

Jacques Bair, HEC–ULG, University of Liege, 4000 Belgium. email:  
j.bair@ulg.ac.be

Piotr Błaszczyk, Institute of Mathematics, Pedagogical University of  
Cracow, Poland. email: pb@up.krakow.pl

Peter Heinig, Germany. email: peter.c.heinig@gmail.com

Vladimir Kanovei, IPPI RAS, Moscow, Russia. email:  
kanovei@googlegmail.com

Mikhail G. Katz, Department of Mathematics, Bar Ilan University, Ramat  
Gan 52900 Israel. email: katzmik@macs.biu.ac.il

Thomas McGaffey, Rice University, US. email:  
thomasmcgaffey@sbcglobal.net

## CAUCHY’S WORK ON INTEGRAL GEOMETRY, CENTERS OF CURVATURE, AND OTHER APPLICATIONS OF INFINITESIMALS

### Abstract

Like his colleagues de Prony, Petit, and Poisson at the *Ecole Polytechnique*, Cauchy used infinitesimals in the Leibniz–Euler tradition both in his research and teaching. Cauchy applied infinitesimals in an 1826 work in differential geometry where infinitesimals are used neither as variable quantities nor as sequences but rather as *numbers*. He also applied infinitesimals in an 1832 article on integral geometry, similarly as numbers. We explore these and other applications of Cauchy’s infinitesimals as used in his textbooks and research articles.

An attentive reading of Cauchy’s work challenges received views on Cauchy’s role in the history of analysis and geometry. We demonstrate

---

Mathematical Reviews subject classification: Primary: 01A55; Secondary: 26E35, 01A85, 03A05, 53C65

Key words: Cauchy–Crofton formula, center of curvature, continuity, infinitesimals, integral geometry, *limite*, *standard part*, de Prony, Poisson

Received by the editors Nov 30, 2019

Communicated by: Marton Elekes

the viability of Cauchy’s infinitesimal techniques in fields as diverse as geometric probability, differential geometry, elasticity, Dirac delta functions, continuity and convergence.

## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Dirac delta, summation of series</b>	<b>3</b>
<b>3</b>	<b>Differentials, infinitesimals, and derivatives</b>	<b>4</b>
<b>4</b>	<b>Integral geometry</b>	<b>5</b>
4.1	Decomposition into infinitesimal segments . . . . .	5
4.2	Analysis of Cauchy’s argument . . . . .	6
<b>5</b>	<b>Centers of curvature, elasticity</b>	<b>7</b>
5.1	<i>Angle de contingence</i> . . . . .	7
5.2	Center of curvature and radius of curvature . . . . .	8
5.3	$\varepsilon$ , <i>nombre infiniment petit</i> . . . . .	8
5.4	Second characterisation of radius and center of curvature . . . . .	9
5.5	Formula for $\rho$ in terms of infinitesimal displacements . . . . .	10
5.6	Elasticity . . . . .	11
<b>6</b>	<b>Continuity in Cauchy</b>	<b>11</b>
6.1	Ambiguity in definition of continuity . . . . .	11
6.2	From variables to infinitesimals . . . . .	12
6.3	The 1853 definition . . . . .	13
6.4	Contingency and determinacy . . . . .	14
<b>7</b>	<b>Reception of Cauchy’s ideas among his colleagues</b>	<b>15</b>
<b>8</b>	<b>Modern infinitesimals in relation to Cauchy’s procedures</b>	<b>17</b>
<b>9</b>	<b>Conclusion</b>	<b>19</b>

## 1 Introduction

Cauchy was one of the founders of rigorous analysis. However, the meaning of *rigor* to Cauchy is subject to debate among scholars. Cauchy used the term *infiniment petit* (infinitely small) both as an adjective and as a noun, but the meaning of Cauchy's term is similarly subject to debate. While Judith Grabiner and some other historians feel that a Cauchyan infinitesimal is a sequence tending to zero (see e.g., [25], 1981), others argue that there is a difference between null sequences and infinitesimals in Cauchy; see e.g., Laugwitz ([32], 1987), Katz–Katz ([29], 2011), Borovik–Katz ([9], 2012), Smoryński ([38], 2012, pp. 361–373 and [39], 2017, pp. 56, 61), Bair et al. ([2], 2017 and [4], 2019).

Cauchy used infinitesimals in the Leibniz–Euler tradition both in his research and teaching, like his colleagues de Prony, Petit, and Poisson (see Section 7). In the present text we will examine several applications Cauchy makes of infinitesimals, and argue that he uses them as atomic entities (i.e., entities not analyzable into simpler constituents) rather than sequences. We explore Cauchy's use of infinitesimals in areas ranging from Dirac delta to integral geometry.

## 2 Dirac delta, summation of series

We consider Cauchy's treatment of (what will be called later) a Dirac delta function. Cauchy explicitly uses a unit-impulse, infinitely tall, infinitely narrow delta function, as an integral kernel. Thus, in 1827, Cauchy used infinitesimals in his definition of a Dirac delta function:

Moreover one finds, denoting by  $\alpha, \epsilon$  two infinitely small numbers,<sup>1</sup>

$$\frac{1}{2} \int_{a-\epsilon}^{a+\epsilon} F(\mu) \frac{\alpha d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a) \quad (2.1)$$

(Cauchy [14], 1827, p. 289; counter (2.1) added)

A formula equivalent to (2.1) was proposed by Dirac a century later.<sup>2</sup> The expression

$$\frac{\alpha}{\alpha^2 + (\mu - a)^2} \quad (2.2)$$

<sup>1</sup>As discussed in (Laugwitz [33], 1989), a further condition needs to be imposed on  $\alpha$  and  $\epsilon$  in modern mathematics to ensure the correctness of the formula.

<sup>2</sup>The key property of the Dirac delta “function”  $\delta(x)$  is exemplified by the defining formula  $\int_{-\infty}^{\infty} f(x)\delta(x) = f(0)$ , where  $f(x)$  is any continuous function of  $x$  (Dirac [20], 1930/1958, p. 59).

occurring in Cauchy’s formula is known as the *Cauchy distribution* in probability theory. Here Cauchy specifies a function which meets the criteria as set forth by Dirac a century later. Cauchy integrates the function  $F$  against the kernel (2.2) as in formula (2.1) so as to extract the value of  $F$  at the point  $a$ , exploiting the characteristic property of a delta function.

