}_i)_{\boldsymbol{q}}^{|\boldsymbol{u}|}$ such that $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v} \in U_i} [\boldsymbol{v}]$ for all i. Using the regularity assumption and Corollary 4.3, we get a finite set $W \subseteq \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{w}| = |\boldsymbol{u}|$ for all $\boldsymbol{w} \in W$, $\bigcap_i \bigcup_{\boldsymbol{v} \in U_i} [\boldsymbol{v}] = \bigcup_{\boldsymbol{w} \in W} [\boldsymbol{w}]$, and if $i = 1, \dots, n$ and $\boldsymbol{w} \in W$ then $[\boldsymbol{w}] \subseteq [\boldsymbol{v}]$ for some $\boldsymbol{v} \in U_i$ — hence $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}_i$. We conclude that if $\boldsymbol{w} \in W$ then $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}$, hence $\boldsymbol{w} \in \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$. Thus $W \subseteq \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$. However $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{w} \in W} [\boldsymbol{w}]$ by the choice of W. Thus $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$.

(iii) We have $\boldsymbol{\pi} \sqsubset \boldsymbol{\sigma}$ by Corollary 6.1. To check that $\boldsymbol{\sigma}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$, let $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\sigma})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. As $\boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$, there is a finite $U \subseteq \mathbf{MT}(\boldsymbol{\varrho})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$, and $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v} \in U} [\boldsymbol{v}]$. As $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$, by iterated application of Definition 6.2(∗), we get a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and if $\boldsymbol{v} \in U$ then $\boldsymbol{v} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$, where

$$\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|} = \{\boldsymbol{v}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{v}'| = |\boldsymbol{v}| = |\boldsymbol{u}| \wedge \boldsymbol{v}' \cup \boldsymbol{q} \in \boldsymbol{D}\}\,.$$

Note finally that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee U$ by construction, hence $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$ as well.

(iv) We have to check that $\boldsymbol{\varphi}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$. Let $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. There is a finite set $U \subseteq \mathbf{MT}(\boldsymbol{\pi}_\mu)$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$ and $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v} \in U} [\boldsymbol{v}]$. And so on as in the proof of (iii). □

7 Generic refinement of a multiforcing by Jensen

Here we define a splitting/fusion construction of refinements. The construction was originally invented as a method to obtain perfect sets in Polish spaces. Jensen modified it in [19] in order to get refinements of certain countable subforcings of the Sacks forcing. The next definition introduces essentially a product version of Jensen's refinements, applicable to arboreal forcings and multiforcings. As we deal with *finite* support products (see Remark 4.1), the standard technique in the theory of countable-support Sacks products, as *e.g.* in [36] or [37, 38, 39], is not fully applicable. The notion of a *system* in the next definition contains appropriate changes of instrumentarium related to the splitting/fusion construction. That infinite finite-support products of Jensen-style forcing notions are CCC, preserve cardinals (unlike finite-support Sacks products), and admit a suitable version of splitting/fusion technique, was demonstrated in [22, 23].

Definition 7.1. Suppose that $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ is a small multiforcing.

(0) Let a $\boldsymbol{\pi}$-*system* be any indexed set of the form $\varphi = \langle T^\varphi_{\xi k} \rangle_{\langle \xi, k \rangle \in |\varphi|}$, where $|\varphi| \subseteq |\boldsymbol{\pi}| \times \omega$ is finite and $T^\varphi_{\xi k} = \varphi(\xi, k) \in \bigcup^{\mathtt{fin}} \mathbb{P}_\xi$ for all ξ, k. (Recall that $\bigcup^{\mathtt{fin}} \mathbb{P}_\xi$ consists of all finite unions of trees in $\mathbb{P}_\xi$.) We order the set $\mathbf{Sys}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-systems componentwise: $\varphi \leqslant \psi$ (φ extends ψ) iff $|\psi| \subseteq |\varphi|$ and $T^\varphi_{\xi k} \subseteq T^\psi_{\xi k}$ for all $\langle \xi, k \rangle \in |\psi|$. Accordingly the set $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$ is ordered so that $\langle n, \varphi \rangle \preccurlyeq \langle m, \psi \rangle$ iff $|\psi| \subseteq |\varphi|$ and $\langle n, T^\varphi_{\xi k} \rangle \preccurlyeq \langle m, T^\psi_{\xi k} \rangle$ in $\omega \times \mathbf{PT}$ (Section 3) for all ξ, k; this implies $m \le n$.

(1) Let $\mathfrak{M} \in \mathrm{HC}$ be any set.[7] The set $\mathfrak{M}^+$ of all sets $X \in \mathrm{HC}$, $\in$-definable in HC by formulas with sets in $\mathfrak{M}$ as parameters, is still countable. Therefore there exists a $\preccurlyeq$-decreasing sequence $\Phi = \langle \langle n_j, \varphi_j \rangle \rangle_{j<\omega}$ of pairs $\langle n_j, \varphi_j \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, $\mathfrak{M}^+$-*generic* in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, open dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.[8] Let us fix any such a $\mathfrak{M}^+$-generic sequence Φ.

By definition, each φ_j has the form $\varphi_j = \langle T^{\varphi_j}_{\xi k} \rangle_{\langle \xi, k \rangle \in |\varphi_j|}$, where $|\varphi_j| \subseteq |\boldsymbol{\pi}| \times \omega$ is finite, and each tree $T^{\varphi_j}_{\xi k}$ belongs to $\bigcup^{\mathtt{fin}} \mathbb{P}_\xi$. We have $n_j \to \infty$ by the genericity, so that it can be wlog assumed that $n_0 < n_1 < n_2 < \dots$ **strictly**.

[7] Recall that HC = all *hereditarily countable* sets, *i.e.* those having at most countable transitive closures.

[8] The density means that for any $\langle m, \psi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ there is $\langle n, \varphi \rangle \in D$ with $\langle n, \varphi \rangle \preccurlyeq \langle m, \psi \rangle$. The openness means that if $\langle m, \psi \rangle \in D$ and $\langle n, \varphi \rangle \preccurlyeq \langle m, \psi \rangle$ then $\langle n, \varphi \rangle \in D$.

(2) Let $\xi \in |\boldsymbol{\pi}|$, $k < \omega$. By the genericity assumption, there is a number $j(\xi, k)$ such that if $j \ge j(\xi, k)$ then $\langle \xi, k \rangle \in |\varphi_j|$, hence the tree $\varphi_j(\xi, k) = T^{\varphi_j}_{\xi k} \in \bigcup^{\mathtt{fin}} \mathbb{P}_\xi$ is defined, and we have

$$\ldots \preccurlyeq \langle n_{j(\xi,k)+2}, T^{\varphi_{j(\xi,k)+2}}_{\xi k} \rangle \preccurlyeq \langle n_{j(\xi,k)+1}, T^{\varphi_{j(\xi,k)+1}}_{\xi k} \rangle \preccurlyeq \langle n_{j(\xi,k)}, T^{\varphi_{j(\xi,k)}}_{\xi k} \rangle \,,$$

with $n_{j(\xi,k)} < n_{j(\xi,k)+1} < n_{j(\xi,k)+2} < \ldots$ strictly, by (1) above.

(3) Then it follows by Lemma 3.2 that each intersection $\boldsymbol{Q}^{\Phi}_{\xi k} = \bigcap_{j \ge j(\xi,k)} T^{\varphi_j}_{\xi k}$ is a tree in **PT** (not necessarily in $\mathbb{P}_\xi$), and the relation $\langle n_j, \boldsymbol{Q}^{\Phi}_{\xi k} \rangle \preccurlyeq \langle n_j, T^{\varphi_j}_{\xi k} \rangle$ holds for all $j \ge j(\xi, k)$. We define $\mathbb{Q}^{\Phi}_\xi = \{ \boldsymbol{Q}^{\Phi}_{\xi k} {\restriction}_s : k < \omega \wedge s \in \boldsymbol{Q}^{\Phi}_{\xi k} \}$.

(4) We finally let $\boldsymbol{\varphi} = \langle \mathbb{Q}^{\Phi}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ and $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varphi} = \langle \mathbb{P}_\xi \cup \mathbb{Q}^{\Phi}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$.

(5) Finally if $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ is obtained this way from an $\mathfrak{M}^+$-generic sequence Φ, then $\boldsymbol{\varphi}$ is called an $\mathfrak{M}$-*generic refinement of* $\boldsymbol{\pi}$. □

Lemma 7.2 (by the countability of $\mathfrak{M}^+$)**.** *If* $\boldsymbol{\pi}$ *is a small multiforcing and* $\mathfrak{M} \in \mathrm{HC}$ *then there is an* $\mathfrak{M}$-*generic refinement* $\boldsymbol{\varphi}$ *of* $\boldsymbol{\pi}$. □

The next theorem is formulated under the assumptions and notation of Definition 7.1. Its goal is to demonstrate that the construction of Definition 7.1 results in refinements of types $\sqsubset_D$ and $\sqsubset_{\boldsymbol{D}}$.

Theorem 7.3. *If* $\mathfrak{M} \in \mathrm{HC}$ *is transitive,* $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathfrak{M}$ *is a small multiforcing, and* $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi] = \langle \mathbb{Q}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ *is an* $\mathfrak{M}$-*generic refinement of* $\boldsymbol{\pi}$*, then:*

(i) $\boldsymbol{\varphi}$ *is a small special multiforcing,* $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$*, and* $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$;

(ii) *if pairs* $\langle \xi, k \rangle \ne \langle \eta, \ell \rangle$ *belong to* $|\boldsymbol{\pi}| = |\boldsymbol{\varphi}|$ *then* $[\boldsymbol{Q}^{\Phi}_{\xi k}] \cap [\boldsymbol{Q}^{\Phi}_{\eta \ell}] = \varnothing$;

(iii) *if* $\xi \in |\boldsymbol{\pi}|$, $S \in \mathbb{Q}_\xi$ *and* $T \in \mathbb{P}_\xi$ *then* $[S] \cap [T]$ *is clopen in* $[S]$ *and* $T \not\subseteq S$*, in particular,* $\mathbb{Q}_\xi \cap \mathbb{P}_\xi = \varnothing$;

(iv) *if* $\xi \in |\boldsymbol{\pi}|$ *then the set* $\mathbb{Q}_\xi$ *is open dense in* $\mathbb{Q}_\xi \cup \mathbb{P}_\xi$;

(v) *if* $\xi \in |\boldsymbol{\pi}|$ *and a set* $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}_\xi$ *is pre-dense in* $\mathbb{P}_\xi$ *then* $\mathbb{P}_\xi \sqsubset_D \mathbb{Q}_\xi$;

(vi) *if in addition* $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\lambda} \boldsymbol{\pi}_\alpha$*, where* $\lambda < \omega_1$ *and* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \mathfrak{M}$ *is a* $\sqsubset$-*increasing sequence of small special multiforcings, then* $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\varphi}$ *for all* $\alpha < \lambda$.

Proof. We argue in the notation of Definition 7.1.

(ii) By Corollary 3.7(iii), the set D of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, where φ is a pairwise AD system and $|\varphi|$ contains both $\langle \xi, k \rangle$, $\langle \eta, \ell \rangle$, is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$, and obviously $D \in \mathfrak{M}^+$. Thus $\langle n_j, \varphi_j \rangle \in D$ for some $j < \omega$. Then $T^{\varphi_j}_{\xi k} \cap T^{\varphi_j}_{\eta \ell} = \varnothing$ since φ_j is AD. But $\boldsymbol{Q}^{\Phi}_{\xi k} \subseteq T^{\varphi_j}_{\xi k}$, $\boldsymbol{Q}^{\Phi}_{\eta \ell} \subseteq T^{\varphi_j}_{\eta \ell}$ by construction.

(iii) Let $S = \boldsymbol{Q}^{\Phi}_{\xi k}$. To prove the clopenness claim, note that the set $D(T)$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k \rangle \in |\varphi|$ and if $s \in 2^n$ then either $T^{\varphi}_{\xi k} {\restriction}_s \subseteq T$ or $[T^{\varphi}_{\xi k}] \cap [T] = \varnothing$, is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$. To prove $T \not\subseteq S$, similarly the set $D'(T)$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k \rangle \in |\varphi|$ and $T \not\subseteq T^{\varphi}_{\xi k}$, is dense. Note that $D(T), D'(T) \in \mathfrak{M}^+$ and argue as above.

(iv) The openness easily follows from (iii). To prove the density, let $T \in \mathbb{P}_\xi$. The set $\Delta(T)$ of all pairs $\langle n, \varphi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T^{\varphi}_{\xi k} = T$ for some k, belongs to $\mathfrak{M}^+$ and is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.

(i) By construction, the sets $\boldsymbol{\varphi}(\xi) = \mathbb{Q}^{\Phi}_{\xi}$ are special arboreal forcings, and hence $\boldsymbol{\varphi}$ is a small special multiforcing, and $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$. To establish $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, let $\xi \in |\boldsymbol{\pi}|$. We have to prove that $\mathbb{P}_\xi \sqsubset \mathbb{Q}_\xi$. Condition (1) of Definition 5.1 follows from (iv), condition (3) from (iii), and (2) holds since $\boldsymbol{Q}^{\Phi}_{\xi k} \subseteq T^{\varphi_j}_{\xi k} \in \bigcup^{\mathtt{fin}} \mathbb{P}_\xi$ for some j.

(v) Assume that $\xi \in |\boldsymbol{\pi}|$, $k < \omega$, $D \in \mathfrak{M}^+$ is pre-dense in $\mathbb{P}_\xi$. Then the set $D' = \{T \in \mathbb{P}_\xi : \exists\, S \in D (T \subseteq S)\}$ is open dense in $\mathbb{P}_\xi$, and hence the set $\Delta \in \mathfrak{M}^+$ of all pairs $\langle n, \varphi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T^{\varphi}_{\xi k}{\restriction}_s \in D'$ for all $s \in 2^n \cap T^{\varphi}_{\xi k}$, is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$ by Lemma 3.6. Thus $\langle n_j, \varphi_j\rangle \in \Delta$ for some j, and this implies $\boldsymbol{Q}^{\Phi}_{\xi k} \subseteq T^{\varphi_j}_{\xi k} \subseteq^{\mathtt{fin}} \bigcup D$.

(vi) We have to prove that $\boldsymbol{\pi}_\alpha(\xi) \sqsubset \boldsymbol{\varphi}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}_\alpha|$. And as $\boldsymbol{\pi}(\xi) \sqsubset \boldsymbol{\varphi}(\xi)$ has been checked, it suffices to prove that $\boldsymbol{Q}^{\Phi}_{\xi k} \subseteq^{\mathtt{fin}} \bigcup \boldsymbol{\pi}_\alpha(\xi)$. However $D = \boldsymbol{\pi}_\alpha(\xi)$ is pre-dense in $\boldsymbol{\pi}(\xi) = \mathbb{P}_\xi$ by Lemma 5.2(iv), and still $D \in \mathfrak{M}^+$, hence we can refer to (v). □

Corollary 7.4. *In the assumptions of Lemma 7.2, if $|\boldsymbol{\pi}| \subseteq Z \subseteq \omega_1$ and Z is countable then there is a small special multiforcing $\boldsymbol{\varphi}$ such that $|\boldsymbol{\varphi}| = Z$ and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$.*

Proof. If $|\boldsymbol{\pi}| = Z$ then let $\mathfrak{M} \in \mathrm{HC}$ be any countable set containing $\boldsymbol{\pi}$, pick $\boldsymbol{\varphi}$ by Lemma 7.2, and apply Theorem 7.3. If $|\boldsymbol{\pi}| \subsetneqq Z$ then we trivially extend the construction by $\boldsymbol{\varphi}(\xi) = \mathbb{P}_{\mathrm{coh}}$ (see Example 3.4) for all $\xi \in Z \smallsetminus |\boldsymbol{\pi}|$. □

8 Generic refinement: sealing dense sets

This Section proves a special consequence of $\mathfrak{M}^+$-genericity of multiforcing refinements, the relation $\sqsubset_{\boldsymbol{D}}$ of Definition 6.2 between a multiforcing and its refinement, via a dense set $\boldsymbol{D}$.

Theorem 8.1. *Under the assumptions of Theorem 7.3, if $\boldsymbol{D} \in \mathfrak{M}^+$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, and $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$.*

Proof. We suppose that $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ is obtained from a decreasing $\mathfrak{M}^+$-generic sequence Φ of pairs $\langle n_j, \varphi_j\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, as in Definition 7.1(1), and argue in the notation of 7.1. Suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, as in $(*)$ of Definition 6.2; the extra condition $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$ holds automatically as we have $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$. We have to find a multitree $\boldsymbol{q}$ which witnesses 6.2$(*)$ for $\boldsymbol{u}$.

Each term $T^{\boldsymbol{u}}_\xi$ of $\boldsymbol{u}$ ($\xi \in |\boldsymbol{u}|$) is equal to some $\boldsymbol{Q}^{\Phi}_{\xi, k_\xi}{\restriction}_{t_\xi}$, where $t_\xi \in \boldsymbol{Q}^{\Phi}_{\xi, k_\xi}$. We can wlog assume that simply $t_\xi = \Lambda$, so that $T^{\boldsymbol{u}}_\xi = \boldsymbol{Q}^{\Phi}_{\xi, k_\xi}$, $\forall\, \xi$.

Definition 8.2. If $n < \omega$ then let $\mathbf{Sys}_n(\boldsymbol{\pi})$ contain all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k_\xi\rangle \in |\varphi|$ for all $\xi \in |\boldsymbol{u}|$, and $T^{\varphi}_{\xi k}{\restriction}_t \in \mathbb{P}_\xi = \boldsymbol{\pi}(\xi)$ (not just $\in \bigcup^{\mathtt{fin}} \mathbb{P}_\xi$!) for all $\langle \xi, k\rangle \in |\varphi|$ and $t \in 2^n \cap T^{\varphi}_{\xi k}$. If $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ then let $\mathbf{S}^n_\varphi$ contain all multistrings $\mathbf{s} = \langle s_\xi\rangle_{\xi \in |\boldsymbol{u}|}$ such that $s_\xi \in 2^n \cap T^{\varphi}_{\xi, k_\xi}$, $\forall\, \xi \in |\boldsymbol{u}|$. If $\mathbf{s} = \langle s_\xi\rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^n_\varphi$ then define $\boldsymbol{v}^{\mathbf{s}}_\varphi \in \mathbf{MT}(\boldsymbol{\pi})$ by $|\boldsymbol{v}^{\mathbf{s}}_\varphi| = |\boldsymbol{u}|$ and $T^{\boldsymbol{v}^{\mathbf{s}}_\varphi}_\xi = T^{\varphi}_{\xi, k_\xi}{\restriction}_{s_\xi}$ for all $\xi \in |\boldsymbol{u}|$. □

Lemma 8.3. *Let $n < \omega$ and $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$. There exists a system $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ satisfying $\langle n, \psi \rangle \preccurlyeq \langle n, \varphi \rangle$.*

Proof. Add each absent $\langle \xi, k_\xi \rangle \notin |\varphi|$ to $|\psi|$ and define $T^\psi_{\xi,k_\xi} \in \mathbb{P}_\xi$ arbitrarily. If $\langle \xi, k \rangle \in |\psi|$ and $t \in 2^n \cap T^\psi_{\xi k}$, but $T^\psi_{\xi k}\restriction_t \in \bigcup^{\mathtt{fin}} \mathbb{P}_\xi \smallsetminus \mathbb{P}_\xi$, then shrink $T^\psi_{\xi k}$ to a tree in $\mathbb{P}_\xi$ by Lemma 3.7(i), and do this for all triples ξ, k, t as indicated. □ (Lemma) □

Lemma 8.4. *If $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$, then the set $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ and there is $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and* (1) *if $\mathbf{s} \in \mathbf{S}^n_\varphi$ then $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q} \in \boldsymbol{D}$, — is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.*

Lemma. Let $\langle n, \psi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$. We'll find a pair $\langle n, \varphi \rangle \in \Delta_{\boldsymbol{r}}$ (same n!) with $\langle n, \varphi \rangle \preccurlyeq \langle n, \psi \rangle$. We wlog assume that $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$, by Lemma 8.3.

