

## Cubic hypersurfaces with an odd number of singular points

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Let us discuss the computational complexity of smoothness recognition for complex projective cubic hypersurfaces. In particular, it is interesting to consider hypersurfaces with only finitely many singular points. The well-known set partition problem is  $NP$ -complete. Indirectly, its complexity is confirmed by S. Margulies *et al* (2015). Initially we have proved that this problem can be reduced to recognition of singular points on a cubic hypersurface constructed in polynomial time. Applying the probabilistic method proposed by L. Valiant and V. Vazirani (1986) one can prove that searching for the unique singular point is hard too. On the other hand, smoothness is equivalent to inconsistency of the system of algebraic equations and can be tested in exponential time (A. L. Chistov 2012).

Firstly let us consider the projective cubic hypersurface given by the equation  $x_0^2(\alpha_1x_1 + \dots + \alpha_nx_n) = g(x_1, \dots, x_n)$ , where  $g$  is a cubic form in  $n$  variables. This form generalizes Weierstrass normal form for plane curves. It is important that this form is easy to find two hyperplanes containing all singular points, if any. At each singular point the left side of the equation vanishes. If it vanishes identically, then the hypersurface is a cone. Contrariwise, each singular point belongs to the union of two hyperplanes given by the equations  $\alpha_1x_1 + \dots + \alpha_nx_n = 0$  and  $x_0 = 0$ , respectively.

The cubic form is invariant under the reflection  $x_0 \mapsto -x_0$ . Suppose the number of singular points is odd. If the point  $P$  with homogeneous coordinates  $(1 : 0 : \dots : 0)$  is not a singular point on the hypersurface, then at least one of these points belongs to the hyperplane given by  $x_0 = 0$ . And the number of such singular points is odd too. Moreover, the hyperplane  $x_0 = 0$  cannot be tangent to the cubic hypersurface at a smooth point. So we either have a singular point  $P$  or can restrict searching by the hyperplane section. For an even number of singular points, we cannot distinguish smooth and singular cubic hypersurfaces. For example, the plane curve given by the equation  $x_0^2x_1 = x_1x_2^2 - x_1^3$  has exactly two singular points with homogeneous coordinates  $(\pm 1 : 0 : 1)$ . But these points do not remain fixed under the reflection  $x_0 \mapsto -x_0$ .

Secondly it is proved that a generic cubic form over the field of complex numbers can be transformed into the form without monomials of exactly two variables by means of a non-degenerate linear transformation of coordinates. This form generalizes Hesse normal form for plane curves. The normal form

for cubic surfaces has been discovered by A. Emch (1931 and 1939).

If the coefficients of the monomials in only one variable are equal to one, and the remaining coefficients are owned small enough polydisc near zero, the transformation can be approximated by means of an iterative algorithm. Under these restrictions the same result holds over the reals. The result generalizes the Levy–Desplanques theorem on strictly diagonally dominant matrices.

P. Gordan and M. Nöther (1876) constructed the first cubic hypersurface with vanishing hessian but not a cone. The examples of cubic hypersurfaces with vanishing hessian have been revisited by R. Gondim and F. Russo (2015). All such hypersurfaces are singular. We have a simple but interesting result. No cubic form in  $n + 1$  variables with vanishing hessian is equivalent to the form  $x_0^3 + \dots + x_n^3 + g(x_0, \dots, x_n)$ , where  $g$  is a multilinear form. Moreover, for each  $n$  the set of reducible forms of this type is finite.

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