From a modern viewpoint, formula (2.1) holds up to an infinitesimal error. For obvious reasons, Cauchy was unfamiliar with modern set-theoretic foundational *ontology* of analysis (with or without infinitesimals), but his *procedures* find better proxies in modern infinitesimal frameworks than Weierstrassian ones.<sup>3</sup> From the modern viewpoint, the right hand side of (2.1), which does not contain infinitesimals, is the *standard part* (see Section 8) of the left hand side, which does contain infinitesimals. Thus, a Cauchy distribution with an infinitesimal scale parameter  $\alpha$  produces an entity with Dirac-delta behavior, exploited by Cauchy already in 1827; see Katz–Tall ([31], 2013) for details.

Similarly, in his article (Cauchy [18], 1853) on the convergence of series of functions, infinitesimals are handled as atomic inputs to functions. Here Cauchy studies the series

$$u_0 + \dots + u_n + \dots$$

Cauchy proceeds to choose “une valeur infiniment grande” (an infinitely large value) for the index  $n$  in [18, p. 456]. He then states his convergence theorem modulo a hypothesis that the sum  $u_n + u_{n+1} + \dots + u_{n'-1}$  should be

toujours infiniment petite pour des valeurs infiniment grandes des  
nombres entiers  $n$  et  $n' > n \dots$ <sup>4</sup> (Cauchy [18], 1853, p. 457;  
emphasis added).

Cauchy’s proof of the continuity of the sum exploits the condition that the sum  $u_n + u_{n+1} + \dots + u_{n'-1}$  should be infinitesimal for atomic infinitesimal inputs. Cauchy writes down such an input in the form  $x = \frac{1}{n}$ ; see e.g., [18, p. 457]. For further details on [18] see Section 6.3; see also Bascelli et al. ([6], 2018).

### 3 Differentials, infinitesimals, and derivatives

In his work in analysis, Cauchy carefully distinguishes between differentials  $ds, dt$  which to Cauchy are noninfinitesimal variables, on the one hand, and infinitesimal increments  $\Delta s, \Delta t$ , on the other:

<sup>3</sup>On the procedures/ontology distinction see Błaszczuk et al. ([8], 2017).

<sup>4</sup>Translation: “always infinitely small for infinitely large values of whole numbers  $n$  and  $n' > n$ .”

[S]oit  $s$  une variable distincte de la variable primitive  $t$ . En vertu des définitions adoptées, le rapport entre les différentielles  $ds$ ,  $dt$ , sera la limite du rapport entre les *accroissements infiniment petits*  $\Delta s$ ,  $\Delta t$ .<sup>5</sup> (Cauchy [16], 1844, p. 11; emphasis added)

Cauchy goes on to express such a relation by means of a formula in terms of the infinitesimals  $\Delta s$  and  $\Delta t$ :

On aura donc<sup>6</sup>

$$\frac{ds}{dt} = \lim. \frac{\Delta s}{\Delta t} \quad (3.1)$$

(ibid., equation (1); significantly, the period after *lim* as in “*lim.*” is in the original; counter (3.1) added)

In modern infinitesimal frameworks, the passage from the ratio of infinitesimals such as  $\frac{\Delta s}{\Delta t}$  to the value of the derivative is carried out by the *standard part function*; see equations (8.3) and (8.5) in Section 8. Paraphrasing Cauchy's definition of the derivative as in (3.1) in Archimedean terms would necessarily involve elements that are inexplicit in the original definition. Thus Cauchy's “*lim.*” finds a closer proxy in the notion of standard part, as in formula (8.7), than in any notion of limit in the context of an Archimedean continuum; see also Bascelli et al. ([5], 2014).

## 4 Integral geometry

An illuminating use of infinitesimals occurs in Cauchy's article in a field today called *integral geometry* (also known as *geometric probability*); see Hykšová et al. ([28], 2012, pp. 3–4) for a discussion.

### 4.1 Decomposition into infinitesimal segments

Cauchy proved a formula known today as the Cauchy–Crofton formula (or the Crofton formula; see e.g., Tabachnikov ([40], 2005, p. 37)) in his article ([17], 1850; originally presented as [15], 1832). Here Cauchy exploits a decomposition of a curve into infinitesimal length elements (respectively, of a surface into infinitesimal area elements) in an essential way in proving a formula for the

---

<sup>5</sup>Let  $s$  be a variable distinct from the primitive variable  $t$ . By virtue of the definitions chosen, the ratio between the differentials  $ds$ ,  $dt$  will be the limit of the ratio between the infinitely small increments  $\Delta s$ ,  $\Delta t$ .”

<sup>6</sup>Translation: “One will then have.”

length of a plane curve (respectively, area of a surface in 3-space). Thus, Cauchy proves the formula

$$S = \frac{1}{4} \int_{-\pi}^{\pi} A dp \quad (4.1)$$

for the length of a plane curve, in his Théorème I in [17, p. 167–168]. In formula (4.1),  $p$  is the polar angle (usually denoted  $\theta$  today), whereas  $A$  is the sum of the orthogonal projections of the length elements onto a rotating line with parameter  $p$ . Note that this is an exact formula (rather than an approximation), typical of modern *integral geometry*.

In his Théorème II, Cauchy goes on to prove a constructive version, or a discretisation, of his Théorème I. Here Cauchy replaces *integrating* with respect to the variable-line differential  $dp$ , by *averaging* over a system of  $n$  equally spaced lines (i.e., such that successive lines form equal angles). Cauchy then obtains the approximation

$$S = \frac{\pi}{2} M \quad (4.2)$$

where  $M$  is the average. Here the equality sign appears in [17, p. 169] as in our formula (4.2), and denotes approximation. Cauchy also provides an explicit error bound of

$$\frac{\pi M}{2n^2} \quad (4.3)$$

for the approximation, in [17, p. 169]. Cauchy first proves the result for a straight line segment, and then writes:

Le théorème II étant ainsi démontré pour le cas particulier où la quantité  $S$  se réduit à une longueur rectiligne  $s$ , il suffira, pour le démontrer dans le cas contraire, de décomposer  $S$  en *éléments infiniment petits*.<sup>7</sup> (Cauchy [17], 1850, p. 171; emphasis added)

Thus Cauchy obtains a sequence of error bounds of the form (4.3) that improve (become smaller) as  $n$  increases.