Let $\mathbf{s} = \langle s_\xi \rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^n_\psi$. Consider the multitree $\boldsymbol{v}^{\mathbf{s}}_\psi \in \mathbf{MT}(\boldsymbol{\pi})$. As $\boldsymbol{D}$ is dense, there are multitrees $\boldsymbol{r}', \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{r}'| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{r}' \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{r}' \in \boldsymbol{D}$. Define a system $\psi' \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\psi'| = |\psi|$, that extends ψ by shrinking each tree $T^\psi_{\xi,k_\xi}\restriction_{s_\xi}$ to $T^{\boldsymbol{v}}_\xi$, so that $T^{\psi'}_{\xi,k_\xi}\restriction_{s_\xi} = T^{\boldsymbol{v}}_\xi$, but $T^{\psi'}_{\xi,k_\xi}\restriction_t = T^\psi_{\xi,k_\xi}\restriction_t$ for all $t \in 2^n \cap T^\psi_{\xi,k_\xi}$, $t \neq s_\xi$, and $T^{\psi'}_{\eta k} = T^\psi_{\eta k}$ whenever $\langle \eta, k \rangle \in |\psi|$ does not have the form $\langle \xi, k_\xi \rangle$, where $\xi \in |\boldsymbol{u}|$. We have $\langle n, \psi' \rangle \preccurlyeq \langle n, \psi \rangle$ by construction, therefore $\mathbf{S}^n_{\psi'} = \mathbf{S}^n_\psi$.

This construction can be iterated, so that all strings $\mathbf{s} \in \mathbf{S}^n_\psi$ are considered one by one. This results in a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $|\varphi| = |\psi|$ and $\langle n, \varphi \rangle \preccurlyeq \langle n, \psi \rangle$ — and then $\mathbf{S}^n_\varphi = \mathbf{S}^n_\psi$, and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{q} \leqslant \boldsymbol{r}$ and still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, such that if $\mathbf{s} \in \mathbf{S}^n_\psi$ then the multitree $\boldsymbol{v}^{\mathbf{s}}_\psi$, satisfies $\boldsymbol{v}^{\mathbf{s}}_\psi \cup \boldsymbol{q} \in \boldsymbol{D}$. Then $\boldsymbol{q}$ witnesses that $\langle n, \varphi \rangle \in \Delta_r$. □ (Lemma) □

By the lemma, we have $\langle n_j, \varphi_j \rangle \in \Delta_{\boldsymbol{p}}$ for some j. Let this be witnessed by a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, so that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and (1) of Lemma 8.4 holds for $n = n_j$, $\varphi = \varphi_j$. We easily conclude that $[\boldsymbol{u}] \subseteq \bigcup_{\mathbf{s} \in \mathbf{S}^n_{\varphi_j}} [\boldsymbol{v}^{\mathbf{s}}_{\varphi_j}]$. Yet $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, $\forall \mathbf{s}$, by (1). □ (Theorem) □

Corollary 8.5. *Under the assumptions of Theorem 7.3, if a set $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi})$, then it remains pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varphi})$.*

Proof. Assume wlog that $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$. (If not then consider $\boldsymbol{D}' = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists \boldsymbol{q} \in \boldsymbol{D}\,(\boldsymbol{p} \leqslant \boldsymbol{q})\}$.) Note that $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$ by Theorem 8.1, and use Lemma 6.3(i). □

9 Real names and direct forcing

Our next goal is to introduce a suitable notation related to names of reals in 2^ω in the context of forcing notions of the form $\mathbf{MT}(\boldsymbol{\pi})$.

Definition 9.1. A *real name* is any set $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times 2)$ such that the sets $K^{\mathbf{c}}_{ni} = \{\boldsymbol{p} \in \mathbf{MT} : \langle \boldsymbol{p}, n, i \rangle \in \mathbf{c}\}$ satisfy the following: if $n < \omega$ and $\boldsymbol{p} \in K^{\mathbf{c}}_{n0}$, $\boldsymbol{q} \in K^{\mathbf{c}}_{n1}$, then $\boldsymbol{p}, \boldsymbol{q}$ are SAD.[9] Let $K^{\mathbf{c}}_n = K^{\mathbf{c}}_{n0} \cup K^{\mathbf{c}}_{n1} \subseteq \mathbf{MT}(\boldsymbol{\pi})$

A real name $\mathbf{c}$ is *small* if each $K^{\mathbf{c}}_n$ is at most countable — then the set $|\mathbf{c}| = \bigcup_n \bigcup_{\boldsymbol{p} \in K^{\mathbf{c}}_n} |\boldsymbol{p}|$, and $\mathbf{c}$ itself, are countable, too.

Let $\boldsymbol{\pi}$ be a multiforcing. A real name $\mathbf{c}$ is $\boldsymbol{\pi}$*-complete* if every set $K^{\mathbf{c}}_n \!\uparrow\! \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K^{\mathbf{c}}_n \, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$ (the $\boldsymbol{\pi}$-cone of $K^{\mathbf{c}}_n$) is pre-dense in $\mathbf{MT}(\boldsymbol{\pi})$. In this case, if a set (a filter) $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is $\mathbf{MT}(\boldsymbol{\pi})$-generic over the family of all sets $K^{\mathbf{c}}_n$, then we define a real $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_{ni} \neq \varnothing$.

We do not require in this case that $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$, or equivalently, $K^{\mathbf{c}}_n \subseteq \mathbf{MT}(\boldsymbol{\pi})$ for all n, but if this inclusion indeed holds then this will be explicitly mentioned. □

Assume that $\mathbf{c}$ is a real name, in the sense of 9.1. Say that a multitree $\boldsymbol{p}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff there is a multitree $\boldsymbol{q} \in K^{\mathbf{c}}_{ni}$ such that $\boldsymbol{p} \leqslant \boldsymbol{q}$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\boldsymbol{p}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus T$ such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$.

The definition of direct forcing is not explicitly associated with any concrete forcing notion, but in fact it is compatible with any multiforcing.

Lemma 9.2. *Let $\boldsymbol{\pi}$ be a multiforcing, $\mathbf{c}$ a $\boldsymbol{\pi}$-complete real name, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$. If $n < \omega$ then there exists $i = 0, 1$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which directly forces $\mathbf{c}(n) = i$. If $T \in \mathbf{PT}$ then there exists $s \in T$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which directly forces $\mathbf{c} \notin [T \!\restriction_s]$.*

Proof. To prove the first claim use the density of sets $K^{\mathbf{c}}_n$ by Definition 9.1 above. To prove the second claim, pick n such that $T \cap 2^n$ contains at least two strings. By the first claim, there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and a string $t \in T \cap 2^n$ such that $\boldsymbol{q}$ directly forces $t \subset \mathbf{c}$. Now take any $s \in T \cap 2^n$, $s \neq t$. □

10 Sealing real names and avoiding refinements

The next definition extends Definition 6.2 to real names.

Definition 10.1. Assume that $\boldsymbol{\pi}, \boldsymbol{\varrho}$ are multiforcings, $\mathbf{c}$ is a real name, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$. Say that $\boldsymbol{\varrho}$ *seals* $\mathbf{c}$ *over* $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$, if $\boldsymbol{\varrho}$ seals, over $\boldsymbol{\pi}$, each set $K^{\mathbf{c}}_n \!\uparrow\! \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K^{\mathbf{c}}_n \, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$, in the sense of Definition 6.2. □

[9] Recall that the condition of somewhere almost disjointness SAD (Definition 4.2) is equivalent to the incompatibility of $\boldsymbol{p}, \boldsymbol{q}$ in $\mathbf{MT}$ and in any set of the form $\mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi}$ is a regular multiforcing, by corollary 4.3.

Corollary 10.2. *Under the assumptions of Theorem 7.3, if* $\mathbf{c} \in \mathfrak{M}^+$ *and* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-complete real name then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$.

Proof. Each set $K_n^{\mathbf{c}}{\uparrow}\boldsymbol{\pi}$ belongs to $\mathfrak{M}^+$ (as so do $\mathbf{c}$ and $\boldsymbol{\pi}$) and is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, so it remains to apply Theorem 8.1. □

Lemma 10.3. *Let* $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ *be multiforcings and* $\mathbf{c}$ *be a real name. Then*

(i) *if* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$ *then* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-complete and a* $(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$*-complete real name*;

(ii) *if* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$ *then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\sigma}$;

(iii) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, *and* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho} = \bigcup^{\mathtt{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. (i) By definition, we have $\boldsymbol{\pi} \sqsubset_{K_n^{\mathbf{c}}\uparrow\boldsymbol{\pi}} \boldsymbol{\varrho}$ for each n, therefore $K_n^{\mathbf{c}}{\uparrow}\boldsymbol{\pi}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$ (then obviously open dense) and pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$ by Lemma 6.3(i). It follows that $K_n^{\mathbf{c}}{\uparrow}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$ is dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$.

To check (ii), (iii) apply (iii), (iv) of Lemma 6.3. □

If $\boldsymbol{\pi}$ is a multiforcing then $\mathbf{MT}(\boldsymbol{\pi})$ adds a collection of *principal generic reals* $x_\xi = x_\xi[G] \in 2^\omega$, $\xi \in |\boldsymbol{\pi}|$, where each x_ξ is $\boldsymbol{\pi}(\xi)$-generic over the ground set universe, see Remark 4.5. Obviously many more reals are added, and given a $\boldsymbol{\pi}$-complete real name $\mathbf{c}$, one can elaborate different requirements for a condition $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ to force that $\mathbf{c}$ is a name of a real of the form $x_{\xi k}$ or to force the opposite. The next definition provides such a condition related to the "opposite" direction.

Definition 10.4. Let $\boldsymbol{\pi}$ be a multiforcing, $\xi \in |\boldsymbol{\pi}|$. A real name $\mathbf{c}$ is *non-principal over* $\boldsymbol{\pi}$ *at* ξ, if the following set is open dense in $\mathbf{MT}(\boldsymbol{\pi})$:

$$\boldsymbol{D}^{\boldsymbol{\pi}}_\xi(\mathbf{c}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_\xi]\} .$$ □

We'll show below (Theorem 12.2(i)) that the non-principality implies $\mathbf{c}$ being **not** a name of the real $x_\xi[\underline{G}]$. And further, the avoidance condition in the next definition will be shown to imply $\mathbf{c}$ being a name of a non-generic real.

Definition 10.5. Let $\boldsymbol{\pi}, \boldsymbol{\varrho}$ be multiforcings, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$, $\xi \in |\boldsymbol{\pi}|$; $\boldsymbol{\varrho}$ *avoids a real name* $\mathbf{c}$ *over* $\boldsymbol{\pi}$ *at* ξ, in symbol $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_\xi \boldsymbol{\varrho}$, if for each $Q \in \boldsymbol{\varrho}(\xi)$, $\boldsymbol{\varrho}$ seals the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{r}| \wedge \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\} ,$$

over $\boldsymbol{\pi}$ in the sense of Definition 6.2 — that is formally $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})} \boldsymbol{\varrho}$. □

Lemma 10.6. *Assume that* $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ *are multiforcings,* $\xi \in |\boldsymbol{\pi}|$, *and* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-complete real name. Then:*

(i) *if* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_\xi \boldsymbol{\varrho}$ *and* $Q \in \boldsymbol{\varrho}(\xi)$ *then the set* $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi})$ *and pre-dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$;

(ii) *if* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_\xi \boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$ *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_\xi \boldsymbol{\sigma}$;

(iii) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, *and* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varrho} = \bigcup^{\mathtt{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. (i) Apply Lemma 6.3(i). To prove (ii) let $S \in \boldsymbol{\sigma}(\xi)$. Then, as $\boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$, there is a finite set $\{Q_1, \dots, Q_m\} \subseteq \boldsymbol{\varrho}(\xi)$ such that $S \subseteq Q_1 \cup \dots \cup Q_m$. We have $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\varrho}$ for all i as $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varrho}$, thus $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\sigma}$, $\forall\, i$, by Lemma 6.3(iii). Note that $\bigcap_i \boldsymbol{D}(\mathbf{c}, Q_i, \boldsymbol{\pi}) \subseteq \boldsymbol{D}(\mathbf{c}, S, \boldsymbol{\pi})$ since $S \subseteq \bigcup_i Q_i$. We conclude that $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},S,\boldsymbol{\pi})} \boldsymbol{\sigma}$ by Lemma 6.3(ii). Therefore $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\sigma}$, as required.

To prove (iii) make use of Lemma 6.3(iv) the same way. □

11 Generic refinement avoids non-principal names

The following theorem says that generic refinements as in Section 7 avoid nonprincipal names. It resembles Theorem 8.1 to some extent, yet the latter is not directly applicable here as both the multitree Q and the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ depend on $\boldsymbol{\varrho}$, and hence the sets $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ do not necessarily belong to $\mathfrak{M}^+$. However the proof will be based on rather similar arguments.

Theorem 11.1. *Under the assumptions of Theorem 7.3, if* $\eta \in |\boldsymbol{\pi}| \subseteq \mathfrak{M}$ *and* $\mathbf{c} \in \mathfrak{M}$ *is a* $\boldsymbol{\pi}$*-complete real name non-principal over* $\boldsymbol{\pi}$ *at* η *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\eta} \boldsymbol{\varrho}$.

Proof. Assume that $\boldsymbol{\varrho} = \boldsymbol{\varrho}[\Phi]$ is obtained from an $\mathfrak{M}^+$-generic sequence Φ in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$, as in Definition 7.1. We stick to the notation of 7.1.

Let $Q \in \boldsymbol{\varrho}(\eta) = \mathbb{P}^{\boldsymbol{\varrho}}_{\eta}$; we have to prove that $\boldsymbol{\varrho}$ seals the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ over $\boldsymbol{\pi}$. By construction $Q = \boldsymbol{Q}^{\Phi}_{\eta K}\restriction_s$ for some $K < \omega$ and $s \in \boldsymbol{Q}^{\Phi}_{\eta K}$; it can be assumed that simply $Q = \boldsymbol{Q}^{\Phi}_{\eta K}$. Following the proof of Theorem 8.1, we suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, and $T^{\boldsymbol{u}}_{\xi} = \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$, for each $\xi \in |\boldsymbol{u}|$. We have to find a multitree $\boldsymbol{q}$ which witnesses 6.2(∗) for $\boldsymbol{u}$, $\boldsymbol{p}$, $\boldsymbol{D} = \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$. Note that η may or may not belong to the set $|\boldsymbol{u}|$, and even if $\eta \in |\boldsymbol{u}|$, so k_η is defined, then K may or may not be equal to k_η. In the remainder of the proof, we use **the notation of Definition 8.2**, in particular, $\mathbf{Sys}_n(\boldsymbol{\pi})$, $\mathbf{S}^n_{\varphi}$, $\boldsymbol{v}^{\mathbf{s}}_{\varphi}$.

Assume that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$. Consider the set $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, such that $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ (see Def. 8.2), $\langle \eta, K \rangle \in |\varphi|$, and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, still $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and

(1′) if $\mathbf{s} \in \mathbf{S}^n_{\varphi}$ and $t \in T^{\varphi}_{\eta K} \cap 2^n$ then $\boldsymbol{v}^{\mathbf{s}}_{\varphi} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi}_{\eta K}\restriction_t]$.

Condition (1′) is similar to (1) of Lemma 8.4, of course. Note that direct forcing of $\mathbf{c} \notin [Q]$ cannot be used in (1′) because Q is not necessarily an element of $\mathfrak{M}$, but $\mathbf{c} \notin [T^{\varphi}_{\eta K}]$ will be an effective replacement.

Lemma 11.2. *If* $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$, *then* $\Delta_{\boldsymbol{r}}$ *is dense in* $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.

Proof. We follow the proof of Lemma 8.4. Let $\langle n, \psi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$. We wlog assume that $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ (see Lemma 8.4), so $\langle \xi, k_\xi \rangle \in |\psi|$ for all $\xi \in |\boldsymbol{u}|$ and $T^{\psi}_{\xi k}\restriction_t \in \mathbb{P}_{\xi}$ for all $\langle \xi, k \rangle \in |\psi|$ and $t \in 2^n \cap T^{\psi}_{\xi k}$, and $\langle \eta, K \rangle \in |\psi|$ as well.

We have to define a system $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ such that $\langle n, \varphi\rangle \preccurlyeq \langle n, \pi\rangle$ and $\varphi \in \Delta_r$. As in the proof of Lemma 8.4, it suffices to fulfill (1′) for **one particular pair** of $\mathbf{s} = \langle s_\xi\rangle_{\xi\in|\boldsymbol{u}|} \in \mathbf{S}^n_\psi$ and $t \in T^\psi_{\eta K} \cap 2^n$; the final goal is then achieved by simple iteration through all such pairs. We have two cases.

Case 1: $\eta \in |\boldsymbol{u}|$, $K = k_\eta$, $t = s_\eta$. Consider the multitree $\boldsymbol{v}^{\mathbf{s}}_\psi \in \mathbf{MT}(\boldsymbol{\pi})$. The set $\boldsymbol{D}^{\boldsymbol{\pi}}_\eta(\mathbf{c})$, as in Definition 10.4, is dense by the non-principality of $\mathbf{c}$. It follows that there are multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{q} \in \boldsymbol{D}^{\boldsymbol{\pi}}_\eta(\mathbf{c})$. Therefore $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{q}}_\eta]$. Define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$, from ψ by:

(a) shrinking each tree $T^\psi_{\xi,k_\xi}\restriction_{s_\xi}$ $(\xi \in |\boldsymbol{u}|)$ to $T^{\boldsymbol{v}}_\xi$, so that $T^\varphi_{\xi,k_\xi}\restriction_{s_\xi} = T^{\boldsymbol{v}}_\xi$,

(b) in particular, shrinking $T^\psi_{\eta K}\restriction_t$ to $T^{\boldsymbol{v}}_\eta$, so that $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$,

and no other changes. We have $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$, $\boldsymbol{v}^{\mathbf{s}}_\varphi = \boldsymbol{v}$, and $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$ by construction. In particular, $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^\varphi_{\eta K}\restriction_t]$, thus (1′) holds.

Case 2: not Case 1. By Lemma 9.2, there exist multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ and a tree $T \in \mathbb{P}_\eta$ such that $T \subseteq T^\psi_{\eta K}\restriction_t$, $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T]$. Define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$, that extends ψ by (a) above and:

(c) shrinking $T^\psi_{\eta K}\restriction_t$ to T, so that $T^\varphi_{\eta K}\restriction_t = T$,

and no other changes. Note that (a) and (c) do not contradict each other since $\langle \eta, T, t\rangle \neq \langle \xi, k_\xi, s_\xi\rangle$ for all $\xi \in \boldsymbol{u}$ by the Case 2 hypothesis. We have $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$, $\boldsymbol{v}^{\mathbf{s}}_\varphi = \boldsymbol{v}$, and $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$ by construction. In particular, $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^\varphi_{\eta K}\restriction_t]$, thus (1′) holds. □ (Lemma) □

Come back to the theorem. As $\Delta_{\boldsymbol{p}} \in \mathfrak{M}^+$, we have $\langle n_j, \varphi_j\rangle \in \Delta_{\boldsymbol{p}}$ for some j by the lemma. Let this be witnessed by a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, so that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and (1′) holds for $n = n_j$, $\varphi = \varphi_j$. In particular, as $T^{\varphi_j}_{\eta K} = \bigcup_{t \in T^{\varphi_j}_{\eta K} \cap 2^n} T^{\varphi_j}_{\eta K}\restriction_t$, the multitree $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi_j}_{\eta K}]$ whenever $\mathbf{s} \in \mathbf{S}^n_{\varphi_j}$, hence directly forces $\mathbf{c} \notin [Q]$ as well, because $Q = \boldsymbol{Q}^\Phi_{\eta K} \subseteq T^{\varphi_j}_{\eta K}$ by construction. Thus if $\mathbf{s} \in \mathbf{S}^n_{\varphi_j}$ then $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \cup \boldsymbol{q} \in \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$, and hence $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \in \boldsymbol{D}(\mathbf{c}, Q, \varphi_j)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. On the other hand, $[\boldsymbol{u}] \subseteq \bigcup_{\mathbf{s} \in \mathbf{S}^n_{\varphi_j}} [\boldsymbol{v}^{\mathbf{s}}_{\varphi_j}]$, so that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, as required. □

12 Consequences for generic extensions

We first prove a lemma on adequately representation of reals in $\mathbf{MT}(\boldsymbol{\pi})$-generic extensions by real names. Then Theorem 12.2 will show corollaries for non-principal names.

