Some applications of finite-support products of Jensen's minimal $\mathbf{\Delta}^1_3$ forcing

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Jensen 1970 defined a forcing $\boldsymbol{J} \in \mathrm{L}$ such that any $\boldsymbol{J}$-generic real a:

- does not belong to L;
- is the only $\boldsymbol{J}$-generic real in $\mathrm{L}[a]$,
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It is the most elementary example of a Goedel-nonconstructible definable real !

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3. a definable Groszek-Laver pair of Vitali classes;
4. a $\varPi^1_2$ set $P \subseteq \mathbb{R} \times \mathbb{R}$, such that:
   1) P is non-uniformizable by ROD sets, and
   2) each cross-section $P_x = \{y : \langle x, y\rangle \in P\}$ is a Vitali class;

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5. in a choiceless model: a countable sequence of Vitali classes whose union is uncountable.

Section 1.
Jensen's basic result

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Jensen's forcing $\boldsymbol{J}$ consists of **perfect sets** $X \subseteq \mathbb{R}$, a subset of the Sacks forcing

Section 2.
Countable product of Jensen's forcing

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It is true in any $\boldsymbol{J}^\omega$-generic extension of L that

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Here Π^1_2 **is the best possible** since any non-empty Σ^1_2 set of reals surely contains a definable element.

Section 3.
Variation: Vitali-invariant forcing

Definable Vitali class w/o definable elements TOC Back

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Compare the forcing notions J and K TOC Back

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2. $\mathbf{K}$ consists of Vitali-large trees.

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2. K consists of Vitali-large trees.

3. K is **Vitali-invariant**, that is, is invariant under rational shifts.

Thus there are two methods of getting a countable Π^1_2 set $X \subseteq \mathbb{R}$ containing no definable elements:

(1) $X = \{x_n : n < \omega\}$, where x_n are independently $\boldsymbol{J}$-generic reals added by the finite-support product $\boldsymbol{J}^\omega$ of Jensen's forcing $\boldsymbol{J}$;

(2) X is the Vitali class of a $\boldsymbol{K}$-generic real x.

Section 4.
Groszek – Laver pairs of Vitali classes

A *Groszek – Laver pair* is any pair of sets $X \neq Y \subseteq \mathbb{R}$ inseparable by an OD (ordinal-definable) set, that is, if $S \subseteq \mathbb{R}$ is OD then

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Example 1. If $\langle x, y \rangle$ is a **Cohen** $\times$ **Cohen** generic pair of reals over L then their L-degrees $X = [x]_{\mathrm{L}} \cap \mathbb{R}$ and $Y = [y]_{\mathrm{L}} \cap \mathbb{R}$ form a Groszek – Laver pair in $\mathrm{L}[x, y]$.

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The sets X, Y is this example are obviously uncountable.
Is there an OD Groszek – Laver pair of **countable** sets in $\mathbb{R}$?

Theorem 3 (Golshani + K + Lyubetsky, MLQ 2016)

There is a special Vitali-connected version of the forcing product $\boldsymbol{K} \times \boldsymbol{K}$ *in* L*, which adds a pair of reals* x, y *such that*

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The Vitali-connected product consists of all pairs $\langle X, Y\rangle$ of sets $X, Y \in \boldsymbol{K}$ such that $X + \mathbb{Q} = Y + \mathbb{Q}$.

An old idea of Harrington – Marker – Shelah, 1990.

Section 5.
Infinite products of large trees

Countable non-uniformization

TOC Back

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The forcing is essentially a finite-support product $\prod_{\alpha<\omega_1} \boldsymbol{K}_\alpha$, where each $\boldsymbol{K}_\alpha$ is a clone of the forcing notion $\boldsymbol{K}$.

Theorem 4 (K and Lyubetsky)

It is consistent with **ZF** *(no axiom of choice!) that there is a countable sequence* $\{X_n\}_{n<\omega}$ *of Vitali classes* $X_n \subseteq \mathbb{R}$*, such the union* $X = \bigcup_n X_n$ *is not countable.*

Section 6.
Final remarks

Theorem (still work in progress, with Ali Enayat)

There is a generic model in which

- *every analytically definable non-empty set of reals contains an analytically definable element*;
- *there is no projective wellordering of the reals.*

Problem (countable definable sets not of reals)

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Is it true in the Solovay model that every countable definable non-empty set X **of any kind** contains a definable element ?

Yes if X is a **set of reals**.

The most elementary open case: X is a ctble **set of sets of reals**.

The speaker thanks **everybody** for patience

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Definition

A set $X \subseteq \mathbb{R}$ is Vitali-large if the Vitali equivalence restricted to X has no Borel transversal.

A **transversal** for Vitali restricted to X is a set $Y \subseteq X$ which meets any Vitali class in X in exactly one point.

Example

The whole set $\mathbb{R}$ is Vitali-large.

$\boldsymbol{J}$ is defined in L in the form $\boldsymbol{J} = \bigcup_{\alpha<\omega_1} \boldsymbol{J}_\alpha$, where each $\boldsymbol{J}_\alpha$ is a ctble set of trees, pre-dense in $\boldsymbol{J}$, and generic, in some sense, over a certain ctble model M_α containing the union $\boldsymbol{J}_{<\alpha} = \bigcup_{\gamma<\alpha} \boldsymbol{J}_\gamma$ of previous steps. This implies:

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(∗) If $n < \omega$, a set $D \in M_\alpha$, $D \subseteq (\boldsymbol{J}_{<\alpha})^n$ is dense in $(\boldsymbol{J}_{<\alpha})^n$, and trees $T_1, \dots, T_n \in \boldsymbol{J}_\alpha$ are 2wise different, then there is a finite subset $D' \subseteq D$ such that $T_1 \times \dots \times T_n \subseteq \bigcup D'$.

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The standard corollary is that *in a $\boldsymbol{J}^n$-generic extension, the only $\boldsymbol{J}$-generic reals are the obvious ones.*

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The **main technical problem** for Theorem 1 was to adapt $(*)$ to the case of the infinite product $(\boldsymbol{J}_{<\alpha})^\omega$.

Forcing is a tool to define set theoretic universes.

A universe is a structure of sets which satisfies the axioms of ZFC, Zermelo – Fraenkel axiomatic system, with the axiom of choice.

The minimal universe is the universe L of Goedel-constructible sets.

More universes can be obtained as extensions of L by forcing, called **forcing extensions**.

The properties of a forcing extension depend on the choice of a partially-ordered set, called **a forcing notion**, or just **a forcing**.

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2. **DEP** holds for all definable sets X in the Goedel constructible universe L — because of the canonical OD wellordering of L.
3. **DEP** fails in many set theoretic universes for instance for the set $X = \mathbb{R} \smallsetminus \mathrm{L}$ of all non-constructible reals.

The problem was mentioned in *Sinq Lettres*, an exchange between BAIRE, BOREL, HADAMARD, LEBESGUE in 1905.

DEP: every definable set of reals $\varnothing \neq X \subseteq \mathbb{R}$ contains an ordinal-definable (OD, in brief) element.

1. **DEP** holds for Σ^1_2 sets of reals: Luzin – Novikov choice method, aka uniformization / scale / absoluteness theorems.
2. **DEP** holds for all definable sets X in the Goedel constructible universe L — because of the canonical OD wellordering of L.
3. **DEP** fails in many set theoretic universes for instance for the set $X = \mathbb{R} \smallsetminus \mathrm{L}$ of all non-constructible reals.

However until recently all known counterexamples to **DEP** have been rather large sets, definitely uncountable. Therefore one can ask: is there a countable counterexample to **DEP** in some universe? **