## 4.2 Analysis of Cauchy’s argument

Cauchy exploits two entities which need to be carefully distinguished to keep track of the argument:

---

<sup>7</sup>Translation: “Theorem II having been proved for the special case when the quantity  $S$  is a straight line segment  $s$ , it would be sufficient, to prove it in the contrary [i.e., the general] case, to decompose  $S$  into infinitely small elements.”

1. the curve itself, and
2. the circle (or in modern terminology, the Grassmannian) of directions parametrized by  $p$  (counterpart of the modern polar angle  $\theta$ ).

Note that Cauchy treats the curve and the Grassmannian differently. Namely, the curve is subdivided into infinitely many infinitesimal elements of length. Meanwhile, as far as the Grassmannian is concerned, Cauchy works with a finite  $n$ , chooses  $n$  directions that are equally spaced, and is interested in the asymptotic behavior of the *sequence* of error estimates (4.3) as  $n$  tends to infinity.

If an infinitesimal merely meant a variable quantity or sequence to Cauchy, then there shouldn't be any difference in Cauchy's treatment of the curve and the Grassmannian; both should be sequences. However, Cauchy does treat them differently:

- the curve is viewed as an aggregate of infinitely many infinitesimal elements;
- the circle of directions is decomposed into  $n$  segments, and the focus is on the asymptotic behavior of the error bound as a function of  $n$ .

The approach in the 1832/1850 paper on integral geometry indicates that Cauchy's infinitesimal (the element of length decomposing the curve) is not a variable quantity or a sequence, but rather an atomic entity, as discussed in Section 2.

## 5 Centers of curvature, elasticity

In studying the geometry of curves, Cauchy routinely exploits infinitesimals and related notions such as infinite proximity. We will analyze Cauchy's book *Leçons sur les applications du calcul infinitésimal à la géométrie* ([13], 1826).

### 5.1 *Angle de contingence*

Cauchy starts by defining the *angle de contingence*  $\pm\Delta\tau$  as the angle between the two tangent lines of an arc  $\pm\Delta s$  at its extremities. To follow the mathematics it is helpful to think of parameter  $\tau$  as the angle measured counter-clockwise between the positive direction of the  $x$ -axis and the tangent vector to the curve. He then considers the normals to the curve at the extremities of the arc starting at the point  $(x, y)$ .

## 5.2 Center of curvature and radius of curvature

Cauchy goes on to give *two* definitions of both the center of curvature and the radius of curvature. Thus, he writes:

la distance du point  $(x, y)$  au point de rencontre des deux normales est sensiblement équivalente au rayon d'un cercle qui aurait la même courbure que la courbe.<sup>8</sup> (Cauchy [13], 1826, p. 98)

Notice that Cauchy mentions two items:

- (Ca1) the intersection of the two normal lines produces a point which will generate the *center* of curvature, and
- (Ca2) the distance between  $(x, y)$  and the point defined in item (Ca1) which will generate the *radius* of curvature.

Thus the *radius* of curvature is naturally defined in terms of the *center* of curvature (namely, as the distance between the point  $(x, y)$  and the center of curvature).

## 5.3 $\varepsilon$ , *nombre infiniment petit*

Next, Cauchy chooses an infinitesimal number  $\varepsilon$  and exploits the law of sines to write down a relation that will give an expression for the radius of curvature:

[S]i l'on désigne par  $\varepsilon$  un *nombre* infiniment petit, on aura<sup>9</sup>

$$\frac{\sin\left(\frac{\pi}{2} \pm \varepsilon\right)}{r} = \frac{\sin(\pm\Delta\tau)}{\sqrt{\Delta x^2 + \Delta y^2}} \quad (5.1)$$

(Cauchy [13], 1826, p. 98; emphasis and counter “(5.1)” added).

Note that Cauchy describes his infinitesimal  $\varepsilon$  neither as a *sequence* nor as a *variable quantity* but rather as an infinitely small *number* (“nombre”). At the next stage, Cauchy passes to the limit to obtain:

On en conclura, en passant aux *limites*,

$$\frac{1}{\rho} = \pm \frac{d\tau}{\sqrt{dx^2 + dy^2}} \quad (5.2)$$

[13, p. 99] (emphasis and counter “(5.2)” added)

---

<sup>8</sup>Translation: “the distance from the point  $(x, y)$  to the intersection point of two normals is appreciably equivalent to the radius of a circle which would have the same curvature as the curve.”

<sup>9</sup>Translation: “If one denotes by  $\varepsilon$  an infinitely small number, one will obtain”

It is instructive to analyze what happens exactly in passing from formula (5.1) to formula (5.2). Here Cauchy replaces the infinitesimals  $\Delta x$ ,  $\Delta y$ , and  $\sin \Delta\tau$  by the corresponding differentials  $dx$ ,  $dy$ , and  $d\tau$ . The expression  $\sin(\frac{\pi}{2} \pm \varepsilon)$  is infinitely close to 1 whereas  $r$  is infinitely close to  $\rho$ , justifying the replacements in the left-hand side of Cauchy's equation.

As in Cauchy's definition of derivative analyzed in Section 3, Cauchy's *limite* here admits of a close proxy in the standard part function (see Section 8). Meanwhile, any attempt to interpret Cauchy's procedure in the context of an Archimedean continuum will have to deal with the nettlesome issue of the absence of Cauchyan infinitesimals like  $\Delta x$ ,  $\Delta y$ ,  $\Delta\tau$ , and  $\varepsilon$  in such a framework.

#### 5.4 Second characterisation of radius and center of curvature

Using his formula for  $\rho$ , Cauchy goes on to give his *second* characterisation of the radius of curvature and center of curvature:

Ce rayon, porté à partir du point  $(x, y)$  sur la normale qui renferme ce point, est ce qu'on nomme le *rayon de courbure* de la courbe proposée, relatif au point dont il s'agit, et l'on appelle *centre de courbure* celle des extrémités du rayon de courbure que l'on peut considérer comme le point de rencontre de deux normales infiniment voisines. (Cauchy [13], p. 99, emphasis in the original)

Here Cauchy notes that the center of curvature can be viewed as the other endpoint of the vector of length  $\rho$  starting at the point  $(x, y)$  and normal to the curve. Cauchy then reiterates the earlier definition of the center of curvature of a plane curve in terms of the intersection point of a pair of *infinitely close* normals, as Leibniz may have done; see Katz–Sherry ([30], 2013). Note that neither the center of curvature nor the radius of curvature are defined using a notion of limit in the context of an Archimedean continuum.