Lemma 12.1. *Suppose that $\boldsymbol{\pi}$ is a regular multiforcing and $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground set universe $\mathbf{V}$.*

If $x \in \mathbf{V}[G] \cap 2^\omega$ then there is a $\boldsymbol{\pi}$-complete real name $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, such that $x = \mathbf{c}[G]$.

If $\mathbf{MT}(\boldsymbol{\pi})$ *is a CCC forcing*[10] *in* $\mathbf{V}$*, and* $\mathbf{c}\in\mathbf{V}$*,* $\mathbf{c}\subseteq\mathbf{MT}(\boldsymbol{\pi})\times\omega\times 2$ *is a* $\boldsymbol{\pi}$*-complete real name, then there is a* ***small*** $\boldsymbol{\pi}$*-complete real name* $\mathbf{d}\in\mathbf{V}$*,* $\mathbf{d}\subseteq\mathbf{MT}(\boldsymbol{\pi})\times\omega\times 2$*, such that* $\mathbf{MT}(\boldsymbol{\pi})$ *forces* $\mathbf{c}[\underline{G}]=\mathbf{d}[\underline{G}]$ *over* $\mathbf{V}$*.*

Proof. The first claim is an instance of a general forcing theorem. To prove the second one, extend each set $K_n^{\mathbf{c}}\subseteq\mathbf{MT}(\boldsymbol{\pi})$ to an open dense set $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}=\{\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi}):\exists\,\boldsymbol{q}\in K_n^{\mathbf{c}}\,(\boldsymbol{p}\leqslant\boldsymbol{q})\}$, choose maximal antichains $A_n\subseteq K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ in those sets — which are countable by CCC, and then let $A_{ni}=\{\boldsymbol{p}\in A_n:\exists\,\boldsymbol{q}\in K_{ni}^{\mathbf{c}}\,(\boldsymbol{p}\leqslant\boldsymbol{q})\}$ and $\mathbf{d}=\{\langle\boldsymbol{p},n,i\rangle:\boldsymbol{p}\in A_{ni}\}$. □

Theorem 12.2. *Let* $\boldsymbol{\pi}$ *be a regular multiforcing and* $\xi\in|\boldsymbol{\pi}|$*. Then*

(i) *if* $\mathbf{MT}(\boldsymbol{\pi})$ *is CCC, a set* $G\subseteq\mathbf{MT}(\boldsymbol{\pi})$ *is generic over the ground set universe* $\mathbf{V}$*, and* $x\in\mathbf{V}[G]\cap 2^\omega$*, then* $x\neq x_\xi[G]$ *if and only if there is a small* $\boldsymbol{\pi}$*-complete real name* $\mathbf{c}\subseteq\mathbf{MT}(\boldsymbol{\pi})\times(\omega\times 2)$*, non-principal over* $\boldsymbol{\pi}$ *at* ξ *and such that* $x=\mathbf{c}[G]$*;*

(ii) *if* $\mathbf{c}\subseteq\mathbf{MT}(\boldsymbol{\pi})\times(\omega\times 2)$ *is a* $\boldsymbol{\pi}$*-complete real name,* $\boldsymbol{\varphi}$ *is a multiforcing,* $\boldsymbol{\pi}\sqsubset_\xi^{\mathbf{c}}\boldsymbol{\varphi}$*, and a set* $G\subseteq\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$ *is generic over* $\mathbf{V}$ *then* $\mathbf{c}[G]\notin\bigcup_{Q\in\boldsymbol{\varphi}(\xi)}[Q]$*.*

Proof. (i) Let $x\neq x_\xi[G]$. By a known forcing theorem, there is a $\boldsymbol{\pi}$-complete real name $\mathbf{c}$ such that $x=\mathbf{c}[G]$ and $\mathbf{MT}(\boldsymbol{\pi})$ forces that $\mathbf{c}\neq x_\xi[\underline{G}]$, and, by Lemma 12.1, $\mathbf{c}$ is small since $\mathbf{MT}(\boldsymbol{\pi})$ is CCC. It remains to show that $\mathbf{c}$ is a non-principal name over $\boldsymbol{\pi}$ at ξ, that is, the set

$$\boldsymbol{D}_\xi^{\boldsymbol{\pi}}(\mathbf{c})=\{\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi}):\xi\in|\boldsymbol{p}|\wedge\boldsymbol{p}\text{ directly forces }\mathbf{c}\notin[T_\xi^{\boldsymbol{p}}]\}\,.$$

is open dense in $\mathbf{MT}(\boldsymbol{\pi})$. The openness is clear, let us prove the density. Consider any $\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi})$. Then $\boldsymbol{q}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c}\neq x_\xi[\underline{G}]$ by the choice of $\mathbf{c}$, hence we can assume that, for some n, $\mathbf{c}(n)\neq x_\xi[\underline{G}](n)$ is $\mathbf{MT}(\boldsymbol{\pi})$-forced by $\boldsymbol{q}$. Then by Lemma 9.2 there is a multitree $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{p}\leqslant\boldsymbol{q}$, and $s\in\omega^{n+1}$, such that $\boldsymbol{p}$ directly forces $s\subseteq\mathbf{c}$. Now it suffices to show that $s\notin T_\xi^{\boldsymbol{p}}$. Suppose otherwise: $s\in T_\xi^{\boldsymbol{p}}$. Then the tree $T=T_\xi^{\boldsymbol{p}}\!\restriction_s$ still belongs to $\mathbf{MT}(\boldsymbol{\pi})$. Therefore the multitree $\boldsymbol{r}$ defined by $T_\xi^{\boldsymbol{r}}=T$ and $T_{\xi'}^{\boldsymbol{r}}=T_{\xi'}^{\boldsymbol{p}}$ for each $\xi'\neq\xi$, belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and satisfies $\boldsymbol{r}\leqslant\boldsymbol{p}\leqslant\boldsymbol{q}$. However $\boldsymbol{r}$ directly forces both $\mathbf{c}(n)$ and $x_\xi[\underline{G}](n)$ to be equal to one and the same value $\ell=s(n)$, which contradicts to the choice of n.

To prove the converse let $\mathbf{c}\subseteq\mathbf{MT}(\boldsymbol{\pi})\times(\omega\times 2)$ be a $\boldsymbol{\pi}$-complete real name non-principal over $\boldsymbol{\pi}$ at ξ, and $x=\mathbf{c}[G]$. Assume to the contrary that $x=x_\xi[G]$. There is a multitree $\boldsymbol{q}\in G$ which $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c}=x_\xi[\underline{G}]$. As $\mathbf{c}$ is non-principal, there is a multitree $\boldsymbol{p}\in G\cap\boldsymbol{D}_\xi^{\boldsymbol{\pi}}(\mathbf{c})$, $\boldsymbol{p}\leqslant\boldsymbol{q}$. Thus $\boldsymbol{p}$ directly forces $\mathbf{c}\notin[T_\xi^{\boldsymbol{p}}]$, and hence $\mathbf{MT}(\boldsymbol{\pi})$-forces the same statement. Yet $\boldsymbol{p}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $x_\xi[\underline{G}]\in[T_\xi^{\boldsymbol{p}}]$, of course, and this is a contradiction.

(ii) Suppose towards the contrary that $Q\in\boldsymbol{\varphi}(\xi)$ and $\mathbf{c}[G]\in[Q]$. By definition, $\boldsymbol{\varphi}$ seals, over $\boldsymbol{\pi}$, the set

$$\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})=\{\boldsymbol{r}\in\mathbf{MT}(\boldsymbol{\pi}):\xi\in|\boldsymbol{r}|\wedge\boldsymbol{r}\text{ directly forces }\mathbf{c}\notin[Q]\}\,.$$

Therefore $\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$ by Lemma 6.3, and hence $G\cap\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})\neq\varnothing$. In other words, there is a multitree $\boldsymbol{r}\in\mathbf{MT}(\boldsymbol{\pi})$ which directly forces $\mathbf{c}\notin[Q]$. It easily follows that $\mathbf{c}[G]\notin[Q]$, which is a contradiction. □

[10] The CCC property means that every antichain $A\subseteq\mathbf{MT}(\boldsymbol{\pi})$ is at most countable.

13 Combining refinement types

Here we summarize the properties of generic refinements considered above. The next definition combines the refinement types $\sqsubset_D$, $\sqsubset_{\boldsymbol{D}}$, $\sqsubset_{\mathbf{c}}$, $\sqsubset^{\mathbf{c}}_{\xi}$.

Definition 13.1. Suppose that $\boldsymbol{\pi} \sqsubseteq \boldsymbol{\varrho}$ are multiforcings and $\mathfrak{M} \in \mathrm{HC}$ is any set. Let $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho}$ mean that the four following requirements hold:

(1) if $\xi \in |\boldsymbol{\pi}|$, $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi)$, D is pre-dense in $\boldsymbol{\pi}(\xi)$, then $\boldsymbol{\pi}(\xi) \sqsubset_D \boldsymbol{\varrho}(\xi)$;

(2) if $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$;

(3) if $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$;

(4) if $\xi \in |\boldsymbol{\pi}|$ and $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name, *non-principal over* $\boldsymbol{\pi}$ *at* ξ, then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varrho}$. □

Corollary 13.2 (of lemmas 5.4, 6.3, 10.3, 10.6)**.** *Let* $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ *be multiforcings and* $\mathfrak{M}$ *be a countable set. Then:*

(i) *if* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho} \sqsubseteq \boldsymbol{\sigma}$ *then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\sigma}$;

(ii) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubseteq$*-increasing sequence in* $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, *and* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho} = \bigcup^{\mathtt{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$. □

Corollary 13.3. *If* $\boldsymbol{\pi}$ *is a small multiforcing,* $\mathfrak{M} \in \mathrm{HC}$*, and* $\boldsymbol{\varrho}$ *is an* $\mathfrak{M}$*-generic refinement of* $\boldsymbol{\pi}$ (exists by Lemma 7.2!)*, then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho}$.

Proof. We have $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho}$ by a combination of 7.3(v), 8.1, 8.5, and 11.1. □

14 Increasing sequences of multiforcings

Recall that $\mathbf{MF}$ is the collection of all multiforcings (Section 4). Let

$$\begin{aligned} \mathbf{sMF} &= \{\boldsymbol{\pi} \in \mathbf{MF} : \boldsymbol{\pi} \text{ is a } \textit{small} \text{ multiforcing}\}; \\ \mathbf{spMF} &= \{\boldsymbol{\pi} \in \mathbf{MF} : \boldsymbol{\pi} \text{ is a small and } \textit{special} \text{ multiforcing}\}. \end{aligned}$$

Thus a multiforcing $\boldsymbol{\pi} \in \mathbf{MF}$ belongs to $\mathbf{sMF}$ if $|\boldsymbol{\pi}| \subseteq \omega_1$ is (at most) countable and if $\xi \in |\boldsymbol{\pi}|$ then $\boldsymbol{\pi}(\xi)$ is a countable forcing in $\mathbf{AF}$, and $\boldsymbol{\pi} \in \mathbf{spMF}$ requires that in addition each $\boldsymbol{\pi}(\xi)$ is special (Definition 3.3).

Definition 14.1. If $\kappa \le \omega_1$ then let $\overrightarrow{\mathbf{MF}}_\kappa$ be the set of all $\sqsubseteq$-increasing sequences $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$ of multiforcings $\boldsymbol{\pi}_\alpha \in \mathbf{spMF}$, *domain-continuous* in the sense that if $\lambda < \kappa$ is a limit ordinal then $|\boldsymbol{\pi}_\lambda| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$. Let $\overrightarrow{\mathbf{MF}} = \bigcup_{\kappa<\omega_1} \overrightarrow{\mathbf{MF}}_\kappa$.

We order $\overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$ by the usual relations $\subseteq$ and $\subset$ of extension of sequences. Thus $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varrho}}$ iff $\kappa = \mathtt{dom}(\vec{\boldsymbol{\pi}}) < \lambda = \mathtt{dom}(\vec{\boldsymbol{\varrho}})$ and $\boldsymbol{\pi}_\alpha = \boldsymbol{\varrho}_\alpha$ for all $\alpha < \kappa$. In this case, if $\mathfrak{M}$ is any set, and $\boldsymbol{\varrho}_\kappa$ (the first term of $\vec{\boldsymbol{\varrho}}$ absent in $\vec{\boldsymbol{\pi}}$) satisfies $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varrho}_\kappa$, where $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$, then we write $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varrho}}$.

If $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_\kappa$ then let $\mathbf{MT}(\vec{\boldsymbol{\pi}}) = \mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$ (componentwise union). Accordingly, a $\vec{\boldsymbol{\pi}}$*-complete real name* means a $\boldsymbol{\pi}$-complete real name. □

Lemma 14.2. *If $\vec{\boldsymbol\pi}, \vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$, $\mathbf{c} \in \mathfrak{M}$ is a $\vec{\boldsymbol\pi}$-complete real name, and $\vec{\boldsymbol\pi} \subset_{\{\mathbf{c}\}} \vec{\boldsymbol\varphi}$, then $\mathbf{c}$ is a $\vec{\boldsymbol\varphi}$-complete real name.*

Proof. Let $\kappa = \mathtt{dom}(\vec{\boldsymbol\pi}) < \lambda = \mathtt{dom}(\vec{\boldsymbol\varphi})$ and $\boldsymbol\pi = \bigcup^{\mathtt{cw}} \vec{\boldsymbol\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \boldsymbol\pi_\alpha$. Then by definition $\boldsymbol\pi \sqsubset\!\!\sqsubset_{\{\mathbf{c}\}} \boldsymbol\varphi_\kappa$, hence $\boldsymbol\pi \sqsubset_{\mathbf{c}} \boldsymbol\varphi_\kappa$ because $\mathbf{c}$ is a $\boldsymbol\pi$-complete real name. However $\boldsymbol\pi \sqsubset_{\mathbf{c}} \boldsymbol\varphi = \bigcup^{\mathtt{cw}}_{\kappa\le\alpha<\lambda} \boldsymbol\varphi_\alpha$ by Lemma 10.3(iii). Therefore $\mathbf{c}$ is a $(\boldsymbol\pi \cup^{\mathtt{cw}} \boldsymbol\varphi)$-complete name by Lemma 10.3(i). However, $\boldsymbol\pi \cup^{\mathtt{cw}} \boldsymbol\varphi = \bigcup^{\mathtt{cw}}_{\alpha<\lambda} \boldsymbol\varphi_\alpha = \bigcup^{\mathtt{cw}} \vec{\boldsymbol\varphi}$. □

Definition 14.3. Let $\mathbf{ZFL}^-$ be the subtheory of **ZFC** including all axioms except for the power set axiom, plus the axiom of constructibility $\mathbf{V} = \mathbf{L}$, and plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1, HC, and generally sets related to the continuum, like 2^ω, **PT**, exist, too.) The axiom of choice is included in $\mathbf{ZFL}^-$ in the form of the wellorderability principle.

If $x \in \mathrm{HC}$ (HC= hereditarily countable sets, Footnote 7) then let $\mathfrak{L}(x)$ be the least *countable* transitive model (CTM) of $\mathbf{ZFL}^-$ containing x and satisfying $x \in (\mathrm{HC})^{\mathfrak{L}(x)}$. It necessarily has the form $\mathfrak{L}(x) = \mathbf{L}_\mu$ for some $\mu = \mu_x < \omega_1$.

An ordinal $\xi < \kappa$ is a *crucial ordinal* of a sequence $\vec{\boldsymbol\pi} = \langle \boldsymbol\pi_\alpha \rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}_\kappa$ if $(\bigcup^{\mathtt{cw}}_{\alpha<\xi} \boldsymbol\pi_\alpha) \sqsubset\!\!\sqsubset_{\mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)} \boldsymbol\pi_\xi$ holds. This is equivalent to $\vec{\boldsymbol\pi}\restriction\xi \subset_{\mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)} \vec{\boldsymbol\pi}$. □

Lemma 14.4. *Suppose that $\kappa \le \omega_1$ and $\vec{\boldsymbol\pi} = \langle \boldsymbol\pi_\alpha \rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}_\kappa$. Then:*

(i) $\boldsymbol\pi = \bigcup^{\mathtt{cw}} \vec{\boldsymbol\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \boldsymbol\pi_\alpha$ *is a regular multiforcing;*

(ii) *if $\kappa < \lambda \le \omega_1$ and $\mathfrak{M} \in \mathrm{HC}$ then there is a sequence $\vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$ satisfying $\mathtt{dom}(\vec{\boldsymbol\varphi}) = \lambda$ and $\vec{\boldsymbol\pi} \subset_{\mathfrak{M}} \vec{\boldsymbol\varphi}$;*

(iii) *if $\xi < \kappa$ is a crucial ordinal of $\vec{\boldsymbol\pi}$, $\boldsymbol\pi_{<\xi} = \bigcup^{\mathtt{cw}}_{\alpha<\xi} \boldsymbol\pi_\alpha$, $\boldsymbol\pi_{\ge\xi} = \bigcup^{\mathtt{cw}}_{\xi\le\beta<\kappa} \boldsymbol\pi_\beta$, then $\boldsymbol\pi_{<\xi} \sqsubset\!\!\sqsubset_{\mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)} \boldsymbol\pi_{\ge\xi}$ and $\boldsymbol\pi_{<\xi} \sqsubset\!\!\sqsubset_{\mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)} \boldsymbol\pi_\beta$ for $\xi \le \beta < \kappa$, hence*

  (a) $\mathbf{MT}(\boldsymbol\pi_{\ge\xi})$ *is open dense in* $\mathbf{MT}(\vec{\boldsymbol\pi})$,

  (b) *if $\boldsymbol D \in \mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)$, $\boldsymbol D \subseteq \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\xi)$, $\boldsymbol D$ is open dense in $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\xi)$, then $\boldsymbol D$ is pre-dense in $\mathbf{MT}(\boldsymbol\pi_{<\xi} \cup^{\mathtt{cw}} \boldsymbol\pi_{\ge\xi}) = \mathbf{MT}(\vec{\boldsymbol\pi})$.*

Proof. (i) Make use of Lemma 5.2(iv).

(ii) We define terms $\boldsymbol\varphi_\alpha$ of the sequence $\vec{\boldsymbol\varphi}$ required by induction.

Naturally put $\boldsymbol\varphi_\alpha = \boldsymbol\pi_\alpha$ for each $\alpha < \kappa$. To define the crucial term $\boldsymbol\varphi_\kappa$, we wlog assume that $\mathfrak{M}$ contains $\vec{\boldsymbol\pi}$ and satisfies $\kappa \subseteq \mathfrak{M}$ (otherwise take a bigger set). By Lemma 7.2, there is an $\mathfrak{M}$-generic refinement $\boldsymbol\pi'$ of $\boldsymbol\pi = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \boldsymbol\pi_\alpha$. By Theorem 7.3, $\boldsymbol\pi'$ is a small special multiforcing, $\boldsymbol\pi \sqsubset \boldsymbol\pi'$, and $\boldsymbol\pi_\alpha \sqsubset \boldsymbol\pi'$ for all $\alpha < \kappa$. In addition $\boldsymbol\pi \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol\pi'$ by Corollary 13.3. We let $\boldsymbol\varphi_\kappa = \boldsymbol\pi'$. The extended sequence $\vec{\boldsymbol\varphi}_+ = \langle \boldsymbol\varphi_\alpha \rangle_{\alpha<\kappa+1}$ belongs to $\overrightarrow{\mathbf{MF}}_{\kappa+1}$ and satisfies $\vec{\boldsymbol\pi} \subset_{\mathfrak{M}} \vec{\boldsymbol\varphi}_+$.

The following steps are pretty similar, except that we can take $\mathfrak{M} = \varnothing$.

To prove the main claim of (iii) make use of Corollary 13.2.

To prove (iii)(a) apply Corollary 6.1.

(iii)(b) As $\boldsymbol\pi_{<\xi} \sqsubset\!\!\sqsubset_{\mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)} \boldsymbol\pi_{\ge\xi}$ and $\boldsymbol D \in \mathfrak{L}(\vec{\boldsymbol\pi}\restriction\xi)$, we have $\boldsymbol\pi_{<\xi} \sqsubset_{\boldsymbol D} \boldsymbol\pi_{\ge\xi}$, therefore $\boldsymbol D$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol\pi})$ by Lemma 6.3(i). □

15 The key sequence

In this section we define the forcing notion to prove Theorem 1.2. It will have the form $\mathbf{MT}(\mathbb{\Pi})$, for a certain multiforcing $\mathbb{\Pi}$ with $|\mathbb{\Pi}| = \omega_1$. The multiforcing $\mathbb{\Pi}$ will be equal to the componentwise union of terms of a certain sequence $\vec{\mathbb{\Pi}} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$. The construction of this sequence in $\mathbf{L}$, the constructble universe, will employ some ideas related to diamond-style constructions, as well as to some sort of *definable genericity*. The following definition introduces another important notion involved in the construction.

Definition 15.1. A sequence $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ *blocks* a set W if either $\vec{\pi} \in W$ (*positive* block) or there is no sequence $\vec{\varphi} \in W$ extending $\vec{\pi}$ (*negative* block). □

Recall that HC = all hereditarily countable sets, Footnote 7.

Definition 15.2. We use standard notation Σ_n^{HC}, Π_n^{HC}, Δ_n^{HC} (slanted Σ, Π, Δ) for classes of *lightface* definability in HC (no parameters allowed), and $\mathbf{\Sigma}_n(\mathrm{HC})$, $\mathbf{\Pi}_n(\mathrm{HC})$, $\mathbf{\Delta}_n(\mathrm{HC})$ for *boldface* definability in HC (parameters in HC allowed). It is well-known that if $n \ge 1$ and $X \subseteq 2^\omega$ then

$$X \in \Sigma_n^{\mathrm{HC}} \iff X \in \Sigma_{n+1}^1, \quad \text{and} \quad X \in \mathbf{\Sigma}_n(\mathrm{HC}) \iff X \in \mathbf{\Sigma}_{n+1}^1,$$

and the same for Π, $\mathbf{\Pi}$, Δ, $\mathbf{\Delta}$. □

Theorem 15.3 (in $\mathbf{L}$). *Let $\mathtt{n} \ge 3$. There exists a sequence $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying the following requirements*:

(i) *the sequence $\vec{\mathbb{\Pi}}$ belongs to the definability class $\Delta_{\mathtt{n}-2}^{\mathrm{HC}}$* ;

(ii) $|\bigcup^{\mathsf{cw}} \vec{\mathbb{\Pi}}| = \omega_1$;

(iii) *if $\mathtt{n} \ge 4$ and $W \subseteq \overrightarrow{\mathbf{MF}}$ is a boldface $\mathbf{\Sigma}_{\mathtt{n}-3}(\mathrm{HC})$ set then there is an ordinal $\gamma < \omega_1$ such that the sequence $\vec{\mathbb{\Pi}} \restriction \gamma$ blocks W* ;

(iv) *there is a closed unbounded set $\mathbb{C} \subseteq \omega_1$ such that every $\gamma \in \mathbb{C}$ is a crucial ordinal for $\vec{\mathbb{\Pi}}$ in the sense of Definition 14.3.*

Proof. We argue under $\mathbf{V} = \mathbf{L}$. If $\mathtt{n} \ge 4$ then let $\mathbf{un}_{\mathtt{n}}(p, x)$ be a canonical universal $\Sigma_{\mathtt{n}-3}$ formula, so that the family of all boldface $\mathbf{\Sigma}_{\mathtt{n}-3}(\mathrm{HC})$ sets $X \subseteq \mathrm{HC}$ is equal to the family of all sets of the form $\Upsilon_{\mathtt{n}}(p) = \{x \in \mathrm{HC} : \mathrm{HC} \models \mathbf{un}_{\mathtt{n}}(p, x)\}$, $p \in \mathrm{HC}$.

Claim. *If $\mathtt{n} \ge 4$ then the set $\{\langle \vec{\pi}, p \rangle : \vec{\pi} \in \overrightarrow{\mathbf{MF}} \wedge p \in \mathrm{HC} \wedge \vec{\pi}$ blocks $\Upsilon_{\mathtt{n}}(p)\}$ is $\Delta_{\mathtt{n}-2}^{\mathrm{HC}}$.*

Claim. We skip a routine verification that $\overrightarrow{\mathbf{MF}}$ is Δ_1^{HC}. Further, if $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ and $p \in \mathrm{HC}$ then for $\vec{\pi}$ to block $\Upsilon_{\mathtt{n}}(p)$ it is necessary and sufficient that

$$\underbrace{\vec{\pi} \in \Upsilon_{\mathtt{n}}(p)}_{\Sigma_{\mathtt{n}-3}^{\mathrm{HC}}} \vee \underbrace{\neg \exists \vec{\varphi} \, (\underbrace{\vec{\varphi} \in \overrightarrow{\mathbf{MF}} \wedge \vec{\varphi} \text{ extends } \vec{\pi}}_{\Delta_1^{\mathrm{HC}}} \wedge \underbrace{\vec{\varphi} \in \Upsilon_{\mathtt{n}}(p)}_{\Sigma_{\mathtt{n}-3}^{\mathrm{HC}}})}_{\Pi_{\mathtt{n}-3}^{\mathrm{HC}}},$$

so this is a disjunction of $\Sigma_{\mathtt{n}-3}^{\mathrm{HC}}$ and $\Pi_{\mathtt{n}-3}^{\mathrm{HC}}$, hence, $\Delta_{\mathtt{n}-2}^{\mathrm{HC}}$. □ (Claim) □

For $\alpha < \omega_1$, define a sequence $\vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}$ by induction as follows.

We let $\vec{\pi}[0] = \varnothing$, the empty sequence.

Step $\alpha \to \alpha + 1$**.** Suppose that $\vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}$ is defined, $\kappa = \mathtt{dom}\,\vec{\pi}[\alpha]$, $\mathfrak{M} = \mathfrak{L}(\vec{\pi}[\alpha])$, and p_α is the α-th element of $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in the sense of the Goedel wellordering $\leqslant_{\mathbf{L}}$. By Lemma 14.4(ii), there is a sequence $\vec{\tau} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ satisfying $\vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\tau}$. By Corollary 7.4, there is a sequence $\vec{\varphi} \in \overrightarrow{\mathbf{MF}}_{\kappa+2}$ satisfying $\vec{\tau} \subset \vec{\varphi}$ and $\alpha \in |\vec{\varphi}(\kappa+1)|$. Finally if $\mathtt{n} \geq 4$ then there is a sequence $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ satisfying $\vec{\varphi} \subset \vec{\pi}$ and blocking the set $\Upsilon_{\mathtt{n}}(p_\alpha)$, while if $\mathtt{n} = 3$ then simply put $\vec{\pi} = \vec{\varphi}$. Thus overall we have:

$$(*)\quad \vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\pi},\ \kappa+1 < \mathtt{dom}\,\vec{\pi},\ \alpha \in |\vec{\varphi}(\kappa+1)|, \text{ and if } \mathtt{n} \geq 4 \text{ then } \vec{\pi} \text{ blocks } \Upsilon_{\mathtt{n}}(p_\alpha).$$

Let $\vec{\pi}[\alpha+1]$ be the $\leqslant_{\mathbf{L}}$-least one of sequences $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ satisfying $(*)$. Note that the axiom $\mathbf{V} = \mathbf{L}$ is a sine qua non of this construction since otherwise the $\leqslant_{\mathbf{L}}$-least choice of $\vec{\pi}[\alpha+1]$ would not be necessarily possible.

Limit step. If $\lambda < \omega_1$ is limit then we naturally define $\vec{\pi}[\lambda] = \bigcup_{\alpha<\lambda} \vec{\pi}[\alpha]$.

We have $\alpha < \beta \implies \vec{\pi}[\alpha] \subset \vec{\pi}[\beta]$ by construction, hence $\vec{\Pi} = \bigcup_\alpha \vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}_{\omega_1}$. To prove (i), note first of all that the relation $R(\vec{\pi}, \vec{\varphi}, \mathfrak{M}) :=$ "$\vec{\pi} \subset_{\mathfrak{M}} \vec{\varphi}$" is absolute for all transitive models of $\mathbf{ZFL}^-$, hence R is Δ_1^{HC}. Easily the assignment $\vec{\pi} \mapsto \mathfrak{L}(\vec{\pi})$ is Δ_1^{HC} as well. Finally "to block $\Upsilon_{\mathtt{n}}(p)$" is a $\Delta_{\mathtt{n}-2}^{\mathrm{HC}}$ relation by the claim above. Using these facts, a routine estimation shows that $(*)$ is a $\Delta_{\mathtt{n}-2}^{\mathrm{HC}}$ relation (in $\mathbf{L}$). On the other hand, it is known that, under $\mathbf{V} = \mathbf{L}$, choosing the $\leqslant_{\mathbf{L}}$-least element in every non-empty section of a Δ_k^{HC} set, $k \geq 1$, results in a set (transversal) of the same class Δ_k^{HC}. This completes the verification of (i).

To check (ii), note that $\alpha \in |\bigcup^{\mathtt{cw}} \vec{\pi}[\alpha+1]|$ by construction.

To check (iii) ($\mathtt{n} \geq 4$), note that any boldface $\mathbf{\Sigma}_{\mathtt{n}-3}(\mathrm{HC})$ set $W \subseteq \overrightarrow{\mathbf{MF}}$ is equal to $\Upsilon_{\mathtt{n}}(p_\alpha)$ for some $\alpha < \omega_1$, so $\gamma = \mathtt{dom}\,\vec{\pi}[\alpha+1]$ is as required.

(iv) The set $\mathbb{C} = \{\mathtt{dom}\,\vec{\pi}[\alpha] : \alpha < \omega_1\}$ is closed unbounded by the limit step of the construction. Moreover if $\gamma = \mathtt{dom}\,\vec{\pi}[\alpha] \in \mathbb{C}$ then $\vec{\Pi} \restriction \gamma = \vec{\pi}[\alpha]$, and hence γ is crucial for $\vec{\Pi}$ by construction. □

Blanket Assumption 15.4 (in $\mathbf{L}$). From now on we fix **a number** $\mathtt{n} \geq 3$ as in Theorem 1.2. We also **fix a sequence** $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying (i) – (iv) of Theorem 15.3 for this $\mathtt{n}$. We call this fixed $\vec{\Pi}$ *the key sequence*. □

Lemma 15.5. *If* $\mathtt{n} \geq 4$ *and* $W \subseteq \overrightarrow{\mathbf{MF}}$ *is a* $\mathbf{\Sigma}_{\mathtt{n}-3}(\mathrm{HC})$ *set dense in* $\overrightarrow{\mathbf{MF}}$ *then there is an ordinal* $\gamma < \omega_1$ *such that* $\vec{\Pi} \restriction \gamma \in W$*.*

Proof. By 15.4, $\vec{\Pi}$ satisfies (iii) of Theorem 15.3, hence there is an ordinal $\gamma < \omega_1$ such that $\vec{\Pi} \restriction \gamma$ blocks W. The negative block is impossible by the density of W, hence in fact $\vec{\Pi} \restriction \gamma \in W$. □

16 Key forcing notion

We continue to argue in $\mathbf{L}$, and we'll make use of the key sequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1}$ introduced by 15.4.

Definition 16.1 (in $\mathbf{L}$)**.** Define the multiforcings

$$\begin{array}{rclcll} \mathbb{\Pi} & = & \bigcup^{\mathtt{cw}}_{\alpha<\omega_1} \mathbb{\Pi}_\alpha & \in & \mathbf{MF}, & \\ \mathbb{\Pi}_{<\gamma} & = & \bigcup^{\mathtt{cw}}_{\alpha<\gamma} \mathbb{\Pi}_\alpha & \in & \mathbf{sMF}, & \text{for each } \gamma<\omega_1, \\ \mathbb{\Pi}_{\ge\gamma} & = & \bigcup^{\mathtt{cw}}_{\gamma\le\alpha<\omega_1} \mathbb{\Pi}_\alpha & \in & \mathbf{MF}, & \text{for each } \gamma<\omega_1. \end{array}$$

We further define $\mathbb{P} = \mathbf{MT}(\mathbb{\Pi}) = \mathbf{MT}(\vec{\mathbb{\Pi}})$, and, for all $\gamma < \omega_1$,

$$\mathbb{P}_{<\gamma} = \mathbf{MT}(\mathbb{\Pi}_{<\gamma}) = \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)\,, \quad \mathbb{P}_{\ge\gamma} = \mathbf{MT}(\mathbb{\Pi}_{\ge\gamma}) = \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction(\omega_1\smallsetminus\gamma))\,. \qquad \square$$

The set $\mathbb{P} = \mathbf{MT}(\mathbb{\Pi})$ will be our **key forcing notion**.

Corollary 16.2 (in $\mathbf{L}$, by 15.3(ii))**.** *$\mathbb{\Pi}$ is a regular multiforcing and $|\mathbb{\Pi}| = \omega_1$, thus $\mathbb{P} = \prod_{\xi<\omega_1} \mathbb{\Pi}(\xi)$ (with finite support).* $\square$

If $\xi < \omega_1$ then, following the corollary, let $\alpha(\xi) < \omega_1$ be the least ordinal α satisfying $\xi \in |\mathbb{\Pi}_\alpha|$. Thus a forcing $\mathbb{\Pi}_\alpha(\xi) \in \mathbf{AF}$ is defined whenever α satisfies $\alpha(\xi) \le \alpha < \omega_1$, and $\langle \mathbb{\Pi}_\alpha(\xi)\rangle_{\alpha(\xi)\le\alpha<\omega_1}$ is a $\sqsubset$-increasing sequence of special forcings in $\mathbf{AF}$. Note that $\mathbb{\Pi}(\xi) = \bigcup_{\alpha(\xi)\le\alpha<\omega_1} \mathbb{\Pi}_\alpha(\xi)$ by construction.

Corollary 16.3 (in $\mathbf{L}$)**.** *The sequence of ordinals $\langle\alpha(\xi)\rangle_{\xi<\omega_1}$ and the sequence of forcings $\langle \mathbb{\Pi}_\alpha(\xi)\rangle_{\xi<\omega_1,\,\alpha(\xi)\le\alpha<\omega_1}$ are $\varDelta^{\mathrm{HC}}_{\mathbb{n}-2}$.*

Proof. By construction the following double equivalence holds:

$$\begin{array}{rcl} \alpha < \alpha(\xi) & \Longleftrightarrow & \exists\,\boldsymbol{\pi}\,(\boldsymbol{\pi} = \mathbb{\Pi}_\alpha \wedge \xi \in \mathtt{dom}\,\boldsymbol{\pi}) \quad \Longleftrightarrow \\ & \Longleftrightarrow & \forall\,\boldsymbol{\pi}\,(\boldsymbol{\pi} = \mathbb{\Pi}_\alpha \Longrightarrow \xi \in \mathtt{dom}\,\boldsymbol{\pi}) \quad . \end{array}$$

However $\boldsymbol{\pi} = \mathbb{\Pi}_\alpha$ is a $\varDelta^{\mathrm{HC}}_{\mathbb{n}-2}$ relation by Theorem 15.3(i). It follows that so is the sequence $\langle\alpha(\xi)\rangle_{\xi<\omega_1}$. The second claim is similar. $\square$

Corollary 16.4 (in $\mathbf{L}$, of Lemma 5.2(iv))**.** *If $\xi < \omega_1$ and $\alpha(\xi) \le \alpha < \omega_1$ then the set $\mathbb{\Pi}_\alpha(\xi)$ is pre-dense in $\mathbb{\Pi}(\xi)$ and in $\mathbb{\Pi}$.* $\square$

In spite of Corollary 16.2, the sets $|\mathbb{\Pi}_{<\gamma}|$ can be quite arbitrary (countable) subsets of ω_1. However we get the next corollary:

Corollary 16.5 (in $\mathbf{L}$, of Corollary 16.2)**.** *$\mathbb{C}' = \{\gamma < \omega_1 : |\mathbb{\Pi}_{<\gamma}| = \gamma\}$ is closed unbounded in ω_1.* $\square$

To prove the CCC property, we'll need the following result.

Lemma 16.6 (in $\mathbf{L}$)**.** *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ then the set $\mathscr{O}_X$ of all ordinals $\gamma < \omega_1$, such that $\langle \mathbf{L}_\gamma\,; X\cap\mathbf{L}_\gamma\rangle$ is an elementary submodel of $\langle\mathbf{L}_{\omega_1}\,; X\rangle$ and $X\cap\mathbf{L}_\gamma \in \mathfrak{L}(\vec{\mathbb{\Pi}}\restriction\gamma)$, is stationary, hence unbounded in ω_1.*

More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set $\mathscr{O}$ of all ordinals $\gamma < \omega_1$, such that $\langle\mathbf{L}_\gamma\,; \langle X_n\cap\mathbf{L}_\gamma\rangle_{n<\omega}\rangle$ is an elementary submodel of $\langle\mathbf{L}_{\omega_1}\,; \langle X_n\rangle_{n<\omega}\rangle$ and $\langle X_n\cap\mathbf{L}_\gamma\rangle_{n<\omega} \in \mathfrak{L}(\vec{\mathbb{\Pi}}\restriction\gamma)$, is stationary, hence unbounded in ω_1.

Proof. Let $C \subseteq \omega_1$ be a closed unbounded set. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing $C, \omega_1, X, \vec{\sqcap}$, and such that $M \cap \mathbf{L}_{\omega_1}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and $\gamma = \phi(\omega_1)$. Then

$$\gamma < \lambda < \omega_1, \quad \phi(X) = X \cap \mathbf{L}_\gamma, \quad \phi(C) = C \cap \gamma, \quad \phi(\vec{\sqcap}) = \vec{\sqcap} \restriction \gamma$$

by the choice of M. It follows that $\langle \mathbf{L}_\gamma ; X \cap \mathbf{L}_\gamma, C \cap \gamma, \vec{\sqcap} \restriction \gamma \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1} ; X, C, \vec{\sqcap} \rangle$, so $\gamma \in \mathscr{O}_X$. Moreover, γ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. (See Definition 14.3 on models $\mathfrak{L}(\vec{\boldsymbol{\pi}}) \models \mathbf{ZFL}^-$.) We conclude that $X \cap \mathbf{L}_\gamma \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$ since $X \cap \mathbf{L}_\gamma \in \mathbf{L}_\lambda$ by construction. On the other hand, $C \cap \gamma$ is unbounded in γ by the elementarity, therefore $\gamma \in C$, as required.

The second, more general claim does not differ much. □

Corollary 16.7 (in $\mathbf{L}$). *The forcing $\mathbb{P}$ satisfies CCC. Therefore $\mathbb{P}$-generic extensions of $\mathbf{L}$ preserve cardinals.*

Proof. Suppose that $A \subseteq \mathbb{P} = \mathbf{MT}(\vec{\sqcap})$ is a maximal antichain. By 15.4 and Theorem 15.3(iv), there is a closed unbounded set $\mathbb{C} \subseteq \omega_1$ such that every $\gamma \in \mathbb{C}$ is a crucial ordinal for $\vec{\sqcap}$. By Lemma 16.6, there is an ordinal $\gamma \in \mathbb{C}$ such that $A' = A \cap \mathbb{P}_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma} = \mathbf{MT}(\vec{\sqcap} \restriction \gamma)$ and $A' \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. It follows that the set $D(A') = \{\boldsymbol{p} \in \mathbb{P}_{<\gamma} : \exists\, \boldsymbol{q} \in A\, (p \leqslant q)\} \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$ is open dense in $\mathbb{P}_{<\gamma}$.