Cauchy's presentation of infinitesimal techniques here contains no trace of the variable quantities or sequences exploited in his textbooks in the definitions of infinitesimals. To adapt Cauchy's definition of center of curvature to modern custom, it is certainly possible to paraphrase it in the context of an Archimedean continuum. This can be done for example by taking a suitable sequence of (pairs of) normals and passing to a limit. However, such a paraphrase would not be faithful to Cauchy's own *procedure*. We will follow the continuation of Cauchy's analysis in Section 5.5.

### 5.5 Formula for $\rho$ in terms of infinitesimal displacements

At this stage, Cauchy's goal is to develop a formula for the radius of curvature  $\rho$  of the curve at a point  $(x, y)$ . Cauchy seeks to express  $\rho$  in terms of the distance between the pair of points obtained from  $(x, y)$  by means of equal infinitesimal displacements, one along the curve and the other along the tangent. To this end, Cauchy starts by choosing an infinitesimal  $i$ :

Ajoutons que, si, à partir du point  $(x, y)$ , on porte sur la courbe donnée et sur sa tangente, prolongées dans le même sens que l'arc  $s$ , des longueurs égales et infiniment petites représentées par  $i$ , on trouvera, pour les coordonnées de l'extrémité de la seconde longueur,

$$x + i \frac{dx}{ds}, \quad y + i \frac{dy}{ds}$$

et, pour les coordonnées de l'extrémité de la première,

$$x + i \frac{dx}{ds} + \frac{i^2}{2} \left( \frac{d^2x}{ds^2} + I \right), \quad y + i \frac{dy}{ds} + \frac{i^2}{2} \left( \frac{d^2y}{ds^2} + J \right),$$

$I, J$  désignant des quantités infiniment petites. (Cauchy [13], 1826, p. 105)

Cauchy refers to the endpoints of the infinitesimal segment of the curve itself as the *extremities*. He denotes the distance between the two extremities by  $\gamma$ .<sup>10</sup> Then straightforward calculations produce the following formula for  $\rho$ :

De cette dernière formule . . . on tire

$$\rho = \lim \frac{i^2}{2\gamma}. \tag{5.3}$$

(ibid.; counter “(5.3)” added)

Note that in Cauchy's formula that we labeled (5.3), the symbol “lim” is applied to a ratio of two infinitesimals. Therefore the use of lim here is analogous to the use of the standard part function as in (8.7). Cauchy employed a similar technique in the definition of derivative analyzed in Section 3, and in passing from formula (5.1) to formula (5.2) as analyzed in Section 5.3. Cauchy concludes as follows:

---

<sup>10</sup>Actually Cauchy uses a slightly different symbol not available in modern fonts.

En conséquence, *pour obtenir le rayon de courbure d'une courbe en un point donné, il suffit de porter sur cette courbe et sur sa tangente, prolongées dans le même sens, des longueurs égales et infiniment petites, et de diviser le carré de l'une d'elles par le double de la distance comprise entre les deux extrémités.* La limite du quotient est la valeur exacte du rayon de courbure. (op. cit., pp. 105–106; emphasis in the original)

Cauchy's *limite* here again plays the role of the standard part (8.3).

### 5.6 Elasticity

Another example of Cauchy's application of infinitesimals is his foundational article on elasticity ([12], 1823) where *un élément infiniment petit* is exploited on page 302. The article is mentioned by Freudenthal in ([23], 1971, p. 378); for details see Belhoste ([7], 1991, p. 94).

## 6 Continuity in Cauchy

In his *Cours d'Analyse* ([11], 1821), Cauchy comments as follows concerning the continuity of a few functions, including the function  $\frac{a}{x}$  in the range between 0 and infinity:

[E]ach of these functions is continuous in the neighborhood of any finite value given to the variable  $x$  if that finite value is contained . . . , for the function  $\frac{a}{x}$  . . . , between the limits  $x = 0$  and  $x = \infty$ . (Cauchy as translated by Bradley and Sandifer<sup>11</sup> [10], 2009, p. 27)

Here Cauchy asserts the continuity of the function  $\frac{a}{x}$  for finite values of  $x$  contained between the limits 0 and  $\infty$ .

### 6.1 Ambiguity in definition of continuity

From a modern viewpoint, the above definition is ambiguous. Interpreting it as continuity on  $(0, \infty)$  would rule out the possibility of interpreting Cauchy's continuity as *uniform continuity*, since  $\frac{1}{x}$  is not uniformly continuous on  $(0, \infty)$ . Interpreting it as saying that for every real  $x$  there is a neighborhood of  $x$  where the function is continuous, would not rule out a *uniform* interpretation; e.g., the function  $\frac{1}{x}$  is uniformly continuous in a suitable neighborhood of each nonzero real point.

<sup>11</sup>Reinhard Siegmund-Schultze writes: "By and large, with few exceptions to be noted below, the translation is fine" ([37], 2009).

Cauchy on occasion mentions that  $x$  is a *real* (as opposed to *complex*) variable. However, identifying Cauchy's notion with the modern notion of real number would clearly be problematic. Cauchy seems not to have elaborated a distinction between finer types of continuity we are familiar in modern mathematics, such as ordinary pointwise continuity *vs* uniform continuity.