Yet γ is a crucial ordinal for $\vec{\sqcap}$, therefore by Lemma 14.4(iii)(b) both the set $D(A')$, and hence A' itself as well, remain pre-dense in the whole set $\mathbb{P} = \mathbf{MT}(\vec{\sqcap})$. We conclude that $A = A'$ is countable. □

Corollary 16.8 (in $\mathbf{L}$). *If a set $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$ then there is an ordinal $\gamma < \omega_1$ such that $D \cap \mathbb{P}_{<\gamma}$ is already pre-dense in $\mathbb{P}$.*

Proof. We can assume that in fact D is dense. Let $A \subseteq D$ be a maximal antichain in D; then A is a maximal antichain in $\mathbb{P}$ because of the density of D. Then $A \subseteq \mathbb{P}_{<\gamma}$ for some $\gamma < \omega_1$ by Lemma 16.7. But A is pre-dense in $\mathbb{P}$. □

17 Basic generic extension

Recall that the key sequence $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha < \omega_1}$ of small special multiforcings $\sqcap_\alpha$ is defined in $\mathbf{L}$ by 15.4, the componentwise union $\sqcap = \bigcup^{\text{cw}}_{\alpha < \omega_1} \sqcap_\alpha$ is a multiforcing, $|\sqcap| = \omega_1$ in $\mathbf{L}$, and $\mathbb{P} = \mathbf{MT}(\vec{\sqcap}) = \mathbf{MT}(\sqcap) \in \mathbf{L}$ is our key forcing notion, equal to the finite-support product $\prod_{\xi < \omega_1} \sqcap(\xi)$ of arboreal forcings $\sqcap(\xi)$ in $\mathbf{L}$. See Section 16, where some properties of $\mathbb{P}$ are established, including CCC and definability of the factors $\sqcap(\xi)$ in $\mathbf{L}$. Our goal will be to show that certain submodels of $\mathbb{P}$-generic models prove Theorem 1.2.

Remark 17.1. From now on, we'll typically argue in $\mathbf{L}$ and in $\omega_1^{\mathbf{L}}$-preserving generic extensions $\mathbf{L}$ (this includes, *e.g.*, $\mathbb{P}$-generic extensions by Corollary 16.7). Thus it will always be the case that $\omega_1^{\mathbf{L}} = \omega_1$. This allows us to still think that $|\sqcap| = \omega_1$ (rather than $\omega_1^{\mathbf{L}}$). □

Definition 17.2. Let a set $G \subseteq \mathbb{P}$ be generic over the constructible set universe $\mathbf{L}$. If $\xi < \omega_1$ then following Remark 4.5, we

– define $G(\xi) = \{T^{\boldsymbol{p}}_\xi : \boldsymbol{p} \in G \wedge \xi \in |\boldsymbol{p}|\} \subseteq \sqcap(\xi)$;

– let $x_\xi = x_\xi[G] \in 2^\omega$ be the only real in $\bigcap_{T \in G(\xi)} [T]$.

– let $\boldsymbol{X} = \boldsymbol{X}[G] = \langle x_\xi[G] \rangle_{\xi<\omega_1} = \{\langle \xi, x_\xi[G] \rangle : \xi < \omega_1\}$..

Thus $\mathbb{P}$ adjoins an array $\boldsymbol{X}[G]$ of reals to $\mathbf{L}$, where each $x_\xi = x_\xi[G] \in 2^\omega \cap \mathbf{L}[G]$ is a $\sqcap(\xi)$-generic real over $\mathbf{L}$, and $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}[G]]$.

If $\Delta \subseteq \omega_1$ then let $\mathbb{P} \restriction \Delta = \mathbf{MT}(\sqcap \restriction \Delta) = \{\boldsymbol{p} \in \mathbb{P} : |\boldsymbol{p}| \subseteq \Delta\}$. □

The next lemma makes use of the product structure of $\mathbb{P}$.

Lemma 17.3. *Suppose that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1$. Then $\mathbb{P} = \mathbf{MT}(\sqcap)$ is equal to the product $(\mathbb{P} \restriction \Delta) \times (\mathbb{P} \restriction \Delta')$, where $\Delta' = \omega_1 \smallsetminus \Delta$. If $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$, then the set $G \restriction \Delta = \{\boldsymbol{p} \in G : |\boldsymbol{p}| \subseteq \Delta\}$ is $(\mathbb{P} \restriction \Delta)$-generic over $\mathbf{L}$.*

If $\xi < \omega_1$, $\xi \notin \Delta$, then $x_\xi[G] \notin \mathbf{L}[G \restriction \Delta]$. □

18 Definability of generic reals

Recall that the factors $\sqcap(\xi)$ of the forcing notion $\mathbb{P} = \mathbf{MT}(\sqcap) = \prod_{\xi<\omega_1} \sqcap(\xi)$ are defined by $\sqcap(\xi) = \bigcup_{\alpha(\xi) \le \alpha < \omega_1} \sqcap_\alpha(\xi)$, where $\alpha(\xi) < \omega_1$, and the sets $\sqcap_\alpha(\xi)$ are countable sets of perfect trees, whose definability in $\mathbf{L}$ is determined by Corollary 16.3. We'll freely use the notation introduced by Definition 17.2.

Theorem 18.1. *Assume that a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, $\xi < \omega_1$, and $x \in \mathbf{L}[G] \cap 2^\omega$. The following are equivalent:*

(1) $x = x_\xi[G]$;

(2) *x is $\sqcap(\xi)$-generic over $\mathbf{L}$*;

(3) $x \in \bigcap_{\alpha(\xi) \le \alpha < \omega_1} \bigcup_{T \in \sqcap_\alpha(\xi)} [T]$.

Proof. (1) $\Longrightarrow$ (2) is a routine (see Remark 4.5). To check (2) $\Longrightarrow$ (3) recall that each set $\sqcap_\alpha(\xi)$ is pre-dense in $\sqcap(\xi)$ by Lemma 5.2(iv). It remains to establish (3) $\Longrightarrow$ (1). Suppose that $x \in \mathbf{L}[G] \cap 2^\omega$ but (1) fails, that is, $x \ne x_\xi[G]$. By Theorem 12.2(i) there is a small ($\mathbb{P} = \mathbf{MT}(\sqcap)$ is CCC by 16.7) $\sqcap$-complete real name $\mathbf{c} \in \mathbf{L}$, such that $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$, $x = \mathbf{c}[G]$, and $\mathbf{c}$ is non-principal over $\sqcap$ at ξ, meaning that the set

$$\boldsymbol{D}^{\sqcap}_\xi(\mathbf{c}) = \{\boldsymbol{p} \in \mathbb{P} = \mathbf{MT}(\sqcap) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_\xi]\}\,.$$

is open dense in $\mathbb{P} = \mathbf{MT}(\sqcap)$. By the smallness of $\mathbf{c}$, there is an ordinal $\gamma < \omega_1$ such that $\mathbf{c}$ is a $\sqcap_{<\gamma}$-complete real name, and we can assume, by Corollary 16.8, that $\boldsymbol{D}^{\sqcap}_{\xi k}(\mathbf{c}) \cap \mathbb{P}_{<\gamma}$ is pre-dense in $\mathbb{P}$, therefore, open dense in $\mathbb{P}_{<\gamma}$ — and then $\mathbf{c}$ is non-principal over $\sqcap_{<\gamma}$ at ξ. We can further assume that $\mathbf{c} \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. And finally, we can assume that γ belongs to the set $\mathbb{C}$ of Theorem 15.3(iv), in other words, γ is crucial for $\vec{\boldsymbol{\pi}}$, that is, $\sqcap_{<\gamma} \sqsubset_{\mathfrak{L}(\vec{\sqcap} \restriction \gamma)} \sqcap_\gamma$. It follows that $\sqcap_{<\gamma} \sqsubset_{\mathfrak{L}(\vec{\sqcap} \restriction \gamma)} \sqcap_{\ge\gamma}$ by Lemma 14.4(iii).

Then $\mathbb{\Pi}_{<\gamma} \sqsubset^{\mathbf{c}}_{\xi} \mathbb{\Pi}_{\geq\gamma}$ holds as well by 13.1(4), since $\mathbf{c} \in \mathfrak{L}(\overrightarrow{\mathbb{\Pi}}\restriction\gamma)$ and because of the non-principality of $\mathbf{c}$. Now Theorem 12.2(ii) with $\boldsymbol{\pi} = \mathbb{\Pi}_{<\gamma}$ and $\boldsymbol{\varphi} = \mathbb{\Pi}_{\geq\gamma}$ (note that $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varphi} = \mathbb{\Pi}$) implies $x = \mathbf{c}[G] \notin \bigcup_{Q\in\mathbb{\Pi}_{\geq\gamma}(\xi)}[Q]$, in particular, $x \notin \bigcup_{Q\in\mathbb{\Pi}_{\gamma}(\xi)}[Q]$. In other words, (3) fails as well. □

Corollary 18.2. *Assume that $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, and M is a generic extension of $\mathbf{L}$ satisfying $2^\omega \cap M \subseteq \mathbf{L}[G]$. Then $\boldsymbol{X}[G] \cap M$ is a set of definability class $\Pi^{\mathrm{HC}}_{\mathtt{n}-2}$ in M.*

Recall that $\boldsymbol{X}[G]$ was introduced in Definition 17.2.

Proof. By the theorem, it holds in M that $\langle\xi, x\rangle \in \boldsymbol{X}[G]$ iff

$$\forall\,\alpha < \omega_1\,\exists\,T \in \mathbb{\Pi}_\alpha(\xi)\big(\alpha(\xi) \leq \alpha \Longrightarrow x \in [T]\big),$$

which can be re-written as

$$\forall\,\alpha < \omega_1\,\forall\,\mu < \omega_1\,\forall\,Y\,\exists\,T \in Y\big(\mu = \alpha(\xi) \wedge Y = \mathbb{\Pi}_\alpha(\xi) \wedge \mu \leq \alpha \Longrightarrow x \in [T]\big).$$

Here the equality $\mu = \alpha(\xi)$ is $\Delta^{\mathrm{HC}}_{\mathtt{n}-2}$ by Corollary 16.3, and so is the equality $Y = \mathbb{\Pi}_\alpha(\xi)$ by Corollary 16.3. It follows that the whole relation is $\Pi^{\mathrm{HC}}_{\mathtt{n}-2}$, since the quantifier $\exists\,T \in Y$ is bounded. □

Corollary 18.3. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then it holds in $\mathbf{L}[G]$ that there is a "good" $\Delta^1_{\mathtt{n}}$ wellordering of 2^ω of length ω_1.*

Proof. If $\gamma < \omega_1$ then let $\boldsymbol{X}_\gamma = \langle x_\xi[G]\rangle_{\xi<\gamma}$. The equality $Y = \boldsymbol{X}_\gamma$ is a $\Pi^{\mathrm{HC}}_{\mathtt{n}-2}$ relation in $\mathbf{L}[G]$ (with γ, Y as arguments) by Corollary 18.2. If $x \in 2^\omega \cap \mathbf{L}[G]$ then let $\gamma(x)$ be the least $\gamma < \omega_1$ such that $x \in \mathbf{L}[\boldsymbol{X}_\gamma]$, and $\nu(x) < \omega_1$ be the index of x in the canonical wellordering of 2^ω in $\mathbf{L}[\boldsymbol{X}_\gamma]$. We wellorder $2^\omega \cap \mathbf{L}[G]$ according to the lexicographical ordering of the triples $\langle \mathtt{max}\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x)\rangle$. This is $\Delta^{\mathrm{HC}}_{\mathtt{n}-1}$ by the above, hence $\Delta^1_{\mathtt{n}}$. The "goodness" (that is, the set of all coded proper initial segments has to be $\Sigma^1_{\mathtt{n}}$) can be easily verified. □

19 The non-separation model

The model for Theorem 1.2 will be defined on the base of a $\mathbb{P}$-generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. More exactly, it will have the form $\mathbf{L}[G\restriction\Delta]$, where $\Delta \subseteq \omega_1^{\mathbf{L}}$ will itself be a generic set over $\mathbf{L}[G]$.

Let $\mathbb{Q} = \{1,2,3\}^{\omega_1^{\mathbf{L}}} \cap \mathbf{L}$ with *countable* support; a typical element of $\mathbb{Q}$ is a partial map $q \in \mathbf{L}$ from $\omega_1^{\mathbf{L}}$ to the 3-element set $\{1,2,3\}$, with a domain $\mathtt{dom}\,q \subseteq \omega_1^{\mathbf{L}}$ countable in $\mathbf{L}$, that is, just bounded in $\omega_1^{\mathbf{L}}$. (The choice of the 3-element set $\{1,2,3\}$ is explained by later considerations, see Definition 19.3.) We order $\mathbb{Q}$ opposite to extension, that is, let $q \leqslant q'$ (q is stronger) iff $q' \subseteq q$. Thus $\mathbb{Q} \in \mathbf{L}$, and, inside $\mathbf{L}$, $\mathbb{Q}$ is equal to the product $\{1,2,3\}^{\omega_1}$ with countable support. Accordingly a $\mathbb{Q}$-generic object is a full $\mathbb{Q}$-generic map $H : \omega_1^{\mathbf{L}} \to \{1,2,3\}$.

Recall that $\mathbb{P}$ is a CCC forcing in $\mathbf{L}$ by Corollary 16.7.

Lemma 19.1. *$\mathbb{P}$ remains CCC in any $\mathbb{Q}$-generic extension $\mathbf{L}[H]$ of $\mathbf{L}$, therefore $\mathbb{P}\times\mathbb{Q}$ preserves cardinals over $\mathbf{L}$.*

Proof. Suppose towards the contrary that some $q' \in \mathbb{Q}$ forces that C is an uncountable antichain in $\mathbb{P}$, C being a $\mathbb{Q}$-name. Note that, in $\mathbf{L}$, $\mathbb{Q}$ is *countably complete*: if $q_0 \geqslant q_1 \geqslant q_2 \geqslant \dots$ is a sequence in $\mathbb{Q}$ then there is a condition $q = \bigcup_k q_k \in \mathbb{Q}$; $q \leqslant q_k$, $\forall\, k$. Therefore, *arguing in* $\mathbf{L}$, we can define by induction a decreasing sequence $\langle q_\xi \rangle_{\xi<\omega_1}$ in $\mathbb{Q}$ and a sequence of pairwise incompatible conditions $p_\xi \in \mathbb{P}$, such that $q_0 \leqslant q'$ and each q_ξ forces that $p_\xi \in C$. But then $A = \{p_\xi : \xi < \omega_1\} \in \mathbf{L}$ is an uncountable antichain in $\mathbb{P}$, a contradiction. □

Lemma 19.2. *Assume that a set $G \times H$ is $\mathbb{P}\times\mathbb{Q}$-generic over $\mathbf{L}$. Then*

(i) $2^\omega \cap \mathbf{L}[G,H] \subseteq \mathbf{L}[G]$, *hence* $\omega_1^{\mathbf{L}} = \omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}[G,H]}$;

(ii) *if* $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}}$ *then* $\mathbf{L}[G \restriction \Delta, H] \cap 2^\omega \subseteq \mathbf{L}[G\restriction\Delta]$;

(iii) *if* $\Delta \in \mathbf{L}[H]$, $\Delta \subseteq \omega_1^{\mathbf{L}}$, *and* $\xi < \omega_1^{\mathbf{L}}$ *then* $x_\xi[G] \in \mathbf{L}[G\restriction\Delta]$ *iff* $\xi \in \Delta$.

Proof. Note that $\mathbb{Q}$ may not be countably complete in $\mathbf{L}[G]$ any more, so that the most elementary way to prove (i) does not work. However consider $\mathbf{L}[G,H]$ as a $\mathbb{P}$-generic extension $\mathbf{L}[H][G]$ of $\mathbf{L}[H]$. Let $x \in 2^\omega \cap \mathbf{L}[H][G]$. As $\mathbb{P} = \mathbf{MT}(\sqcap)$ is CCC in $\mathbf{L}[H]$ by Lemma 19.1, there exists a small $\sqcap$-complete real name $\mathbf{c} \in \mathbf{L}[H]$, such that $\mathbf{c} \subseteq \mathbb{P}\times\omega\times 2$ and $x = \mathbf{c}[G]$. Because of the smallness, $\mathbf{c}$ is effectively coded by a real, hence $\mathbf{c} \in \mathbf{L}$ because $\mathbf{L}[H]$ has just the same reals as $\mathbf{L}$. Thus $\mathbf{c} \in \mathbf{L}$ and $x = \mathbf{c}[G] \in \mathbf{L}[G]$.

The proof of (ii) is similar.

(iii) In the nontrivial direction, suppose that $\xi \notin \Delta$. Consider the set $\Delta' = \omega_1^{\mathbf{L}} \smallsetminus \{\xi\} \in \mathbf{L}$. As obviously $G\restriction\Delta \in \mathbf{L}[G\restriction\Delta', H]$, any real in $\mathbf{L}[G\restriction\Delta]$ belongs to $\mathbf{L}[G\restriction\Delta']$ by (ii). But $x_\xi[G] \notin \mathbf{L}[G\restriction\Delta']$ by Lemma 17.3. □

Recall that if $\nu \in \mathbf{Ord}$ then the ordinal product 2ν is considered as the ordered sum of ν copies of $2 = \{0,1\}$. (Contrary to $\nu 2 = \nu + \nu$.) Thus if $\nu = \lambda + m$, where λ is a limit ordinal or 0 and $m < \omega$, then $2\nu = \lambda + 2m$ and $2\nu + 1 = \lambda + 2m + 1$, and $\langle \nu, i \rangle \mapsto 2\nu + i$ is a bijection of $\omega_1 \times 2$ onto ω_1.

The next definition uses the 6-fold splitting construction of Harrington [18, Part 2].

Definition 19.3. If $H : \omega_1^{\mathbf{L}} \to \{1,2,3\}$ then define sets

$$\mathbb{1}_H = \{2\nu : H(2\nu) = 1\}\,, \quad \mathbb{2}_H = \{2\nu : H(2\nu) = 2\}\,, \quad \mathbb{3}_H = \{2\nu : H(2\nu) = 3\}\,,$$

$$\mathbb{4}_H = \{2\nu + 1 : H(2\nu+1) = 1\}\,, \quad \mathbb{5}_H = \{2\nu + 1 : H(2\nu+1) = 2\}\,,$$

and $\mathbb{6}_H = \{2\nu + 1 : H(2\nu + 1) = 3\}$, and further

$$\begin{aligned} \Delta_H \;=\; & \{4\nu : 2\nu \in \mathbb{1}_H \cup \mathbb{3}_H\} \cup \{4\nu + 1 : 2\nu \in \mathbb{2}_H \cup \mathbb{3}_H\} \cup \\ & \cup \{4\nu + 2 : 2\nu + 1 \in \mathbb{4}_H\} \cup \{4\nu + 3 : 2\nu + 1 \in \mathbb{5}_H\}\,. \end{aligned}$$ □

Note that $\mathbf{L}[G\restriction\Delta_H] \subseteq \mathbf{L}[G]$ is not necessarily true since the set Δ_H does not necessarily belong to $\mathbf{L}[G]$, but we have $\mathbf{L}[G\restriction\Delta_H] \subseteq \mathbf{L}[G][H]$, of course.

20 Non-separation theorem: the HC version

Now we prove the following result, the HC-definability version of Theorem 1.2.

Theorem 20.1. *Let a set $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathbf{L}$ and $H : \omega_1^{\mathbf{L}} \to \{1,2,3\}$ be a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$. Then it is true in $\mathbf{L}[G \upharpoonright \Delta_H]$ that*

(i) $\mathbb{1}_H$, $\mathbb{2}_H$ *are disjoint* $\mathit{\Pi}^{\mathrm{HC}}_{\mathbb{n}-1}$ *sets, not separable by disjoint* $\mathbf{\Sigma}_{\mathbb{n}-1}(\mathrm{HC})$ *sets;*

(ii) $\mathbb{4}_H$, $\mathbb{5}_H$ *are disjoint* $\mathit{\Sigma}^{\mathrm{HC}}_{\mathbb{n}-1}$ *sets, not separable by disjoint* $\mathbf{\Pi}_{\mathbb{n}-1}(\mathrm{HC})$ *sets.*

The proof of Theorem 20.1 below in this Section includes a reference to the following result, which will have its own lengthy proof in the remainder.

Theorem 20.2 (will be proved in Section 26)**.** *Assume that $X \in \mathbf{L}$, $X \subseteq \omega_1^{\mathbf{L}}$ is unbounded in $\omega_1^{\mathbf{L}}$, and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then $\mathbf{L}[G \upharpoonright X] \cap 2^\omega$ is an elementary submodel of $\mathbf{L}[G] \cap 2^\omega$ w.r.t. all $\mathit{\Sigma}^1_{\mathbb{n}-1}$ formulas with real parameters in $\mathbf{L}[G \upharpoonright X]$.*

Corollary 20.3. *Under the assumptions of Theorem 20.1,* $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]}$ *is an elementary submodel of* $\mathrm{HC}^{\mathbf{L}[G]}$ *w.r.t. all* $\mathit{\Sigma}_{\mathbb{n}-2}$ *formulas.*

Note that $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]} \subseteq \mathrm{HC}^{\mathbf{L}[G]}$ by Lemma 19.2, while $\mathbf{L}[G \upharpoonright \Delta_H] \not\subseteq \mathbf{L}[G]$.

Corollary. We have $\omega_1^{\mathbf{L}} = \omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}[G \upharpoonright \Delta_H]}$ and $\Delta_H \cap \lambda \in \mathbf{L}$ for any $\lambda < \omega_1^{\mathbf{L}}$ by Lemma 19.2. It remains to cite theorem 20.2, having in mind that $\mathit{\Sigma}^{\mathrm{HC}}_{\mathbb{n}-2}$-definability corresponds to $\mathit{\Sigma}^1_{\mathbb{n}-1}$-definability.

□ (Corollary 20.3 from Theorem 20.2) □

Theorem 20.1. (i) To check that, say, $\mathbb{1}_H$ is $\mathit{\Pi}^{\mathrm{HC}}_{\mathbb{n}-1}$ in $\mathbf{L}[G \upharpoonright \Delta_H]$, it suffices to prove that the equality

$$\mathbb{1}_H = \{2\nu < \omega_1 : \neg \exists x \, (\langle 4\nu+1, x\rangle \in \boldsymbol{X})\}$$

holds in $\mathbf{L}[G \upharpoonright \Delta_H]$, where $\boldsymbol{X} = \boldsymbol{X}[G] \cap \mathbf{L}[G \upharpoonright \Delta_H]$ is a $\mathit{\Pi}^{\mathrm{HC}}_{\mathbb{n}-2}$ set in $\mathbf{L}[G \upharpoonright \Delta_H]$ by Corollary 18.2. (For $\mathbb{2}_H$ it would be $\langle 4\nu, x\rangle \in \boldsymbol{X}$ in the displayed formula.)

First suppose that $\nu < \omega_1^{\mathbf{L}}$, $\xi = 4\nu+1$, $x \in \mathbf{L}[G \upharpoonright \Delta_H] \cap 2^\omega$, and $\langle \xi, x\rangle \in \boldsymbol{X}$; prove that $2\nu \notin \mathbb{1}_H$. Now, by definition $x = x_\xi[G]$, and $\xi \in \Delta_H$ by Lemma 19.2(iii). But then $2\nu \in \mathbb{2}_H \cup \mathbb{3}_H$, so $2\nu \notin \mathbb{1}_H$, as required.

To prove the converse, let $2\nu \notin \mathbb{1}_H$, so that $2\nu \in \mathbb{2}_H \cup \mathbb{3}_H$. Then $\xi = 4\nu+1 \in \Delta_H$, and hence $x = x_\xi \in \mathbf{L}[G \upharpoonright \Delta_H]$ and $\langle \xi, x\rangle = \langle 4\nu+1, x\rangle \in \boldsymbol{X}$, as required.

To prove the non-separability claim, suppose towards the contrary that, in $\mathbf{L}[G \upharpoonright \Delta_H]$, the sets $\mathbb{1}_H, \mathbb{2}_H$ are separated by disjoint $\mathbf{\Sigma}_{\mathbb{n}-1}(\mathrm{HC})$ sets $A, B \subseteq \omega_1 = \omega_1^{\mathbf{L}}$. The sets A, B are defined, in the set $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]}$, by $\mathit{\Sigma}_{\mathbb{n}-1}$ formulas, resp., $\varphi(a,\xi)$, $\psi(a,\xi)$, with a real parameter $a \in \mathbf{L}[G \upharpoonright \Delta_H] \cap 2^\omega$; hence, $a \in \mathbf{L}[G]$ by Lemma 19.2. Let $\lambda < \omega_1^{\mathbf{L}}$ be a limit ordinal such that $a \in \mathbf{L}[G \upharpoonright \Delta_{H\lambda}]$, where $\Delta_{H\lambda} = \Delta_H \cap \lambda \in \mathbf{L}$.

If $K : \omega_1^{\mathbf{L}} \to \{1,2,3\}$ (for instance, $K = H$), then let

$$A^*_K = \{\xi < \omega_1^{\mathbf{L}} : \varphi(a,\xi)^{\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_K]}}\}, \quad B^*_K = \{\xi < \omega_1^{\mathbf{L}} : \psi(a,\xi)^{\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_K]}}\}. \qquad (*)$$

Then by definition $\mathbb{1}_H \subseteq A = A^*_H$, $\mathbb{2}_H \subseteq B = B^*_H$, and $A^*_H \cap B^*_H = \varnothing$. Fix a condition $q_0 \in \mathbb{Q}$ compatible with H (here meaning that simply $q_0 \subset H$), which forces the mentioned properties of A, B, so that,

(†) if $K : \omega_1^{\mathbf{L}} \to \{1, 2, 3\}$ is a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$ and compatible with q_0, then $\mathbb{1}_K \subseteq A^*_K$, $\mathbb{2}_K \subseteq B^*_K$, and $A^*_K \cap B^*_K = \varnothing$.

We may assume that $\mathtt{dom}\, q_0 \subseteq \lambda$, otherwise just increase λ.

Let ν_0 be any ordinal, $\lambda \le \nu_0 < \omega_1$. Consider the maps $H_1, H_2, H_3 : \omega_1^{\mathbf{L}} \to \{1, 2, 3\}$, generic over $\mathbf{L}[G]$, compatible with q_0, and satisfying $H_i(2\nu_0) = i$, $i = 1, 2, 3$, and $H_1(\alpha) = H_2(\alpha) = H_3(\alpha)$ for all $\alpha \ne 2\nu_0$. Then $\Delta_{H_3} = \Delta_{H_1} \cup \{4\nu_0 + 1\}$ by Definition 19.3, hence, $\mathbf{L}[G \restriction \Delta_{H_1}] \subseteq \mathbf{L}[G \restriction \Delta_{H_3}]$. It follows by Corollary 20.3 that $A^*_{H_1} \subseteq A^*_{H_3}$. Therefore $\mathbb{1}_{H_1} \subseteq A^*_{H_1} \subseteq A^*_{H_3}$ by (†). We conclude that $2\nu_0 \in A^*_{H_3}$, just because $2\nu_0 \in \mathbb{1}_{H_1}$ by the choice of H_1.

And we have $2\nu_0 \in B^*_{H_3}$ by a similar argument (with H_2). Thus $A^*_{H_3} \cap B^*_{H_3} \ne \varnothing$, contrary to (†). The contradiction ends the proof of (i).

The proof of 20.1(ii) is pretty similar.

□ (Theorem 20.1 modulo Theorem 20.2) □

21 The main theorem modulo theorem 20.2

Theorem 1.2 modulo theorem 20.2. (i) We argue under the assumptions of Theorem 20.1. To define a non-separable pair of $\varPi^1_{\mathfrak{n}}$ sets in $\mathbf{L}[G \restriction \Delta_H]$, let $\mathbf{WO} \subseteq 2^\omega$ be the $\varPi^1_1$ set of codes of countable ordinals, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w. As $\omega_1^{\mathbf{L}} = \omega_1$ by Corollary 16.7, for any $\xi < \omega_1$ there is a code $w \in \mathbf{WO} \cap \mathbf{L}$ with $|w| = \xi$. Let w_ξ be the $\leqslant_{\mathbf{L}}$-least of those, and $X = \{w_\xi : \xi \in \mathbb{1}_H\}$, $Y = \{w_\xi : \xi \in \mathbb{2}_H\}$.

The sets $X, Y \subseteq \mathbf{WO} \cap \mathbf{L}$ are $\varPi^{\mathrm{HC}}_{\mathfrak{n}-1}$ in $\mathbf{L}[G \restriction \Delta_H]$ together with $\mathbb{1}_H$ and $\mathbb{2}_H$, and hence $\varPi^1_{\mathfrak{n}}$, and $X \cap Y = \varnothing$. Suppose to the contrary that, in $\mathbf{L}[G \restriction \Delta_H]$, $X', Y' \subseteq 2^\omega$ are disjoint sets in $\mathbf{\Sigma}^1_{\mathfrak{n}}$, hence $\mathbf{\Sigma}_{\mathfrak{n}-1}(\mathrm{HC})$, such that $X \subseteq X'$ and $Y \subseteq Y'$. Then, in $\mathbf{L}[G \restriction \Delta_H]$,

$$A = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in X'\} \quad \text{and} \quad B = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in Y'\}$$

are disjoint sets in $\mathbf{\Sigma}_{\mathfrak{n}-1}(\mathrm{HC})$, and we have $\mathbb{1}_H \subseteq A$ and $\mathbb{2}_H \subseteq B$ by construction, contrary to Theorem 20.1. The contradiction ends the proof of (i). The proof of 1.2(ii) is pretty similar.

□ (Theorem 1.2 modulo Theorem 20.2) □

22 Auxiliary forcing relation

Here we begin a lengthy proof of Theorem 20.2. It involves an auxiliary forcing relation, not explicitly connected with any particular forcing notion, in particular, with the key forcing $\mathbb{P}$.

Blanket Assumption 22.1. We'll assume that $\mathfrak{n} \ge 4$, since if $\mathfrak{n} = 3$ then Theorem 20.2 holds by the Shoenfield absoluteness. □

We argue in L. Consider 2nd order arithmetic language, with variables $k, l, m, n, \dots$ of type 0 over ω and variables $a, b, x, y, \dots$ of type 1 over 2^ω, whose atomic formulas are those of the form $x(k) = n$. Let $\mathscr{L}$ be the extension of this language, which allows to substitute variables of type 0 with natural numbers and variables of type 1 with **small real names** (Definition 9.1) $\mathbf{c} \in \mathbf{L}$.

We define natural classes $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ $(n \ge 1)$ of $\mathscr{L}$-formulas. Let $\mathscr{L}(\Sigma\Pi)^1_1$ be the closure of $\mathscr{L}\Sigma^1_1 \cup \mathscr{L}\Pi^1_1$ under $\neg, \wedge, \vee$ and quantifiers over ω. If φ is a formula in $\mathscr{L}\Sigma^1_n$ (resp., $\mathscr{L}\Pi^1_n$), then let φ^- be the result of canonical transformation of $\neg\,\varphi$ to the $\mathscr{L}\Pi^1_n$ (resp., $\mathscr{L}\Sigma^1_n$) form.

Now we define a relation $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ between multitrees $\boldsymbol{p}$, sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, and closed $\mathscr{L}$-formulas φ in $\mathscr{L}(\Sigma\Pi)^1_1$ or $\mathscr{L}\Sigma^1_n \cup \mathscr{L}\Pi^1_n$, $n \ge 2$, which will suitably approximate the true $\mathbb{P}$-forcing relation. The definition goes on by induction on the complexity of φ.

1°. Let $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}$ (not necessarily $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$), and φ is a closed $\mathscr{L}(\Sigma\Pi)^1_1$ formula. We define $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ iff there is a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ (recall Definition 14.3 on $\mathbf{ZFL}^-$), an ordinal $\vartheta < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$, and a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$, such that

(1) $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ (meaning: $\boldsymbol{p}_0$ is weaker),

(2) $\mathfrak{M}$ contains $\vec{\boldsymbol{\pi}} \restriction \vartheta$ (then contains $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ and $\boldsymbol{p}_0$ as well),

(3) every name $\mathbf{c}$ in φ belongs to $\mathfrak{M}$ and is $\vec{\boldsymbol{\pi}} \restriction \vartheta$-complete,

(4) $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\pi}}$ — therefore $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\{\mathbf{c}\}} \vec{\boldsymbol{\pi}}$ for any name $\mathbf{c}$ in φ, and

(5) $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the usual sense.[11]

2°. If $\varphi(x)$ is a $\mathscr{L}\Pi^1_n$ formula, $n \ge 1$, then we define $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \exists\, x\, \varphi(x)$ iff there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi(\mathbf{c})$.

3°. If φ is a closed $\mathscr{L}\Pi^1_n$ formula, $n \ge 2$, then we define $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ iff there is no sequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ and multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\tau}})$ such that $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\tau}}$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol{\tau}}}\ \varphi^-$.

Remark 22.2. The condition "$\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$" in 1° does not depend on the choice of a CTM $\mathfrak{M}$ containing $\vec{\boldsymbol{\pi}} \restriction \vartheta$ and φ, since if φ is $\mathscr{L}(\Sigma\Pi)^1_1$ then all transitive models agree on the formula $\varphi[G]$ by the Mostowski absoluteness theorem [20, Theorem 25.4]. □

Lemma 22.3. *Assume that sequences $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}}$ belong to $\overrightarrow{\mathbf{MF}}$, $\boldsymbol{q}, \boldsymbol{p} \in \mathbf{MT}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, φ is an $\mathscr{L}$-formula. Then $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ implies $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\ \varphi$.*

Proof. If φ is a $\mathscr{L}(\Sigma\Pi)^1_1$ formula, $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, and this is witnessed by $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ as in 1°, then the exactly same $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ witness $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\ \varphi$.

The induction step $\exists$, as in 2°, is pretty elementary.

Now the induction step $\forall$, as in 3°. Let φ be a closed $\mathscr{L}\Pi^1_n$-formula, $n \ge 2$, and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$. Assume that $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\ \varphi$ fails. Then by 3° there exist: a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\ \varphi^-$. But then $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}}'$ and $\boldsymbol{q}' \leqslant \boldsymbol{p}$, hence $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ fails by 3°. □

[11] Item 1° not only requires $\varphi[\underline{G}]$ to be forced, but also suitably seals this status by $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\pi}}$. This will help us to prove the consistency of $\mathtt{forc}$ in Lemma 22.7.

Definition 22.4. If K is one of the classes $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ ($n \ge 2$), then let $\mathbf{FORC}[K]$ consist of all triples $\langle \vec{\boldsymbol\pi}, \boldsymbol p, \varphi\rangle$ such that $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi$. □

Then $\mathbf{FORC}[K]$ is a subset of HC.

Lemma 22.5 (definability, in $\mathbf{L}$). *$\mathbf{FORC}[\mathscr{L}(\Sigma\Pi)^1_1] \in \Delta^{\mathrm{HC}}_1$. If $n \ge 2$ then $\mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ belongs to $\Sigma^{\mathrm{HC}}_{n-1}$ and $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$ belongs to Π^{HC}_{n-1}.*

Proof. Relations like $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, "being a formula in $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$", $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\rho})$, forcing over a CTM, etc. are definable in HC by bounded formulas, hence Δ^{HC}_1. On the top of this, the model $\mathfrak{M}$ can be tied by both $\exists$ and $\forall$ in 1°, see Remark 22.2. This wraps up the Δ^{HC}_1 estimation for $\mathscr{L}(\Sigma\Pi)^1_1$.

The inductive step by 2° is quite simple.

Now the step by 3°. Assume that $n \ge 2$, and it is already established that $\mathbf{FORC}[\mathscr{L}\Sigma^1_n] \in \Sigma^{\mathrm{HC}}_{n-1}$. Then $\langle \vec{\boldsymbol\pi}, \boldsymbol p, \varphi\rangle \in \mathbf{FORC}[\mathscr{L}\Pi^1_n]$ iff $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol p \in \mathbf{MT}$, φ is a closed $\mathscr{L}\Pi^1_n$ formula, and, by 3°, there exist no triple $\langle \vec{\boldsymbol\tau}, \boldsymbol p', \psi\rangle \in \mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ such that $\vec{\boldsymbol\tau} \in \overrightarrow{\mathbf{MF}}$, $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\tau}$, $\boldsymbol p' \in \mathbf{MT}(\vec{\boldsymbol\tau})$, $\boldsymbol p' \leqslant \boldsymbol p$, and ψ is φ^-. We easily get the required estimation Π^{HC}_{n-1} of $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$. □

Lemma 22.6 (in $\mathbf{L}$). *Let $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\pi})$, φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$.*

(i) *If $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$, $\mathfrak{N} \models \mathbf{ZFL}^-$ is a TM containing $\vec{\boldsymbol\varphi}$ and φ, and $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi$, then $\boldsymbol p$ $\mathbf{MT}(\vec{\boldsymbol\varphi})$-forces $\varphi[\underline{G}]$ over $\mathfrak{N}$ in the usual sense.*

(ii) *If a TM $\mathfrak{N} \models \mathbf{ZFL}^-$ contains $\vec{\boldsymbol\pi}$, each name $\mathbf{c}$ in φ belongs to $\mathfrak{N}$ and is $\vec{\boldsymbol\pi}$-complete, and $\boldsymbol p$ $\mathbf{MT}(\vec{\boldsymbol\pi})$-forces $\varphi[\underline{G}]$ over $\mathfrak{N}$, then there exists $\vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol\pi} \subset_{\mathfrak{N}} \vec{\boldsymbol\varphi}$ and $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\varphi}}\; \varphi$.*

Proof. (i) By definition there is an ordinal $\vartheta < \mathtt{dom}\,\vec{\boldsymbol\pi}$, a multitree $\boldsymbol p_0 \in \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$, and a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ containing $\vec{\boldsymbol\pi}\restriction\vartheta$ and such that $\boldsymbol p \leqslant \boldsymbol p_0$, every name $\mathbf{c}$ in φ belongs to $\mathfrak{M}$ and is $\vec{\boldsymbol\pi}\restriction\vartheta$-complete, $\vec{\boldsymbol\pi}\restriction\vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol\pi}$, and $\boldsymbol p_0$ $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. We can w.l.o.g. assume that $\mathfrak{M} \subseteq \mathfrak{N}$. (Otherwise $\mathfrak{N} \subseteq \mathfrak{M}$, and we replace $\mathfrak{N}$ by $\mathfrak{M}$.)

Now suppose that $G \subseteq \mathbf{MT}(\vec{\boldsymbol\varphi})$ is a set $\mathbf{MT}(\vec{\boldsymbol\varphi})$-generic over $\mathfrak{N}$ and $\boldsymbol p \in G$ — then $\boldsymbol p_0 \in G$, too. We have to prove that $\varphi[G]$ is true in $\mathfrak{N}[G]$.

We claim that the set $G' = G \cap \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$ is $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-generic over $\mathfrak{M}$. Indeed, let a set $\boldsymbol D \in \mathfrak{M}$, $\boldsymbol D \subseteq \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$, be open dense in $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$. Then, as $\vec{\boldsymbol\pi}\restriction\vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol\varphi}$, $\boldsymbol D$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol\varphi})$ by 14.4(iii)(b), and hence $G \cap \boldsymbol D \ne \varnothing$ by the choice of G. It follows that $G' \cap \boldsymbol D \ne \varnothing$.