## 6.2 From variables to infinitesimals

There was a transformation in Cauchy's thinking about continuity from an 1817 treatment in terms of *variables* to an 1821 treatment in terms of *infinitesimals*. In 1817, Cauchy defined continuity of  $f$  in terms of commutation of taking limit and evaluating the function:

*La limite d'une fonction continue de plusieurs variables est la même fonction de leur limite. Conséquence de ce Théorème relativement à la continuité des fonctions composées qui ne dépendent que d'une seule variable.*<sup>12</sup> (Cauchy as quoted by Guitard [26], 1986, p. 34; emphasis added; cf. Belhoste [7], 1991, p. 309)

Four years later in his *Cours d'Analyse*, Cauchy defined continuity as follows:

Among the objects related to the study of infinitely small quantities, we ought to include ideas about the continuity and the discontinuity of functions. In view of this, let us first consider functions of a single variable. Let  $f(x)$  be a function of the variable  $x$ , and suppose that for each value of  $x$  between two given limits, the function always takes a unique finite value. If, beginning with a value of  $x$  contained between these limits, we add to the variable  $x$  an *infinitely small increment*  $\alpha$ , the function itself is incremented by the difference  $f(x + \alpha) - f(x)$ , which depends both on the new variable  $\alpha$  and on the value of  $x$ . Given this, the function  $f(x)$  is a *continuous* function of  $x$  between the assigned limits if, for each value of  $x$  between these limits, the numerical value of the difference  $f(x + \alpha) - f(x)$  decreases indefinitely with the numerical value of  $\alpha$ . (Cauchy as translated in [10, p. 26]; emphasis on “continuous” in the original; emphasis on “infinitely small increment” added)

---

<sup>12</sup>Translation: “The limit of a continuous function of several variables is [equal to] the same function of their limit. Consequences of this Theorem with regard to the continuity of composite functions dependent on a single variable.” The reference for this particular lesson in the Archives of the Ecole Polytechnique is as follows: Le 4 Mars 1817, la leçon 20. Archives E. P., X II C7, Registre d'instruction 1816–1817.

This definition can be thought of as an intermediary one between the 1817 definition purely in terms of variables, and his second 1821 definition stated purely in terms of infinitesimals. Cauchy's second definition summarizes the definition just given as follows:

In other words, *the function  $f(x)$  is continuous with respect to  $x$  between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.* (ibid.; emphasis in the original)

Cauchy's second definition just quoted can be compared with one of the first modern ones; see formula (8.4). Cauchy concludes his discussion of continuity and discontinuity as follows:

We also say that the function  $f(x)$  is a continuous function of the variable  $x$  in a neighborhood of a particular value of the variable  $x$  whenever it is continuous between two limits of  $x$  that enclose that particular value, even if they are very close together. Finally, whenever the function  $f(x)$  ceases to be continuous in the neighborhood of a particular value of  $x$ , we say that it becomes discontinuous, and that there is *solution*<sup>13</sup> of continuity for this particular value. (ibid.; emphasis in the original)

Three salient points emerge from these passages:

1. Cauchy makes it clear at the outset that in his mind continuity is “among the objects related to the study of infinitely small quantities”;
2. the infinitely small  $\alpha$  is used conspicuously in the definitions;
3. conspicuously absent from Cauchy's multiple definitions of continuity is the notion of *limit*.<sup>14</sup>

Yushkevich observes in this connection that “the definition of continuity in Cauchy is as far from the *Epsilontik* as his definition of limit” ([42], 1986, p. 69).

### 6.3 The 1853 definition

Some three decades later in 1853, Cauchy defined continuity in a similar fashion:

<sup>13</sup>meaning *dissolution*, i.e., absence (of continuity).

<sup>14</sup>The word *limit* itself does occur in Cauchy's definitions here but in an entirely different sense of *endpoint of an interval* where inputs to the function originate (what we would call today the *domain* of the function); cf. Smoryński ([39], 2017, p. 52, note 48).

... une fonction  $u$  de la variable réelle  $x$  sera *continue*, entre deux limites données de  $x$ , si, cette fonction admettant pour chaque valeur intermédiaire de  $x$  une valeur unique et finie, un accroissement infiniment petit attribué à la variable produit toujours, entre les limites dont il s'agit, un accroissement infiniment petit de la fonction elle-même. (Cauchy [18], 1853; emphasis in the original)

Cauchy's 1853 definition echoes the 1821 definition given in Section 6.2, where Cauchy denoted his infinitely small  $\alpha$  and required the difference  $f(x+\alpha)-f(x)$  to be infinitesimal as a criterion for continuity of the function  $f$ .

Cauchy's 1821 example of the function  $\frac{1}{x}$  between 0 and infinity suggests that Cauchy's definition of continuity is, from a modern viewpoint, somewhat ambiguous, as discussed in Section 6.1. Resolving the ambiguity by attributing uniform continuity to Cauchy may not preserve such inherent ambiguity.

A possible interpretation of Cauchy's comments is available in the context of an infinitesimal-enriched continuum. Here one can interpret  $x$  as referring to an assignable value (i.e., what we refer to today as a *real* value), and  $\alpha$  an (inassignable) infinitesimal. Then  $\frac{1}{x}$  is continuous in a neighborhood of  $x$  in the sense that for each infinitesimal  $\alpha$  the difference  $f(x+\alpha)-f(x)$  is also infinitesimal.

#### 6.4 Contingency and determinacy

We wish to suggest, following Hacking ([27], 2014, pp. 72–75), the possibility of alternative courses for the development of analysis (a *Latin model* as opposed to a *butterfly model*).<sup>15</sup> From such a standpoint, the traditional assumption that the historical development inexorably led to modern classical analysis (as formalized by Weierstrass and others) remains merely a hypothesis. A reader of Dani–Papadopoulos [19] may be surprised to learn that

Cauchy gave a faultless definition of continuous function, using the notion of 'limit' for the first time. Following Cauchy's idea, Weierstrass popularized the  $\epsilon$ - $\delta$  argument in the 1870's. ([19], 2019, p.283)

Such views fit well with a deterministic *butterfly* model leading from Cauchy to Weierstrass. However, such views are not merely anachronistic but contrary to fact, as we saw in Sections 6.2 and 6.3. Cauchy did write: “Lorsque les valeurs numériques successives d'une même variable décroissent indéfiniment,

<sup>15</sup>Hacking contrasts a model of a deterministic biological development of animals like butterflies (the egg–larva–cocoon–butterfly sequence), as opposed to a model of a contingent historical evolution of languages like Latin.

de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme un *infinitement petit* ou une quantité *infinitement petite*. Une variable de cette espèce a zéro pour limite" [11, p. 4] (emphasis in the original). However, interpreting Cauchy's wording as an anticipation of the modern *Epsilontik* notion of limit would be anachronistic, since Cauchy's wording here echoes formulations provided by his teacher Lacroix, and even earlier formulations found in Leibniz, not to speak of the ancient method of exhaustion; see Bair et al. ([4], 2019) for details.

We argue, following Robinson ([35], 1966) and Laugwitz [32], 1987), that the procedures of Leibniz, Euler and Cauchy were closer to the procedures in Robinson's framework than the procedures in a Weierstrassian framework. On this view, interpretation of the work of Leibniz, Euler, and Cauchy in analysis is more successful in a modern infinitesimal framework than a modern Archimedean one; see e.g., Bair et al. ([1], 2018) on Leibniz and Bair et al. ([3], 2017) on Euler. For a survey of infinitesimal mathematics and its history see e.g., Robinson ([35], 1966, chapter 10, pp. 260–282).

## 7 Reception of Cauchy's ideas among his colleagues

For historians advocating an externalist approach to the history of mathematics, it is important to consider the reception of Cauchy's ideas among his contemporaries. Cauchy's contemporaries and colleagues at the *École Polytechnique* (Poisson and others; see below) had specific ideas about what *infinitely small* meant. One cannot provide a proper analysis of Cauchy's notion without taking into account the ideas on the subject among his contemporaries. There seems to be little reason to doubt that the notion of infinitely small in the minds of Poisson, de Prony, Petit, and others was solidly in the Leibniz–l'Hôpital–Bernoulli–Euler school. If so, the question arises how modern commentators could assume that Cauchy meant something else by *infinitely small* than was customary in his natural scientific milieu.

How was the notion perceived by Cauchy's contemporaries like Poisson as well as a majority of Cauchy's colleagues at the *École Polytechnique*? Their comments (see below) indicate that their work was a natural habitat for infinitesimals in the sense of the founders of the calculus. Thus, Cauchy's colleague Petit, a professor of physics, requested that

this material [on differential calculus] be presented without certain notions from algebra, which mainly had to do with series and which, he alleged, the students would never have occasion to use in the [engineering] services. Moreover, he insisted that the method

of *infinitesimals* be used. (Petit as translated by Belhoste in [7], 1991, p. 65; emphasis added)

In a similar vein, de Prony reported:

I will finish my observations on the course in pure analysis by manifesting the desire to see the use of the algorithm of imaginaries [i.e., complex numbers] reduced to what is strictly necessary. I have been astonished, for instance, to see the expression of the element of a curve, given in polar coordinates, derived [by Cauchy] from an analysis using this algorithm; it follows much more quickly and with greater ease from a consideration of *infinitesimals*. (de Prony as translated in [7, p. 83]; emphasis added)

Poisson and de Prony both championed the use of infinitesimals through their influence on the *Conseil de Perfectionnement* (CP) of the *École*, as noted by Gilain:

[O]n trouve dans le programme officiel, adopté par le CP, une modification significative : l'ajout dans les applications géométriques du calcul différentiel et intégral, et dans le programme de mécanique, de l'instruction d'utiliser les *infinitement petits*. Même si l'auteur de cette proposition n'est pas mentionné dans les Procès-verbaux, on peut penser qu'elle émane des examinateurs de mathématiques, *Poisson* et *de Prony*, qui animaient en général la commission programme du CP, et dont on connaît les convictions en faveur de la méthode des *infinitement petits*. (Gilain [24], 1989, §32; emphasis added)

Like Cauchy, his contemporaries de Prony, Petit, and Poisson saw infinitesimals as a natural tool both in teaching and in research, though they were critical of what they saw as excessive rigor in Cauchy's teaching.

Paying proper attention to the scientific context of the period goes hand-in-hand with looking at Cauchy's practices and procedures on his own terms, or as close as possible to his own terms, without necessarily committing oneself to an Archimedean interpretation thereof. Thus, Ferraro writes:

Cauchy uses infinitesimal neighborhoods of  $x$  in a decisive way . . . Infinitesimals are not thought as a mere *façon de parler*, but they are conceived as numbers, though a theory of infinitesimal numbers is lacking. (Ferraro [21], 2008, p. 354)

This comment by Ferraro is remarkable for two reasons:

1. it displays a clear grasp of the distinction between procedure and ontology (see Błaszczyk et al. ([8], 2017);
2. it is a striking admission concerning the *bona fide* nature of Cauchy's infinitesimals.

Ferraro's comment is influenced by Laugwitz's perceptive analysis of Cauchy's sum theorem in ([32], 1987), a paper cited several times on Ferraro's page 354.

## 8 Modern infinitesimals in relation to Cauchy's procedures

While set-theoretic justifications for a modern framework, Archimedean or otherwise, are obviously not to be found in Cauchy, Cauchy's *procedures* exploiting infinitesimals find closer proxies in Robinson's framework for analysis with infinitesimals than in a Weierstrassian framework. In this section we outline a construction of a hyperreal extension

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R}, \quad (8.1)$$

and point out specific similarities between procedures using the hyperreals, on the one hand, with Cauchy's procedures, on the other.

Let  $\mathbb{R}^{\mathbb{N}}$  denote the ring of sequences of real numbers, with arithmetic operations defined termwise. Then we have

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \text{MAX}$$

where MAX is the maximal ideal consisting of all "negligible" sequences  $(u_n)$ . Here a sequence is negligible if it vanishes for a set of indices of full measure  $\xi$ , namely,  $\xi(\{n \in \mathbb{N} : u_n = 0\}) = 1$ . Here

$$\xi : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}$$

is a finitely additive probability measure taking the value 1 on cofinite sets, where  $\mathcal{P}(\mathbb{N})$  is the set of subsets of  $\mathbb{N}$ . The subset  $\mathcal{F}_\xi \subseteq \mathcal{P}(\mathbb{N})$  consisting of sets of full measure  $\xi$  is called a free ultrafilter. These originate with Tarski ([41], 1930). The set-theoretic presentation of an infinitesimal-enriched continuum was therefore not available prior to that date.

The embedding (8.1) uses constant sequences. We can therefore define the subring

$${}^{\flat}\mathbb{R} \subseteq {}^*\mathbb{R} \quad (8.2)$$

to be the set of the finite elements of  ${}^*\mathbb{R}$ ; i.e., elements smaller in absolute value than some real number, relying on the embedding (8.1). The subring (8.2) admits a map  $\mathbf{st}$  to  $\mathbb{R}$ , known as *standard part*

$$\mathbf{st}: {}^b\mathbb{R} \rightarrow \mathbb{R}, \quad (8.3)$$

which rounds off each finite hyperreal number to its nearest real number. This enables one, for instance, to define continuity and the derivative as follows. Following Robinson, we say that a function

$f(x)$  is continuous in [an open interval]  $(a, b)$  if

$$f(x_0 + \eta) =_1 f(x_0) \quad (8.4)$$

for all *standard*  $x_0$  in the open interval and for all infinitesimal  $\eta$ . (Robinson [34], 1961, p. 436; emphasis in the original; counter (8.4) added)

Robinson's symbol " $=_1$ " denotes the relation of infinite proximity. Robinson's notation  $=_1$  in [34] for infinite proximity was replaced by  $\simeq$  in his books and by  $\approx$  in most modern sources in infinitesimal analysis.