Now if $\mathbf{c}$ is a name in φ then $\mathbf{c} \in \mathfrak{M}$ and $\mathbf{c}$ is $\vec{\boldsymbol\pi}\restriction\vartheta$-complete. It follows by the above that $\mathbf{c}[G'] \in 2^\omega$ is defined. Therefore $\mathbf{c}[G] = \mathbf{c}[G']$, because $G' \subseteq G$. Thus $\varphi[G]$ coincides with $\varphi[G']$. Note also that $\boldsymbol p_0 \in G'$. We conclude that $\varphi[G']$ holds in $\mathfrak{M}[G']$ as $\boldsymbol p_0$ forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. The same formula $\varphi[\underline{G}]$ is holds $\mathfrak{N}[G]$ by the Mostowski absoluteness.

(ii) Lemma 14.4(ii) yields $\vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol\pi} \subset_{\mathfrak{N}} \vec{\boldsymbol\varphi}$. □

Lemma 22.7 (in $\mathbf{L}$). *Let $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\pi})$, φ be a formula in $\mathscr{L}(\Sigma\Pi)^1_1$ or $\mathscr{L}\Sigma^1_n$, $n \ge 2$. Then $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi$ and $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi^-$ cannot hold together.*

Proof. Let $\varphi \in \mathscr{L}(\Sigma\Pi)^1_1$. If both $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi$ and $\boldsymbol p \;\mathtt{forc}_{\vec{\boldsymbol\pi}}\; \varphi^-$ then, by Lemma 22.6, $\boldsymbol p$ $\mathbf{MT}(\vec{\boldsymbol\pi})$-forces both $\varphi[\underline{G}]$ and $\varphi^-[\underline{G}]$ over a large enough CTM $\mathfrak{M}$, a contradiction. If $\varphi \in \mathscr{L}\Sigma^1_n$ then the result follows by 3°. □

23 Tail invariance

Invariance theorems are very typical for all kinds of forcing. We prove two major invariance theorems on the auxiliary forcing. The first one shows tail invariance, while the other one (Section 24) explores the permutational invariance.

If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{MF}}$ and $\gamma < \lambda = \mathtt{dom}\,\vec{\pi}$ then let the γ-*tail* $\vec{\pi}\restriction_{\geq\gamma}$ be the restriction $\vec{\pi}\restriction[\gamma,\lambda)$ to the ordinal semiinterval $[\gamma,\lambda) = \{\alpha : \gamma \leq \alpha < \lambda\}$. Then the set $\mathbf{MT}(\vec{\pi}\restriction_{\geq\gamma}) = \bigcup^{\mathtt{cw}}_{\gamma\leq\alpha<\lambda} \vec{\pi}(\alpha)$ is open dense in $\mathbf{MT}(\vec{\pi})$ by Lemma 14.4(iii)(a). Therefore it can be expected that if $\vec{\varphi}$ is another sequence of the same length $\lambda = \mathtt{dom}\,\vec{\varphi}$, and $\vec{\varphi}\restriction_{\geq\gamma} = \vec{\pi}\restriction_{\geq\gamma}$, then the relation $\mathtt{forc}_{\vec{\pi}}$ coincides with $\mathtt{forc}_{\vec{\varphi}}$. And indeed this turns out to be the case (almost).

Theorem 23.1. *Assume that* $\vec{\pi}, \vec{\varphi}$ *are sequences in* $\overrightarrow{\mathbf{MF}}$, $\gamma < \lambda = \mathtt{dom}\,\vec{\pi} = \mathtt{dom}\,\vec{\varphi}$, $\vec{\varphi}\restriction_{\geq\gamma} = \vec{\pi}\restriction_{\geq\gamma}$, $\boldsymbol{p} \in \mathbf{MT}$, $n \geq 2$, *and* φ *is a formula in* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$. *Then* $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi$ *iff* $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varphi}}\ \varphi$.

Proof. **Part 1:** the $\mathscr{L}\Pi^1_2$ case. Let $\psi(x)$ be a $\mathscr{L}\Sigma^1_1$ formula. Suppose that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varphi}}\ \forall x\, \psi(x)$ **fails**, so there is $\vec{\varphi}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\varphi}')$ such that $\vec{\varphi} \subseteq \vec{\varphi}'$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\varphi}'}\ \exists x\, \psi^-(x)$. We can assume that $\boldsymbol{q} \in \mathbf{MT}(\vec{\varphi}'\restriction_{\geq\gamma})$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{q}\ \mathtt{forc}_{\vec{\varphi}'}\ \psi^-(\mathbf{c})$.

Let $\lambda' = \mathtt{dom}\,\vec{\varphi}'$. Define a sequence $\vec{\pi}'$ so that $\mathtt{dom}\,\vec{\pi}' = \lambda' = \mathtt{dom}\,\vec{\varphi}'$, $\vec{\pi} \subseteq \vec{\pi}'$, and $\vec{\pi}'\restriction_{\geq\lambda} = \vec{\varphi}'\restriction_{\geq\lambda}$. Then $\vec{\pi}'\restriction_{\geq\gamma} = \vec{\varphi}'\restriction_{\geq\gamma}$, hence $\boldsymbol{q} \in \mathbf{MT}(\vec{\pi}'\restriction_{\geq\gamma}) \subseteq \mathbf{MT}(\vec{\pi}')$.

Consider any CTM $\mathfrak{N} \models \mathbf{ZFL}^-$ containing ψ, $\mathbf{c}$, $\vec{\pi}'$, $\vec{\varphi}'$. Then $\boldsymbol{q}$ $\mathbf{MT}(\vec{\varphi}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$ by Lemma 22.6. However the forcing notions $\mathbf{MT}(\vec{\pi}')$, $\mathbf{MT}(\vec{\varphi}')$ contain one and the same dense set $\mathbf{MT}(\vec{\pi}'\restriction_{\geq\gamma}) = \mathbf{MT}(\vec{\varphi}'\restriction_{\geq\gamma})$. Therefore $\boldsymbol{q}$ also $\mathbf{MT}(\vec{\pi}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$. Then by definition $\boldsymbol{q}\ \mathtt{forc}_{\vec{\pi}'}\ \psi^-(\mathbf{c})$ and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\pi}'}\ \exists x\, \psi^-(x)$, hence $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \forall x\, \psi(x)$ **fails**, as required.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $n \geq 2$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$. Assume that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \exists x\, \varphi(x)$. By definition (see 2° in Section 22), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi(\mathbf{c})$. Then we have $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varphi}}\ \varphi(\mathbf{c})$ by the inductive hypothesis, thus $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varphi}}\ \exists x\, \psi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geq 3$. Let φ be a $\mathscr{L}\Pi^1_n$ formula, and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varphi}}\ \varphi$ **fails**. Then by 3° of Section 22, there is a sequence $\vec{\varphi}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\varphi}')$ such that $\vec{\varphi} \subseteq \vec{\varphi}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\varphi}'}\ \varphi^-$. By Lemma 14.4(iii)(a), there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\varphi}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\ \mathtt{forc}_{\vec{\varphi}'}\ \varphi^-$. Define a sequence $\vec{\pi}' \in \overrightarrow{\mathbf{MF}}$ by $\mathtt{dom}\,\vec{\pi}' = \lambda' = \mathtt{dom}\,\vec{\varphi}'$, $\vec{\pi} \subseteq \vec{\pi}'$, and $\vec{\pi}'\restriction_{\geq\lambda} = \vec{\varphi}'\restriction_{\geq\lambda}$. Then $\boldsymbol{r} \in \mathbf{MT}(\vec{\pi}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}$, and also $\boldsymbol{r}\ \mathtt{forc}_{\vec{\pi}'}\ \varphi^-$ by the inductive hypothesis. We conclude that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi$ **fails** as well. □

24 Permutations

Still **arguing in L**, we let PERM be the set of all bijections $\boldsymbol{h} : \omega_1 \xrightarrow{\text{onto}} \omega_1$, such that $\boldsymbol{h} = \boldsymbol{h}^{-1}$ and the *non-identity domain* $\mathbf{NID}(\boldsymbol{h}) = \{\xi : \boldsymbol{h}(\xi) \neq \xi\}$ is at most countable. Elements of PERM will be called *permutations*.

Let $\boldsymbol{h} \in \mathrm{PERM}$. We extend the action of $\boldsymbol{h}$ as follows.

- if $\boldsymbol{p}$ is a multitree then $\boldsymbol{hp}$ is a multitree, $|\boldsymbol{hp}| = \boldsymbol{h}``\boldsymbol{p} = \{\boldsymbol{h}(\xi) : \xi \in |\boldsymbol{p}|\}$, and $(\boldsymbol{hp})(\boldsymbol{h}(\xi)) = \boldsymbol{p}(\xi)$ whenever $\xi \in |\boldsymbol{p}|$, in other words, $\boldsymbol{hp}$ coincides with the superposition $\boldsymbol{p} \circ (\boldsymbol{h}^{-1})$;
- if $\boldsymbol{\pi} \in \mathbf{MT}$ is a multiforcing then $\boldsymbol{h}\cdot\boldsymbol{\pi} = \boldsymbol{\pi} \circ (\boldsymbol{h}^{-1})$ is a multiforcing, $|\boldsymbol{h}\cdot\boldsymbol{\pi}| = \boldsymbol{h}``\boldsymbol{\pi}$ and $(\boldsymbol{h}\cdot\boldsymbol{\pi})(\boldsymbol{h}(\xi)) = \boldsymbol{\pi}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}|$;
- if $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then put $\boldsymbol{h}\mathbf{c} = \{\langle \boldsymbol{hp}, n, i\rangle : \langle \boldsymbol{p}, n, i\rangle \in \mathbf{c}\}$, thus easily $\boldsymbol{h}\mathbf{c}$ is a real name as well;
- if $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}$, then $\boldsymbol{h}\vec{\boldsymbol{\pi}} = \langle \boldsymbol{h}\cdot\boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$, still a sequence in $\overrightarrow{\mathbf{MF}}$;
- if $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is a $\mathscr{L}$-formula (with all names explicitly indicated), then $\boldsymbol{h}\varphi$ is $\varphi(\boldsymbol{h}\mathbf{c}_1, \dots, \boldsymbol{h}\mathbf{c}_n)$.

Many notions and relations defined above are clearly PERM-invariant, *e.g.*, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ iff $\boldsymbol{hp} \in \mathbf{MT}(\boldsymbol{h}\cdot\boldsymbol{\pi})$, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$ iff $\boldsymbol{h}\cdot\boldsymbol{\pi} \sqsubset \boldsymbol{h}\cdot\boldsymbol{\varrho}$, *et cetera*. The invariance also takes place with respect to the relation $\mathtt{forc}$.

Theorem 24.1. *Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{h} \in \mathrm{PERM}$, $n \geq 2$, *and* φ *belongs to* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$. *Then* $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ *iff* $(\boldsymbol{hp})\ \mathtt{forc}_{\boldsymbol{h}\vec{\boldsymbol{\pi}}}\ (\boldsymbol{h}\varphi)$.

Proof. Let $\vec{\boldsymbol{\varrho}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$, $\boldsymbol{q} = \boldsymbol{hp}$.

Part 1: the $\mathscr{L}\Pi^1_2$ case. Assume that $\varphi(x)$ is a $\mathscr{L}\Sigma^1_1$ formula, $\psi(x) := \boldsymbol{h}\varphi(x)$, and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}}\ \forall x\, \psi(x)$ **fails**. Then by definition (Section 22) there is a sequence $\vec{\boldsymbol{\varrho}}' \in \overrightarrow{\mathbf{MF}}$, a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varrho}}')$, and a small real name $\mathbf{d}$, such that $\vec{\boldsymbol{\varrho}} \subset \vec{\boldsymbol{\varrho}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}'}\ \psi^-(\mathbf{d})$. The sequence $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varrho}}'$ then satisfies $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varrho}}$, the multitree $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ belongs to $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\mathbf{c} = \boldsymbol{h}^{-1}\mathbf{d}$ is a small real name. However we cannot now claim that $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-(\mathbf{c})$, since the existence of $\mathfrak{M}$, ϑ as in 1° in Section 22 is not necessarily preserved by the action of $\boldsymbol{h}^{-1}$ or $\boldsymbol{h}$.

To circumwent this difficulty, let $\mathfrak{M} \models \mathbf{ZFL}^-$ be a CTM containing $\vec{\boldsymbol{\pi}}'$, $\vec{\boldsymbol{\varrho}}'$, $\boldsymbol{h}$, $\mathbf{c}$, $\mathbf{d}$ and (all names in) φ, ψ. Then $\boldsymbol{q}'$ $\mathbf{MT}(\vec{\boldsymbol{\varrho}}')$-forces $\psi^-(\mathbf{d})[\underline{G}]$ over $\mathfrak{M}$ by Lemma 22.6(i). Then $\boldsymbol{p}'$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\varphi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{M}$, by the standard theorems of forcing. Lemma22.6(ii) yields a sequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ with $\vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\tau}}$, such that $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol{\tau}}}\ \varphi^-(\mathbf{c})$, hence $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol{\tau}}}\ \exists x\, \varphi^-(x)$ by 2°. However $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\tau}}$ and $\boldsymbol{p}' \leqslant \boldsymbol{p}$, therefore, $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \forall x\, \varphi(x)$ **fails** by 3°, as required.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $n \geq 2$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$ and $\psi(x) := \boldsymbol{h}\varphi(x)$. Assume that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \exists x\, \varphi(x)$. By definition (2° in Section 22), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi(\mathbf{c})$. Then we have $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}}\ \psi(\mathbf{d})$ by inductive assumption, where $\mathbf{d} = \boldsymbol{h}\mathbf{c}$ is a small real name itself. Thus $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}}\ \exists x\, \psi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geq 3$. Let φ be a formula in $\mathscr{L}\Pi^1_n$, and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}}\ \psi$ **fails**, where $\boldsymbol{q} = \boldsymbol{hp}$, $\vec{\boldsymbol{\varrho}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$, and ψ is $\boldsymbol{h}\varphi$, as above. By 3°, there is a sequence $\vec{\boldsymbol{\varrho}}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varrho}}')$ such that $\vec{\boldsymbol{\varrho}} \subseteq \vec{\boldsymbol{\varrho}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \mathtt{forc}_{\vec{\boldsymbol{\varrho}}'}\ \psi^-$. Now let $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ and $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varrho}}'$, so that $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$. We have $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-$ by inductive assumption. We conclude that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ **fails**, as required. □

25 Forcing inside the key sequence

The following Theorem 25.3 will show that the forcing relation $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$, considered with countable initial segments $\vec{\boldsymbol{\pi}} = \vec{\Pi} \restriction \alpha$ of the key sequence $\vec{\Pi}$, coincides with the true $\mathbb{P}$-forcing relation up to level $\mathfrak{n}-1$.

We argue in L. Recall that the key sequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$, satisfying (i), (ii), (iii), (iv) of Theorem 15.3, was introduced by 15.4, and $\mathbb{P} = \mathbf{MT}(\vec{\Pi})$ is our forcing notion. In addition, $\mathfrak{n} \geq 4$ by 22.1.

Definition 25.1. We write $\boldsymbol{p}\ \mathtt{forc}_\alpha\ \varphi$ instead of $\boldsymbol{p}\ \mathtt{forc}_{\vec{\Pi}\restriction\alpha}\ \varphi$, for the sake of brevity. Let $\boldsymbol{p}\ \mathtt{forc}\ \varphi$ mean: $\boldsymbol{p}\ \mathtt{forc}_\alpha\ \varphi$ for some $\alpha < \omega_1$. □

Lemma 25.2 (in **L**). *Assume that* $\boldsymbol{p} \in \mathbb{P}$, $\alpha < \omega_1$, *and* $\boldsymbol{p}\ \mathtt{forc}_\alpha\ \varphi$. *Then:*

(i) *if* $\alpha \leq \beta < \omega_1$, $\boldsymbol{q} \in \mathbb{P}_{<\beta} = \mathbf{MT}(\vec{\Pi}\restriction\beta)$, *and* $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}\ \mathtt{forc}_\beta\ \varphi$;

(ii) *if* $\boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}\ \mathtt{forc}_\beta\ \varphi$ *for some* β; $\alpha \leq \beta < \omega_1$;

(iii) *if* $\boldsymbol{q} \in \mathbb{P}$ *and* $\boldsymbol{q}\ \mathtt{forc}\ \varphi^-$ *then* $\boldsymbol{p}, \boldsymbol{q}$ *are* SAD;

(iv) *therefore, 1st, if* $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, *and* $\boldsymbol{p}\ \mathtt{forc}\ \varphi$ *then* $\boldsymbol{q}\ \mathtt{forc}\ \varphi$, *and 2nd,* $\boldsymbol{p}\ \mathtt{forc}\ \varphi$, $\boldsymbol{p}\ \mathtt{forc}\ \varphi^-$ *cannot hold together.*

Proof. To prove (i) apply Lemma 22.3. To prove (ii) pick β such that $\alpha < \beta < \omega_1$ and $\boldsymbol{q} \in \mathbf{MT}(\vec{\Pi}\restriction\beta)$, and apply (i). To prove (iii) note that $\boldsymbol{p}, \boldsymbol{q}$ are incompatible in $\mathbb{P}$, as otherwise (i) leads to contradiction, but the incompatibility in $\mathbb{P}$ implies being SAD by Corollary 4.3. □

Theorem 25.3. *If* φ *is a closed* $\mathscr{L}$*-formula in* $\mathscr{L}(\Sigma\Pi)^1_1 \cup \mathscr{L}\Sigma^1_2 \cup \mathscr{L}\Pi^1_2 \cup \ldots \cup \mathscr{L}\Sigma^1_{\mathfrak{n}-2} \cup \mathscr{L}\Pi^1_{\mathfrak{n}-2} \cup \mathscr{L}\Sigma^1_{\mathfrak{n}-1}$ *and* $\boldsymbol{p} \in \mathbb{P}$, *then* $\boldsymbol{p}$ $\mathbb{P}$*-forces* $\varphi[\underline{G}]$ *over* **L** *in the usual sense, if and only if* $\boldsymbol{p}\ \mathtt{forc}\ \varphi$.

Proof. Let $\Vdash$ denote the usual $\mathbb{P}$-forcing relation over **L**.

Part 1: φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$. If $\boldsymbol{p}\ \mathtt{forc}\ \varphi$ then $\boldsymbol{p}\ \mathtt{forc}_{\vec{\Pi}\restriction\gamma}\ \varphi$ for some $\gamma < \omega_1$, and then $\boldsymbol{p} \Vdash \varphi[\underline{G}]$ by Lemma 22.6 with $\vec{\boldsymbol{\varphi}} = \vec{\Pi}$ and $\mathfrak{N} = \mathbf{L}$.

Suppose now that $\boldsymbol{p} \Vdash \varphi[\underline{G}]$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\Pi}\restriction\gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\Pi}\restriction\gamma_0)$. (Recall Definition 14.3 on models $\mathfrak{L}(x) \models \mathbf{ZFL}^-$.) The set U of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\gamma_0 < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ and there is an ordinal ϑ, $\gamma_0 < \vartheta < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$, such that $\vec{\boldsymbol{\pi}}\restriction\vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$, is dense in $\overrightarrow{\mathbf{MF}}$ by Lemma 14.4(ii), and is $\mathbf{\Delta}_1(\mathrm{HC})$. Therefore by Corollary 15.5 there is an ordinal $\gamma < \omega_1$ such that $\vec{\boldsymbol{\pi}} = \vec{\Pi}\restriction\gamma \in U$. Let this be witnessed by an ordinal ϑ, $\gamma_0 < \vartheta < \gamma = \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ and $\vec{\boldsymbol{\pi}}\restriction\vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$. We claim that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$ in the usual sense — then by definition $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, and we are done.