We also define the derivative of  $t = f(s)$  as

$$f'(s) = \mathbf{st} \left( \frac{\Delta t}{\Delta s} \right) \quad (8.5)$$

where  $\Delta s \neq 0$ , or equivalently  $f'(s)$  is the standard real number such that

$$f'(s) \approx \frac{\Delta t}{\Delta s}. \quad (8.6)$$

Such a definition parallels Cauchy's definition (3.1) of derivative, more closely than any *Epsilontik* definition. Limit is defined in terms of standard part, e.g., by setting

$$\lim_{s \rightarrow 0} f(s) = \mathbf{st}(f(\epsilon)) \quad (8.7)$$

where  $\epsilon$  is a nonzero infinitesimal. This definition of limit via the standard part is analogous to Cauchy's *limite*, similarly defined in terms of infinitesimals, as analyzed in Section 3. For more details on Robinson's framework see e.g., Fletcher et al. ([22], 2017).

## 9 Conclusion

We have argued that Cauchy's work on integral geometry, centers of curvature, and other applications exploits infinitesimals as atomic entities not reducible to simpler ones (such as terms in a sequence).

The oft-repeated claim, as documented e.g., in Bair et al. ([2], 2017) and Bascelli et al. ([6], 2018), that “Cauchy's infinitesimal is a variable with limit 0” is a reductionist view of Cauchy's foundational stance, at odds with much compelling evidence in Cauchy's writings, as we argued in Sections 2 through 6. Cauchy's notion of infinitesimal was therefore close to that of his contemporary scientists including Poisson, as we saw in Section 7.

While Cauchy did give an occasional *Epsilontik* proof that today would be interpreted in the context of an Archimedean continuum, his techniques relying on infinitesimals find better proxies in a modern framework exploiting an infinitesimal-enriched continuum. Cauchy's infinitesimal techniques in fields as diverse as geometric probability, differential geometry, continuity and convergence are just as viable as his *Epsilontik* techniques.

Robinson first proposed an interpretation of Cauchy's *procedures* in the framework of a modern theory of infinitesimals in ([35], 1966, chapter 10). A *set-theoretic foundation* for infinitesimals could not have been provided by Cauchy for obvious reasons, but Cauchy's *procedures* find closer proxies in modern infinitesimal frameworks than in modern Archimedean ones.

## Acknowledgments

We thank Peter Fletcher, Elías Fuentes Guillén, Karel Hrbacek, Taras Kudryk, David Pierce, and David Sherry for helpful suggestions. V. Kanovei was partially supported by RFBR Grant 17-01-00705.

## References

- [1] J. Bair, P. Błaszczyk, R. Ely, P. Heinig, and M. Katz, *Leibniz's well-founded fictions and their interpretations*, Mat. Stud. **49**(2) (2018), 186–224. See <http://dx.doi.org/10.15330/ms.49.2.186-224> and <https://arxiv.org/abs/1812.00226>
- [2] J. Bair, P. Błaszczyk, R. Ely, V. Henry, V. Kanovei, K. Katz, M. Katz, T. Kudryk, S. Kutateladze, T. McGaffey, T. Mormann, D. Schaps, and D. Sherry, *Cauchy, infinitesimals and ghosts of departed quantifiers*, Mat. Stud. **47**(2) (2017), 115–144. See <https://arxiv.org/abs/1712.00226> and <http://matstud.org.ua/texts/2017/47-2/115-144.pdf>

- [3] J. Bair, P. Błaszczyk, R. Ely, V. Henry, V. Kanovei, K. Katz, M. Katz, S. Kutateladze, T. McGaffey, P. Reeder, D. Schaps, D. Sherry, and S. Shnider, *Interpreting the infinitesimal mathematics of Leibniz and Euler*, Journal for General Philosophy of Science **48**(2) (2017), 195–238. See <http://dx.doi.org/10.1007/s10838-016-9334-z> and <https://arxiv.org/abs/1605.00455>
- [4] J. Bair, P. Błaszczyk, P. Heinig, V. Kanovei, M. Katz, *19th century real analysis, forward and backward*, Antiquitates Mathematicae **13** (2019). See <https://arxiv.org/abs/1907.07451>
- [5] T. Bascelli, Bottazzi, E. Herzberg, F.; V. Kanovei, K. Katz, M. Katz, T. Nowik, D. Sherry, and S. Shnider, *Fermat, Leibniz, Euler, and the gang: The true history of the concepts of limit and shadow*, Notices of the American Mathematical Society **61**(8) (2014), 848–864. See <http://www.ams.org/notices/201408/rnoti-p848.pdf> and <https://arxiv.org/abs/1407.0233>
- [6] T. Bascelli, P. Błaszczyk, A. Borovik, V. Kanovei, K. Katz, M. Katz, S. Kutateladze, T. McGaffey, D. Schaps, and D. Sherry, *Cauchy's infinitesimals, his sum theorem, and foundational paradigms*, Foundations of Science **23**(2) (2018), 267–296. See <http://dx.doi.org/10.1007/s10699-017-9534-y> and <https://arxiv.org/abs/1704.07723>
- [7] B. Belhoste, *Augustin-Louis Cauchy. A biography*, Translated from the French and with a foreword by Frank Ragland. Springer-Verlag, New York, 1991.
- [8] P. Błaszczyk, V. Kanovei, K. Katz, M. Katz, S. Kutateladze, and D. Sherry, *Toward a history of mathematics focused on procedures*, Foundations of Science **22**(4) (2017), 763–783. See <http://dx.doi.org/10.1007/s10699-016-9498-3> and <https://arxiv.org/abs/1609.04531>
- [9] A. Borovik and M. Katz, *Who gave you the Cauchy-Weierstrass tale? The dual history of rigorous calculus*, Foundations of Science **17**(3) (2012), 245–276. See <http://dx.doi.org/10.1007/s10699-011-9235-x> and <https://arxiv.org/abs/1108.2885>
- [10] R. Bradley and C. Sandifer, *Cauchy's Cours d'analyse. An annotated translation*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer, New York, 2009.

- [11] A. L. Cauchy, *Cours d'Analyse de L'École Royale Polytechnique. Première Partie. Analyse algébrique*. Imprimerie Royale, Paris, 1821. Translation in Bradley–Sandifer ([10], 2009).
- [12] A. L. Cauchy, *Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques*, Oeuvres complètes Série 2, Tome 2, pp. 300–304. Reprinted from *Bulletin de la Société philomatique*, p. 9–13, 1823.
- [13] A. L. Cauchy, *Leçons sur les applications du calcul infinitésimal à la géométrie*. Paris, Imprimerie Royale, 1826. In *Oeuvres Complètes*, Série 2, Tome 5.
- [14] A. L. Cauchy (1815), *Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie* (composed in 1815; published in 1827, with additional Notes), Oeuvres Complètes, Série 1, Tome 1, 4–318.
- [15] A. L. Cauchy, *Mémoire sur la rectification des courbes et la quadrature des surfaces courbes*, Paris, 1832. Published as (Cauchy [17], 1850).
- [16] A. L. Cauchy, *Exercices d'analyse et de physique mathématique (vol. 3)*, Paris, Bachelier, 1844.
- [17] A. L. Cauchy, *Mémoire sur la rectification des courbes et la quadrature des surfaces courbes*, Mémoires de l'Académie des Sciences Paris **22** (1850), 3–15 (published version of [15]). Reprinted in *Oeuvres complètes 1/II*, Paris: Gauthier-Villars (1908), 167–177.
- [18] A. L. Cauchy, *Note sur les séries convergentes dont les divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données*, 1853, in *Oeuvres complètes*, Série 1, Tome 12, pp. 30–36. Gauthier–Villars, Paris, 1900.
- [19] S. Dani and A. Papadopoulos (Eds.), *Geometry in History*, Springer, 2019. See <http://dx.doi.org/10.1007/978-3-030-13609-3>
- [20] P. Dirac, *The Principles of Quantum Mechanics*, 4th edition, Oxford, Clarendon Press, 1958 (first published in 1930).
- [21] G. Ferraro, *The rise and development of the theory of series up to the early 1820s*, Sources and Studies in the History of Mathematics and Physical Sciences. Springer, New York, 2008.

- [22] P. Fletcher, K. Hrbacek, V. Kanovei, M. Katz, C. Lobry, and S. Sanders, *Approaches to analysis with infinitesimals following Robinson, Nelson, and others*, Real Analysis Exchange **42**(2) (2017), 193–252. See <https://arxiv.org/abs/1703.00425> and <http://msupress.org/journals/issue/?id=50-21D-61F>
- [23] H. Freudenthal, *Did Cauchy plagiarise Bolzano?*, Archive for History of Exact Sciences **7** (1971), 375–392.
- [24] C. Gilain, *Cauchy et le cours d'analyse de l'École polytechnique*, with an editorial preface by Emmanuel Grison, Bull. Soc. Amis Bibl. École Polytech., no. 5, 1989.
- [25] J. Grabiner, *The origins of Cauchy's rigorous calculus*, MIT Press, Cambridge, Mass.–London, 1981.
- [26] T. Guitard, *La querelle des infiniment petits à l'École Polytechnique au XIXe siècle*, Historia Sci. **30** (1986), 1–61.
- [27] T. Hacking, *Why is there philosophy of mathematics at all?*, Cambridge University Press, Cambridge, 2014.
- [28] M. Hykšová, A. Kalousová, and I. Saxl, *Early history of geometric probability and stereology*, Image Anal. Stereol. **31**(1) (2012), 1–16.
- [29] K. Katz and M. Katz, *Cauchy's continuum*, Perspectives on Science **19**(4) (2011), 426–452. See [http://dx.doi.org/10.1162/POSC\\_a\\_00047](http://dx.doi.org/10.1162/POSC_a_00047) and <https://arxiv.org/abs/1108.4201>
- [30] M. Katz and D. Sherry, *Leibniz's Infinitesimals: Their Fictionality, Their Modern Implementations, And Their Foes From Berkeley To Russell And Beyond*, Erkenntnis **78**(3) (2013), 571–625. See <http://dx.doi.org/10.1007/s10670-012-9370-y> and <https://arxiv.org/abs/1205.0174>
- [31] M. Katz and D. Tall, *A Cauchy-Dirac delta function*, Foundations of Science **18**(1) (2013), 107–123. See <http://dx.doi.org/10.1007/s10699-012-9289-4> and <https://arxiv.org/abs/1206.0119>
- [32] D. Laugwitz, *Infinitely small quantities in Cauchy's textbooks*, Historia Mathematica **14** (1987), 258–274.
- [33] D. Laugwitz, *Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820*, Archive for History of Exact Sciences **39**(3) (1989), 195–245.

- [34] A. Robinson, *Non-standard analysis*, Nederl. Akad. Wetensch. Proc. Ser. A, **64** = Indag. Math., **23** (1961), 432–440. Reprinted in *Selected Papers of Abraham Robinson* [36], pp. 3–11.
- [35] A. Robinson, *Non-standard analysis*, North-Holland Publishing, Amsterdam, 1966.
- [36] A. Robinson, *Selected papers of Abraham Robinson. Vol. II. Non-standard analysis and philosophy*. Edited and with introductions by W. A. J. Luxemburg and S. Körner. Yale University Press, New Haven, Conn., 1979.
- [37] R. Siegmund-Schultze, Review of Bradley–Sandifer [10] for Zentralblatt (2009). See <https://zbmath.org/?q=an%3A1189.26001>
- [38] C. Smoryński, *Adventures in formalism*, Texts in Mathematics, 2, College Publications, London, 2012.
- [39] C. Smoryński, *MVT: a most valuable theorem*. Springer, Cham, 2017.
- [40] S. Tabachnikov, *Geometry and billiards*, Student Mathematical Library, 30. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005.
- [41] A. Tarski, *Une contribution à la théorie de la mesure*, Fundamenta Mathematicae, **15** (1930), 42–50.
- [42] A. Yushkevich, *The development of the concept of the limit up to K. Weierstrass*, Istor.-Mat. Issled. **30** (1986), 11–81.