To prove the claim, assume otherwise. Then there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\Pi}\restriction\vartheta)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-forces $\neg\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$. Then by definition (1° in Section 22) $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \neg\varphi$ holds, hence $\boldsymbol{q}\ \mathtt{forc}\ \neg\varphi$, and then $\boldsymbol{q} \Vdash \neg\varphi[\underline{G}]$ (see above), with a contradiction to $\boldsymbol{p} \Vdash \varphi[\underline{G}]$.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$ $(n \geq 1)$. Consider a $\mathscr{L}\Pi^1_n$ formula $\varphi(x)$. Assume $\boldsymbol{p}$ $\texttt{forc}$ $\exists x\,\varphi(x)$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}$ $\texttt{forc}$ $\varphi(\mathbf{c})$. By inductive hypothesis, $\boldsymbol{p} \Vdash \varphi(c)[\underline{G}]$, that is, $\boldsymbol{p} \Vdash \exists x\,\varphi(x)[\underline{G}]$. Conversely, assume that $\boldsymbol{p} \Vdash \exists x\,\varphi(x)[\underline{G}]$. As $\mathbb{P}$ is CCC, there is a small real name $\mathbf{c}$ (in $\mathbf{L}$) such that $\boldsymbol{p} \Vdash \varphi(\mathbf{c})[\underline{G}]$. We have $\boldsymbol{p}$ $\texttt{forc}$ $\varphi(\mathbf{c})$ by the inductive hypothesis, hence $\boldsymbol{p}$ $\texttt{forc}$ $\exists x\,\varphi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$ $(2 \leq n \leq \mathfrak{n}-2)$. Assume that φ is a closed $\mathscr{L}\Sigma^1_n$ formula, and $\boldsymbol{p}$ $\texttt{forc}$ φ^-. By Lemma 25.2(iv), there is no multitree $\boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, with $\boldsymbol{q}$ $\texttt{forc}$ φ. This implies $\boldsymbol{p} \Vdash \varphi^-$ by the inductive hypothesis.

Conversely, let $\boldsymbol{p} \Vdash \varphi^-$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\sqcap}\restriction\gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\sqcap}\restriction\gamma_0)$. Consider the set U of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\texttt{dom}\,\vec{\boldsymbol{\pi}} > \gamma_0$ and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}$ $\texttt{forc}_{\vec{\boldsymbol{\pi}}}$ φ. Then U belongs to $\mathbf{\Sigma}_{n-1}(\text{HC})$ (φ, $\boldsymbol{p}_0$ as parameters) by Lemma 22.5, hence to $\mathbf{\Sigma}_{\mathfrak{n}-3}(\text{HC})$, where $\mathfrak{n} \geq 4$ by 22.1. Therefore by 15.4 (and (iii) of Theorem 15.3) there is $\gamma < \omega_1$ such that $\vec{\sqcap}\restriction\gamma$ blocks U.

Case 1: $\vec{\sqcap}\restriction\gamma \in U$. Let this be witnessed by a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, so that in particular $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\gamma > \gamma_0$. Thus $\boldsymbol{q} \in \mathbf{MT}(\vec{\sqcap}\restriction\gamma)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}$ $\texttt{forc}_{\vec{\sqcap}\restriction\gamma}$ φ, that is, $\boldsymbol{q} \Vdash \varphi[\underline{G}]$ by the inductive hypothesis, contrary to the choice of $\boldsymbol{p}$. Therefore Case 1 cannot happen, and we have:

Case 2: no sequence in U extends $\vec{\sqcap}\restriction\gamma$. We can assume that $\gamma > \gamma_0$. (If not, replace γ by $\gamma_0 + 1$.) We claim that $\boldsymbol{p}$ $\texttt{forc}_\gamma$ φ^-. Indeed otherwise by 3° there is a sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, such that $\vec{\sqcap}\restriction\gamma \subseteq \vec{\boldsymbol{\pi}}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}$ $\texttt{forc}_{\vec{\boldsymbol{\pi}}}$ φ. But then $\vec{\boldsymbol{\pi}}$ belongs to U. On the other hand, $\vec{\sqcap}\restriction\gamma \subseteq \vec{\boldsymbol{\pi}}$, contrary to the Case 2 assumption. Thus indeed $\boldsymbol{p}$ $\texttt{forc}$ φ^-, as required. □

26 Elementary equivalence theorem

Theorem 20.2. Suppose the contrary. Then there is a $\Pi^1_{\mathfrak{n}-2}$ formula $\varphi(r, x)$ with $r \in 2^\omega \cap \mathbf{L}[G\restriction X]$ as the only parameter, and a real $x_0 \in 2^\omega \cap \mathbf{L}[G]$ such that $\varphi(r, x_0)$ is true in $\mathbf{L}[G]$ but there is no $x \in 2^\omega \cap \mathbf{L}[G\restriction X]$ such that $\varphi(r, x)$ is true in $\mathbf{L}[G]$. By a version of Lemma 12.1, we have $r = \mathbf{c}_0[G]$, where $\mathbf{c}_0 \subseteq \mathbf{MT}(\sqcap\restriction X) \times \omega \times 2$ is a small $(\sqcap\restriction X)$-complete real name. (See Section 17 on notation.) And there is a small $\mathbb{P}$-complete real name $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$ such that $x_0 = \mathbf{c}[\underline{G}]$.

By Theorem 25.3, there is a multitree $\boldsymbol{p}_0 \in G$ such that

(1) $\boldsymbol{p}_0$ $\mathbb{P}$-forces '$\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}[\underline{G}]) \wedge \neg\,\exists\, x \in \mathbf{L}[\underline{G}\restriction X]\,\varphi(\mathbf{c}_0[\underline{G}], x)$' over $\mathbf{L}$;

(2) $\boldsymbol{p}_0$ $\texttt{forc}$ $\varphi(\mathbf{c}_0, \mathbf{c})$, that is, $\boldsymbol{p}_0$ $\texttt{forc}_{\vec{\sqcap}\restriction\gamma_0}$ $\varphi(\mathbf{c}_0, \mathbf{c})$, where $\gamma_0 < \omega_1$ — and we can assume that $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\sqcap}\restriction\gamma_0)$ as well.

As $\mathbf{c}, \mathbf{c}_0$ are small names, there is an ordinal $\delta < \omega_1$ satisfying

(3) $|\mathbf{c}_0| \subseteq \delta \cap X$, $|\mathbf{c}| \subseteq \delta$, and $|\boldsymbol{p}_0| \subseteq \delta$.

As $|\vec{\sqcap}| = \omega_1$ by Corollary 16.2, we can enlarge γ_0, if necessary, to make sure that

(4) $\delta \subseteq |\vec{\sqcap}\restriction\gamma_0|$, that is, if $\eta < \delta$ then $\eta \in |\sqcap_{\alpha'}|$ for some $\alpha' = \alpha'(\eta) < \gamma_0$.

We start from here towards a contradiction. Let $D = \delta \smallsetminus X$.

Let U consist of all sequences $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, such that $\vec{\mathbb{\Pi}}\restriction\gamma_0 \subset \vec{\boldsymbol\pi}$, and hence $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol\pi})$ by (2), and there is $\zeta < \mathtt{dom}\,\vec{\boldsymbol\pi}$ and $\boldsymbol{h} \in$ PERM such that

(A) $\mathbf{NID}(\boldsymbol{h}) \cap (\delta \cap X) = \varnothing$, and $\boldsymbol{h}$ maps D onto a set $R \subseteq X \smallsetminus \delta$;

(B) $\gamma_0 \le \zeta < \mathtt{dom}\,\vec{\boldsymbol\pi}$ and $(\boldsymbol{h}\vec{\boldsymbol\pi})\restriction_{\ge\zeta} = \vec{\boldsymbol\pi}\restriction_{\ge\zeta}$, that is, $\boldsymbol{h}(\vec{\boldsymbol\pi}(\alpha)) = \vec{\boldsymbol\pi}(\alpha)$ whenever $\zeta \le \alpha < \mathtt{dom}\,\vec{\boldsymbol\pi}$.

It holds by routine estimations that U is a $\boldsymbol\Sigma_1(\mathrm{HC})$ set (with $\vec{\mathbb{\Pi}}\restriction\gamma_0$, δ as parameters), hence a $\boldsymbol\Sigma_{\mathfrak{n}-3}(\mathrm{HC})$ set because $\mathfrak{n} \ge 4$ by 22.1. Therefore by 15.4 there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{\Pi}}\restriction\gamma$ blocks U.

Case 1: $\vec{\mathbb{\Pi}}\restriction\gamma \in U$, so that (A), (B) hold for $\vec{\boldsymbol\pi} = \vec{\mathbb{\Pi}}\restriction\gamma$, via some $\zeta \in [\gamma_0, \gamma)$ and $\boldsymbol{h} \in$ PERM. In particular, by (B), $\boldsymbol{h}(\mathbb{\Pi}_\alpha) = \mathbb{\Pi}_\alpha$ whenever $\zeta \le \alpha < \gamma$. By Lemma 22.3 and (2), we have $\boldsymbol{p}_0 \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c})$. Let $\mathbf{c}' = \boldsymbol{h}\mathbf{c}$, $\boldsymbol{p}_0' = \boldsymbol{h}\boldsymbol{p}_0$. Note that $\boldsymbol{h}\mathbf{c}_0 = \mathbf{c}_0$ since $|\mathbf{c}_0| \cap \mathbf{NID}(\boldsymbol{h}) = \varnothing$ by (A). Now Theorem 24.1 implies $\boldsymbol{p}_0' \;\mathtt{forc}_{\boldsymbol{h}\cdot(\vec{\mathbb{\Pi}}\restriction\gamma)}\; \varphi(\mathbf{c}_0, \mathbf{c}')$. Thus $\boldsymbol{p}_0' \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c}')$ holds by Theorem 23.1 and (B). But the common domain $|\boldsymbol{p}_0| \cap |\boldsymbol{p}_0'|$ does not intersect $\mathbf{NID}(\boldsymbol{h})$ by (A) since $|\boldsymbol{p}_0| \subseteq \delta$. It follows that $\boldsymbol{p}_0, \boldsymbol{p}_0'$ are compatible, basically $\boldsymbol{p} = \boldsymbol{p}_0 \cup \boldsymbol{p}_0' \in \mathbf{MT}$ (not necessarily $\in \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)$) and $\boldsymbol{p} \preccurlyeq \boldsymbol{p}_0'$, hence still $\boldsymbol{p} \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c}')$.

Unfortunately Theorem 25.3 is not applicable immediately to conclude that $\boldsymbol{p}$ $\mathbb{P}$-forces $\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}'[\underline{G}])$ over $\mathbf{L}$, simply because $\boldsymbol{p}$ may not belong to $\mathbb{P}$. We need an additional argument. Recall that $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma_0)$, hence $\boldsymbol{p}_0' \in \mathbf{MT}(\boldsymbol{h}\cdot(\vec{\mathbb{\Pi}}\restriction\gamma_0))$. As $\zeta > \gamma_0$, there is a multitree $\boldsymbol{q}_0 \in \mathbf{MT}(\boldsymbol{h}\cdot\mathbb{\Pi}_\zeta)$ satisfying $|\boldsymbol{q}_0| = |\boldsymbol{p}_0'|$ and $\boldsymbol{q}_0 \preccurlyeq \boldsymbol{p}_0'$. Then still $\boldsymbol{q}_0 \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c}')$ (because $\boldsymbol{p}_0' \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c}')$), and $\boldsymbol{q}_0 \in \mathbf{MT}(\mathbb{\Pi}_\zeta)$ since $\boldsymbol{h}\cdot\mathbb{\Pi}_\zeta = \mathbb{\Pi}_\zeta$. Thus $\boldsymbol{q}_0 \in \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)$. Moreover $\boldsymbol{q}_0$ is compatible with $\boldsymbol{p}_0$ in $\mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)$ because $|\boldsymbol{q}_0| = |\boldsymbol{p}_0'|$ and $\boldsymbol{q}_0 \preccurlyeq \boldsymbol{p}_0'$, and $\boldsymbol{p}_0'$ coinsides with $\boldsymbol{p}_0$ on the common domain $|\boldsymbol{p}_0| \cap |\boldsymbol{p}_0'| = \delta \cap X$. Thus there exists $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)$ with $\boldsymbol{q} \preccurlyeq \boldsymbol{p}_0$, $\boldsymbol{q} \preccurlyeq \boldsymbol{q}_0$. Then $\boldsymbol{q} \;\mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\; \varphi(\mathbf{c}_0, \mathbf{c}')$ holds, and we conclude that

(5) $\boldsymbol{q}$ $\mathbb{P}$-forces $\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}'[\underline{G}])$ over $\mathbf{L}$

by Theorem 25.3. However $|\mathbf{c}'| \subseteq (\delta \cap X) \cup R \subseteq X$ by construction, and hence $\mathbf{c}'[\underline{G}] \in \mathbf{L}[\underline{G}\restriction X]$ is forced. Thus $\boldsymbol{q}$ $\mathbb{P}$-forces $\exists x \in \mathbf{L}[\underline{G}\restriction X]\, \varphi(\mathbf{c}_0[\underline{G}], x)$ over $\mathbf{L}$ by (5), contrary to (1). The contradiction closes Case 1.

Case 2: no sequence in U extends $\vec{\mathbb{\Pi}}\restriction\gamma$. We can assume that $\gamma > \gamma_0$. (Otherwise replace γ by $\gamma_0 + 1$.) Pick any set $R \subseteq X \smallsetminus \delta$ satisfying $\mathtt{card}\, R = \mathtt{card}\, D$. However $D \subseteq \delta$, hence $D \cap R = \varnothing$, so there is a permutation $\boldsymbol{h} \in$ PERM, $\boldsymbol{h} : D \xrightarrow{\text{onto}} R$, satisfying $\mathbf{NID}(\boldsymbol{h}) = D \cup R$, hence (A).

Pick any ordinal λ, $\gamma < \lambda < \omega_1$. Our plan is now to somewhat modify $\vec{\mathbb{\Pi}}\restriction\lambda$ in order to fulfill (B) as well, with $\zeta = \gamma$. The modification will replace the R-part of $\vec{\mathbb{\Pi}}\restriction\lambda$ above γ by the $\boldsymbol{h}$-copy of its D-part. To render this in detail, recall that $\vec{\mathbb{\Pi}}\restriction\lambda = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\lambda}$, where each $\mathbb{\Pi}_\alpha$ is a small special multiforcing, whose domain $d_\alpha = |\mathbb{\Pi}_\alpha| \subseteq \omega_1$ is countable. If $\alpha < \gamma$ then put $\boldsymbol\pi_\alpha = \mathbb{\Pi}_\alpha$. Suppose that $\gamma \le \alpha < \lambda$. Then $D \subseteq |\mathbb{\Pi}_\alpha|$ by (4). On the base of $\mathbb{\Pi}_\alpha$, define a modified multiforcing $\boldsymbol\pi_\alpha$ such that

(a) $|\boldsymbol{\pi}_\alpha| = d_\alpha \cup R$ — note that $D \subseteq d_\alpha = |\mathbb{\Pi}_\alpha| \subseteq |\boldsymbol{\pi}_\alpha|$ in this case because $D \subseteq \delta \subseteq |\vec{\mathbb{\Pi}}\restriction\gamma|$ by (4) (as $\gamma_0 \le \gamma$);

(b) if $\xi \in d_\alpha \smallsetminus R$ then $\boldsymbol{\pi}_\alpha(\xi) = \mathbb{\Pi}_\alpha(\xi)$,

(c) if $\xi \in D$, so $\boldsymbol{h}(\xi) = \eta \in R$, then $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\xi)$.

We assert that $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{MF}}$, that is, if $\alpha < \beta < \lambda$ then $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_\beta$. This amounts to the following: if $\eta \in |\boldsymbol{\pi}_\alpha|$ then $\boldsymbol{\pi}_\alpha(\eta) \sqsubset \boldsymbol{\pi}_\beta(\eta)$.

If $\eta \notin R$ then $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\eta)$ by construction. It remains to check that $\boldsymbol{\pi}_\alpha(\eta) \sqsubset \boldsymbol{\pi}_\beta(\eta)$ whenever $\alpha < \beta < \lambda$, $\eta = \boldsymbol{h}(\xi) \in R \cap |\boldsymbol{\pi}_\alpha|$, and $\xi \in D$. If now $\alpha < \gamma$ then $R \cap |\boldsymbol{\pi}_\alpha| = \varnothing$ by the choice of R, so it remains to consider the case when $\gamma \le \alpha$. Then $\xi, \eta \in |\boldsymbol{\pi}_\alpha|$ by construction, and we have $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\xi)$ and $\boldsymbol{\pi}_\beta(\eta) = \mathbb{\Pi}_\beta(\xi)$. Therefore $\boldsymbol{\pi}_\alpha(\xi) \sqsubset \boldsymbol{\pi}_\beta(\xi)$, and we are done.

We claim that the sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ satisfies $\vec{\mathbb{\Pi}}\restriction\gamma \subseteq \vec{\boldsymbol{\pi}}$ and (A), (B). Indeed $\vec{\mathbb{\Pi}}\restriction\gamma \subseteq \vec{\boldsymbol{\pi}}$ as $\gamma \ge \gamma_0$. (A) hold by construction. We claim that (B) is satisfied with $\zeta = \gamma$, that is, if $\gamma \le \alpha < \lambda$ then $\boldsymbol{h}\cdot\boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\alpha$. Indeed $D \cup R \subseteq |\boldsymbol{\pi}_\alpha|$ by (a), and hence $\boldsymbol{h}\cdot\boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\alpha$ holds by (b), (c).

Thus $\vec{\boldsymbol{\pi}} \in U$ and $\vec{\mathbb{\Pi}}\restriction\gamma \subset \vec{\boldsymbol{\pi}}$. But this contradicts to the Case 2 assumption.

To conclude, either case leads to a contradiction. □ (Theorem 20.2) □

□ (Theorem 1.2, see the end of Section 20)

27 Remarks and problems

One may ask what happens with the separation theorem at other projective levels $m \ne \mathbb{n}$ in the model of Section 19. As for the above levels, it happens that, in the model $\mathbf{L}[G\restriction\Delta_H]$ of Theorem 20.1, there is a "good" $\varDelta^1_{\mathbb{n}+1}$ wellordering of the reals, of length ω_1. (The gaps in Δ_H do not allow the wellorder construction of Corollary 18.3 does not go through at level $\mathbb{n}$!) It follows by standard arguments that the separation theorem holds for $\mathbf{\Pi}^1_m$ and fails for $\mathbf{\Sigma}^1_m$, for all $m > \mathbb{n}$, in the model $\mathbf{L}[G\restriction\Delta_H]$. As for the levels $3 \le m < \mathbb{n}$, we conjecture that separation holds for $\mathbf{\Pi}^1_m$ and fails for $\mathbf{\Sigma}^1_m$ in $\mathbf{L}[G\restriction\Delta_H]$, but this problem is open.

Let $\mathbb{P}_{\mathbb{n}}$ be the forcing notion $\mathbb{P}$ defined in Section 16 for a given $\mathbb{n} \ge 3$. Using a certain amalgamation of all $\mathbb{P}_{\mathbb{n}}$, $\mathbb{n} \ge 3$, defined by a rather sophisticated product-like construction, first applied in [18, Part 1] and [40], a generic extension of $\mathbf{L}$ can be defined, in which the separation theorem **fails** simultaneously for all classes $\mathbf{\Sigma}^1_{\mathbb{n}}$, $\mathbf{\Pi}^1_{\mathbb{n}}$, $\mathbb{n} \ge 3$.

Finally, it is an interesting and perhaps very difficult problem to define a generic extension of $\mathbf{L}$ in which the separation theorem **holds** for a given class $\mathbf{\Sigma}^1_{\mathbb{n}}$, $\mathbb{n} \ge 3$, beginning with say $\mathbf{\Sigma}^1_3$. This problem has been open since early years of forcing, see [16, Problem 3029]. In this regard, we can mention a recent preprint by Hoffelner [41] with interesting results.

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